WRRC-23-J
16 February 1955

UNCLASSIFIED

SCATTERING BY A PROLATE SPHEROID

by
F. V. Schultz*
Willow Run Research Center
of the
Engineering Research Institute
UNIVERSITY OF MICHIGAN

The problem of the scattering of a plane electromagnetic wave by a perfectly conducting prolate spheroid is solved for the case in which the incident wave strikes the spheroid nose-on. The solution involves setting up for the scattered wave a series in terms of two sets of solutions of the vector Helmholtz equation. The undetermined coefficients used in this series are evaluated by using the boundary conditions on the surface of the spheroid. The scattering cross-section is then determined.

---

*Now Professor, Department of Electrical Engineering, at the University of Tennessee, Knoxville, Tennessee.

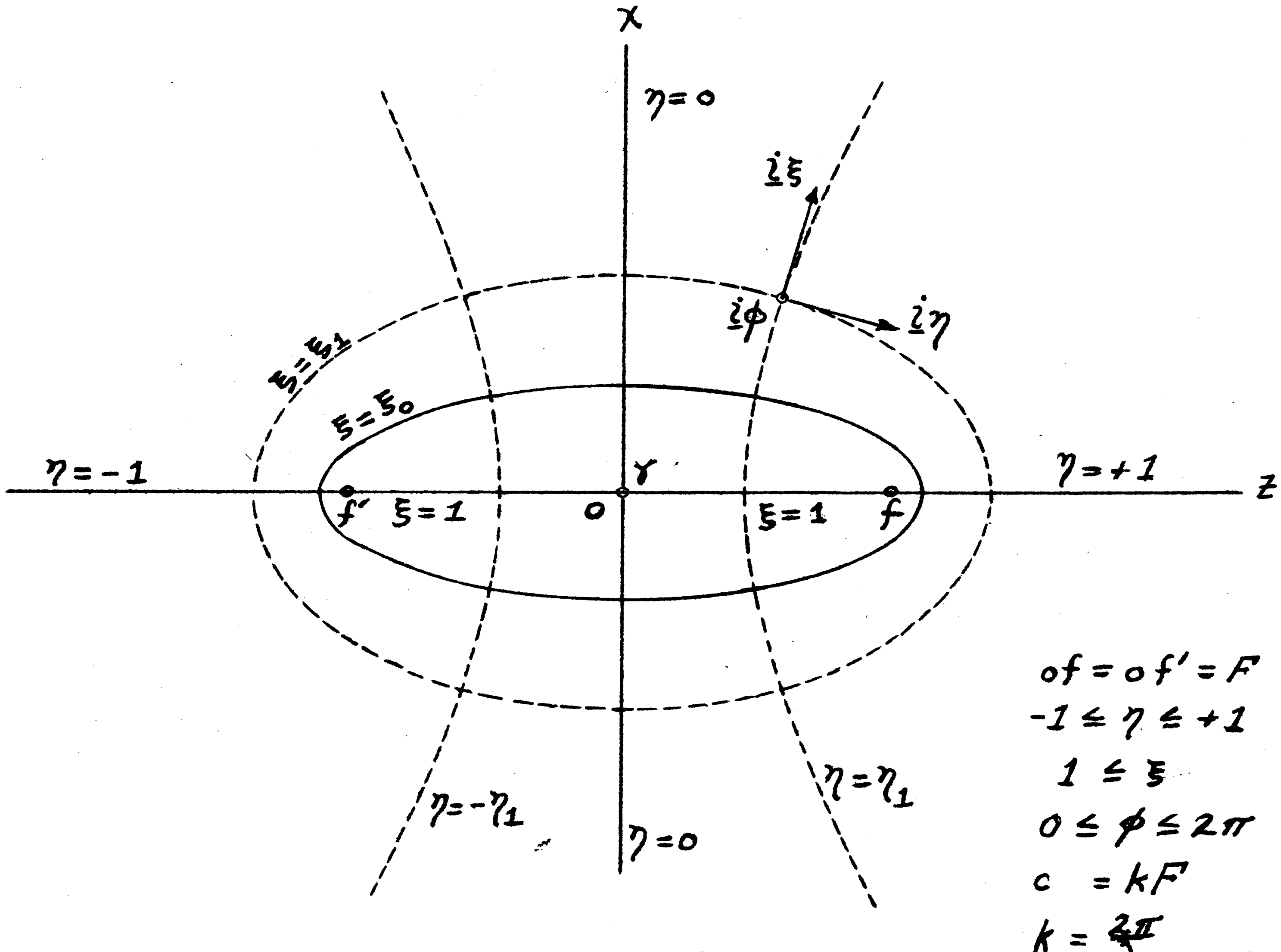

Figure 1
Prolate Spheroidal Coordinates

I. INTRODUCTION

The problem solved in the present work is that of determining the scattering of a single-frequency plane electromagnetic wave striking nose-on a perfectly conducting prolate spheroid, which is embedded in free space. Rationalized MKS units are used. In accordance with engineering practice, $\sqrt{-1}$ is indicated by "j", and time variation is assumed to be of the form $e^{+j\omega t}$.

For the solution of the present problem the use of prolate spheroidal coordinates (η, ξ, ϕ) is indicated. It is assumed that the origin of the prolate spheroidal coordinate system coincides with the origin of the rectangular coordinate system and that the z-axis of the rectangular coordinates is the axis of rotation of the prolate spheroidal coordinates. The two coordinate systems (Fig. 1) are related by these equations:

$$x = F\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi, \tag{1}$$

$$y = F\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi, \tag{2}$$

$$z = F\xi\eta, \tag{3}$$

where F is the semi-focal distance of the prolate spheroidal coordinate system.

II. METHOD OF SOLUTION

The expression for the electric vector of the scattered wave must meet the following conditions:

1. It must satisfy the vector wave equation (or the vector Helmholtz equation for a monochromatic wave, as we have here).

2. Its divergence must be zero.

3. At very large distances from the prolate spheroid the scattered wave must take on the behavior of a spherical diverging wave with the center of the spheroid as its center.

4. The resultant of the incident and scattered waves must satisfy the proper boundary conditions over the surface of the spheroid.

