Studies in Radar Cross-Sections – I

Scattering by a Prolate Spheroid

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STUDIES IN RADAR CROSS-SECTIONS

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IX Electromagnetic Scattering by an Oblate Spheroid, by L. M. Rauch (October 1953).

X The Radar Cross-Section of a Sphere by M. E. Anderson, R. K. Ritt, and H. Weil (to be published).

XI The Numerical Determination of the Radar Cross-Section of a Prolate Spheroid by K. M. Siegel, B. H. Gere, I. Marx, and F. B. Sleator (to be published).

XII A classified summary report by K. M. Siegel, M. E. Anderson, R. R. Bonkowski, and W. C. Orthwein (to be published).

PREFACE

The problem of determining the scattering of an incident electromagnetic wave by a sphere was first solved in 1908 by Mie. Contributions to the solution of the problem of determining the scattering of an incident electromagnetic wave by an ellipsoidal body were made by Herzfeld in 1911, Gans in 1920, and Möglich in 1927 (Ref. 13). As Stratton (Ref. 16, p. 573) points out, however, none of this work on ellipsoids was carried through to a completely useful solution and the many practical applications of the theory show the need for obtaining a more useable solution. Such a solution, it is believed, is contained in the following pages for a plane wave striking nose-on a prolate spheroid. The problem is solved for both vector (electromagnetic) and scalar (sound) waves. It is believed that it would be a little more difficult to obtain a similar solution for the case of broadside incidence. Dr. Leigh Page and his colleagues have done much work on this problem of broadside incidence. The problem of determining the scattering for arbitrary angles of incidence is considerably more difficult.

As pointed out in the Conclusions of this work the usefulness of the results obtained is limited by the lack of extensive tables either of the prolate spheroidal functions or of the series coefficients of these functions. Because of the usefulness of such tables it is hoped that those already existing will be greatly extended.

In conclusion the writer wishes to express his appreciation to Mr. Gunnar Hok for many illuminating discussions and helpful suggestions, and to Mr. Yuji Morita for his careful checking of mathematical details and carrying out of the lengthy numerical calculations. Dr. L. J. Chu, of the Massachusetts Institute of Technology, very kindly spent several hours with the writer in investigating the possibilities of finding some special vector functions for the expansions of the vector fields in order to simplify the resulting expression for the scattered wave.

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I. INTRODUCTION

The present paper is concerned primarily with the scattering of a plane electromagnetic wave by a perfectly conducting prolate spheroid. For purposes of comparison, however, the problem of the scattering of a scalar wave by a smooth, perfectly reflecting prolate spheroid, is also solved.

Five methods are known to be available for the solution of scattering problems involving electromagnetic waves. If the wavelength of the electromagnetic waves being scattered is very much shorter than the significant dimensions of the scattering object, the reflection laws of geometrical optics may be used. This method was widely utilized during World War II for the calculation of the scattering of microwave radar waves (Ref. 1,2,3,4,5). At somewhat longer wavelengths, yet under conditions where the wavelength still is short compared with the important dimensions of the scatterer, the methods of physical optics are applicable (Ref. 6,7,8). If the wavelength is very long compared with the major dimensions of the scatterer the celebrated Rayleigh scattering law applies. At wavelengths of the same order of magnitude as the significant dimensions of the scattering object, electromagnetic field theory must be used. This is an exact method of attack which involves no physical approximations. It often is necessary, however, to introduce mathematical approximations in order to achieve solutions. Fortunately, at wavelengths of the same order of magnitude as, or greater than, the significant dimensions of the scatterer, the mathematical problems arising from the application of electromagnetic field theory are most amenable to solution, so this method has its greatest applicability where the approximate methods of geometrical optics and physical optics fail.

During the war Schwinger applied a variational method to the solution of the problem of determining the effects caused by discontinuities in waveguides, under conditions where the vector problem can be reduced to a scalar problem (Ref. 9). Recently this method has been applied to the diffraction of electromagnetic and scalar waves through an opening in a plane screen (Ref. 10,11,12). So far the method has not been applied to the scattering of either scalar or vector waves by a three-dimensional object.

The principal problem solved in the present work is that of determining the scattering of a single-frequency plane electromagnetic wave striking nose-on a perfectly conducting prolate spheroid, which is embedded in free space. The method of attack is that of electromagnetic field theory, although the results of applying the methods of geometrical optics and of physical optics also are included. Rationalized MKS units are used. Following engineering practice, $\sqrt{-1}$ is indicated by "j", and time variations by the expression $e^{+j\omega t}$. The latter step is valid since we are concerned with only the single-frequency, or monochromatic, case. This is not a serious limitation since, of course, any non-sinusoidal wave, including pulses, can be expressed as a sum of sine waves. Vector quantities are underlined.

Möglich (Ref. 13) has attacked the problem of determining the scattering of an electromagnetic wave by a triaxial ellipsoid. His approach is that of determining solutions of an integral equation which arises from the use of Whittaker's (Ref. 14) integral solution of the scalar wave equation. However, Möglich's results are believed to be considerably less amenable to numerical calculation than are those contained herein. In fact the only numerical results obtained by Möglich are those for the scattering caused by a circular disc, which is a limiting case of an oblate spheroid.

II. PRELIMINARIES

For the solution of the present problem the use of prolate spheroidal coordinates is indicated. It is assumed that the origin of the prolate spheroidal coordinate system coincides with the origin of the rectangular coordinate system and that the z-axis of the rectangular coordinates is the axis of rotation of the prolate spheroidal coordinates. The two coordinate systems (Fig. 1) are related by these equations:

$$x = F\sqrt{(\xi^2-1)(1-\eta^2)}\,\cos\phi\,, \tag{1}$$

$$y = F\sqrt{(\xi^2-1)(1-\eta^2)}\,\sin\phi\,, \tag{2}$$

$$z = F\xi\eta\,. \tag{3}$$

The surfaces of constant ξ are confocal prolate spheroids with common foci at f and f' on the z-axis. The surfaces of constant η are two-sheeted hyperboloids of revolution about the z-axis with the common foci f and f'.

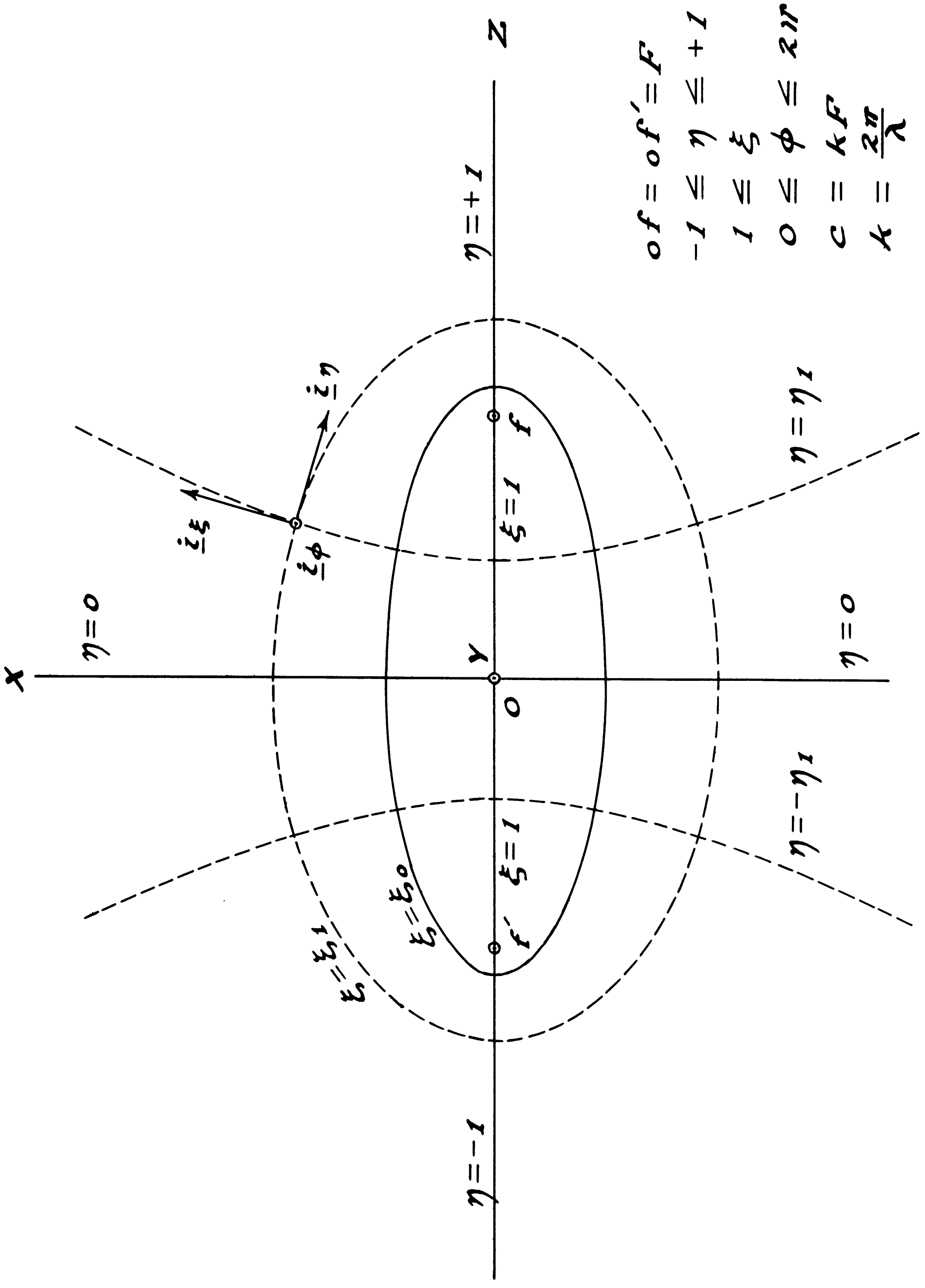

FIG. 1 PROLATE SPHEROIDAL COORDINATES

Half-planes through the z-axis are surfaces of constant ϕ, and ϕ is the angle between these planes and the upper half of the xz-plane, measured in a counterclockwise direction when looking in the direction of negative z. F is the semifocal distance. These prolate spheroidal coordinates form a right-handed system when taken in the order η, ξ, ϕ.

If "a" is the semimajor axis of the spheroid, "b" the semiminor axis, and "e" the eccentricity of the generating ellipse, it may be shown that the following relations are valid:

$$b = a\sqrt{1-e^2}\,, \tag{4}$$

$$\xi = \frac{a}{F} = \frac{1}{e}\quad . \tag{5}$$

The distance to any point in space from the origin is given by

$$r = \sqrt{x^2+y^2+z^2}\,, \tag{6}$$

$$r = F\sqrt{\xi^2-1+\eta^2}\quad . \tag{7}$$

At large distances from the origin, ξ becomes very large compared with unity and η so, under these conditions,

$$\lim_{\xi, r \to \infty} r = F\xi \tag{8}$$

The angle θ between the radius vector r and the z-axis is given by

$$\theta = \cos^{-1}\frac{z}{r}\quad . \tag{9}$$

In the limit, as r and ξ become very large,

$$\lim_{\xi, r \to \infty} \theta = \cos^{-1}\eta\,. \tag{10}$$

Solutions of the scalar Helmholtz equation,

$$\nabla^2 \psi + k^2 \psi = 0 , \tag{11}$$

are used as presented by Stratton, Morse, Chu and Hutner (Ref. 15).

In solving equation (11) by the method of separation of variables, three second-order linear ordinary differential equations result; one in η, one in ξ and one in ϕ. Hence there are two independent solutions of each equation. The separation constants are m and b. Actually a quantity l, which is related to b and which assumes integral values, is used instead of b.

Solutions of the equation in ϕ are $e^{\pm jm\phi}$, $\sin m\phi$ and $\cos m\phi$. Obviously m must be an integer in the present work.

The differential equations in η and ξ have regular singular points at ± 1 and an irregular singular point at infinity. Consequently, when the equations are solved in terms of a series of orthogonal functions, a three-term recurrence relation results for the determination of the series coefficients and these coefficients then can be found only as numerics.

Due to the range of η $(-1 \leqq \eta \leqq +1)$, it is necessary to use only the first solution of the equation in η, since only this solution is regular throughout the range of η. The solution given by Stratton, Morse, Chu and Hutner is

$$S_{ml}^{(1)}(\eta) = \sum_{n=0,1}^{\infty}{}' d_n^{ml} P_{n+m}^{m}(\eta) . \tag{12}$$

The numerical coefficients d_n^{ml} are tabulated by Stratton, Morse, Chu and Hutner for values of $c = \frac{2\pi}{\lambda}F = kF$ from 0 to 5 and for values of $m + l$ from 0 to 3. The prime on the summation symbol indicates that the summation is to be over even values of n, if l is even, and over odd values of n, if l is odd. $P_{n+m}^{m}(\eta)$ are associated Legendre functions.

ξ has the range $1 \leqq \xi \leqq \infty$. Stratton, Morse, Chu and Hutner give the following solution which is regular for all finite values of ξ :

$$R_{ml}^{(1)}(\xi) = \frac{(\xi^2-1)^{m/2}}{\xi^m \sum_{n=0,1}^{\infty}{}' d_n^{ml} \frac{(n+2m)!}{n!}} \sum_{n=0,1}^{\infty}{}' j^{l-n} d_n^{ml} \frac{(n+2m)!}{n!} \mathrm{j}_{m+n}(c\xi), \tag{13}$$

where $\mathrm{j}_{m+n}(c\xi)$ is the spherical Bessel function of the first kind.

$$\mathrm{j}_{m+n}(c\xi) = \sqrt{\frac{\pi}{2c\xi}}\, J_{m+n+1/2}(c\xi). \tag{14}$$

A second solution which has logarithmic singularities at $\xi = \pm 1$ is also given:

$$R_{ml}^{(2)}(\xi) = \frac{(\xi^2-1)^{m/2}}{\xi^m \sum_{n=0,1}^{\infty}{}' d_n^{ml} \frac{(n+2m)!}{n!}} \sum_{n=0,1}^{\infty}{}' j^{l-n} d_n^{ml} \frac{(n+2m)!}{n!} n_{m+n}(c\xi), \tag{15}$$

where $n_{m+n}(c\xi)$ is the spherical Bessel function of the second kind.

$$n_{m+n}(c\xi) = \sqrt{\frac{\pi}{2c\xi}}\, N_{n+m+\frac{1}{2}}(c\xi) = (-1)^{n+m+1}\sqrt{\frac{\pi}{2c\xi}}\, J_{-n-m-\frac{1}{2}}(c\xi). \tag{16}$$

When the above expression for $R_{ml}^{(2)}(\xi)$ converges very slowly ($\xi \cong 1$ or m large), the following equations may be used:

l even:

$$R_{ml}^{(2)}(\xi) = \frac{2c^{m-1}\Gamma\left(\frac{l+2m+1}{2}\right)}{\Gamma\left(\frac{l+2}{2}\right)\Gamma\left(m-\frac{1}{2}\right) d_{-2m}^{ml} \sum_{n=0}^{\infty}{}' d_n^{ml} \frac{(n+2m)!}{n!}} \sum_{n=-\infty}^{+\infty}{}' d_n^{ml} Q_{m+n}^{m}(\xi), \tag{17}$$

l odd:

$$R^{(2)}_{ml}(\xi) = \frac{-8c^{m-2}\,\Gamma\left(\frac{l+2m+2}{2}\right)}{\Gamma\left(\frac{l+1}{2}\right)\Gamma\left(m-\frac{3}{2}\right) d^{ml}_{1-2m} \sum\limits_{n=1}^{\infty}{}' d^{ml}_{n} \frac{(n+2m)!}{n!}} \sum_{n=-\infty}^{+\infty}{}' d^{ml}_{n} Q^{m}_{m+n}(\xi) . \tag{18}$$

The $Q^{m}_{m+n}(\xi)$ are Legendre functions of the second kind.

Two other useful solutions of the differential equation in ξ can be formed as follows:

$$R^{(3)}_{ml}(\xi) = R^{(1)}_{ml}(\xi) + j R^{(2)}_{ml}(\xi) , \tag{19}$$

$$R^{(4)}_{ml}(\xi) = R^{(1)}_{ml}(\xi) - j R^{(2)}_{ml}(\xi) . \tag{20}$$

These are noteworthy for having the following asymptotic behavior:

$$\lim_{c\xi \to \infty} R^{(3)}_{ml}(\xi) = \frac{1}{c\xi} e^{j\left(c\xi - \frac{l+m+1}{2}\pi\right)} , \tag{21}$$

$$\lim_{c\xi \to \infty} R^{(4)}_{ml}(\xi) = \frac{1}{c\xi} e^{-j\left(c\xi - \frac{l+m+1}{2}\pi\right)} . \tag{22}$$

The solutions of equation (11) which are used in the present paper are

$$\psi^{(\ )}_{ml} = S^{(1)}_{ml}(\eta)\, R^{(\ )}_{ml}(\xi) \cos m\phi . \tag{23}$$

The blank superscripts on $\psi^{(\ )}_{ml}$ and $R^{(\ )}_{ml}(\xi)$ indicate that any one of the integers 1, 2, 3 or 4 may be used. Of course, the equation holds only for a single value of this superscript used throughout the equation.