In order to obtain an expression for the electric vector of the scattered wave which meets the listed conditions, the following procedures are followed:

a. Solutions of the vector Helmholtz equation in prolate spheroidal coordinates are formed from solutions of the scalar Helmholtz equation, by ~~means~~ of the method described by Stratton.[1] Solutions are formed which have zero divergence.

[1] J.A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, (1941).

b. The incident wave is expressed in terms of appropriate solutions of the vector Helmholtz equation in prolate spheroidal coordinates.

c. The electric vector of the scattered wave is expressed as an infinite series, with undetermined coefficients, of appropriate solutions of the vector Helmholtz equation in prolate spheroidal coordinates. The solutions used have the proper behavior at infinitely great distances from the origin of coordinates to ensure the satisfaction of condition 3 above.

d. The undetermined coefficients in the series for the electric vector of the scattered wave are evaluated with the aid of the boundary condition on the surface of the scattering prolate spheroid.

III. SOLUTIONS OF THE VECTOR HELMHOLTZ EQUATION

As mentioned previously, solutions of the vector Helmholtz equation are to be formed from solutions of the scalar Helmholtz equation by means of the procedure set forth by Stratton.

Solutions of the scalar Helmholtz equation,

$$\nabla^2\psi + k^2\psi = 0, \tag{4}$$

are used as presented by Stratton, Morse, Chu, and Hutner.[2] When equation (4) is solved by the method of separation of variables, three second-order linear ordinary differential equations result: one in η, one in ξ, and one in ϕ. The separation constants are m and b. Actually a quantity n, which is related to b and which assumes integral values, is used instead of b.

Solutions of the equation in ϕ are $e^{jm\phi}$, $\sin m\phi$, and $\cos m\phi$. Obviously m must be an integer in the present work.

Because of the range of $\eta\,(-1 \leq \eta \leq +1)$, only the first solution of the equation in η may be used, since only this solution is regular throughout the range of η. The solution given by Stratton, Morse, Chu, and Hutner is

$$S^{(1)}_{mn}(\eta) = \sum_{k=0,1}^{\infty}{}' d^{mn}_k \; P^m_{k+m}(\eta). \tag{5}$$

The numerical coefficients d^{mn}_k are tabulated by Stratton, Morse, Chu, and Hutner for values of $c = (2\pi/\lambda)F = kF$ from 0 to 5 and for values of $m + n$ from 0 to 3. The prime on the summation symbol indicates that the summation is to be over even values of k if n is even, and over odd values of k if n is odd. The functions $P^m_{k+m}(\eta)$ are associated Legendre functions.

[2] J.A. Stratton ., P. M. Morse, L. J. Chu, and R. A. Hutner, Elliptic Cylinder and Spheroidal Wave Functions, Wiley, New York, (1941).

The variable ξ has the range $1 \leq \xi < \infty$. Stratton, Morse, Chu, and Hutner give the following solution which is regular for all finite values of ξ:

$$R^{(1)}_{mn}(\xi) = \frac{(\xi^2-1)^{m/2}}{\xi^m \sum_{k=0,1}^{\infty}{}' d_k^{mn} \frac{(k+2m)!}{k!}} \sum_{k=0,1}^{\infty}{}' j^{n-k} d_k^{mn} \frac{(k+2m)!}{k!} j_{m+k}(c\xi), \qquad (6)$$

where $j_{m+k}(c\xi)$ is the spherical Bessel function of the first kind. It is to be observed that the coefficients d_k^{mn} used in equation (6) are the same as those used in equation (5). A second solution which has logarithmic singularities at $\xi = \pm 1$ is also given,

$$R^{(2)}_{mn}(\xi) = \frac{(\xi^2-1)^{m/2}}{\xi^m \sum_{k=0,1}^{\infty}{}' d_k^{mn} \frac{(k+2m)!}{k!}} \sum_{k=0,1}^{\infty}{}' j^{n-k} d_k^{mn} \frac{(k+2m)!}{k!} k_{m+k}(c\xi), \qquad (7)$$

where $k_{m+k}(c\xi)$ is the spherical Bessel function of the second kind.

When the above expression for $R^{(2)}_{mn}(\xi)$ converges very slowly ($\xi \approx 1$ or m large), the following equations may be used:

n even:

$$R^{(2)}_{mn}(\xi) = \frac{2c^{m-1}\Gamma\left(\frac{n+2m+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(m-\frac{1}{2}\right) d_{-2m}^{mn} \sum_{k=0}^{\infty}{}' d_k^{mn} \frac{(k+2m)!}{k!}} \sum_{k=-\infty}^{+\infty}{}' d_k^{mn} Q^m_{m+k}(\xi), \qquad (8)$$

n odd:

$$R^{(2)}_{mn}(\xi) = \frac{-8c^{m-2}\Gamma\left(\frac{n+2m+2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(m-\frac{3}{2}\right) d_{1-2m}^{mn} \sum_{k=1}^{\infty}{}' d_k^{mn} \frac{(k+2m)!}{k!}} \sum_{k=-\infty}^{+\infty}{}' d_k^{mn} Q^m_{m+k}(\xi). \qquad (9)$$

The functions $Q^{m}_{m+k}(\xi)$ are Legendre functions of the second kind.

Two other useful solutions of the differential equation in ξ can be formed as follows:

$$R^{(3)}_{mn}(\xi) = R^{(1)}_{mn}(\xi) + jR^{(2)}_{mn}(\xi), \tag{10}$$

$$R^{(4)}_{mn}(\xi) = R^{(1)}_{mn}(\xi) - jR^{(2)}_{mn}(\xi). \tag{11}$$

These are noteworthy for having the following asymptotic behavior for large $c\xi$:

$$R^{(3)}_{mn}(\xi) \approx \frac{1}{c\xi} e^{j\left(c\xi - \frac{n+m+1}{2}\pi\right)}, \tag{12}$$

$$R^{(4)}_{mn}(\xi) \approx \frac{1}{c\xi} e^{-j\left(c\xi - \frac{n+m+1}{2}\pi\right)}. \tag{13}$$

The solutions of equation (4) which are used in the present paper are

$$\psi^{(h)}_{mn} = S^{(1)}_{mn}(\eta)\, R^{(h)}_{mn}(\xi) \cos m\phi. \tag{14}$$

The superscript "h" may take the values 1, 2, 3, or 4.

From these solutions of the scalar Helmholtz equation the following solutions of the vector Helmholtz equation may be formed ($\underline{a}$ being an arbitrary constant vector):

$$\underline{L}^{(h)}_{mn} = \nabla \psi^{(h)}_{mn}, \tag{15}$$

$$\underline{M}^{(h)}_{mn} = \underline{L}^{(h)}_{mn} \times \underline{a} = \nabla \psi^{(h)}_{mn} \times \underline{a}, \tag{16}$$

$$\underline{N}^{(h)}_{mn} = \frac{1}{k} \nabla \times M^{(h)}_{mn}. \tag{17}$$

Series in terms of $\underline{L}_{mn}$, $\underline{M}_{mn}$, and $\underline{N}_{mn}$ may be used to express any ordinary solution of the vector wave equation. Only $\underline{M}_{mn}$ and $\underline{N}_{mn}$ are of use here since the propagation of electromagnetic waves in free space is under consideration. This physical condition requires that

$$\nabla \cdot \underline{E} = 0, \tag{18}$$

$$\nabla \cdot \underline{H} = 0 ; \tag{19}$$

and from the defining equations given above it is clear that

$$\nabla \cdot \underline{L}_{mn}^{(h)} \neq 0, \tag{20}$$

$$\nabla \cdot \underline{M}_{mn}^{(h)} = \nabla \cdot \underline{N}_{mn}^{(h)} = 0. \tag{21}$$

The vectors $\underline{E}$ and $\underline{H}$ can be expressed in infinite series of the functions $\underline{M}_{mn}^{(h)}$ and $\underline{N}_{mn}^{(h)}$.

IV. EXPRESSION FOR INCIDENT WAVE

Let it be assumed that the incident plane wave is moving along the z-axis in the negative z-direction, that the electric vector has a magnitude E_o and points in the positive y-direction, and that the magnetic vector has a magnitude H_o and points in the positive x-direction. This is the situation for a plane electromagnetic wave striking the spheroid nose-on.

With the help of an expansion developed by Morse[3] the incident electric and magnetic field vectors may be expressed as follows:

$$\underline{I_E} = -j\frac{I_{E_o}}{k}\sum_{n=0}^{\infty} A_{on}\ {}^{x}\underline{M}_{on}^{(1)}, \tag{22}$$

$$\underline{I_H} = -j\frac{I_{H_o}}{k}\sum_{n=0}^{\infty} A_{on}\ {}^{y}\underline{M}_{on}^{(1)}, \tag{23}$$

where

$$A_n = \frac{2j^n}{N_{on}}\sum_{k=0,1}^{\infty}{}' d_k^{on}, \tag{24}$$

$$N_n = \sum_{k=0,1}^{\infty}{}' \frac{2}{2k+1}\left(d_k^{on}\right)^2. \tag{25}$$

The vector expressions ${}^{x}\underline{M}_m^{(1)}$, ${}^{y}\underline{M}_m^{(1)}$, ${}^{z}\underline{M}_m^{(1)}$ are found as follows. By definition,

$${}^{x}\underline{M}_m^{(h)} = \nabla\psi_{mn}^{(h)} \times \underline{i}_x, \tag{26}$$

$${}^{y}\underline{M}_m^{(h)} = \nabla\psi_{mn}^{(h)} \times \underline{i}_y, \tag{27}$$

$${}^{z}\underline{M}_m^{(h)} = \nabla\psi_{mn}^{(h)} \times \underline{i}_z, \tag{28}$$

---

[3] P.M. Morse, "Addition Formulae for Spheroidal Functions", Proc. Nat. Acad. Sci., Vol. 21, pp. 56-62, (1935).

where $\underline{i}_x$, $\underline{i}_y$, and $\underline{i}_z$ are the unit vectors in the x-, y-, and z-directions, respectively. For h = 1, 2, 3, or 4, the function $\psi_{mn}^{(h)}$ is expressed in prolate spheroidal coordinates and its gradient is found.[4] The unit vector $\underline{i}_x$, $\underline{i}_y$, or $\underline{i}_z$ is also expressed in prolate spheroidal coordinates. The results are:

$$\begin{aligned}
x_{\underline{M}_{mn}}^{(h)} = {} & \underline{i}_\eta F^{-1}\Big\{ m\xi(\xi^2-1)^{-1/2}(\xi^2-\eta^2)^{-1/2} S_{mn}^{(1)}(\eta) R_{mn}^{(h)}(\xi)\cos\phi \sin m\phi \\
& -(\xi^2-1)^{1/2}(\xi^2-\eta^2)^{-1/2} S_{mn}^{(1)}(\eta)\left[\frac{d}{d\xi} R_{mn}^{(h)}(\xi)\right] \sin\phi \cos m\phi \Big\} \\
& + \underline{i}_\xi F^{-1}\Big\{(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1/2}\left[\frac{d}{d\eta} S_{mn}^{(1)}(\eta)\right] R_{mn}^{(h)}(\xi) \sin\phi\cos m\phi \\
& + m\eta(1-\eta^2)^{-1/2}(\xi^2-\eta^2)^{-1/2} S_{mn}^{(1)}(\eta) R_{mn}^{(h)}(\xi) \cos\phi \sin m\phi \Big\} \\
& + \underline{i}_\phi F^{-1}\Big\{\eta(\xi^2-1)(\xi^2-\eta^2)^{-1} S_{mn}^{(1)}(\eta)\left[\frac{d}{d\xi} R_{mn}^{(h)}(\xi)\right]\cos\phi\cos m\phi \\
& + \xi(1-\eta^2)(\xi^2-\eta^2)^{-1}\left[\frac{d}{d\eta} S_{mn}^{(1)}(\eta)\right] R_{mn}^{(h)}(\xi) \cos\phi\cos m\phi \Big\}.
\end{aligned} \tag{29}$$

$$\begin{aligned}
z_{\underline{M}_{mn}}^{(h)} = {} & \underline{i}_\eta F^{-1}\Big\{ m\eta(1-\eta^2)^{-1/2}(\xi^2-\eta^2)^{-1/2} S_{mn}^{(1)}(\eta) R_{mn}^{(h)}(\xi)\sin m\phi \Big\} \\
& + \underline{i}_\xi F^{-1}\Big\{ -m\xi(\xi^2-1)^{-1/2}(\xi^2-\eta^2)^{-1/2} S_{mn}^{(1)}(\eta) R_{mn}^{(h)}(\xi) \sin m\phi \Big\} \\
& + \underline{i}_\phi F^{-1}\Big\{ -\xi(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1} S_{mn}^{(1)}(\eta)\left[\frac{d}{d\xi} R_{mn}^{(h)}(\xi)\right]\cos m\phi \\
& + \eta(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}(\xi^2-\eta^2)^{-1}\left[\frac{d}{d\eta} S_{mn}^{(1)}(\eta)\right] R_{mn}^{(h)}(\xi) \cos m\phi \Big\}.
\end{aligned} \tag{30}$$