From these solutions of the scalar Helmholtz equation the following solutions of the vector Helmholtz equation may be formed (Ref. 16) ($\underline{a}$ being any arbitrary constant unit vector):

$$\underline{L}_{ml}^{(\ )} = \nabla \psi_{ml}^{(\ )} \ , \tag{24}$$

$$\underline{M}_{ml}^{(\ )} = \underline{L}_{ml}^{(\ )} \times \underline{a} = \nabla \psi_{ml}^{(\ )} \times \underline{a} \ , \tag{25}$$

$$\underline{N}_{ml}^{(\ )} = \frac{1}{k} \nabla \times \underline{M}_{ml}^{(\ )} \ . \tag{26}$$

Morse and Feshbach (Ref. 17) show that each pair of independent sets of solutions of the vector Helmholtz equation forms a complete set of orthogonal functions. Therefore we may use series in terms of $\underline{L}_{ml}$, $\underline{M}_{ml}$ and $\underline{N}_{ml}$ to express any ordinary vector function. Only $\underline{M}_{ml}$ and $\underline{N}_{ml}$ are of use here since the propagation of electromagnetic waves in free space is under consideration, which requires that

$$\nabla \cdot \underline{E} = 0 \ , \tag{27}$$

$$\nabla \cdot \underline{H} = 0 \ , \tag{28}$$

and from the defining equations given above it is clear that

$$\nabla \cdot \underline{L}_{ml}^{(\ )} \neq 0 \ , \tag{29}$$

$$\nabla \cdot \underline{M}_{ml}^{(\ )} = \nabla \cdot \underline{N}_{ml}^{(\ )} = 0 \ . \tag{30}$$

Therefore, $\underline{E}$ and $\underline{H}$ may be expressed in terms of infinite series of $\underline{M}_{ml}^{()}$ and $\underline{N}_{ml}^{()}$ functions and, because of equations (27), (28) and (29), $\underline{L}_{ml}^{()}$ will be of no use to us.

III. EXPRESSION FOR INCIDENT WAVE

Morse (Ref. 18) has developed an expansion, in terms of prolate spheroidal functions, for a plane scalar wave of unit amplitude. The following form of this expansion will be useful for expressing the plane electromagnetic wave incident upon the prolate spheroid:

$$e^{jkX} = 2\sum_{m,l} \frac{j^{m+l}(2-\delta_{o,m})}{N_{ml}} \cos\left[m(\phi-\zeta)\right] S_{ml}^{(1)}(\cos\rho) \times S_{ml}^{(1)}(\eta) R_{ml}^{(1)}(\xi), \tag{31}$$

where

$$X = x\sin\rho\cos\zeta + y\sin\rho\sin\zeta + z\cos\rho, \tag{32}$$

$$X = F\left[\xi\eta\cos\rho + \sqrt{(\xi^2-1)(1-\eta^2)}\ \sin\rho\cos(\phi-\zeta)\right], \tag{33}$$

and

$$N_{ml} = \sum_{n=o,1}^{\infty}{}' \frac{2(2m+n)!}{(2m+2n+1)\,n!}\left(d_n^{ml}\right)^2. \tag{34}$$

The prime on this summation sign again signifies that the summation is over even values of n, if l is even, and over odd values of n, if l is odd. The d_n^{ml} are the numerical coefficients tabulated by Stratton, Morse, Chu and Hutner.

Now X measures distance along the line of propagation. Since $e^{j\omega t}$ is being used as the time factor, the expression e^{jkX} indicates wave propagation in the direction of decreasing X. ρ is the angle between the positive portion of the line of propagation and the positive z-axis. ζ is the angle between the positive x-axis and the projection on the xy-plane of the positive portion of the line of propagation. ζ is positive when measured in a counter-clockwise direction from the positive x-axis, when looking along the z-axis in the direction of decreasing z. Due to the symmetry about the z-axis, the coordinate system can be rotated about the z-axis until the line of propagation lies in the xz-plane with the result that ζ is zero. The resulting form of equation (31) is

$$e^{jkX} = \sum_{m,l} A_{ml}\, \psi_{ml}^{(1)}(\eta, \xi, \phi)\,, \tag{35}$$

where

$$\psi_{ml}^{(1)}(\eta,\xi,\phi) = S_{ml}^{(1)}(\eta) R_{ml}^{(1)}(\xi) \cos m\phi \tag{36}$$

and

$$A_{ml} = \frac{2 j^{m+l}(2-\delta_{o,m})}{N_{ml}}\, S_{ml}^{(1)}(\cos\rho)\,. \tag{37}$$

The electric field intensity vector ${}^{I}\underline{E}^{t}$ and the magnetic field intensity vector ${}^{I}\underline{H}^{t}$ of an incident monochromatic plane electromagnetic wave being propagated along the line of X in a negative direction, can be expressed thus:

$$\begin{aligned} {}^{I}\underline{E}^{t}(x,y,z,t) &= {}^{I}\underline{E}(x,y,z)\, e^{j\omega t} = {}^{I}\underline{E}_{o}\, e^{jkX} e^{j\omega t} = \\ &\qquad (\underline{i}_x {}^{I}E_x + \underline{i}_y {}^{I}E_y + \underline{i}_z {}^{I}E_z)\, e^{jkX} e^{j\omega t}\,, \end{aligned} \tag{38}$$

$${}^{I}\underline{E} = {}^{I}\underline{E}_{o}\, e^{jkX} = (\underline{i}_x {}^{I}E_x + \underline{i}_y {}^{I}E_y + \underline{i}_z {}^{I}E_z) \sum_{m,l} A_{ml}\, \psi_{ml}^{(1)}\,, \tag{39}$$

$$ {}^{I}\underline{H} = {}^{I}\underline{H}_o e^{jkX} = (\underline{i}_x {}^{I}H_x + \underline{i}_y {}^{I}H_y + \underline{i}_z {}^{I}H_z) \sum_{m,l} A_{ml} \psi_{ml}^{(1)} . \tag{40} $$

${}^{I}\underline{E}^{t}$, ${}^{I}\underline{H}^{t}$ and the direction of propagation (along the line of which X is measured) must be mutually perpendicular. Equations (38) and (39) make evident the meaning of the notations used to represent the various forms of the field vectors. $\underline{i}_x$, $\underline{i}_y$ and $\underline{i}_z$ are the unit vectors in the x, y and z directions, respectively.

By making use of Maxwell's equations for a monochromatic wave being propagated in free space it is not difficult to derive the following relations:

$$ \underline{E} = -j \frac{1}{k} \sqrt{\frac{\mu}{\epsilon}} \, \nabla \times \underline{H} , \tag{41} $$

$$ \underline{H} = j \frac{1}{k} \sqrt{\frac{\epsilon}{\mu}} \, \nabla \times \underline{E} . \tag{42} $$

By using equation (40) in equation (41) one obtains

$$ \begin{aligned} {}^{I}\underline{E} = -j \frac{1}{k} \sqrt{\frac{\mu}{\epsilon}} \Big[& \nabla \times (\underline{i}_x {}^{I}H_x \sum_{m,l} A_{ml} \psi_{ml}^{(1)}) \\ & + \nabla \times (\underline{i}_y {}^{I}H_y \sum_{m,l} A_{ml} \psi_{ml}^{(1)}) \\ & + \nabla \times (\underline{i}_z {}^{I}H_z \sum_{m,l} A_{ml} \psi_{ml}^{(1)}) \Big] , \end{aligned} \tag{43} $$

$$
{}^{I}\underline{E} = -j\,\frac{1}{k}\sqrt{\frac{\mu}{\epsilon}}\Big[{}^{I}H_x \sum_{m,l} A_{ml}\,(\nabla \psi^{(1)}_{ml} \times \underline{i}_x) + {}^{I}H_y \sum_{m,l} A_{ml}\,(\nabla \psi^{(1)}_{ml} \times \underline{i}_y) + {}^{I}H_z \sum_{m,l} A_{ml}\,(\nabla \psi^{(1)}_{ml} \times \underline{i}_z)\Big] \,. \tag{44}
$$

Equation (25) can be used to simplify equation (44).

$$
{}^{I}\underline{E} = -j\,\frac{1}{k}\sqrt{\frac{\mu}{\epsilon}}\Big[{}^{I}H_x \sum_{m,l} A_{ml}\,{}^{x}\underline{M}^{(1)}_{ml} + {}^{I}H_y \sum_{m,l} A_{ml}\,{}^{y}\underline{M}^{(1)}_{ml} + {}^{I}H_z \sum_{m,l} A_{ml}\,{}^{z}\underline{M}^{(1)}_{ml}\Big] \,. \tag{45}
$$

Here ${}^{x}\underline{M}^{(1)}_{ml}$ is used to signify that the constant unit vector $\underline{i}_x$ was used for $\underline{a}$ to form ${}^{x}\underline{M}^{(1)}_{ml}$ and similarly for ${}^{y}\underline{M}^{(1)}_{ml}$ and ${}^{z}\underline{M}^{(1)}_{ml}$. The superscript "$1$" indicates that $R^{(1)}_{ml}(\xi)$ was used in forming $\psi^{(1)}_{ml}$.

In an analogous way the following expression for ${}^{I}\underline{H}$ is found:

$$
{}^{I}\underline{H} = j\,\frac{1}{k}\sqrt{\frac{\epsilon}{\mu}}\Big[{}^{I}E_x \sum_{m,l} A_{ml}\,{}^{x}\underline{M}^{(1)}_{ml} + {}^{I}E_y \sum_{m,l} A_{ml}\,{}^{y}\underline{M}^{(1)}_{ml} + {}^{I}E_z \sum_{m,l} A_{ml}\,{}^{z}\underline{M}^{(1)}_{ml}\Big] \,. \tag{46}
$$

Now let it be assumed that the incident plane wave is moving along the z-axis in the negative z-direction, that the electric vector has a magnitude E_0 and points in the positive y-direction, and that the magnetic vector has a magnitude H_0 and points in the positive x-direction. This is the situation for a plane electromagnetic wave striking the spheroid nose-on and it is illustrated in figure 2. A_{ml}, as given by equation (37), then becomes

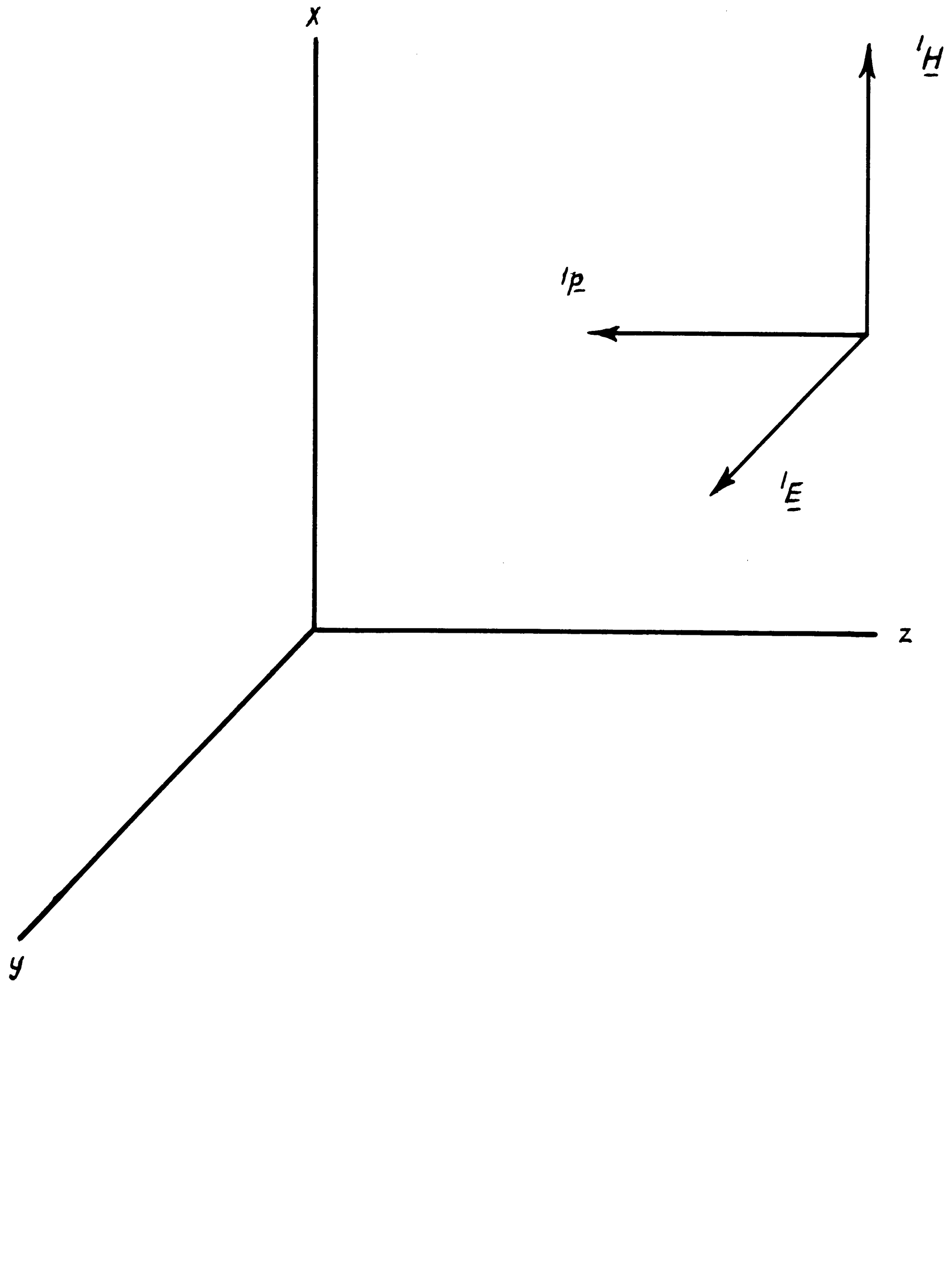

FIG. 2 INCIDENT ELECTROMAGNETIC WAVE

$$A_{ml} = \frac{2j^{m+l}(2-\delta_{o,m})}{N_{ml}} \, S_{ml}^{(1)}(1) \; . \tag{47}$$

The following relations hold (Ref. 15, p. 68):

$$m=0: \qquad S_{ol}^{(1)}(1) = \sum_{n=0,1}^{\infty}{}' d_n^{ol} \; , \tag{48}$$

$$m>0: \qquad S_{ml}^{(1)}(1) = 0 \; . \tag{49}$$

Therefore, the expression for A_{ml} becomes

$$m=0: \qquad A_{ol} = \frac{2j^l}{N_{ol}} \sum_{n=0,1}^{\infty}{}' d_n^{ol} \; , \tag{50}$$

$$m>0: \qquad A_{ml} = 0 \; , \tag{51}$$

and N_{ml} becomes

$$N_{ol} = \sum_{n=0,1}^{\infty}{}' \frac{2}{2n+1} \left(d_n^{ol}\right)^2 \; . \tag{52}$$

Equations (45) and (46) then take the forms

$$^{I}\underline{E} = -j \, \frac{^{I}E_o}{k} \sum_{l=0}^{\infty} A_{ol} \, {}^{x}\underline{M}_{ol}^{(1)} \; , \tag{53}$$

$$ {}^{I}\underline{H} = j\,\frac{{}^{I}H_o}{k}\sum_{l=o}^{\infty} A_{ol}\;{}^{y}\underline{M}_{ol}^{(1)} . \tag{54}$$

Of course ${}^{I}\underline{E}$ can be expressed in the following form directly, by making use of equation (35):

$$ {}^{I}\underline{E} = \underline{i}_y\,{}^{I}E_o\,e^{jkz} = \underline{i}_y\,{}^{I}E_o \sum_{l=o}^{\infty} A_{ol}\;\psi_{ol}^{(1)}(\eta,\xi,\phi) , \tag{55}$$

and then $\underline{i}_y$ can be expressed in terms of $\underline{i}_\eta$, $\underline{i}_\xi$, and $\underline{i}_\phi$ by means of equation (65). Here $\underline{i}_\eta$, $\underline{i}_\xi$ and $\underline{i}_\phi$ are the unit vectors in the prolate spheroidal coordinate system and the subscript of each indicates along which coordinate direction it points. Equation (53) is used for ${}^{I}\underline{E}$ in this analysis because this form of ${}^{I}\underline{E}$ makes simpler the evaluation of the integrals which arise in satisfying the boundary conditions.

The vector expressions ${}^{x}\underline{M}_{ml}^{(1)}$, ${}^{y}\underline{M}_{ml}^{(1)}$ and ${}^{z}\underline{M}_{ml}^{(1)}$ are found as follows: by definition

$$ {}^{x}\underline{M}_{ml}^{(1)} = \nabla\,\psi_{ml}^{(1)} \times \underline{i}_x . \tag{56}$$

$\psi_{ml}^{(\,)}$ is expressed in prolate spheroidal coordinates and its gradient may be found by using equation (78) on page 49 of reference 16. Using the value of $\psi_{ml}^{(\,)}$ given in equation (23), the gradient is found to be

$$ \begin{aligned} \nabla\,\psi_{ml}^{(\,)} = {} & \underline{i}_\eta\left\{\frac{1}{F}\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\left[\frac{d}{d\eta}\,S_{ml}^{(1)}(\eta)\right] R_{ml}^{(\,)}(\xi)\cos m\phi\right\} \\ & + \underline{i}_\xi\left\{\frac{1}{F}\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\;S_{ml}^{(1)}(\eta)\left[\frac{d}{d\xi}\,R_{ml}^{(\,)}(\xi)\right]\cos m\phi\right\} \\ & + \underline{i}_\phi\left\{\frac{1}{F}\,\frac{(-m)}{\sqrt{(\xi^2-1)(1-\eta^2)}}\;S_{ml}^{(1)}(\eta)\,R_{ml}^{(\,)}(\xi)\sin m\phi\right\} . \end{aligned} \tag{57}$$

Here, again, the blank superscripts occurring on $\psi_{ml}^{()}$ and $R_{ml}^{()}(\xi)$ indicate that any one of the integers 1, 2, 3 or 4 may be used, so long as the same integer is used throughout.