The expression for $y_{\underline{M}_{mn}}^{(h)}$ is not listed since it is not used in the present work.

---

[4] Stratton, Reference 1, p. 49, Eq. (78).

V. EXPRESSION FOR SCATTERED WAVE

The expression for the electric vector of the scattered wave is found for the special case of incident wave discussed in Section IV, namely nose-on incidence.

The M-vectors used in Section IV to express the incident waves satisfy the vector Helmholtz equation and have zero divergence, as is pointed out in Section III. Therefore, if the scattered wave is expressed as a linear combination of these vectors with constant coefficients, the resulting expression will satisfy conditions 1 and 2 of Section II. Actually an infinite series of M-vectors, with undetermined coefficients, will be used for expressing the scattered wave. For the representation of the M-vectors, the following solution of the scalar Helmholtz equation (in prolate spheroidal coordinates) is used:

$$\psi_{mn}^{(h)} = S_{mn}^{(1)} (\eta) R_{mn}^{(h)} (\xi) \cos m\phi \quad . \tag{31}$$

As was pointed out in Section III, four different forms of $R_{mn}^{(h)}(\xi)$ are available and their pertinent properties are given there. Since the incident plane wave exists throughout finite space, in the expression for it given by equation (22) it is necessary to use the function $R_{mn}^{(1)}(\xi)$, which is regular for all finite values of ξ. The scattered wave exists only outside the perfectly conducting prolate spheroid. The locus $\xi = 1$ is a straight line extending from $z = -F$ to $z = F$ (Fig. 1). For all prolate spheroids whose minor axis is different from zero, the values of ξ lie in the range $1 < \xi < \infty$. Thus for representation of the scattered wave in the case at hand, any one of the four forms of $R_{mn}^{(h)}(\xi)$ may be used. Because of the asymptotic behavior of $R_{mn}^{(h)}(\xi)$ for very large values of $c\xi$, as given by formula (13), this function, and therefore $\psi^{(4)}_{mn}$, is used to represent the scattered wave. The final expression for the scattered wave will then satisfy condition 3 of Section II.

Condition 4 of Section II will be satisfied since it is used to calculate the coefficients in the series for the scattered wave. The prolate spheroid is assumed to be perfectly conducting, so that this boundary condition is expressed by

$$[\underline{n} \times {}^{T}\underline{E}]_{\xi=\xi_0} = [\,\underline{i}_{\xi} \times ({}^{I}\underline{E} + {}^{S}\underline{E})]_{\xi=\xi_0} = 0, \qquad (32)$$

where ${}^{T}\underline{E}$ is the electric vector of the total electromagnetic field and $\underline{n}$ is the unit vector normal to the surface of the prolate spheroid. The number ξ_0 is the value of ξ on the surface of the spheroid. The boundary condition implies that

$$[{}^{I}E_{\eta} + {}^{S}E_{\eta}]_{\xi=\xi_0} = 0, \qquad (33)$$

$$[{}^{I}E_{\phi} + {}^{S}E_{\phi}]_{\xi=\xi_0} = 0, \qquad (34)$$

where ${}^{I}E_{\eta}$ is the η-component of ${}^{I}\underline{E}$ and ${}^{I}E_{\phi}$ is the ϕ-component, and similarly for ${}^{S}\underline{E}$.

Since there are two boundary equations to be satisfied, it is to be expected that it will be necessary to use for ${}^{S}\underline{E}$ the sum of two series of vector functions, each with undetermined coefficients. At least eight sets of such vector functions are available, the individual terms of each set satisfying the vector Helmholtz equation and having zero divergence. These sets are

$$\underline{x}_{M\,mn}^{(h)},\ \underline{y}_{M\,mn}^{(h)},\ \underline{z}_{M\,mn}^{(h)},\ \underline{r}_{M\,mn}^{(h)},\ \underline{x}_{N\,mn}^{(h)},\ \underline{y}_{N\,mn}^{(h)},\ \underline{z}_{N\,mn}^{(h)},\ \underline{r}_{N\,mn}^{(h)}\ . \qquad (35)$$

Two of these sets of vector functions have not been previously discussed. They are

$$ {}^{r}\underline{M}^{(h)}_{mn} = \nabla \psi^{(h)}_{mn} \times \underline{i}_r \,, \tag{36} $$

and

$$ {}^{r}\underline{N}^{(h)}_{mn} = \frac{1}{k} \nabla \times \; {}^{r}\underline{M}^{(h)}_{mn} \,, \tag{37} $$

where $\underline{i}_r$ is the unit radius vector. Of course $\underline{i}_r$ is not a constant vector but Stratton (Ref. 1) shows that nevertheless ${}^{r}\underline{M}$ and ${}^{r}\underline{N}$ satisfy the vector Helmholtz equation. In choosing two of the eight listed sets of vector functions to represent the scattered wave, one chooses functions whose three components vary with ϕ in the same way as do the corresponding components of the incident wave. For the E-vector of the scattered wave the vector functions which have the proper variation with ϕ are

$$ {}^{x}\underline{M}^{(h)}_{0n},\ {}^{z}\underline{M}^{(h)}_{1n},\ {}^{z}\underline{N}^{(h)}_{1n},\ {}^{r}\underline{M}^{(h)}_{1n},\ {}^{r}\underline{N}^{(h)}_{1n} \,. \tag{38} $$

Of these, ${}^{x}\underline{M}^{(4)}_{0n}$ and ${}^{z}\underline{M}^{(4)}_{1n}$ were chosen for reasons of simplicity. The resulting expression for the scattered wave is

$$ {}^{S}\underline{E} = \sum_{n=0}^{\infty} \left[\alpha_n \; {}^{x}\underline{M}^{(4)}_{0n} + \beta_n \; {}^{z}\underline{M}^{(4)}_{1n} \right] \tag{39} $$

where α_n and β_n are undetermined coefficients. Expressions for ${}^{x}\underline{M}^{(4)}_{0n}$ and ${}^{z}\underline{M}^{(4)}_{1n}$ are given by equations (29) and (30), respectively.