$\underline{i}_x$ also must be expressed in prolate spheroidal coordinates. One may use the methods of tensor analysis to make this transformation.

$$\begin{aligned} \underline{i}_x = \underline{i}_\eta &\left[-\eta(\xi^2-1)^{1/2}(\xi^2-\eta^2)^{-1/2}\cos\phi\right] \\ + \underline{i}_\xi &\left[\xi(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1/2}\cos\phi\right] \\ + \underline{i}_\phi &\left[-\sin\phi\right] \ . \end{aligned} \tag{58}$$

Then there results

$$ {}^x\underline{M}_{ml}^{()} = \nabla\psi_{ml}^{()} \times \underline{i}_x \ , \tag{59}$$

$$\begin{aligned} {}^x\underline{M}_{ml}^{()} = \underline{i}_\eta F^{-1} &\Big\{ m\xi(\xi^2-1)^{-1/2}(\xi^2-\eta^2)^{-1/2} S_{ml}^{(1)}(\eta)R_{ml}^{()}(\xi)\cos\phi\sin m\phi \\ &- (\xi^2-1)^{1/2}(\xi^2-\eta^2)^{-1/2} S_{ml}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{ml}^{()}(\xi)\right]\sin\phi\cos m\phi \Big\} \\ + \underline{i}_\xi F^{-1} &\Big\{ (1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1/2}\left[\frac{d}{d\eta}S_{ml}^{(1)}(\eta)\right]R_{ml}^{()}(\xi)\sin\phi\cos m\phi \\ &+ m\eta(1-\eta^2)^{-1/2}(\xi^2-\eta^2)^{-1/2} S_{ml}^{(1)}(\eta)R_{ml}^{()}(\xi)\cos\phi\sin m\phi \Big\} \\ + \underline{i}_\phi F^{-1} &\Big\{ \eta(\xi^2-1)(\xi^2-\eta^2)^{-1} S_{ml}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{ml}^{()}(\xi)\right]\cos\phi\cos m\phi \\ &+ \xi(1-\eta^2)(\xi^2-\eta^2)^{-1}\left[\frac{d}{d\eta}S_{ml}^{(1)}(\eta)\right]R_{ml}^{()}(\xi)\cos\phi\cos m\phi \Big\} . \end{aligned} \tag{60}$$

In a similar way one finds

$$ {}^{y}\underline{M}_{ml}^{(\,)} = \nabla \psi_{ml}^{(\,)} \times \underline{i}_y \quad , \tag{61} $$

$$ \begin{aligned} {}^{y}\underline{M}_{ml}^{(\,)} = {} & \underline{i}_\eta F^{-1} \Big\{ m\xi(\xi^2-1)^{-1/2}(\xi^2-\eta^2)^{-1/2} S_{ml}^{(1)}(\eta) R_{ml}^{(\,)}(\xi) \sin\phi \sin m\phi \\ & + (\xi^2-1)^{1/2}(\xi^2-\eta^2)^{-1/2} S_{ml}^{(1)}(\eta)\Big[\frac{d}{d\xi} R_{ml}^{(\,)}(\xi)\Big] \cos\phi \cos m\phi \Big\} \\ & + \underline{i}_\xi F^{-1} \Big\{ -(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1/2}\Big[\frac{d}{d\eta} S_{ml}^{(1)}(\eta)\Big] R_{ml}^{(\,)}(\xi) \cos\phi \cos m\phi \\ & + m\eta(1-\eta^2)^{-1/2}(\xi^2-\eta^2)^{-1/2} S_{ml}^{(1)}(\eta) R_{ml}^{(\,)}(\xi) \sin\phi \sin m\phi \Big\} \\ & + \underline{i}_\phi F^{-1} \Big\{ \eta(\xi^2-1)(\xi^2-\eta^2)^{-1} S_{ml}^{(1)}(\eta)\Big[\frac{d}{d\xi} R_{ml}^{(\,)}(\xi)\Big] \sin\phi \cos m\phi \\ & + \xi(1-\eta^2)(\xi^2-\eta^2)^{-1}\Big[\frac{d}{d\eta} S_{ml}^{(1)}(\eta)\Big] R_{ml}^{(\,)}(\xi) \sin\phi \cos m\phi \Big\} \quad , \end{aligned} \tag{62} $$

$$ {}^{z}\underline{M}_{ml}^{(\ )} = \nabla \psi_{ml}^{(\ )} \times \underline{i}_{z} \quad , \tag{63} $$

$$ \begin{aligned} {}^{z}\underline{M}_{ml}^{(\ )} &= \underline{i}_{\eta} F^{-1}\left\{ m\,\eta\,(1-\eta^2)^{-1/2}(\xi^2-\eta^2)^{-1/2}\, S_{ml}^{(1)}(\eta) R_{ml}^{(\ )}(\xi)\, \sin m\phi \right\} \\ &+ \underline{i}_{\xi} F^{-1}\left\{ -m\,\xi\,(\xi^2-1)^{-1/2}(\xi^2-\eta^2)^{-1/2}\, S_{ml}^{(1)}(\eta) R_{ml}^{(\ )}(\xi) \sin m\phi \right\} \\ &+ \underline{i}_{\phi} F^{-1}\left\{ -\xi(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1} S_{ml}^{(1)}(\eta)\left[\frac{d}{d\xi} R_{ml}^{(\ )}(\xi)\right]\cos m\phi \right. \\ &+ \left. \eta(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1}\left[\frac{d}{d\eta} S_{ml}^{(1)}(\eta)\right] R_{ml}^{(\ )}(\xi) \cos m\phi \right\} . \end{aligned} \tag{64} $$

Obvious simplifications result in the values of the three M-vectors when m = 0.

For the sake of completeness the following expressions for $\underline{i}_y$ and $\underline{i}_z$ are included:

$$ \begin{aligned} \underline{i}_y &= \underline{i}_{\eta}\left[-\eta(\xi^2-1)^{1/2}(\xi^2-\eta^2)^{-1/2} \sin\phi \right] \\ &+ \underline{i}_{\xi}\left[\xi(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1/2} \sin\phi \right] \\ &+ \underline{i}_{\phi}\left[\cos\phi \right] , \end{aligned} \tag{65} $$

$$ \begin{aligned} \underline{i}_z &= \underline{i}_{\eta}\left[\xi(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1/2} \right] \\ &+ \underline{i}_{\xi}\left[\eta(\xi^2-1)^{1/2}(\xi^2-\eta^2)^{-1/2} \right] . \end{aligned} \tag{66} $$

IV. EXPRESSION FOR SCATTERED WAVE

The expression for the electric vector of the scattered wave will be found for the special case of incident wave discussed in Section III, namely, nose-on incidence.

The expression for the electric vector of the scattered wave must meet the following conditions:

1. It must satisfy the vector wave equation (or the vector Helmholtz equation for a monochromatic wave, as we have here).

2. Its divergence must be zero.

3. At very large distances from the prolate spheroid the scattered wave must take on the behavior of a spherical diverging wave with the center of the spheroid as its center.

4. The resultant of the incident and scattered waves must satisfy the proper boundary conditions over the surface of the spheroid.

Conditions 1 and 2 insure that the expression satisfies Maxwell's equations and conditions 3 and 4 are the boundary conditions.

The M-vectors used in Section III to express the incident waves satisfy the vector Helmholtz equation and have zero divergence, as is pointed out in Section II. Therefore, if the scattered wave is expressed as a linear combination of these vectors with constant coefficients, the resulting expression will satisfy conditions 1 and 2, listed above. Actually an infinite series of M-vectors, with undetermined coefficients, will be used for expressing the scattered wave.

In forming the M-vectors the following solution of the scalar Helmholtz equation (in prolate spheroidal coordinates) was used:

$$\psi_{ml}^{(\;)} = S_{ml}^{(1)}(\eta) R_{ml}^{(\;)}(\xi) \cos m\phi \;. \tag{67}$$

As pointed out in Section II, four different forms of $R_{ml}^{()}(\xi)$ are available and their properties are given there. Since the incident plane wave exists throughout finite space, in the expression for it given by equation (31) it is necessary to use $R_{ml}^{(1)}(\xi)$ which is regular for all finite values of ξ. The scattered wave exists only outside the perfectly conducting prolate spheroid. Now a value of ξ equal to unity corresponds to a mathematical straight line extending from $z=-F$ to $z=+F$. For all prolate spheroids, the minor axis of which is different from zero, $1<\xi\leq\infty$. Therefore, in representing the scattered wave in the case at hand anyone of the four forms of $R_{ml}^{()}(\xi)$ may be used. Due to the asymptotic behavior of $R_{ml}^{(4)}(\xi)$ for very large values of $c\,\xi$, as given by equation (22), one is led to try this particular $R_{ml}^{()}(\xi)$ in forming $\psi_{ml}^{()}$ for use in expressing the scattered wave. The final expression for the scattered wave will have to be checked to see if it does satisfy the third condition listed at the beginning of the section.

The fourth condition will be satisfied by using it to calculate the undetermined coefficients in the series for the scattered wave. Since the prolate spheroid is assumed to be perfectly conducting, this boundary condition is expressed by

$$\left[\underline{n}\times{}^{T}\underline{E}\right]_{\xi=\xi_o}=\left[\underline{i}_{\xi}\times({}^{I}\underline{E}+{}^{S}\underline{E})\right]_{\xi=\xi_o}=0\quad, \tag{68}$$

where ${}^{T}\underline{E}$ is the electric vector of the total electromagnetic field and $\underline{n}$ is the unit vector normal to the surface of the prolate spheroid. ξ_o is the value of ξ on the surface of the spheroid.

Then we have

$$\left[{}^{I}E_{\eta}+{}^{S}E_{\eta}\right]_{\xi=\xi_o}=0\quad, \tag{69}$$

$$\left[{}^{I}E_{\phi}+{}^{S}E_{\phi}\right]_{\xi=\xi_o}=0\quad, \tag{70}$$

where ${}^{()}E_{\eta}$ is the η -component of ${}^{()}\underline{E}$ and ${}^{()}E_{\phi}$ is the ϕ -component.

Since there are two boundary equations to be satisfied, it is to be expected that it will be necessary to use for ${}^{s}\underline{E}$ the sum of two series of vector functions, each with undetermined coefficients. At least eight sets of such vector functions are available, the individual terms of each set satisfying the vector Helmholtz equation and having zero divergence. These sets are ${}^{x}\underline{M}_{ml}^{()}$, ${}^{y}\underline{M}_{ml}^{()}$, ${}^{z}\underline{M}_{ml}^{()}$, ${}^{r}\underline{M}_{ml}^{()}$, ${}^{x}\underline{N}_{ml}^{()}$, ${}^{y}\underline{N}_{ml}^{()}$, ${}^{z}\underline{N}_{ml}^{()}$, ${}^{r}\underline{N}_{ml}^{()}$.

Two of these sets of vector functions have not been previously discussed. They are

$$ {}^{r}\underline{M}_{ml}^{()} = \nabla \psi_{ml}^{()} \times \underline{i}_{r} \quad , \tag{71} $$

and

$$ {}^{r}\underline{N}_{ml}^{()} = \frac{1}{k} \nabla \times {}^{r}\underline{M}_{ml}^{()} \quad , \tag{72} $$

where $\underline{i}_{r}$ is the unit radius vector. Of course $\underline{i}_{r}$ is not a constant vector, but Stratton (Ref. 16) shows that nevertheless ${}^{r}\underline{M}$ and ${}^{r}\underline{N}$ satisfy the vector Helmholtz equation. These two sets of vector functions prove very useful in solving the problem of scattering of an electromagnetic wave by a sphere and it is to be expected that the two sets,

$$ {}^{\xi}\underline{M}_{ml}^{()} = \nabla \psi_{ml}^{()} \times \underline{i}_{\xi} \quad , \tag{73} $$

and

$$ {}^{\xi}\underline{N}_{ml}^{()} = \frac{1}{k} \nabla \times {}^{\xi}\underline{M}_{ml}^{()} \quad , \tag{74} $$

would be useful for the problem under consideration here. Unfortunately, however, as Stratton infers and as calculation shows, these two sets of vector functions do not satisfy the vector Helmholtz equation.

In choosing two of the eight sets of vector functions, listed above, to represent the scattered wave, it is highly desirable to choose functions each of the three components of which vary with ϕ in the same way as do the corresponding components of the incident wave. For setting up an expression for the E-vector of the scattered wave the vector functions which have the proper variation with ϕ are ${}^{x}\underline{M}_{ol}^{()}$, ${}^{z}\underline{M}_{1l}^{()}$, ${}^{y}\underline{N}_{ol}^{()}$, ${}^{z}\underline{N}_{1l}^{()}$, ${}^{r}\underline{M}_{1l}^{()}$, and ${}^{r}\underline{N}_{1l}^{()}$. Of these ${}^{x}\underline{M}_{ol}^{(4)}$ and ${}^{z}\underline{M}_{1l}^{(4)}$ were chosen for reasons of simplicity. The resulting expression for the scattered wave is

$$ {}^{s}\underline{E} = \sum_{l=o}^{\infty} \left[\alpha_{ol}\, {}^{x}\underline{M}_{ol}^{(4)} + \beta_{1l}\, {}^{z}\underline{M}_{1l}^{(4)} \right] , \tag{75} $$

where α_{ol} and β_{1l} are undetermined coefficients. Expressions for ${}^{x}\underline{M}_{ol}^{(4)}$ and ${}^{z}\underline{M}_{1l}^{(4)}$ are given near the end of Section III.

It is necessary to investigate this expression for ${}^{s}\underline{E}$ to determine whether or not it satisfies the third condition listed at the beginning of this section. In Stratton, Morse, Chu and Hutner (Ref. 15) it is shown that

$$ \lim_{c\xi \to \infty} R_{ml}^{(4)}(\xi) = \frac{1}{c\xi}\, e^{-j\left(c\xi - \frac{l+m+1}{2}\pi\right)} . \tag{76} $$

It also can be shown that

$$ \lim_{c\xi \to \infty} \left[\frac{d}{d\xi} R_{ml}^{(4)}(\xi) \right] = \frac{1}{\xi}\, e^{-j\left(c\xi - \frac{l+m}{2}\pi\right)} . \tag{77} $$

By using equations (76) and (77) there results

$$ \lim_{c\xi \to \infty} {}^{x}\underline{M}_{ol}^{(4)} = \left[-\underline{i}_{\eta}\, S_{ol}^{(1)}(\eta) \sin\phi + \underline{i}_{\phi}\, \eta\, S_{ol}^{(1)}(\eta) \cos\phi \right] \times \frac{1}{F\xi}\, e^{-j\left(c\xi - \frac{l}{2}\pi\right)} . \tag{78} $$

Now, in Section II it was shown that

$$ c\xi = \frac{2\pi}{\lambda} F\xi \tag{79} $$

and

$$\lim_{c\xi \to \infty} (F\xi) = r \ , \tag{80}$$

so equation (78) becomes

$$\lim_{c\xi \to \infty} {}^{x}\underline{M}_{ol}^{(4)} = \left[-\underline{i}_{\eta} S_{ol}^{(1)}(\eta) \sin\phi + \underline{i}_{\phi} \eta S_{ol}^{(1)}(\eta) \cos\phi \right] \times \frac{1}{r} e^{-j\left(\frac{2\pi}{\lambda} r - \frac{l}{2}\pi\right)} . \tag{81}$$

In a similar way it may be shown that

$$\lim_{c\xi \to \infty} {}^{z}\underline{M}_{1l}^{(4)} = \left[-\underline{i}_{\phi} (1-\eta^2)^{1/2} S_{1l}^{(1)}(\eta) \cos\phi \right] \times \frac{1}{r} e^{-j\left(\frac{2\pi}{\lambda} r - \frac{l+1}{2}\pi\right)} . \tag{82}$$

Equations (81) and (82) show that each term of equation (75) for ${}^{s}\underline{E}$ has the correct behavior at large distances from the prolate spheroid, so ${}^{s}\underline{E}$ will also behave correctly and the third condition listed at the beginning of this section is satisfied.

The expression for ${}^{s}\underline{E}$, equation (75), therefore satisfies the first three of the four necessary conditions. The fourth requirement, that of satisfying the boundary conditions over the surface of the spheroid, will be taken care of in the next section by using these conditions to determine the α_{ol} and β_{1l} of equation (75).

V. SATISFYING BOUNDARY CONDITIONS

The boundary conditions are expressed by equations (69) and (70). By using the expressions for ${}^{I}\underline{E}$ and ${}^{s}\underline{E}$ given by equations (53) and (75), respectively, in equation (69) there results

$$
\begin{aligned}
j\,\frac{{}^{I}E_o}{k}\sum_{l=0}^{\infty} A_{ol}(\xi_o^2-1)^{1/2}\, S_{ol}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o} = \\
\sum_{l=0}^{\infty}\alpha_{ol}\,(\xi_o^2-1)^{1/2}\, S_{ol}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o} \\
-\sum_{l=0}^{\infty}\beta_{1l}\,\eta(1-\eta^2)^{-1/2}\, S_{1l}^{(1)}(\eta)R_{1l}^{(4)}(\xi_o)\ .
\end{aligned}
\tag{83}
$$

Similarly by using equation (70), one obtains,

$$
\begin{aligned}
j\,\frac{{}^{I}E_o}{k}\sum_{l=0}^{\infty}A_{ol}\left\{\eta(\xi_o^2-1)S_{ol}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o}+\ \xi_o(1-\eta^2)\left[\frac{d}{d\eta}S_{ol}^{(1)}(\eta)\right]R_{ol}^{(1)}(\xi_o)\right\} = \\
\sum_{l=0}^{\infty}\alpha_{ol}\left\{\eta(\xi_o^2-1)\,S_{ol}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o}+\ \xi_o(1-\eta^2)\left[\frac{d}{d\eta}S_{ol}^{(1)}(\eta)\right]R_{ol}^{(4)}(\xi_o)\right\} \\
-\sum_{l=0}^{\infty}\beta_{1l}\left\{\xi_o(\xi_o^2-1)^{1/2}(1-\eta^2)^{1/2}\,S_{1l}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{1l}^{(4)}(\xi)\right]_{\xi=\xi_o}\right. \\
\left.-\eta\,(\xi_o^2-1)^{1/2}\,(1-\eta^2)^{1/2}\left[\frac{d}{d\eta}S_{1l}^{(1)}(\eta)\right]R_{1l}^{(4)}\,(\xi_o)\right\}\ .
\end{aligned}
\tag{84}
$$

These equations must hold for all allowed values of η and may be used to determine α_{ol} and β_{1l} . Now, according to equation (12),

$$
S_{ml}^{(1)}(\eta) = \sum_{n=0,1}^{\infty}{}' d_n^{ml} P_{m+n}^{m}(\eta)\ , \tag{85}
$$

where d_n^{ml} are the previously-mentioned numerical coefficients arising from the solution of the differential equation for $S_{ml}^{(1)}(\eta)$. $P_{m+n}^{m}(\eta)$

are the associated Legendre functions of the first kind and, as before, the prime on the summation sign indicates that the summation is over only odd values of n if l is odd, and over only even values of n if l is even. Since, in this case, m as well as n takes on only integral values, the $P^m_{m+n}(\eta)$ are polynomials, and this fact may be used to solve equations (83) and (84) for α_{ol} and β_{1l}. That is, the $P^m_{m+n}(\eta)$ may be expressed as polynomials and each side of the equations expressed as a power series in η. Then, by equating coefficients, equations would be obtained for calculating the α_{ol} and β_{1l}. This, however, would involve an immense amount of tedious algebra, especially since many of the $P^m_{m+n}(\eta)$ involve $(1-\eta^2)^{1/2}$ as a factor and this would have to be expanded in a power series.