It can be shown that, for large $c\xi$,

$$\left[\frac{d}{d\xi} R_{mn}^{(4)}(\xi)\right] \approx \frac{1}{\xi} e^{-j\left(c\xi - \frac{n+m}{2}\pi\right)}. \tag{40}$$

From equations (13) and (40) results the asymptotic expression

$$x_{\underline{M}_{0n}}^{(4)} \approx \left[-\underline{i}_{\eta} S_{0n}^{(1)}(\eta)\sin\phi + \underline{i}_{\phi}\eta\, S_{0n}^{(1)}(\eta)\cos\phi\right] \times \frac{1}{F\xi} e^{-j\left(c\xi - \frac{n}{2}\pi\right)}, \tag{41}$$

for large $c\xi$.

Now

$$c\xi = \frac{2\pi}{\lambda} F\xi \tag{42}$$

and

$$\lim_{c\xi \to \infty} (F\xi) = r, \tag{43}$$

so that formula (41) becomes

$$x_{\underline{M}_{0n}}^{(4)} \approx \left[-\underline{i}_{\eta} S_{0n}^{(1)}(\eta)\sin\phi + \underline{i}_{\phi}\eta S_{0n}^{(1)}(\eta)\cos\phi\right] x\frac{1}{r} e^{-j\left(\frac{2\pi}{\lambda} r - \frac{n}{2}\pi\right)}. \tag{44}$$

In a similar way it may be shown that

$$z_{\underline{M}_{1n}}^{(4)} \approx \left[-\underline{i}_{\phi}(1-\eta^2)^{\frac{1}{2}} S_{1n}^{(1)}(\eta)\cos\phi\right] x \frac{1}{r} e^{-j\left(\frac{2\pi}{\lambda} r - \frac{n+1}{2}\pi\right)}. \tag{45}$$

Equations (44) and (45) show that each term of equation (39) for $S_{\underline{E}}$ has the correct behavior at large distances from the prolate spheroid, The expression for $^S\underline{E}$, equation (39), therefore satisfies the first three of the four necessary conditions. The fourth requirement, that of satisfying the boundary conditions over the surface of the spheroid, is met when these conditions are used, in the next section, to determine the coefficients α_n and β_n of equation (39).

VI. SATISFYING BOUNDARY CONDITIONS

The boundary conditions are expressed by equations (33) and (34). If one substitutes the expressions for ${}^{I}\underline{E}$ and ${}^{S}\underline{E}$ given by equations (22) and (39), respectively, into equation (33) there results

$$
\begin{aligned}
j\frac{{}^{I}E_0}{k}\sum_{n=0}^{\infty} A_n(\xi_0^2-1)^{1/2} S_{0n}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{0n}^{(1)}(\xi)\right]_{\xi=\xi_0} &= \\
\sum_{n=0}^{\infty}\alpha_n(\xi_0^2-1)^{1/2} S_{0n}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{0n}^{(4)}(\xi)\right]_{\xi=\xi_0} & \\
-\sum_{n=0}^{\infty}\beta_n\,\eta(1-\eta^2)^{-1/2} S_{1n}^{(1)}(\eta) R_{1n}^{(4)}(\xi_0) & .
\end{aligned}
\tag{46}
$$

Similarly, by using equation (34), one obtains

$$
\begin{aligned}
j\frac{{}^{I}E_0}{k}\sum_{n=0}^{\infty} A_n\left\{\eta(\xi_0^2-1) S_{0n}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{0n}^{(1)}(\xi)\right]_{\xi=\xi_0}+\xi_0(1-\eta^2)\left[\frac{d}{d\eta}S_{0n}^{(1)}(\eta)\right]R_{0n}^{(1)}(\xi_0)\right\} &= \\
\sum_{n=0}^{\infty}\alpha_n\left\{\eta(\xi_0^2-1) S_{0n}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{0n}^{(4)}(\xi)\right]_{\xi=\xi_0}+\xi_0(1-\eta^2)\left[\frac{d}{d\eta}S_{0n}^{(1)}(\eta)\right]R_{0n}^{(4)}(\xi_0)\right\} & \\
-\sum_{n=0}^{\infty}\beta_n\left\{\xi_0(\xi_0^2-1)^{1/2}(1-\eta^2)^{1/2} S_{1n}^{(1)}(\eta)\left[\frac{d}{d\xi}R_{1n}^{(4)}(\xi)\right]_{\xi=\xi_0}\right. & \\
\left.-\eta(\xi_0^2-1)^{1/2}(1-\eta^2)^{1/2}\left[\frac{d}{d\eta}S_{1n}^{(1)}(\eta)\right]R_{1n}^{(4)}(\xi_0)\right\} & .
\end{aligned}
\tag{47}
$$

The equations must hold for all allowed values of η and may be used to determine α_n and β_n. Equations (46) and (47) are multiplied by $S_{0N}^{(1)}(\eta)$ (where N is any non-negative integer) and then integrated from $\eta = -1$ to $\eta = +1$. The following results are obtained (a is the semi-major axis of the scattering spheroid):

$$\left({}^{I}E_0 a\right)\left(\frac{j}{c\xi_0}\right)\sum_{n=0}^{\infty} A_n (\xi_0^2-1)^{1/2}\left[\frac{d}{d\xi}R_{0n}^{(1)}(\xi)\right]_{\xi=\xi_0}\int_{-1}^{+1} S_{0n}^{(1)}(\eta)S_{0N}^{(1)}(\eta)d\eta =$$

$$\sum_{n=0}^{\infty}\alpha_n (\xi_0^2-1)^{1/2}\left[\frac{d}{d\xi}R_{0n}^{(4)}(\xi)\right]_{\xi=\xi_0}\int_{-1}^{+1} S_{0n}^{(1)}(\eta)S_{0N}^{(1)}(\eta)d\eta$$

$$-\sum_{n=0}^{\infty}\beta_n R_{1n}^{(4)}(\xi_0)\int_{-1}^{+1}\frac{\eta}{\sqrt{1-\eta^2}}S_{1n}^{(1)}(\eta)S_{0N}^{(1)}(\eta)d\eta , \tag{48}$$

$$\left({}^{I}E_0 a\right)\left(\frac{j}{c\xi_0}\right)\sum_{n=0}^{\infty} A_n\left\{(\xi_0^2-1)\left[\frac{d}{d\xi}R_{0n}^{(1)}(\xi)\right]_{\xi=\xi_0}\int_{-1}^{+1}\eta\, S_{0n}^{(1)}(\eta)S_{0N}^{(1)}(\eta)d\eta\right.$$

$$\left.+\xi_0 R_{0n}^{(1)}(\xi_0)\int_{-1}^{+1}(1-\eta^2)\left[\frac{d}{d\eta}S_{0n}^{(1)}(\eta)\right]S_{0N}^{(1)}(\eta)d\eta\right\} =$$

$$\sum_{n=0}^{\infty}\alpha_n\left\{(\xi_0^2-1)\left[\frac{d}{d\xi}R_{0n}^{(4)}(\xi)\right]_{\xi=\xi_0}\int_{-1}^{+1}\eta\, S_{0n}^{(1)}(\eta)S_{0N}^{(1)}(\eta)d\eta\right.$$

$$\left.+\xi_0 R_{0n}^{(4)}(\xi_0)\int_{-1}^{+1}(1-\eta^2)\left[\frac{d}{d\eta}S_{0n}^{(1)}(\eta)\right]S_{0N}^{(1)}(\eta)d\eta\right\}$$

$$-\sum_{n=0}^{\infty}\beta_n\left\{\xi_0(\xi_0^2-1)^{1/2}\left[\frac{d}{d\xi}R_{1n}^{(4)}(\xi)\right]_{\xi=\xi_0}\int_{-1}^{+1}\sqrt{1-\eta^2}\, S_{0n}^{(1)}(\eta)S_{0N}^{(1)}(\eta)d\eta\right.$$

$$\left.-(\xi_0^2-1)^{1/2}R_{1n}^{(4)}(\xi_0)\int_{-1}^{+1}\eta\sqrt{1-\eta^2}\left[\frac{d}{d\eta}S_{1n}^{(1)}(\eta)\right]S_{0N}^{(1)}(\eta)d\eta\right\} . \tag{49}$$

Stratton, Morse, Chu, and Hutner (Ref.2) show that the functions $S^{(1)}_{mn}(\eta)$ are orthogonal functions, namely that

$$I_1^{Nn} = \int_{-1}^{+1} S^{(1)}_{on}(\eta) S^{(1)}_{oN}(\eta) d\eta = \begin{cases} 0 & , \; n \neq N \\ 2 \sum_{k\, 0,1}^{\infty}{}' \dfrac{(d_k^{oN})^2}{2k+1} & , \; n = N \end{cases} \tag{50}$$

The other integrals are much more difficult to handle. The method that is employed to evaluate the remaining integrals is to use equation (5) to express $S^{(1)}_{mn}(\eta)$ and its derivatives in terms of $P^m_{m+k}(\eta)$ and its derivatives. This method involves the manipulation of double series and the evaluation of integrals involving $P^m_{m+k}(\eta)$ and its derivatives, but the work is straightforward. Only the results are included, and these are listed below.

$$I_2^{Nn} = \int_{-1}^{+1} \frac{\eta}{\sqrt{1-\eta^2}} S^{(1)}_{1n}(\eta) S^{(1)}_{oN}(\eta) d\eta. \tag{51}$$

For $n \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ and $N \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ this integral is zero. For n odd and N even one obtains, since $d_k^{mn} = 0$ for $k + n$ odd,

$$\begin{aligned} I_2^{Nn} = {} & 2d_o^{oN} (d_1^{1n} + d_3^{1n} + d_5^{1n} + \ldots.) \\ & + \frac{4}{5} d_2^{oN} (d_1^{1n} + d_3^{1n} + d_5^{1n} + \ldots.) \\ & + \frac{6}{5} d_2^{oN} (d_3^{1n} + d_5^{1n} + d_7^{1n} + \ldots.) \\ & + \frac{8}{9} d_4^{oN} (d_3^{1n} + d_5^{1n} + d_7^{1n} + \ldots.) \\ & + \frac{10}{9} d_4^{oN} (d_5^{1n} + d_7^{1n} + d_9^{1n} + \ldots.) \end{aligned} \tag{52}$$

.

.

.

For n even and N odd the result is

$$
\begin{aligned}
-I_2^{Nn} &= \frac{2}{3} d_1^{oN} \left(d_0^{1n} + d_2^{1n} + d_4^{1n} + \ldots.\right) \\
&+ \frac{4}{3} d_1^{oN} \left(d_2^{1n} + d_4^{1n} + d_6^{1n} + \ldots.\right) \\
&+ \frac{6}{7} d_3^{oN} \left(d_2^{1n} + d_4^{1n} + d_6^{1n} + \ldots.\right) \\
&+ \frac{8}{7} d_3^{oN} \left(d_4^{1n} + d_6^{1n} + d_8^{1n} + \ldots.\right) \\
&+ \frac{10}{11} d_5^{oN} \left(d_4^{1n} + d_6^{1n} + d_8^{1n} + \ldots.\right) \\
&+ \frac{12}{11} d_5^{oN} \left(d_6^{1n} + d_8^{1n} + d_{10}^{1n} + \ldots.\right) \\
&\quad \vdots
\end{aligned}
\tag{53}
$$

$$
I_3^{Nn} = \int_{-1}^{+1} \eta\, S_{on}^{(1)}(\eta)\, S_{oN}^{(1)}(\eta)\, d\eta\,. \tag{54}
$$