What is believed to be a better method has been used here for finding α_{ol} and β_{1l}. Equations (83) and (84) are multiplied by $S^{(1)}_{oL}(\eta)$ (where L is any non-negative integer) and then integrated from $\eta=-1$ to $\eta=+1$. The following results are obtained (a_o is the semimajor axis of the scattering spheroid):

$$\left({}^{I}E_o a_o\right)\left(\frac{j}{c\xi_o}\right)\sum_{l=0}^{\infty} A_{ol}(\xi_o^2-1)^{1/2}\left[\frac{d}{d\xi}R^{(1)}_{ol}(\xi)\right]_{\xi=\xi_o}\int_{-1}^{+1} S^{(1)}_{ol}(\eta)\,S^{(1)}_{oL}(\eta)\,d\eta =$$

$$\sum_{l=0}^{\infty}\alpha_{ol}(\xi_o^2-1)^{1/2}\left[\frac{d}{d\xi}R^{(4)}_{ol}(\xi)\right]_{\xi=\xi_o}\int_{-1}^{+1} S^{(1)}_{ol}(\eta)\,S^{(1)}_{oL}(\eta)\,d\eta$$

$$-\sum_{l=0}^{\infty}\beta_{1l}\,R^{(4)}_{1l}(\xi_o)\int_{-1}^{+1}\frac{\eta}{\sqrt{1-\eta^2}}\,S^{(1)}_{1l}(\eta)\,S^{(1)}_{oL}(\eta)\,d\eta\;, \qquad (86)$$

$$\left({}^{I}E_o a_o\right)\left(\frac{j}{c\xi_o}\right) \sum_{l=0}^{\infty} A_{ol} \left\{ (\xi_o^2 - 1)\left[\frac{d}{d\xi} R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o} \int_{-1}^{+1} \eta\, S_{ol}^{(1)}(\eta)\, S_{oL}^{(1)}(\eta)\, d\eta \right.$$

$$\left. + \xi_o R_{ol}^{(1)}(\xi_o) \int_{-1}^{+1} (1-\eta^2)\left[\frac{d}{d\eta} S_{ol}^{(1)}(\eta)\right] S_{oL}^{(1)}(\eta)\, d\eta \right\} =$$

$$\sum_{l=0}^{\infty} \alpha_{ol} \left\{ (\xi_o^2 - 1)\left[\frac{d}{d\xi} R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o} \int_{-1}^{+1} \eta\, S_{ol}^{(1)}(\eta)\, S_{oL}^{(1)}(\eta)\, d\eta \right.$$

$$\left. + \xi_o R_{ol}^{(4)}(\xi_o) \int_{-1}^{+1} (1-\eta^2)\left[\frac{d}{d\eta} S_{ol}^{(1)}(\eta)\right] S_{oL}^{(1)}(\eta)\, d\eta \right\}$$

$$-\sum_{l=0}^{\infty} \beta_{1l} \left\{ \xi_o (\xi_o^2 - 1)^{1/2} \left[\frac{d}{d\xi} R_{1l}^{(4)}(\xi)\right]_{\xi=\xi_o} \int_{-1}^{+1} \sqrt{1-\eta^2}\, S_{1l}^{(1)}(\eta)\, S_{oL}^{(1)}(\eta)\, d\eta \right.$$

$$\left. - (\xi_o^2 - 1)^{1/2} R_{1l}^{(4)}(\xi_o) \int_{-1}^{+1} \eta \sqrt{1-\eta^2} \left[\frac{d}{d\eta} S_{1l}^{(1)}(\eta)\right] S_{oL}^{(1)}(\eta)\, d\eta \right\} . \tag{87}$$

Stratton, Morse, Chu and Hutner (Ref. 15, pp. 42,63) show that the $S_{ml}^{(1)}(\eta)$ are orthogonal functions and that

$$I_1 = \int_{-1}^{+1} S_{ol}^{(1)}(\eta)\, S_{oL}^{(1)}(\eta)\, d\eta = \begin{cases} 0 & , \ l \neq L \\ 2 \sum\limits_{n=0,1}^{\infty}{}' \dfrac{(d_n^{oL})^2}{2n+1} & , \ l = L \end{cases} . \tag{88}$$

The other integrals are much more difficult to handle, especially since no recurrence relations have been found for the $S_{ml}^{(1)}(\eta)$ functions. The method which has been used to evaluate the remaining integrals is that of using equation (85) to express $S_{ml}^{(1)}(\eta)$, and its derivatives, in terms of $P_{m+n}^{m}(\eta)$ and its derivatives. This involves manipulation of double series and the evaluation of integrals involving $P_{m+n}^{m}(\eta)$, and its derivatives, but the work is straight-forward and only the results are included. These are listed below.

$$I_2 = \int_{-1}^{+1} \frac{\eta}{\sqrt{1-\eta^2}}\, S_{1l}^{(1)}(\eta)\, S_{oL}^{(1)}(\eta)\, d\eta . \tag{89}$$

For $l \left\{ \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \right\}$ and $L \left\{ \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \right\}$ this integral is zero. For l odd and L even one obtains *

$$
\begin{aligned}
-I_2(L,l) = {} & 2 d_0^{oL}\left(d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots\cdot\right) \\
& + \tfrac{4}{5} d_2^{oL}\left(d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots\cdot\right) \\
& + \tfrac{6}{5} d_2^{oL}\left(d_3^{1l} + d_5^{1l} + d_7^{1l} + \cdots\cdot\right) \\
& + \tfrac{8}{9} d_4^{oL}\left(d_3^{1l} + d_5^{1l} + d_7^{1l} + \cdots\cdot\right) \\
& + \tfrac{10}{9} d_4^{oL}\left(d_5^{1l} + d_7^{1l} + d_9^{1l} + \cdots\cdot\right) \\
& \vdots
\end{aligned}
\tag{90}
$$

For l even and L odd there results

$$
\begin{aligned}
-I_2(L,l) = {} & \tfrac{2}{3} d_1^{oL}\left(d_0^{1l} + d_2^{1l} + d_4^{1l} + \cdots\right) \\
& + \tfrac{4}{3} d_1^{oL}\left(d_2^{1l} + d_4^{1l} + d_6^{1l} + \cdots\right) \\
& + \tfrac{6}{7} d_3^{oL}\left(d_2^{1l} + d_4^{1l} + d_6^{1l} + \cdots\right) \\
& + \tfrac{8}{7} d_3^{oL}\left(d_4^{1l} + d_6^{1l} + d_8^{1l} + \cdots\right) \\
& + \tfrac{10}{11} d_5^{oL}\left(d_4^{1l} + d_6^{1l} + d_8^{1l} + \cdots\right) \\
& + \tfrac{12}{11} d_5^{oL}\left(d_6^{1l} + d_8^{1l} + d_{10}^{1l} + \cdots\right) \\
& \vdots
\end{aligned}
\tag{91}
$$

$$
I_3 = \int_{-1}^{+1} \eta\, S_{ol}^{(1)}(\eta)\, S_{oL}^{(1)}(\eta)\, d\eta \; . \tag{92}
$$

---

* It should be noted that $d_\eta^{ml} = 0$ whenever $(l+\eta)$ is odd.

I_3 is zero for $l \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ and $L \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$. For $l \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ and $L \begin{Bmatrix}\text{odd}\\ \text{even}\end{Bmatrix}$ we find

$$I_3(L,l) = \sum_{n=1}^{\infty}{}' \left[\frac{n+1}{n(n+2)}\right]\left[d^{ol}_{\frac{n-1}{2}}\, d^{oL}_{\frac{n+1}{2}} + d^{ol}_{\frac{n+1}{2}}\, d^{oL}_{\frac{n-1}{2}}\right] . \tag{93}$$

$$I_4 = \int_{-1}^{+1} (1-\eta^2)\left[\frac{d}{d\eta}\, S^{(1)}_{ol}(\eta)\right] S^{(1)}_{oL}(\eta)\, d\eta \quad . \tag{94}$$

I_4, also, is zero for $l \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ and $L \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$. For $l \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ and $L \begin{Bmatrix}\text{odd}\\ \text{even}\end{Bmatrix}$ we obtain

$$I_4(L,l) = \sum_{n=1}^{\infty}{}' \frac{n+1}{2n(n+2)}\left[(n+3)\, d^{ol}_{\frac{n+1}{2}}\, d^{oL}_{\frac{n-1}{2}} - (n-1)\, d^{ol}_{\frac{n-1}{2}}\, d^{oL}_{\frac{n+1}{2}}\right] . \tag{95}$$

$$I_5 = \int_{-1}^{+1} \sqrt{1-\eta^2}\; S^{(1)}_{1l}(\eta)\, S^{(1)}_{oL}(\eta)\, d\eta \quad . \tag{96}$$

This integral is zero if l is $\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ and L is $\begin{Bmatrix}\text{odd}\\ \text{even}\end{Bmatrix}$. For $l \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ and $L \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ oné finds

$$I_5(L,l) = \sum_{n=0}^{\infty}{}' \left[\frac{n+2}{2(n+1)(n+3)}\right]\left[n\, d^{1l}_{\frac{n}{2}-1}\, d^{oL}_{\frac{n}{2}+1} - (n+4)\, d^{1l}_{\frac{n}{2}}\, d^{oL}_{\frac{n}{2}}\right] . \tag{97}$$

$$I_6(L,l) = \int_{-1}^{+1} \eta\sqrt{1-\eta^2}\left[\frac{d}{d\eta}\, S^{(1)}_{1l}(\eta)\right] S^{(1)}_{oL}(\eta)\, d\eta \quad . \tag{98}$$

If l is $\left\{\begin{matrix}\text{even}\\\text{odd}\end{matrix}\right\}$ and L is $\left\{\begin{matrix}\text{odd}\\\text{even}\end{matrix}\right\}$, I_6 is zero. For l $\left\{\begin{matrix}\text{even}\\\text{odd}\end{matrix}\right\}$ and L $\left\{\begin{matrix}\text{even}\\\text{odd}\end{matrix}\right\}$ we obtain

$$I_6(L,l) = {\sum_{n=0}^{\infty}}' \left[\frac{(n+2)^2}{4(n+1)(n+3)}\right]\left[(n+4)\, d_{\frac{n}{2}}^{1l}\, d_{\frac{n}{2}}^{oL} + n\, d_{\frac{n}{2}-1}^{1l}\, d_{\frac{n}{2}+1}^{oL}\right] - \sum\nolimits_2 . \tag{99}$$

For both l and L even, $\sum_2$ is given by

$$\begin{aligned}\sum\nolimits_2 = {} & 2\, d_0^{oL}\left(\tfrac{1}{3} d_0^{1l} + d_2^{1l} + d_4^{1l} + \cdots\right)\\
& + \tfrac{4}{5}\, d_2^{oL}\left(\tfrac{1}{3} d_0^{1l} + d_2^{1l} + d_4^{1l} + \cdots\right)\\
& + \tfrac{6}{5}\, d_2^{oL}\left(\tfrac{3}{7} d_2^{1l} + d_4^{1l} + d_6^{1l} + \cdots\right)\\
& + \tfrac{8}{9}\, d_4^{oL}\left(\tfrac{3}{7} d_2^{1l} + d_4^{1l} + d_6^{1l} + \cdots\right)\\
& + \tfrac{10}{9}\, d_4^{oL}\left(\tfrac{5}{11} d_4^{1l} + d_6^{1l} + d_8^{1l} + \cdots\right)\\
& + \tfrac{12}{13}\, d_6^{oL}\left(\tfrac{5}{11} d_4^{1l} + d_6^{1l} + d_8^{1l} + \cdots\right)\\
& \;\;\vdots\end{aligned} \tag{100}$$

and for both l and L odd, $\sum_2$ is given by

$$\begin{aligned}\sum_2 = \tfrac{2}{3}\, d_1^{o\ell}\big(& \quad d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots\big)\\
+ \tfrac{4}{3}\, d_1^{o\ell}\big(& \tfrac{2}{5}\, d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots\big)\\
+ \tfrac{6}{7}\, d_3^{o\ell}\big(& \tfrac{2}{5}\, d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots\big)\\
+ \tfrac{8}{7}\, d_3^{o\ell}\big(& \tfrac{4}{9}\, d_3^{1l} + d_5^{1l} + d_7^{1l} + \cdots\big)\\
+ \tfrac{10}{11}\, d_5^{o\ell}\big(& \tfrac{4}{9}\, d_3^{1l} + d_5^{1l} + d_7^{1l} + \cdots\big)\\
+ \tfrac{12}{11}\, d_5^{o\ell}\big(& \tfrac{6}{13}\, d_5^{1l} + d_7^{1l} + d_9^{1l} + \cdots\big)\\
+ \tfrac{14}{15}\, d_7^{o\ell}\big(& \tfrac{6}{13}\, d_5^{1l} + d_7^{1l} + d_9^{1l} + \cdots\big)\\
& \vdots\end{aligned} \tag{101}$$

We are now in a position to use equations (86) and (87) to evaluate the coefficients α_{ol} and β_{1l}. In order to handle the equations more easily the following substitutions will be used:

$$B_{\ell l} = \left(\frac{j}{c\xi_o}\right) A_{ol}\,(\xi_o^2-1)^{1/2}\left[\frac{d}{d\xi} R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o} \int_{-1}^{+1} S_{ol}^{(1)}(\eta)\, S_{o\ell}^{(1)}(\eta)\,d\eta\,, \tag{102}$$

$$C_{\ell l} = (\xi_o^2-1)^{1/2}\left[\frac{d}{d\xi} R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o} \int_{-1}^{+1} S_{ol}^{(1)}(\eta)\, S_{o\ell}^{(1)}(\eta)\,d\eta\,, \tag{103}$$

$$D_{\ell l} = -R_{1l}^{(4)}(\xi_o)\int_{-1}^{+1} \frac{\eta}{\sqrt{1-\eta^2}}\, S_{1l}^{(1)}(\eta)\, S_{o\ell}^{(1)}(\eta)\,d\eta\,, \tag{104}$$

$$\begin{aligned} U_{\ell l} = \left(\frac{j}{c\xi_o}\right) A_{ol}\Big\{ & (\xi_o^2-1)\left[\frac{d}{d\xi} R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o} \int_{-1}^{+1}\eta\, S_{ol}^{(1)}(\eta)\, S_{o\ell}^{(1)}(\eta)\,d\eta \\ & + \xi_o R_{ol}^{(1)}(\xi_o)\int_{-1}^{+1}(1-\eta^2)\left[\frac{d}{d\eta} S_{ol}^{(1)}(\eta)\right] S_{o\ell}^{(1)}(\eta)\,d\eta\Big\}\,,\end{aligned} \tag{105}$$

$$\begin{aligned} V_{\ell l} = {} & (\xi_o^2-1)\left[\frac{d}{d\xi} R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o} \int_{-1}^{+1}\eta\, S_{ol}^{(1)}(\eta)\, S_{o\ell}^{(1)}(\eta)\,d\eta \\ & + \xi_o R_{ol}^{(4)}(\xi_o)\int_{-1}^{+1}(1-\eta^2)\left[\frac{d}{d\eta} S_{ol}^{(1)}(\eta)\right] S_{o\ell}^{(1)}(\eta)\,d\eta\,,\end{aligned} \tag{106}$$

$$W_{\mathcal{L}l} = -\xi_o(\xi_o^2-1)^{\frac{1}{2}}\left[\frac{d}{d\xi} R_{1l}^{(4)}(\xi)\right]_{\xi=\xi_o} \int_{-1}^{+1} \sqrt{1-\eta^2}\, S_{1l}^{(1)}(\eta)\, S_{o\mathcal{L}}^{(1)}(\eta)\, d\eta$$

$$+ \ (\xi_o^2-1)^{\frac{1}{2}} R_{1l}^{(4)}(\xi_o) \int_{-1}^{+1} \eta \sqrt{1-\eta^2} \left[\frac{d}{d\eta} S_{1l}^{(1)}(\eta)\right] S_{o\mathcal{L}}^{(1)}(\eta)\, d\eta \, . \tag{107}$$

Using these symbols in equations (86) and (87) we obtain

$$\sum_{l=o}^{\infty} \left(\alpha_{ol}\, C_{\mathcal{L}l} + \beta_{1l}\, D_{\mathcal{L}l}\right) = {}^{I}E_o\, a_o \sum_{l=o}^{\infty} B_{\mathcal{L}l} \, , \tag{108}$$

$$\sum_{l=o}^{\infty} \left(\alpha_{ol}\, V_{\mathcal{L}l} + \beta_{1l}\, W_{\mathcal{L}l}\right) = {}^{I}E_o\, a_o \sum_{l=o}^{\infty} U_{\mathcal{L}l} \, . \tag{109}$$

Here we have $2\mathcal{L}$ equations in $2\,l$ unknowns and both $\mathcal{L}$ and l go from zero to infinity. Now let us expand equations (108) and (109), allowing l and $\mathcal{L}$ to range upward from zero and using the information which we have concerning the values of l and $\mathcal{L}$ for which the integrals, and thus the values of $B_{\mathcal{L}l}$, $C_{\mathcal{L}l} \cdots W_{\mathcal{L}l}$, are zero.