The integral I_3^{Nn} is zero for $n \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ and $N \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$. For $n \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ and $N \begin{Bmatrix}\text{odd}\\ \text{even}\end{Bmatrix}$ we find

$$
I_3^{Nn} = \sum_{k=1}^{\infty}{}' \left[\frac{k+1}{k(k+2)}\right] \left[d^{On}_{\frac{k-1}{2}} \cdot d^{ON}_{\frac{k+1}{2}} + d^{On}_{\frac{k+1}{2}} \cdot d^{ON}_{\frac{k-1}{2}} \right], \tag{55}
$$

$$
I_4^{Nn} = \int_{-1}^{+1} (1-\eta^2) \left[\frac{d}{d\eta} S_{on}^{(1)}(\eta)\right] S_{oN}^{(1)}(\eta)\, d\eta\,. \tag{56}
$$

The integral I_4^{Nn} also is zero for $n \begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$ and $N \begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$. For $n \begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$ and $N \begin{Bmatrix}\text{odd}\\\text{even}\end{Bmatrix}$ one obtains

$$I_4^{Nn} = \sum_{k=1}^{\infty}{}' \frac{k+1}{2k(k+2)}\left[(k+3)\, d^{on}_{\frac{k+1}{2}}\, d^{oN}_{\frac{k-1}{2}} - (k-1)\, d^{on}_{\frac{k-1}{2}}\, d^{oN}_{\frac{k+1}{2}}\right]. \tag{57}$$

$$I_5^{Nn} = \int_{-1}^{+1} \sqrt{1-\eta^2}\; S^{(1)}_{1n}(\eta)\, S^{(1)}_{oN}(\eta)\, d\eta\;. \tag{58}$$

This integral is zero if n is $\begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$ and N is $\begin{Bmatrix}\text{odd}\\\text{even}\end{Bmatrix}$. For $n \begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$ and $N \begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$ one finds

$$I_5^{Nn} = \sum_{k=0}^{\infty}{}' \left[\frac{k+2}{2(k+1)(k+3)}\right]\left[k\, d^{1n}_{\frac{k}{2}-1}\, d^{oN}_{\frac{k}{2}+1} - (k+4)\, d^{1n}_{\frac{k}{2}}\, d^{oN}_{\frac{k}{2}}\right]. \tag{59}$$

$$I_6^{Nn} = \int_{-1}^{+1} \eta\sqrt{1-\eta^2}\left[\frac{d}{d\eta} S^{(1)}_{1n}(\eta)\right] S^{(1)}_{oN}(\eta)\, d\eta\,. \tag{60}$$

If n is $\begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$ and N is $\begin{Bmatrix}\text{odd}\\\text{even}\end{Bmatrix}$, the integral I_6^{Nn} is zero. For $n \begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$ and $N \begin{Bmatrix}\text{even}\\\text{odd}\end{Bmatrix}$ the integral has the value

$$I_6^{Nn} = \sum_{k=0}^{\infty}{}' \left[\frac{(k+2)^2}{4(k+1)(k+3)}\right]\left[(k+4)\, d^{1n}_{\frac{k}{2}}\, d^{oN}_{\frac{k}{2}} + k\, d^{1n}_{\frac{k}{2}-1}\, d^{oN}_{\frac{k}{2}+1}\right] - \sum{}_2\,. \tag{61}$$

For both n and N even, $\sum_2$ is given by

$$\begin{aligned}\sum\nolimits_2 = {} & 2d_0^{0N}\left(\tfrac{1}{3} d_0^{1n} + d_2^{1n} + d_4^{1n} + \ldots\right) \\ & + \tfrac{4}{5} d_2^{0N}\left(\tfrac{1}{3} d_0^{1n} + d_2^{1n} + d_4^{1n} + \ldots\right) \\ & + \tfrac{6}{5} d_2^{0N}\left(\tfrac{3}{7} d_2^{1n} + d_4^{1n} + d_6^{1n} + \ldots\right) \\ & + \tfrac{8}{9} d_4^{0N}\left(\tfrac{3}{7} d_2^{1n} + d_4^{1n} + d_6^{1n} + \ldots\right) \\ & + \tfrac{10}{9} d_4^{0N}\left(\tfrac{5}{11} d_4^{1n} + d_6^{1n} + d_8^{1n} + \ldots\right) \\ & + \tfrac{12}{13} d_6^{0N}\left(\tfrac{5}{11} d_4^{1n} + d_6^{1n} + d_8^{1n} + \ldots\right) \\ & \;\vdots\end{aligned} \tag{62}$$

and for both n and N odd, $\sum_2$ is given by

$$\begin{aligned}\sum\nolimits_2 = {} & \tfrac{2}{3} d_1^{0N}\left(d_1^{1n} + d_3^{1n} + d_5^{1n} + \ldots\right) \\ & + \tfrac{4}{3} d_1^{0N}\left(\tfrac{2}{5} d_1^{1n} + d_3^{1n} + d_5^{1n} + \ldots\right) \\ & + \tfrac{6}{7} d_3^{0N}\left(\tfrac{2}{5} d_1^{1n} + d_3^{1n} + d_5^{1n} + \ldots\right) \\ & + \tfrac{8}{7} d_3^{0N}\left(\tfrac{4}{9} d_3^{1n} + d_5^{1n} + d_7^{1n} + \ldots\right) \\ & + \tfrac{10}{11} d_5^{0N}\left(\tfrac{4}{9} d_3^{1n} + d_5^{1n} + d_7^{1n} + \ldots\right) \\ & + \tfrac{12}{11} d_5^{0N}\left(\tfrac{6}{13} d_5^{1n} + d_7^{1n} + d_9^{1n} + \ldots\right) \\ & + \tfrac{14}{15} d_7^{0N}\left(\tfrac{6}{13} d_5^{1n} + d_7^{1n} + d_9^{1n} + \ldots\right) \\ & \;\vdots\end{aligned} \tag{63}$$