We obtain

$$
\begin{array}{l}
\alpha_{00} C_{00} + 0 + 0 + 0 + \cdots + 0 + \beta_{11} D_{01} + 0 + \beta_{13} D_{03} + \cdots = ({}^{I}E_o\, a_o)(B_{00} + 0 + 0 + 0 + \cdots) \\
0 + \alpha_{01} C_{11} + 0 + 0 + \cdots + \beta_{10} D_{10} + 0 + \beta_{12} D_{12} + 0 + \cdots = ({}^{I}E_o\, a_o)(0 + B_{11} + 0 + 0 + \cdots) \\
0 + 0 + \alpha_{02} C_{22} + 0 + \cdots + 0 + \beta_{11} D_{21} + 0 + \beta_{13} D_{23} + \cdots = ({}^{I}E_o\, a_o)(0 + 0 + B_{22} + 0 + \cdots) \\
0 + 0 + 0 + \alpha_{03} C_{33} + \cdots + \beta_{10} D_{30} + 0 + \beta_{12} D_{32} + 0 + \cdots = ({}^{I}E_o\, a_o)(0 + 0 + 0 + B_{33} + \cdots) \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}
\tag{110}
$$

$$
\begin{array}{l}
0 + \alpha_{01} V_{01} + 0 + \alpha_{03} V_{03} + \cdots + \beta_{10} W_{00} + 0 + \beta_{12} W_{02} + 0 + \cdots = ({}^{I}E_o\, a_o)(0 + U_{01} + 0 + U_{03} + \cdots) \\
\alpha_{00} V_{10} + 0 + \alpha_{02} V_{12} + 0 + \cdots + 0 + \beta_{11} W_{11} + 0 + \beta_{13} W_{13} + \cdots = ({}^{I}E_o\, a_o)(U_{10} + 0 + U_{12} + 0 + \cdots) \\
0 + \alpha_{01} V_{21} + 0 + \alpha_{03} V_{23} + \cdots + \beta_{10} W_{20} + 0 + \beta_{12} W_{22} + 0 + \cdots = ({}^{I}E_o\, a_o)(0 + U_{21} + 0 + U_{23} + \cdots) \\
\alpha_{00} V_{30} + 0 + \alpha_{02} V_{32} + 0 + \cdots + 0 + \beta_{11} W_{31} + 0 + \beta_{13} W_{33} + \cdots = ({}^{I}E_o\, a_o)(U_{30} + 0 + U_{32} + 0 + \cdots) \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}
\tag{111}
$$

Each term $(\alpha_{ol}\ {}^{x}\underline{M}_{ol}^{(4)} + \beta_{1l}\ {}^{z}\underline{M}_{1l}^{(4)})$ in the expression for ${}^{S}\underline{E}$, given by equation (75), represents the reradiation from the spheroid for the forced oscillation of the $l^{\underline{th}}$ mode. From physical reasoning it is known that the amplitude of the higher modes becomes less the higher the order of the mode. Hence the coefficients α_{ol} and β_{1l} must approach zero as l increases. Therefore, in using equations (108) and (109) to find α_{ol} and β_{1l}, reasonably accurate values may be obtained by assuming that α_{ol} and β_{1l} are zero for values of l above a given number l'. We then use $2\,l'$ of these equations to find α_{ol} and β_{1l} for values of l up to l'. When these values of α_{ol} and β_{1l} are used in equation (75) a solution for ${}^{S}\underline{E}$ is obtained which satisfies the four conditions listed at the beginning of Section IV.

It is evident that a large amount of calculating is required in order to obtain numerical results using equation (75). Some consideration has been given to the possibilities of obtaining a simpler solution which would be more amenable to numerical calculation. To represent ${}^{S}\underline{E}$ it would be desirable to have available two sets of vector functions, one of which had no η-component and the other without a ϕ-component. Then only one set of undetermined coefficients would appear in equation (69) and only the other set would be present in equation (70). This would simplify somewhat the process of calculating these coefficients. Further simplification would result if ${}^{I}\underline{E}$ were expressed by equation (55), which contains only one term for each value of l and this term involves, in addition to some algebraic functions of η, only $S_{ol}^{(1)}(\eta)$ and not its derivatives; and if the above-mentioned vector functions used for expressing ${}^{S}\underline{E}$ were to involve only $S_{ol}^{(1)}(\eta)$ with other algebraic functions of η identical to those occurring in the expression for ${}^{I}\underline{E}$. Then, due to the orthogonality integral (equation 88), each coefficient α_{ol} and β_{1l} would be given by a single equation involving no other coefficient. It would be very desirable that the divergence of each of these functions be zero for all values of the eigenvalues l and m; then each term of the series for the scattered wave would be divergenceless with the result that the divergence of the complete expression would be zero, as is necessary. Morse and Feshbach (Ref. 17, p. 445) develop two solutions of the vector Helmholtz equation which have the above-mentioned desirable properties, that one solution lacks an η-component and the other lacks a ϕ-component. However, these two vector functions are independent of ϕ and cannot be used to express the ${}^{I}\underline{E}$ or ${}^{S}\underline{E}$ occurring in the present problem because, due to their polarization, both ${}^{I}\underline{E}$ and ${}^{S}\underline{E}$ are functions of ϕ. No other solutions of the vector Helmholtz equation have been found which have the desired properties.

VI. PHYSICAL PROPERTIES OF SCATTERED WAVE

Equation (75), with the appropriate values of α_{ol} and β_{1l} as calculated from equations (108) and (109), is an expression for the electric field vector of the scattered wave (${}^{S}\underline{E}$) which is valid for all values of $\eta\,(-1 \leq \eta \leq 1)$, of $\xi\,(\xi \geq \xi_o)$, and of ϕ. This means that equation (75) can be used for calculating ${}^{S}\underline{E}$ in the immediate vicinity of the scattering spheroid as well as at great distances from the spheroid. If it is desired to find the magnetic field vector of the scattered wave (${}^{S}\underline{H}$) this can be calculated from ${}^{S}\underline{E}$ by using equation (42).

Usually one is more interested in the behavior of the scattered field at relatively great distances from the spheroid and to most easily deduce this behavior it is well to take the limiting form of ${}^{S}\underline{E}$ as $\xi \to \infty$, assuming that $c = \frac{2\pi}{\lambda} F$ does not equal zero. This latter requirement of course means that neither the eccentricity of the generating ellipse of the prolate spheroid, nor the frequency of the electromagnetic waves under consideration, become zero. To obtain the asymptotic form of equation (75) one uses equations (81) and (82) and there results

$$
\begin{aligned}
{}^{S}\underline{E} = \Bigg\{ & \underline{i}_\eta \left[\sum_{l=0}^{\infty} j^{\,l+2}\, \alpha'_{ol}\, S^{(1)}_{ol}(\eta) \sin\phi \right] \\
& + \underline{i}_\phi \left[\sum_{l=0}^{\infty} j^{\,l}\, \alpha'_{ol}\, \eta\, S^{(1)}_{ol}(\eta) \cos\phi + j^{\,l+3} \beta'_{1l} (1-\eta^2)^{\frac{1}{2}} S^{(1)}_{1l}(\eta) \cos\phi \right] \Bigg\} \\
& \times ({}^{I}E_o a_o) \times \frac{1}{r}\, e^{-j\frac{2\pi}{\lambda} r} \,.
\end{aligned}
\tag{112}
$$

Here $\alpha'_{ol} = \frac{1}{{}^{I}E_o a_o} \alpha_{ol}$ and $\beta'_{1l} = \frac{1}{{}^{I}E_o a_o} \beta_{1l}$. We notice at once the absence of an $\underline{i}_\xi$ component, which is the longitudinal component for large values of ξ. This, of course, is a necessary condition that the wave be a purely transverse wave at a large distance from the spheroid, as it must be.

In order to obtain more information concerning ${}^{s}\underline{E}$ it is best to express the electric field vector in terms of components along the rectangular coordinate axes. This is done by using the following relations:

$$\underline{i}_{\eta} = \underline{i}_x\left[-\eta(\xi^2-1)^{1/2}(\xi^2-\eta^2)^{-1/2}\cos\phi\right]$$

$$+ \underline{i}_y\left[-\eta(\xi^2-1)^{1/2}(\xi^2-\eta^2)^{-1/2}\sin\phi\right]$$

$$+ \underline{i}_z\left[\xi(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1/2}\right] , \tag{113}$$

$$\underline{i}_{\xi} = \underline{i}_x\left[\xi(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1/2}\cos\phi\right]$$

$$+ \underline{i}_y\left[\xi(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1/2}\sin\phi\right]$$

$$+ \underline{i}_z\left[\eta(\xi^2-1)^{1/2}(\xi^2-\eta^2)^{-1/2}\right] , \tag{114}$$

$$\underline{i}_{\phi} = \underline{i}_x\left[-\sin\phi\right] + \underline{i}_y\left[\cos\phi\right] . \tag{115}$$

These expressions are obtainable from equations (58), (65), and (66). The resulting expression for ${}^{s}\underline{E}$ is

$$\begin{aligned}{}^{s}\underline{E} = \Bigg\{ &\underline{i}_x\left[\sum_{l=0}^{\infty} j^{l+1}\beta'_{1l}\sqrt{1-\eta^2}\; S_{1l}^{(1)}(\eta)\sin\phi\cos\phi\right] \\ &+ \underline{i}_y\left[\sum_{l=0}^{\infty} j^{l}\alpha'_{0l}\,\eta\, S_{0l}^{(1)}(\eta) + j^{l+3}\beta'_{1l}\sqrt{1-\eta^2}\; S_{1l}^{(1)}(\eta)\cos^2\phi\right] \\ &+ \underline{i}_z\left[\sum_{l=0}^{\infty} j^{l+2}\alpha'_{0l}\sqrt{1-\eta^2}\; S_{0l}^{(1)}(\eta)\sin\phi\right]\Bigg\} \times \frac{{}^{I}E_0 a_0}{r}\, e^{-j\frac{2\pi}{\lambda}r} .\end{aligned} \tag{116}$$

It will be recalled that the incident wave was taken as moving along the z-axis, in the negative z-direction, with the E-vector pointing along the positive y-axis and the H-vector pointing along the positive x-axis. From equation (116) it can be seen that the back-scattered wave, which is the wave moving along the z-axis in the positive z-direction (thus $\eta = 1$ in equation 116), has a component of the E-vector along the y-direction only. Thus the back-scattered wave has the same polarization as the incident wave, which is a well-known property of back-scattered waves from smooth surfaces.

From equation (116) it may be seen, also, that the E-vector of scattered radiation being propagated in the xz-plane ($\phi = 0$ or π) has only a y-component and that of scattered radiation moving in the yz-plane ($\phi = \pm \frac{\pi}{2}$) has both a y-component and a z-component, except along the y-axis ($\eta = 0$) where the y-component disappears, as it must for a transverse wave.

A property of considerable interest is the scattering cross-section of the prolate spheroid. This scattering cross-section may be defined as the interception cross-section (σ) of an isotropic scatterer which scatters in the direction under consideration the same power density as the prolate spheroid scatters in this direction.

The magnitude of the total power ${}^{I}P$ extracted from the incident wave by the isotropic scatterer of cross-section σ is

$$ {}^{I}P = \sigma \left| {}^{I}\underline{p} \right| = \sigma \left| {}^{I}\underline{E} \times {}^{I}\underline{H} \right| = \sigma \sqrt{\frac{\epsilon_0}{\mu_0}} \left({}^{I}E_0 \right)^2 , \qquad (117) $$

where ${}^{I}\underline{p}$ is the Poynting vector of the incident wave. The magnitude of the Poynting vector of the wave scattered by the isotropic scatterer is given by

$$ \left| {}^{S}\underline{p} \right| = \frac{{}^{I}P}{4\pi r^2} = \frac{1}{4\pi} \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{\left({}^{I}E_0 \right)^2}{r^2} \sigma , \qquad (118) $$

where r is the distance from the center of the isotropic scatterer to the point of observation.

In order to compute the magnitude of the Poynting vector of the wave scattered by the prolate spheroid we use equation (116) in the following form:

$$^{S}\underline{E} = \left\{ \underline{i}_x \, {}^{S}T_x(\eta,\phi) + \underline{i}_y \, {}^{S}T_y(\eta,\phi) + \underline{i}_z \, {}^{S}T_z(\eta,\phi) \right\} \times \frac{{}^{I}E_o a_o}{r} \, e^{-j\frac{2\pi}{\lambda} r} , \tag{119}$$

where ${}^{S}T_x$, ${}^{S}T_y$, and ${}^{S}T_z$ are complex. Now let

$$^{S}\underline{T} \, e^{j\tau} = \underline{i}_x \, {}^{S}T_x + \underline{i}_y \, {}^{S}T_y + \underline{i}_z \, {}^{S}T_z \tag{120}$$

and

$$^{S}\underline{T} = \underline{i}_T \, {}^{S}T \quad , \tag{121}$$

where ${}^{S}\underline{T}$ and ${}^{S}T$ are real.

Then

$$^{S}\underline{E} = \underline{i}_T \, {}^{S}T \, \frac{{}^{I}E_o a_o}{r} \, e^{-j\left(\frac{2\pi}{\lambda} r - \tau\right)} \tag{122}$$

and

$$\left| {}^{S}\underline{E} \right|^2 = \left({}^{S}T \right)^2 \frac{\left({}^{I}E_o \right)^2 a_o^2}{r^2} \quad . \tag{123}$$

Hence the magnitude of the corresponding Poynting vector at the point of observation is

$$\left| {}^{S}\underline{p} \right| = \sqrt{\frac{\epsilon_o}{\mu_o}} \left({}^{S}T \right)^2 \frac{\left({}^{I}E_o \right)^2 a_o^2}{r^2} \quad . \tag{124}$$

Now σ in equation (118) must be so adjusted that the values of $\left| {}^{S}\underline{p} \right|$ given by equation (118) and by equation (124) are equal. The resulting value of σ is

$$\sigma = 4\pi ({}^{s}T)^2 a_o^2 \ . \tag{125}$$

In deriving this result it has been tacitly assumed that the polarization of the antenna receiving the scattered energy is so oriented that its pick-up is a maximum.

It is interesting to note the form which equation (116) takes for $F \to 0$, that is, when the prolate spheroid becomes a sphere. Then

$$c = kF \to 0 \quad , \tag{126}$$

$$c\xi = kF \frac{a}{F} = ka \to kr \quad , \tag{127}$$

since, for $F \to 0$,

$$r \to a \to b \quad . \tag{128}$$

Also

$$\eta = \cos\theta \quad , \tag{129}$$

$$S_{ml}^{(1)}(\eta) = P_{m+l}^{m} (\cos\theta) \quad , \tag{130}$$

where θ is the angle between the radius vector and the z-axis. Then equation (116) becomes

$$\begin{aligned} {}^{s}\underline{E} = \Big\{ & \underline{i}_x \Big[\tfrac{1}{2} \sin\theta \sin 2\phi \sum_{l=0}^{\infty} j^{l-1} \beta_{1l}' P_{l+1}^{1} (\cos\theta) \Big] \\ + & \underline{i}_y \Big[\cos\theta \sum_{l=0}^{\infty} j^{l+2} \alpha_{0l}' P_l (\cos\theta) + \sin\theta \cos^2\phi \sum_{l=0}^{\infty} j^{l+1} \beta_{1l}' P_{l+1}^{1} (\cos\theta) \Big] \\ + & \underline{i}_z \Big[\sin\theta \sin\phi \sum_{l=0}^{\infty} j^{l} \alpha_{0l}' P_l (\cos\theta) \Big] \Big\} \times \frac{{}^{I}E_o r_o}{r} \, e^{-j\frac{2\pi}{\lambda} r} \quad , \end{aligned} \tag{131}$$

where r_o is the radius of the scattering sphere. Thus the functions involved are of the correct type for the solution of the spherical scattering problem. Further, by noting that, for $c = 0$,

$$d_n^{ml} = \delta_{nl} = \begin{cases} 0 & n \neq l \\ 1 & n = l \end{cases} , \tag{132}$$

it can be shown from equations (13) and (15) that

$$R_{ol}^{(1)}(\xi) = j_l(kr) , \tag{133}$$

$$R_{ol}^{(2)}(\xi) = n_l(kr) , \tag{134}$$

$$R_{1l}^{(1)}(\xi) = j_{n+1}(kr) , \tag{135}$$

$$R_{1l}^{(2)}(\xi) = n_{n+1}(kr) . \tag{136}$$

$j_n(z)$ and $n_n(z)$ are spherical Bessel functions of the first and second kinds, respectively.

From these last four equations it can be seen that the quantities $B_{\angle l}$, $C_{\angle l}$, $\cdots$ $W_{\angle l}$, given by equations (102) to (107), inclusive, involve functions of the same type as those which occur in the spherical case, hence, so also do the coefficients α_{ol} and β_{1l} since these are functions of $B_{\angle l}$, $C_{\angle l}$, $\cdots$ $W_{\angle l}$. Therefore equation (131) contains, as is to be expected, only functions of the type which occur in the solution of the spherical scattering problem (Ref. 16, p. 563).

VII. NUMERICAL CALCULATIONS

In order to obtain numerical results from equations (75), (112), (116), or (125), it is necessary to first compute the coefficients α_{ol} and β_{1l} from the sets of equations (108) and (109). In turn, before using equations (108) and (109), one must calculate the quantities $B_{\angle l}$, $C_{\angle l}$, $\cdots$ $W_{\angle l}$, using equations (102) to (107), inclusive.

It is assumed that the major and minor axes of the scattering prolate spheroid and the value of the wavelength are given. Then, using the equations given in Section II, one may calculate ξ_o for the spheroid, as well as k and c.

The following information will be useful in calculating B_{Ll}, C_{Ll}, ... W_{Ll}.

A_{ol} may be computed from equation (50), N_{ol} being given by equation (52). The coefficients d_n^{ml} are tabulated in Stratton, Morse, Chu and Hutner (Ref. 15).

$R_{ml}^{(1)}(\xi)$ is given by equation (13). The derivative of this may be taken and this will involve the derivative of $j_{m+n}(c\xi)$, or of $J_{m+n+\frac{1}{2}}(c\xi)$, which may be expressed in terms of Bessel functions by means of the differential recurrence relations for Bessel functions (Ref. 16, pp. 360 and 406).

$$\frac{d}{d\xi} j_{m+n}(c\xi) = c \frac{d}{d(c\xi)} j_{m+n}(c\xi) =$$

$$\frac{c}{2m+2n+1}\left[(m+n) j_{m+n-1}(c\xi) - (m+n+1) j_{m+n+1}(c\xi)\right], \qquad (137)$$

$$\frac{d}{d\xi} J_{m+n+\frac{1}{2}}(c\xi) = \frac{c}{2}\left[J_{m+n-\frac{1}{2}}(c\xi) - J_{m+n+\frac{3}{2}}(c\xi)\right], \qquad (138)$$

where

$$j_{n+m}(c\xi) = \sqrt{\frac{\pi}{2c\xi}}\, J_{n+m+\frac{1}{2}}(c\xi). \qquad (139)$$

Values of $j_{m+n}(c\xi)$ and of $j_{-m-n}(c\xi)$, useful for computing $R_{ml}^{(1)}(\xi)$ and its derivatives, are tabulated (Ref. 19).

$R_{ml}^{(4)}(\xi)$ is given by equation (20), and the $R_{ml}^{(2)}(\xi)$ used in this equation is given by equation (15), or equations (17) and (18). The $n_{m+n}(c\xi)$ occurring in equation (15) can be computed from the tabulated values of $j_{n+m}(c\xi)$ by using these relations (Ref. 16 and Ref. 20, p. 132):

$$n_{m+n}(c\xi) = \sqrt{\frac{\pi}{2c\xi}}\, N_{n+m+\frac{1}{2}}(c\xi) = (-1)^{n+m-1}\sqrt{\frac{\pi}{2c\xi}}\, J_{-n-m-\frac{1}{2}}. \qquad (140)$$

The derivative of $R_{ml}^{(4)}(\xi)$ involves the derivatives of $R_{ml}^{(1)}(\xi)$ and $R_{ml}^{(2)}(\xi)$. The derivative of $R_{ml}^{(2)}(\xi)$ involves the derivative of $n_{m+n}(c\xi)$ which can be computed by using a differential recurrence relation exactly similar to that given by equation (137).

In the particular calculations contained herein ξ_o is very close to unity (1.00504) and $R_{ml}^{(2)}(\xi)$ must be computed from equations (17) and (18). We make use of equation (1) on page 72 of Stratton, Morse, Chu and Hutner (Ref. 15) and obtain the following expression for $R_{ml}^{(2)}(\xi)$:

$$R_{ml}^{(2)}(\xi) = G(c,m,l)\left\{ \sum_{n=-\infty}^{\substack{-2m-1 \\ \text{or} \\ -2m-2}}{}' \frac{d_n^{ml}}{\rho}\left[P_{-n-m-1}^{m}(\xi)\right] + \sum_{\substack{n=-2m+1 \\ \text{or} \\ -2m}}^{+\infty}{}' d_n^{ml}\, Q_{m+n}^{m}(\xi) \right\}, \tag{141}$$

where, for l even,

$$G(c,m,l) = \frac{2c^{m-1}\Gamma\left(\frac{l+2m+1}{2}\right)\cdot}{\Gamma\left(\frac{l+2}{2}\right)\Gamma\left(m-\frac{1}{2}\right) d_{-2m}^{ml} \sum_{n=0}^{\infty}{}' d_n^{ml}\,\frac{(n+2m)!}{n!}}, \tag{142}$$

and, for l odd,

$$G(c,m,l) = \frac{-8c^{m-2}\Gamma\left(\frac{l+2m+2}{2}\right)}{\Gamma\left(\frac{l+1}{2}\right)\Gamma\left(m-\frac{3}{2}\right) d_{1-2m}^{ml} \sum_{n=1}^{\infty}{}' d_n^{ml}\,\frac{(n+2m)!}{n!}}. \tag{143}$$

Values of $\frac{d_n^{ml}}{\rho}$ are listed by Stratton, Morse, Chu and Hutner for negative values of n , for $|n| > 2m$.

The derivative of $R_{ml}^{(2)}(\xi)$ is easily written:

$$\frac{d}{d\xi}R_{ml}^{(2)}(\xi) = G(c,m,l)\left\{ \sum_{n=-\infty}^{\substack{-2m-1 \\ \text{or} \\ -2m-2}}{}' \frac{d_n^{ml}}{\rho}\left[\frac{d}{d\xi}P_{-n-m-1}^{m}(\xi)\right] + \sum_{\substack{n=-2m+1 \\ \text{or } -2m}}^{+\infty}{}' d_n^{ml}\,\frac{d}{d\xi}Q_{m+n}^{m}(\xi) \right\}. \tag{144}$$

Values of $P_n^m(\xi)$, $Q_n^m(\xi)$, and of their first derivatives, have been tabulated (Ref. 21). However, for values of ξ between 1.0 and 1.1 these tables must be used with care. Actually, for such values of ξ, it is believed that the best way of carrying out the computations is by the use of equations (145) and (146), which are given in this set of tables and are listed below.

$$P_{-n-m-1}^{m}(\xi) = \frac{(\xi^2-1)^{m/2}}{2^{-n-m-1}} \sum_{r=0}^{\left[\frac{-n-2m-1}{2}\right]} \frac{(-1)^r(-2n-2m-2-2r)!}{r!(-n-m-1-r)!(-n-2m-1-2r)!} \quad \xi^{-n-2m-1-2r} \tag{145}$$

The symbol $\left[\frac{-n-2m-1}{2}\right]$ indicates the largest integer contained in $\frac{-n-2m-1}{2}$. In equation (145) m and $-n-m-1$ must be positive integers or zero. In the case under consideration m is either 0 or 1 and from equation (141) it can be seen that n runs from $-\infty$ to $-2m-1$ or $-2m-2$ so that $-n-m-1$ goes from m or $m+1$ to $+\infty$.

For calculating $Q_{m+n}^{m}(\xi)$ we use

$$\begin{aligned} Q_{m+n}^{m}(\xi) &= \frac{1}{2} P_{m+n}^{m}(\xi) \log_e \frac{\xi+1}{\xi-1} - \frac{m}{(\xi^2-1)^{1/2}} P_{m+n}^{m-1}(\xi) \\ &+ \frac{m(m-1)\xi}{\xi^2-1} P_{m+n}^{m-2}(\xi) - \frac{m!(3\xi^2+1)}{3(m-3)!(\xi^2-1)^{3/2}} P_{m+n}^{m-3}(\xi) \\ &+ \frac{m!(\xi^3+\xi)}{(m-4)!(\xi^2-1)^2} P_{m+n}^{m-4}(\xi) - \sum_{j=0}^{\left[\frac{1}{2}(m+n-1)\right]} \frac{2m+2n-1-4j}{(1+2j)(m+n-j)} P_{m+n-1-2j}^{m}(\xi). \end{aligned} \tag{146}$$

This expression for $Q_{m+n}^{m}(\xi)$ is valid for $0 \leq m \leq 4$; for n and m both integers; for $n > 0$; and for $\xi \geq 1$. For both m and n zero, the following equation given in the tables is to be used (Ref. 21, p. xiv):

$$Q_0(\xi) = \frac{1}{2} \log_e\left(\frac{\xi+1}{\xi-1}\right). \tag{147}$$

When $m=1$ and $n=-2$, the following relation may be used:

$$Q_{-1}^{1}(\xi) = -\frac{\xi}{\sqrt{\xi^{2}-1}} \quad . \tag{148}$$

This equation may be derived by application of the following recurrence relation listed at the top of page 62 of Magnus-Oberhettinger (Ref. 22):

$$Q_{\nu-1}^{\mu}(z) - z\,Q_{\nu}^{\mu}(z) = -(\nu-\mu+1)\sqrt{z^{2}-1}\;Q_{\nu}^{\mu-1}(z) \quad . \tag{149}$$

When $m=1$ and $n=-1$ there arises the problem of evaluating $Q_{0}^{1}(\xi)$. Equation (146) is not valid for this calculation so the definition of $Q_{n}^{m}(\xi)$ is used.

$$Q_{n}^{m}(\xi) = (\xi^{2}-1)^{m/2}\,\frac{d^{m}}{d\xi^{m}}\,Q_{n}(\xi) \quad . \tag{150}$$

Then

$$Q_{0}^{1}(\xi) = (\xi^{2}-1)^{1/2}\,\frac{d}{d\xi}\,Q_{0}(\xi) \quad . \tag{151}$$

$Q_{0}(\xi)$ is given by equation (147) and the application of equation (151) gives

$$Q_{0}^{1}(\xi) = -(\xi^{2}-1)^{-1/2} \, . \tag{152}$$

In order to evaluate the derivative of $R_{m\ell}^{(2)}(\xi)$, given by equation (144), it is necessary to compute the derivatives of $P_{-n-m-1}^{m}(\xi)$ and of $Q_{m+n}^{m}(\xi)$. This may be done by using the following differential recurrence relation given on page 115 of Jahnke-Emde (Ref. 20):

$$\frac{d}{d\xi}K_{n'}^{m}(\xi) = (\xi^{2}-1)^{-1}\left[(n'-m+1)K_{n'+1}^{m}(\xi) - (n'+1)\xi K_{n'}^{m}(\xi)\right] . \tag{153}$$

Here $K_n^m(\xi)$ may be either $P_n^m(\xi)$ or $Q_n^m(\xi)$. n' and m must be integers and it is necessary that

$$n' \geqq m \geqq 0 \quad . \tag{154}$$

Then

$$\frac{d}{d\xi} P^m_{-n-m-1}(\xi) = (\xi^2-1)^{-1}\left[(-2m-n)P^m_{-m-n}(\xi) + (n+m)\xi P^m_{-n-m-1}(\xi)\right] . \tag{155}$$

Equation (144) shows that

$$-\infty \leq n \leq \begin{Bmatrix} -2m-1 \\ -2m-2 \end{Bmatrix} \quad . \tag{156}$$

Therefore, in equation (155),

$$\begin{Bmatrix} m \\ m+1 \end{Bmatrix} \leq (-n-m-1) \leq +\infty \quad . \tag{157}$$

Since, in our work, $m \geqq 0$, inequality (157) shows that equation (155) always satisfies condition (154). Now $P^m_{-m-n}(\xi)$ and $P^m_{-n-m-1}(\xi)$ in equation (155) may be evaluated by using equation (145).

From equation (153) we find

$$\frac{d}{d\xi} Q^m_{m+n}(\xi) = (\xi^2-1)^{-1}\left[(n+1)Q^m_{m+n+1}(\xi) - (n+m+1)\xi Q^m_{n+m}(\xi)\right] . \tag{158}$$

This equation is valid for

$$(m+n) \geqq m \geqq 0 \quad . \tag{159}$$

In the calculations contained herein, m is either 0 or 1, so the second part of condition (159) is satisfied. The first part of the condition is met for $n \geqq 0$. From equation (144) it may be seen that for $m = 1$ it is necessary to find the derivative of $Q^m_{m+n}(\xi)$ for $n = -2$ and for $n = -1$. For $n = -2$ we use equation (148) and obtain

$$\frac{d}{d\xi} Q^1_{-1}(\xi) = (\xi^2-1)^{-3/2} \quad . \tag{160}$$

When $\eta = -1$, we find from equation (152) that

$$\frac{d}{d\xi} Q_0^1(\xi) = \xi(\xi^2-1)^{-3/2} . \tag{161}$$

The various $Q_\nu^\mu(\xi)$ occurring in equation (158) can be calculated by the use of equation (146).

Calculations have been carried out for $\frac{a_0}{b_0} = 10$ and $c = \frac{2\pi F}{\lambda} = 2$. Here a_0 is the semimajor axis, and b_0 the semiminor axis of the prolate spheroid. The tables included in Stratton, Morse, Chu and Hutner list values of d_n^{ml} for $l=0,1,2,3$ and $m=0$; and for $l=0,1,2$ and $m=1$. Therefore values of α_{00} to α_{02} and of β_{10} to β_{12}, inclusive, were calculated, assuming that α_{0l} and β_{1l} were zero for $l>2$. The convergence is fairly rapid for $c=2$ so the accuracy of the values of α_{0l} and β_{1l} , which were obtained, is fairly good. Then, using equation (125), the value of σ was found for the back-scattering case, that is, for $\eta = 1$. This value is listed below, along with the value of σ for the back-scattering of a scalar wave as calculated in Appendix A.

Due to the large eccentricity of the scattering spheroid, existing tables of Legendre functions were inadequate for the calculations and all values had to be computed by using the formulas contained in this section. Approximately two months of full-time computing were required to obtain this single value of σ for the prolate spheroid. Very likely no more values will be computed until more extensive tables are available.

McAfee and Wolfe (Ref. 8) have determined the scattering of an electromagnetic wave from an ellipsoid by using physical optics. For comparison purposes, the value of σ calculated from their results is included below for the same physical conditions as were used in calculating the values of σ by using electromagnetic theory and scalar theory.

σ may also be calculated by using the geometrical optics equation

$$\sigma = \pi R_1 R_2 , \tag{162}$$

where R_1 and R_2 are the two principal radii of curvature at the point of reflection.

Then, for $c = \frac{2\pi F}{\lambda} = 2$ and for $\frac{a_o}{b_o} = 10$, we obtain the following results:*

	$\frac{\sigma}{a_o^2}$
electromagnetic theory (exact method)	0.994×10^{-3}
scalar waves (exact method)	0.574×10^{-3}
physical optics (approximate method)	0.499×10^{-3}
geometrical optics (approximate method)	0.314×10^{-3}

The value of the ratio σ/r_o^2 for a sphere (r_o being the radius of the sphere) has been computed using both electromagnetic theory and the method of physical optics (Ref. 23, p. 4). The results obtained using the two methods display the same general type of behavior. The ratio σ/r_o^2 approaches zero as the wavelength increases without limit. As the wavelength decreases from infinity the ratio rises to a maximum value and then drops to a minimum as the wavelength is further decreased. This oscillatory behavior continues as the wavelength is continually decreased but the amplitude of the oscillations becomes less and less and the curve becomes asymptotic to the constant value of $\sigma/r_o^2 = \pi = \sigma_o$ given by geometrical optics. The curve of σ/a_o^2 versus wavelength for a prolate spheroid, obtained using physical optics (Ref. 8), shows the same type of behavior. It happens that the first maximum below $\lambda = \infty$, or above $c = 0$, occurs for $c = 2$.

If the values of σ for either the spheroid or the sphere are divided by the asymptotic value, a dimensionless result is obtained which indicates the deviation of the particular value of σ from the asymptotic value. For purposes of comparison these ratios are listed below for both the prolate spheroid and a sphere. For the sphere there are listed the limits between which the values lie after the first maximum is attained.

	Prolate Spheroid $\frac{\sigma}{a_o^2 \sigma_o}$	Sphere $\frac{\sigma}{r_o^2 \sigma_o}$
electromagnetic theory (exact method)	3.16	0.23 to 3.7
scalar waves (exact method)	1.83	
physical optics (approximate method)	1.59	0.77 to 1.7
geometrical optics (approximate method)	1.00	1.00

* Some of the numerical values appearing on this page are in error. These numbers will be corrected and additional numerical work on the problem will be presented in a forthcoming report, UMM-126.

VIII. CONCLUSIONS

The maximum value of c, for which Stratton, Morse, Chu and Hutner have computed the coefficients d_n^{ml}, is 5. This corresponds to a ratio of semifocal distance (F) to wavelength (λ) of 0.796. Therefore, in order to obtain numerical results for smaller values of this ratio (shorter wavelengths and therefore $c > 5$), it would be necessary to extend the tables of coefficients. Stratton, Morse, Chu and Hutner (Ref. 15, p. 15) indicate the procedure for doing this.

The tables of d_n^{ml} mentioned above list values of d_n^{ml} for: $l=0,1,2,3$ for $m=0$; $l=0,1,2$ for $m=1$; $l=0,1$ for $m=2$; and $l=0$ for $m=3$. For higher values of c (shorter wavelengths) it is to be expected that the values of α_{0l} and β_{1l} will approach zero more slowly as l increases. Therefore, in order to use the results given herein to determine the scattering from a prolate spheroid for $c>5$, it would be necessary that the values of d_n^{ml} be found for values of l greater than 3.

Much of the tedious computing required to obtain numerical results was due to the fact that the values of $P_n^m(\xi)$, $Q_n^m(\xi)$, and of their first derivatives, had to be computed, using equations given in Section VII. A value of $\xi = \xi_o = 1.00504$ was being used. The available tables (Ref. 21) list values of $P_n^m(\xi)$, $Q_n^m(\xi)$ and of their first derivatives only for $\xi = 1.0, 1.1, 1.2, \ldots$ and accurate interpolation is not possible for values of ξ between 1.0 and 1.1. Therefore, a great help in using the results contained herein to obtain numerical results, would be the tabulation of values of $P_n^m(\xi)$, $Q_n^m(\xi)$ and of their first derivatives for $1.0 < \xi < 1.1$.

In order to reduce still more the amount of calculating required to apply our results to specific cases, the tabulation of values of $R_{ml}^{(1)}(\xi)$, $R_{ml}^{(2)}(\xi)$, their derivatives, and $S_{ml}^{(1)}(\eta)$ would be extremely helpful. This should be done for $1 \leq \xi \leq \infty$, $-1 \leq \eta \leq +1$ and for a wide range of values of the parameter c. The formulas included in Sections II and VII are useable for such a purpose.

In view of the wide applicability of the spheroidal functions it appears that the extension in available tables, as discussed above, is justified.

APPENDIX A

SCALAR SCATTERING

The problem of determining the scattering of a scalar wave is considerably simpler to solve than that of a vector wave. A sound wave is such a scalar wave, since it can be completely defined using scalar quantities.

Let Ψ be the velocity potential of such a monochromatic wave. Then

$$\Psi = \psi(\eta, \xi, \phi)\, e^{j\omega t} , \tag{A-1}$$

and ψ satisfies the scalar Helmholtz equation

$$\nabla^2 \psi + k^2 \psi = 0 . \tag{A-2}$$

ψ must also satisfy the two boundary conditions. Over the surface of any smooth rigid body,

$$\left[\nabla \psi\right]_{normal} = 0 , \tag{A-3}$$

where $\left[\nabla \psi\right]_{normal}$ is the normal component of the gradient of ψ over the surface, and thus is the negative of the normal component of velocity over this surface. The second boundary condition is that, for a scattering body of finite size, the scattered wave at great distances from the scatterer must behave as a spherical diverging wave.

Now let it be assumed that a smooth, rigid prolate spheroid is located with its center at the center of a rectangular Cartesian coordinate system and that the z-axis is the axis of rotation of the spheroid. The prolate spheroidal coordinate system discussed in Section II is also used here.

Suppose that a plane scalar wave is moving parallel to the z-axis in the direction of decreasing z. The velocity potential $({}^I\psi)$ of such a wave may be expressed as follows:

$${}^I\psi = {}^I\psi_o\, e^{jkz} = {}^I\psi_o \sum_{l=0}^{\infty} A_{ol}\, S_{ol}^{(1)}(\eta) R_{ol}^{(1)}(\xi) , \tag{A-4}$$

by using the material discussed in Section III. The time factor, $e^{j\omega t}$, has been omitted from both sides of this equation. A_{ol} is given by equation (50). ${}^I\psi_o$ is the maximum amplitude reached by the velocity potential of the incident wave.

Let the amplitude of the velocity potential of the scattered wave be expressed as follows:

$$^S\psi = \sum_{l=o}^{\infty} \gamma_{ol}\, \psi_{ol}^{(4)} = \sum_{l=o}^{\infty} \gamma_{ol}\, S_{ol}^{(1)}(\eta) R_{ol}^{(4)}(\xi) \, , \tag{A-5}$$

where the γ_{ol} are undetermined coefficients and $\psi_{ol}^{(4)}$ is a solution of equation (A-2). The use of $R_{ol}^{(4)}(\xi)$ insures the proper behavior at great distances from the spheroid.

The boundary condition over the surface of the spheroid is given by equation (A-3), which may be written thus

$$\left[\frac{\partial \psi}{\partial \xi}\right]_{\xi=\xi_o} = \left[\frac{\partial\, {}^I\psi}{\partial \xi} + \frac{\partial\, {}^S\psi}{\partial \xi}\right]_{\xi=\xi_o} = 0 \, . \tag{A-6}$$

ξ_o is the value of ξ on the surface of the scattering spheroid. By using the values of ${}^I\psi$ and ${}^S\psi$ given by equations (A-4) and (A-5), respectively, we obtain from equation (A-6)

$${}^I\psi_o \sum_{l=o}^{\infty} A_{ol}\, S_{ol}^{(1)}(\eta)\left[\frac{d}{d\xi} R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o} + \sum_{l=o}^{\infty} \gamma_{ol}\, S_{ol}^{(1)}(\eta)\left[\frac{d}{d\xi} R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o} = 0 \, . \tag{A-7}$$

If this equation is multiplied by $S_{oL}^{(1)}(\eta)$ and each term is integrated with respect to η from -1 to +1, the orthogonality relation given by equation (88) may be applied and the following result is obtained:

$${}^I\psi_o A_{oL}\left[\frac{d}{d\xi} R_{oL}^{(1)}(\xi)\right]_{\xi=\xi_o} + \gamma_{oL}\left[\frac{d}{d\xi} R_{oL}^{(4)}(\xi)\right]_{\xi=\xi_o} = 0 \, . \tag{A-8}$$

Here L may be any non-negative integer. Equation (A-8) may be solved for γ_{oL}.

$$\gamma_{ol} = -{}^{I}\psi_o A_{ol} \frac{\left[\frac{d}{d\xi} R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o}}{\left[\frac{d}{d\xi} R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o}} \tag{A-9}$$

The use of this value of γ_{ol} in equation (A-5) gives a completely determined expression for the scattered velocity potential:

$$^{s}\psi = -{}^{I}\psi_o \sum_{l=0}^{\infty} A_{ol} \frac{\left[\frac{d}{d\xi} R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o}}{\left[\frac{d}{d\xi} R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o}} S_{ol}^{(1)}(\eta) R_{ol}^{(4)}(\xi) \,. \tag{A-10}$$

This expression for $^{s}\psi$ is valid for all values of ξ greater than ξ_o. At great distances from the spheroid the asymptotic form of $R_{ol}^{(4)}(\xi)$ given by equation (22) may be used and then equation (A-10) takes the form

$$^{s}\psi = \left\{ \frac{1}{c\xi_o} \sum_{l=0}^{\infty} A_{ol} \frac{\left[\frac{d}{d\xi} R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o}}{\left[\frac{d}{d\xi} R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o}} S_{ol}^{(1)}(\eta) \right\} \times \frac{{}^{I}\psi_o a_o}{r} e^{-j(kr - \frac{l+1}{2}\pi)} , \tag{A-11}$$

or

$$^{s}\psi = T \frac{{}^{I}\psi_o a_o}{r} e^{-jkr} , \tag{A-12}$$

where

$$T = \frac{1}{c\xi_o} \sum_{l=0}^{\infty} (j)^{l+1} A_{ol} \frac{\left[\frac{d}{d\xi} R_{ol}^{(1)}(\xi)\right]_{\xi=\xi_o}}{\left[\frac{d}{d\xi} R_{ol}^{(4)}(\xi)\right]_{\xi=\xi_o}} S_{ol}^{(1)}(\eta) \,. \tag{A-13}$$

Here a_o is the semimajor axis of the scattering spheroid.

The scattering cross-section (σ) of the spheroid can be calculated here as it was in Section VI. Let ^{s}I be the transmission of power per unit of area of wave-front, or the intensity, of the scattered wave. This can be calculated by using the following equation, given on page 349 of Morse and Feshbach (Ref. 17):

$$^{s}I = \frac{\omega \rho k}{2} T \, T^{*} \left(\frac{{}^{I}\psi_o a_o}{r} \right)^2 , \tag{A-14}$$

where ρ is the density of the transmitting medium, $k = \frac{2\pi}{\lambda}$ and ω is $2\pi f$, f being the frequency.

The total incident power (${}^{I}P$) intercepted by an isotropic scatterer of interception cross-section (σ) is

$$ {}^{I}P = \sigma\, {}^{I}I = \frac{w\rho k}{2} \left({}^{I}\psi_o\right)^2 \sigma \; . \tag{A-16} $$

The intensity ${}^{S}I$ at a distance r from the isotropic scatterer is

$$ {}^{S}I = \frac{{}^{I}P}{4\pi r^2} = \frac{w\rho k}{8\pi} \left(\frac{{}^{I}\psi_o}{r}\right)^2 \sigma \; . \tag{A-17} $$

σ must be so adjusted that the values of ${}^{S}I$ given by equations (A-14) and (A-17) are equal. Then σ must be

$$ \sigma = 4\pi a_o^2\, T\, T^* = 4\pi a_o^2 \left| T \right|^2 \; . \tag{A-18} $$

The calculation of σ is primarily the calculation of T. All the terms appearing in the expression for T have already been computed in Section VI for the case of $a_o/b_o = 10$ for the scattering spheroid, and for a value of c of 2. Under these conditions the evaluation of equation (A-18) gives a value of σ of $0.574 \times 10^{-3} a_o^2$ for the back-scattering cross-section ($\eta = 1$).

APPENDIX B

EVALUATION OF INTEGRALS

In this appendix we shall evaluate the integrals occurring in Section V. The integrals will not be evaluated in the order of their occurrence in equations (86) and (87), but rather in the order of their complexity, the simpler ones being evaluated first.

By equation (92)

$$I_3(L,l) = \int_{-1}^{+1} \eta \, S_{0l}^{(1)}(\eta) \, S_{0L}^{(1)}(\eta) \, d\eta \quad . \tag{B-1}$$

By using equation (85) we obtain

$$I_3(L,l) = \int_{-1}^{+1} \eta \Big[\sum_{\substack{n=0,1 \\ (l)}}^{\infty}{}' d_n^{0l} P_n(\eta) \Big] \Big[\sum_{\substack{n=0,1 \\ (L)}}^{\infty}{}' d_n^{0L} P_n(\eta) \Big] d\eta \quad . \tag{B-2}$$

The letter in parentheses under the summation symbol indicates whether l or L is to be used in determining whether even or odd values of n are to be used in the summation process. In order to carry out the integration of equation (B-2) we express the product of the summations as a Cauchy product (Ref. 24, p. 72):

$$\Big(\sum_{n=0}^{\infty} c_n \Big) \Big(\sum_{n=0}^{\infty} d_n \Big) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_k \, d_{n-k} \quad . \tag{B-3}$$

The result is

$$I_3(L,l) = \sum_{\substack{n=0,1 \\ (L+l)}}^{\infty}{}' \sum_{\substack{k=0,1 \\ (l)}}^{n}{}' d_k^{0l} d_{n-k}^{0L} \int_{-1}^{+1} \eta \, P_k(\eta) P_{n-k}(\eta) \, d\eta \, , \tag{B-4}$$

where k takes on $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ values if l is $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ and n takes on $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ values if $(L+l)$ is $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$. Copson (Ref. 25, p. 300) gives the value of the integral occurring in equation (B-4):

$$\int_{-1}^{+1} \eta P_k(\eta) P_{n-k}(\eta)\, d\eta = \begin{cases} \dfrac{2(k+1)}{(2k+1)(2k+3)} & \text{for } n-k = k+1 \\ \dfrac{2k}{4k^2-1} & \text{for } n-k = k-1 \\ 0 & \text{for all other values of } n \text{ and } k \end{cases} \tag{B-5}$$

It is to be noted that early printings of Copson erroneously include a factor π in the numerator of the evaluations of this integral. The integral is different from zero only if $n-k = k \pm 1$ or for $k = \frac{n \pm 1}{2}$. k must be an integer so n must take on only odd values. This means that $(L+l)$ must be odd and, therefore, that I_3 is different from zero only for $L \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}$ and $l \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$.

We then use equation (B-5) in equation (B-4) and obtain

$$I_3(L,l) = \sum_{n=1}^{\infty}{}' \left[\frac{n+1}{n(n+2)}\right]\left[d^{ol}_{\frac{n-1}{2}}\, d^{oL}_{\frac{n+1}{2}} + d^{ol}_{\frac{n+1}{2}}\, d^{oL}_{\frac{n-1}{2}}\right] . \tag{B-6}$$

We next evaluate I_4 , defined by equation (94).

$$I_4(L,l) = \int_{-1}^{+1} (1-\eta^2)\left[\frac{d}{d\eta} S^{(1)}_{ol}(\eta)\right] S^{(1)}_{oL}(\eta)\, d\eta . \tag{B-7}$$

The use of equation (85) gives the following result:

$$I_4(L,l) = \int_{-1}^{+1} (1-\eta^2) \sum_{\substack{n=0,1 \\ (l)}}^{\infty}{}' d^{ol}_n \frac{d}{d\eta} P_n(\eta) \sum_{\substack{n=0,1 \\ (L)}}^{\infty}{}' d^{oL}_n P_n(\eta)\, d\eta . \tag{B-8}$$

Now use is made of the following differential recurrence relation (Ref. 20, p. 115):

$$(1-\eta^2)\frac{d}{d\eta} P_n(\eta) = n\left[P_{n-1}(\eta) - \eta P_n(\eta)\right] , \tag{B-9}$$

and there results

$$I_4(L,l) = \int_{-1}^{+1} \sideset{}{'}\sum_{\substack{n=0,1\\(l)}}^{\infty} d_n^{ol}\, n P_{n-1}(\eta) \sideset{}{'}\sum_{\substack{n=0,1\\(L)}}^{\infty} d_n^{oL} P_n(\eta)\, d\eta$$

$$- \int_{-1}^{+1} \sideset{}{'}\sum_{\substack{n=0,1\\(l)}}^{\infty} d_n^{ol}\, n \eta P_n(\eta) \sideset{}{'}\sum_{\substack{n=0,1\\(L)}}^{\infty} d_n^{oL} P_n(\eta)\, d\eta \;. \tag{B-10}$$

The use of the Cauchy product formula gives

$$I_4(L,l) = \sideset{}{'}\sum_{\substack{n=0,1\\(L+l)}}^{\infty} \sideset{}{'}\sum_{\substack{k=0,1\\(l)}}^{n} k\, d_k^{ol} d_{n-k}^{oL} \int_{-1}^{+1} P_{k-1}(\eta) P_{n-k}(\eta)\, d\eta$$

$$- \sideset{}{'}\sum_{\substack{n=0,1\\(L+l)}}^{\infty} \sideset{}{'}\sum_{\substack{k=0,1\\(l)}}^{n} k\, d_k^{ol} d_{n-k}^{oL} \int_{-1}^{+1} \eta P_k(\eta) P_{n-k}(\eta)\, d\eta \;. \tag{B-11}$$

The first integral occurring in equation (B-11) is the well-known orthogonality integral for Legendre polynomials (Ref. 20, p. 116):

$$\int_{-1}^{+1} P_{k-1}(\eta) P_{n-k}(\eta)\, d\eta = \begin{cases} 0 & \text{for } k-1 \neq n-k \\ \dfrac{2}{2k-1} & \text{for } k-1 = n-k \end{cases} \;. \tag{B-12}$$

The value of the second integral of equation (B-11) has already been given in equation (B-5). Upon substituting equations (B-5) and (B-12) in equation (B-11) we obtain

$$I_4(L,l) = \sideset{}{'}\sum_{\substack{n=1\\(L+l)}}^{\infty} \frac{n+1}{2n(n+2)} \left[(n+3)\, d_{\frac{n+1}{2}}^{ol}\, d_{\frac{n-1}{2}}^{oL} - (n-1)\, d_{\frac{n-1}{2}}^{ol}\, d_{\frac{n+1}{2}}^{oL} \right] \;. \tag{B-13}$$

Since n must be odd, I_4 is zero for L and l both odd or both even.

I_5 will be evaluated next. It is given by equation (96).

$$I_5(L,l) = \int_{-1}^{+1} \sqrt{1-\eta^2}\; S_{1l}^{(1)}(\eta)\, S_{oL}^{(1)}(\eta)\, d\eta \tag{B-14}$$

The application of equation (85) and of the Cauchy product formula gives us

$$I_5(L,l) = \sum_{\substack{n=0,1\\(L+l)}}^{\infty}{}' \sum_{\substack{k=0,1\\(l)}}^{n}{}' d_k^{1l}\, d_{n-k}^{oL} \int_{-1}^{+1} \sqrt{1-\eta^2}\, P_{k+1}^{1}(\eta) P_{n-k}(\eta)\, d\eta\,. \tag{B-15}$$

Magnus-Oberhettinger (Ref. 22) gives the following recurrence relation which is useful in evaluating the integral occurring in equation (B-15):

$$\sqrt{1-\eta^2}\, P_\nu^{\mu+1}(\eta) = (\nu-\mu+1) P_{\nu+1}^{\mu}(\eta) - (\nu+\mu+1)\eta P_\nu^{\mu}(\eta)\,. \tag{B-16}$$

This relation is based on the following definition of $P_n^m(\eta)$:

$$P_n^m(\eta) = (-1)^m (1-\eta^2)^{\frac{m}{2}} \frac{d^m}{d\eta^m} P_n(\eta)\,, \tag{B-17}$$

which is the definition used by Stratton, Morse, Chu and Hutner.

When the value of $\sqrt{1-\eta^2}\, P_{k+1}^{1}(\eta)$, as given by equation (B-16), is substituted in equation (B-15), we obtain

$$I_5(L,l) = \sum_{\substack{n=0,1\\(L+l)}}^{\infty}{}' \sum_{\substack{k=0,1\\(l)}}^{n}{}' d_k^{1l}\, d_{n-k}^{oL} (k+2) \left[\int_{-1}^{+1} P_{k+2}(\eta) P_{n-k}(\eta)\, d\eta - \int_{-1}^{+1} \eta P_{k+1}(\eta) P_{n-k}(\eta)\, d\eta \right]. \tag{B-18}$$

From previous work we know that

$$\int_{-1}^{+1} \eta P_{k+1}(\eta) P_{n-k}(\eta)\, d\eta = \begin{cases} \dfrac{n+2}{(n+1)(n+3)} & \text{for } k = n-k \text{ and for } k+1 = n-k-1 \\ 0 & \text{for all other values of } n \text{ and } k. \end{cases} \tag{B-19}$$

$$\int_{-1}^{+1} P_{k+2}(\eta) P_{n-k}(\eta)\, d\eta = \begin{cases} \dfrac{2}{n+3} & \text{for } k+2 = n-k \\ 0 & \text{for all other values of } n \text{ and } k \end{cases} \quad . \tag{B-20}$$

Equation (B-18) yields, after the substitution of the values of the integrals given by equations (B-19) and (B-20),

$$I_5(L,l) = \sum_{\substack{n=0\\(L+l)}}^{\infty}{}' \left\{ \frac{n+2}{2(n+1)(n+3)} \left[n\, d^{1l}_{\frac{n}{2}-1}\, d^{oL}_{\frac{n}{2}+1} - (n+4)\, d^{1l}_{\frac{n}{2}}\, d^{oL}_{\frac{n}{2}} \right] \right\} . \tag{B-21}$$

This integral is different from zero only if both L and l are even, or if both are odd.

The evaluation of I_2 will be carried out next. This integral is given by equation (89).

$$I_2(L,l) = \int_{-1}^{+1} \frac{\eta}{\sqrt{1-\eta^2}}\, S^{(1)}_{1l}(\eta)\, S^{(1)}_{oL}(\eta)\, d\eta \quad . \tag{B-22}$$

By the use of equation (85) and of the Cauchy product formula we obtain

$$I_2(L,l) = \sum_{\substack{n=0,1\\(L+l)}}^{\infty}{}' \sum_{\substack{k=0,1\\(l)}}^{n}{}' d^{1l}_{k}\, d^{oL}_{n-k} \int_{-1}^{+1} \frac{\eta}{\sqrt{1-\eta^2}}\, P^{1}_{k+1}(\eta)\, P_{n-k}(\eta)\, d\eta \; . \tag{B-23}$$

In order to change the associated Legendre function occurring in this integral to a Legendre function, the definition of the associated Legendre function given in equation (B-17) is applied. I_2 then becomes

$$I_2(L,l) = -\sum_{\substack{n=0,1\\(L+l)}}^{\infty}{}' \sum_{\substack{k=0,1\\(l)}}^{n}{}' d^{1l}_{k}\, d^{oL}_{n-k} \int_{-1}^{+1} \eta\, P_{n-k}(\eta)\, P'_{k+1}(\eta)\, d\eta \;, \tag{B-24}$$

the prime on $P'_{k+1}(\eta)$ indicating the derivative of $P_{k+1}(\eta)$ with respect to η .

The following table is useful for determining when I_2 is different from zero.

TABLE B-1

	n odd		n even	
	k odd	k even	k odd	k even
n - k:	even	odd	odd	even
ηP_{n-k}:	odd	even	even	odd
P'_{k+1}:	odd	even	odd	even
I_2:	$\neq 0$	$\neq 0$	$= 0$	$= 0$
l:	odd	even	odd	even
L:	even	odd	odd	even

In order to evaluate the integral occurring in equation (B-24) use is made of the following formula for $P'_n(\eta)$ given in example 4 on page 282 of Copson (Ref. 25):

$$P_n'(\eta) = (2n-1)P_{n-1}(\eta) + (2n-5)P_{n-3}(\eta) + (2n-9)P_{n-5}(\eta) + \cdots$$

$$+ \begin{cases} 7P_3(\eta) + 3P_1(\eta) & \text{for n even} . \\ 5P_2(\eta) + P_0(\eta) & \text{for n odd} . \end{cases} \qquad \text{(B-25)}$$

The substitution of this formula in equation (B-24) yields

$$I_2(L,l) =$$

$$-\sum_{\substack{n=1 \\ (L+l)}}^{\infty}{}' \sum_{\substack{k=0,1 \\ (l)}}^{n}{}' d_k^{1l} d_{n-k}^{oL} \int_{-1}^{+1} \eta P_{n-k}(\eta) \Big[(2k+1)P_k(\eta) + (2k-3)P_{k-2}(\eta) + (2k-7)P_{k-4}(\eta) + \cdots$$

$$+ \left. \begin{cases} 7P_3(\eta) + 3P_1(\eta) & k \text{ odd} \\ 5P_2(\eta) + P_0(\eta) & k \text{ even} \end{cases} \right\} \Big] d\eta \; . \qquad \text{(B-26)}$$

It is evident from this equation that I_2 is zero if $(n-k) > (k+1)$, or $k < \frac{n-1}{2}$. For $(k+1) \geq (n-k)$ we evaluate I_2 term-by-term. The small number in parentheses after the integral indicates the value of k which was used. For k even, $-I_2(L, l)$ =

$$n = 1: \quad d_0^{1l} d_1^{0L} \int_{-1}^{+1} \eta P_1 P_0 \, d\eta \ (0)$$

$$n = 3: \quad +5 d_2^{1l} d_1^{0L} \int_{-1}^{+1} \eta P_1 P_2 \, d\eta (2) + d_2^{1l} d_1^{0L} \int_{-1}^{+1} \eta P_1 P_0 \, d\eta \ (2)$$

$$n = 5: \quad +5 d_2^{1l} d_3^{0L} \int_{-1}^{+1} \eta P_3 P_2 \, d\eta (2) + 5 d_4^{1l} d_1^{0L} \int_{-1}^{+1} \eta P_1 P_2 \, d\eta \ (4) + d_4^{1l} d_1^{0L} \int_{-1}^{+1} \eta P_1 P_0 \, d\eta \ (4)$$

$$n = 7: \quad +9 d_4^{1l} d_3^{0L} \int_{-1}^{+1} \eta P_3 P_4 \, d\eta (4) + 5 d_4^{1l} d_3^{0L} \int_{-1}^{+1} \eta P_3 P_2 \, d\eta \ (4) + 5 d_6^{1l} d_1^{0L} \int_{-1}^{+1} \eta P_1 P_2 \, d\eta \ (6)$$
$$+ d_6^{1l} d_1^{0L} \int_{-1}^{+1} \eta P_1 P_0 \, d\eta \ (6)$$

$$n = 9: \quad +9 d_4^{1l} d_5^{0L} \int_{-1}^{+1} \eta P_5 P_4 \, d\eta (4) + 9 d_6^{1l} d_3^{0L} \int_{-1}^{+1} \eta P_3 P_4 \, d\eta \ (6) + 5 d_6^{1l} d_3^{0L} \int_{-1}^{+1} \eta P_3 P_2 \, d\eta \ (6)$$
$$+ 5 d_8^{1l} d_1^{0L} \int_{-1}^{+1} \eta P_1 P_2 \, d\eta (8) + d_8^{1l} d_1^{0L} \int_{-1}^{+1} \eta P_1 P_0 \, d\eta \ (8)$$

$$+ \cdots \qquad . \qquad \text{(B-27)}$$

In equation (B-27) we have omitted terms which are obviously zero because the degrees of the Legendre polynomials occurring in the integral differ by more than unity.

The terms are collected which involve the same integrals, the integrals are evaluated using equation (B-5), and there results (for L odd and l even)

$$
\begin{aligned}
-I_2(L,l) = {} & \tfrac{2}{3}\, d_1^{0L}\left(d_0^{1l} + d_2^{1l} + d_4^{1l} + \cdots\right) \\
& + \tfrac{4}{3}\, d_1^{0L}\left(d_2^{1l} + d_4^{1l} + d_6^{1l} + \cdots\right) \\
& + \tfrac{6}{7}\, d_3^{0L}\left(d_2^{1l} + d_4^{1l} + d_6^{1l} + \cdots\right) \\
& + \tfrac{8}{7}\, d_3^{0L}\left(d_4^{1l} + d_6^{1l} + d_8^{1l} + \cdots\right) \\
& + \tfrac{10}{11}\, d_5^{0L}\left(d_4^{1l} + d_6^{1l} + d_8^{1l} + \cdots\right) \\
& + \tfrac{12}{11}\, d_5^{0L}\left(d_6^{1l} + d_8^{1l} + d_{10}^{1l} + \cdots\right) \\
& \;\vdots
\end{aligned}
\tag{B-28}
$$

It should be noted that $d_\eta^{ml} = 0$ whenever $(l+\eta)$ is odd.

The same procedure is applied when k is odd and the following result is obtained (L even and l odd):

$$
\begin{aligned}
-I_2(L,l) = {} & 2\, d_0^{0L}\left(d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots\right) \\
& + \tfrac{4}{5}\, d_2^{0L}\left(d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots\right) \\
& + \tfrac{6}{5}\, d_2^{0L}\left(d_3^{1l} + d_5^{1l} + d_7^{1l} + \cdots\right) \\
& + \tfrac{8}{9}\, d_4^{0L}\left(d_3^{1l} + d_5^{1l} + d_7^{1l} + \cdots\right) \\
& + \tfrac{10}{9}\, d_4^{0L}\left(d_5^{1l} + d_7^{1l} + d_9^{1l} + \cdots\right) \\
& \;\vdots
\end{aligned}
\tag{B-29}
$$

The last integral to be evaluated is I_6. This is defined by equation (98).

$$I_6(L,l) = \int_{-1}^{+1} \eta \sqrt{1-\eta^2} \left[\frac{d}{d\eta} S_{1l}^{(1)}(\eta)\right] S_{oL}^{(1)}(\eta)\, d\eta \quad . \tag{B-30}$$

Utilization of the Cauchy product formula and of equation (85) yields

$$I_6(L,l) = \sum_{\substack{n=0,1\\(L+l)}}^{\infty}{}' \sum_{\substack{k=0,1\\(l)}}^{n}{}' d_k^{1l} d_{n-k}^{oL} \int_{-1}^{+1} \eta \sqrt{1-\eta^2} \left[\frac{d}{d\eta} P_{k+1}^1(\eta)\right] P_{n-k}(\eta)\, d\eta \, . \tag{B-31}$$

In order to reduce the term $\frac{d}{d\eta} P_{k+1}^1 (\eta)$ to a function of Legendre polynomials, equation (B-17) is resorted to and I_6 becomes

$$I_6(L,l) = \sum_{\substack{n=0,1\\(L+l)}}^{\infty}{}' \sum_{\substack{k=0,1\\(l)}}^{n}{}' d_k^{1l} d_{n-k}^{oL} \left\{ \int_{-1}^{+1} \eta^2 P_{n-k} P_{k+1}' \, d\eta - \int_{-1}^{+1} \eta (1-\eta^2) P_{n-k} P_{k+1}'' \, d\eta \right\} . \tag{B-32}$$

The second derivative of P_{k+1} (P_{k+1}'') is transformed by means of the differential equation which is satisfied by $P_n(\eta)$:

$$(1-\eta^2) \frac{d^2 P_n(\eta)}{d\eta^2} - 2\eta \frac{d P_n(\eta)}{d\eta} + n(n+1) P_n(\eta) = 0 \tag{B-33}$$

and we obtain

$$I_6(L,l) = \sum_{\substack{n=0,1\\(L+l)}}^{\infty}{}' \sum_{\substack{k=0,1\\(l)}}^{n}{}' d_k^{1l} d_{n-k}^{oL} \left\{ (k+1)(k+2) \int_{-1}^{+1} \eta P_{n-k} P_{k+1} \, d\eta - \int_{-1}^{+1} \eta^2 P_{n-k} P_{k+1}' \, d\eta \right\} . \tag{B-34}$$

The second integral occurring in equation (B-34) can be simplified by making use of the following relation (Ref. 26, p. 330):

$$\eta P_n'(\eta) = n P_n(\eta) + (2n-3)P_{n-2}(\eta) + (2n-7)P_{n-4}(\eta) + \cdots. \tag{B-35}$$

The result is

$$I_6(L,l) = \sum_{\substack{n=0,1\\(L+l)}}^{\infty}{}' \sum_{\substack{k=0,1\\(l)}}^{n}{}' d_k^{1l} d_{n-k}^{oL} (k+1)(k+2)\int_{-1}^{+1} \eta P_{n-k} P_{k+1}\, d\eta$$

$$- \sum_{\substack{n=0,1\\(L+l)}}^{\infty}{}' \sum_{\substack{k=0,1\\(l)}}^{n}{}' d_k^{1l} d_{n-k}^{oL} \left\{ + \int_{-1}^{+1} \eta P_{n-k}\left[(k+1)P_{k+1} + (2k-1)P_{k-1} + (2k-5)P_{k-3} + \cdots\right]\right\}. \tag{B-36}$$

Let the first summation be indicated by Σ_1 and the second by Σ_2. Then

$$I_6(L,l) = \sum\nolimits_1 - \sum\nolimits_2 \ . \tag{B-37}$$

Σ_2 is zero if $(n-k) > (k+2)$ or $k < \frac{n-2}{2}$. Also, the construction of a table similar to Table B-1 shows that Σ_2 is different from zero only for both L and l odd or both even, and thus for n even and k either odd or even. Σ_2 then is evaluated by a procedure completely similar to that used for calculating I_2. The results are:

<u>L even, l even</u>

$$\begin{aligned}\sum\nolimits_2 = {} & 2\, d_0^{oL}\left(\tfrac{1}{3} d_0^{1l} + d_2^{1l} + d_4^{1l} + \cdots\right)\\ & + \tfrac{4}{5} d_2^{oL}\left(\tfrac{1}{3} d_0^{1l} + d_2^{1l} + d_4^{1l} + \cdots\right)\\ & + \tfrac{6}{5} d_2^{oL}\left(\tfrac{3}{7} d_2^{1l} + d_4^{1l} + d_6^{1l} + \cdots\right)\\ & + \tfrac{8}{9} d_4^{oL}\left(\tfrac{3}{7} d_2^{1l} + d_4^{1l} + d_6^{1l} + \cdots\right)\\ & + \tfrac{10}{9} d_4^{oL}\left(\tfrac{5}{11} d_4^{1l} + d_6^{1l} + d_8^{1l} + \cdots\right)\\ & + \tfrac{12}{13} d_6^{oL}\left(\tfrac{5}{11} d_4^{1l} + d_6^{1l} + d_8^{1l} + \cdots\right)\\ & \ \vdots \end{aligned} \tag{B-38}$$

L odd, l odd

$$
\begin{aligned}
\sum\nolimits_2 = \; & \tfrac{2}{3} d_1^{0L} \left(\quad d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots \right) \\
+ \; & \tfrac{4}{3} d_1^{0L} \left(\tfrac{2}{5} d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots \right) \\
+ \; & \tfrac{6}{7} d_3^{0L} \left(\tfrac{2}{5} d_1^{1l} + d_3^{1l} + d_5^{1l} + \cdots \right) \\
+ \; & \tfrac{8}{7} d_3^{0L} \left(\tfrac{4}{9} d_3^{1l} + d_5^{1l} + d_7^{1l} + \cdots \right) \\
+ \; & \tfrac{10}{11} d_5^{0L} \left(\tfrac{4}{9} d_3^{1l} + d_5^{1l} + d_7^{1l} + \cdots \right) \\
+ \; & \tfrac{12}{11} d_5^{0L} \left(\tfrac{6}{13} d_5^{1l} + d_7^{1l} + d_9^{1l} + \cdots \right) \\
& \vdots
\end{aligned}
\qquad \text{(B-39)}
$$

The value of the integral occurring in Σ_1 may be determined from equation (B-5) and is:

$$
\int_{-1}^{+1} \eta P_{n-k} P_{k+1} \, d\eta = \begin{cases} \dfrac{n+2}{(n+1)(n+3)} & \text{for } k = \frac{n}{2} \text{ or } k = \frac{n}{2} - 1 . \\ 0 & \text{for all other values of k .} \end{cases} \qquad \text{(B-40)}
$$

The value of Σ_1 is then found to be:

$$
\sum\nolimits_1 = \sum_{n=0}^{\infty}{}' \frac{(n+2)^2}{4(n+1)(n+3)} \left[(n+4) \, d_{\frac{n}{2}}^{1l} \, d_{\frac{n}{2}}^{0L} + n \, d_{\frac{n}{2}-1}^{1l} \, d_{\frac{n}{2}+1}^{0L} \right] . \qquad \text{(B-41)}
$$

It is to be noted that Σ_1 is different from zero only for both L and l even, or for both odd.

REFERENCES

1 Carlson, J. F. and S. A. Goudsmit, Microwave Radar Reflections, M.I.T. Radiation Laboratory Report No. 195, 20 February 1943.

2 Spencer, R. C., Reflections from Smooth Curved Surfaces, M. I. T. Radiation Laboratory Report No. 661, 26 January 1945.

3 Crout, P. D., and F. E. Bothwell, Theoretical Treatment of Radar Target Return, M.I.T. Radiation Laboratory Report No. 719, 31 August 1945.

4 Barker, C. B., and H. Riblet, Reflections from Curved Surfaces, M.I.T. Radiation Laboratory Report No. 976, 1 February 1946.

5 Marcus, P. M., Reflection of Radiation from Curved Surfaces, M.I.T. Radiation Laboratory Report No. 1029, 16 January 1946.

6 Spencer, R. C., Back Scattering from Various Shaped Objects, AMC Cambridge Field Station Report WLCEC-1, 10 June 1946.

7 Spencer, R. C., Back Scattering from Various Shaped Surfaces, AMC Cambridge Field Station Report WLCEC-1, 23 July 1946.

8 McAfee, W. S., and H. Wolfe, Scattering of Electromagnetic Waves by an Ellipsoid, Camp Evans Signal Laboratory Report No. T-16, 8 February 1943.

9 Saxon, D. S., Notes on Lectures by Julian Schwinger, Discontinuities in Waveguides, February, 1945.

10 Levine, H., and J. Schwinger, "On the Theory of Diffraction by an Aperture in an Infinite Plane Screen", Physical Review, 74, 958 (1948) and 75, 1423 (1949).

11 Miles, John W., "On Electromagnetic Diffraction Through a Plane Screen", Physical Review, 75, 695 (1949).

12 Miles, John W., "On the Diffraction of an Electromagnetic Wave through a Plane Screen", Journal of Applied Physics, 20,760, Aug. '49.

13 Möglich, F.,"Beugungserscheinungen an Körpern von Ellipsoidischer Gestalt", Annalen der Physik, 83, 609, 1927.

14 Whittaker, E. T., "On the Partial Differential Equations of Mathematical Physics", Mathematischen Annalen, 57, 333, 1902.

15 Stratton, J. A., P. M. Morse, L. J. Chu and R. A. Hutner, Elliptic Cylinder and Spheroidal Wave Functions, Wiley, New York, 1941.

16 Stratton, J. A., Electromagnetic Theory, McGraw-Hill, New York, 1941.

17 Morse, P. M. and Herman Feshbach, Methods of Theoretical Physics, Technology Press, Cambridge, 1946.

18 Morse, P. M., "Addition Formulae for Spheroidal Functions", Proc. Nat. Acad. Sci., 21, 56, 1935.

19 Mathematical Tables Project, National Bureau of Standards, Tables of Spherical Bessel Functions, Columbia University Press, New York, 1947.

20 Jahnke, E. and F. Emde, Tables of Functions, Dover, New York, 1945.

21 Mathematical Tables Project, National Bureau of Standards, Tables of Associated Legendre Functions, Columbia University Press, New York, 1945.

22 Magnus, W. and F. Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics, Chelsea, New York, 1949.

23 Norton, K. A., and A. C. Omberg,"Maximum Range of a Radar Set", Proceedings of the Institute of Radio Engineers, January 1947.

24 Rainville, E. D., Intermediate Course in Differential Equations, Wiley, New York, 1943.

25 Copson, E. T., Theory of Functions of a Complex Variable, Oxford, London, 1935.

26 Whittaker, E. T. and G. N. Watson, Modern Analysis, Macmillan, New York, 1947.

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