It is now possible to use equations (48) and (49) to evaluate the coefficients α_n and β_n. For easier manipulation of the equations, the following substitutions are employed:

$$B_{Nn} = \left(\frac{j}{c\xi_0}\right) A_n (\xi_0^2-1)^{1/2} \left[\frac{d}{d\xi} R_{on}^{(1)}(\xi)\right]_{\xi=\xi_0} \int_{-1}^{+1} S_{on}^{(1)}(\eta) S_{oN}^{(1)}(\eta) d\eta \,, \tag{64}$$

$$C_{Nn} = (\xi_0^2-1)^{1/2} \left[\frac{d}{d\xi} R_{on}^{(4)}(\xi)\right]_{\xi=\xi_0} \int_{-1}^{+1} S_{on}^{(1)}(\eta) S_{oN}^{(1)}(\eta) d\eta \,. \tag{65}$$

$$D_{Nn} = -R_{1n}^{(4)}(\xi_0) \int_{-1}^{+1} \frac{\eta}{\sqrt{1-\eta^2}} S_{1n}^{(1)}(\eta) S_{oN}^{(1)}(\eta) d\eta \,, \tag{66}$$

$$U_{Nn} = \left(\frac{j}{c\xi_0}\right) A_n \left\{ (\xi_0^2-1) \left[\frac{d}{d\xi} R_{on}^{(1)}(\xi)\right]_{\xi=\xi_0} \int_{-1}^{+1} \eta\, S_{on}^{(1)}(\eta) S_{oN}^{(1)}(\eta) d\eta \right. \tag{67}$$

$$\left. + \xi_0 R_{on}^{(1)}(\xi_0) \int_{-1}^{+1} (1-\eta^2) \left[\frac{d}{d\eta} S_{on}^{(1)}(\eta)\right] S_{oN}^{(1)}(\eta) d\eta \right\} \,,$$

$$V_{Nn} = (\xi_0^2-1) \left[\frac{d}{d\xi} R_{on}^{(4)}(\xi)\right]_{\xi=\xi_0} \int_{-1}^{+1} \eta\, S_{on}^{(1)}(\eta) S_{oN}^{(1)}(\eta) d\eta \tag{68}$$

$$+ \xi_0 R_{on}^{(4)}(\xi_0) \int_{-1}^{+1} (1-\eta^2) \left[\frac{d}{d\eta} S_{on}^{(1)}(\eta)\right] S_{oN}^{(1)}(\eta) d\eta \,,$$

$$W_{Nn} = -\xi_0 (\xi_0^2-1)^{1/2} \left[\frac{d}{d\xi} R_{1n}^{(4)}(\xi)\right]_{\xi=\xi_0} \int_{-1}^{+1} \sqrt{1-\eta^2}\, S_{1n}^{(1)}(\eta) S_{oN}^{(1)}(\eta) d\eta \tag{69}$$

$$+ (\xi_0^2-1)^{1/2} R_{1n}^{(4)}(\xi_0) \int_{-1}^{+1} \eta \sqrt{1-\eta^2} \left[\frac{d}{d\eta} S_{1n}^{(1)}(\eta)\right] S_{oN}^{(1)}(\eta) d\eta \,.$$

Using these symbols in equations (48) and (49) one obtains

$$\sum_{n=0}^{\infty} (\alpha_n C_{Nn} + \beta_n D_{Nn}) = {}^I E_o a \sum_{n=0}^{\infty} B_{Nn} , \tag{70}$$

$$\sum_{n=0}^{\infty} (\alpha_n V_{Nn} + \beta_n W_{Nn}) = {}^I E_o a \sum_{n=0}^{\infty} U_{Nn} . \tag{71}$$

These are 2N equations in 2 n unknowns, and both N and n go from zero to infinity.

Each term ($\alpha_n \, {}^x\underline{M}^{(4)}_{on} + \beta_n \, {}^z\underline{M}^{(4)}_{1n}$) in the expression for ${}^S\underline{E}$, given by equation (39), represents the reradiation from the spheroid for the forced oscillation of the n th mode. From physical reasoning it is known that the amplitudes of the modes, above a certain order depending upon the ratio F/λ, become less the higher the order of the mode. Hence the coefficients α_n and β_n must approach zero as n increases, and the same fact can be deduced mathematically from the convergence of the series. In obtaining α_n and β_n from equation (70) and (71), one may expect reasonably accurate values if it is assumed that α_n and β_n are zero for values of n above a given number n'. Then 2 n' of these equations are employed to find α_n and β_n for values of n up to n'. When these values of α_n and β_n are used in equation (39), a solution for ${}^S\underline{E}$ is obtained which satisfies the four conditions listed at the beginning of Section II.

It is evident that a large amount of calculating is required in order to obtain numerical results from equation (39). Some consideration has been given to the possibility of obtaining a simpler solution which would be more amenable to numerical calculation. To represent ${}^S\underline{E}$ it would be desirable to have available two sets of vector functions, one with no η -component and the other with no ϕ -component. Then only one set of undetermined coefficients would appear in equation (33) and only the other set would be present in equation (34). The process of calculating the coefficients would then be considerably simplified. **Simplification would also** result if ${}^{S}\underline{E}$ were expressed by an equation which contained only one term for each value of n and if this term involved, aside from algebraic functions of η, only $S_{on}^{(1)}(\eta)$ and not its derivatives. In addition it would be helpful if the vector functions used for expressing ${}^{S}\underline{E}$ would involve only $S_{on}^{(1)}(\eta)$ with algebraic functions of η identical to those occurring in the expression for ${}^{S}\underline{E}$. Then, because of the orthogonality property (50), each coefficient α_n and β_n would be given by a single equation involving no other coefficient. It would be very desirable that the divergence of each of these functions be zero for all values of the eigenvalues n and m; then each term of the series for the scattered wave would be divergenceless with the result that the divergence of the complete expression would be zero, as is necessary. As yet, no solutions of the vector Helmholtz equation have been found which have these desired properties.

VII. PHYSICAL PROPERTIES OF SCATTERED WAVE

Equation (39), with the appropriate values α_n and β_n as calculated from equations (70) and (71), is an expression for the electric field vector ${}^S\underline{E}$ of the scattered wave which is valid for all values of $\eta\,(-1 \leq \eta \leq +1)$, of $\xi\,(\xi \geq \xi_0)$, and of ϕ. This means that equation (39) can be used for calculating ${}^S\underline{E}$ in the immediate vicinity of the scattering spheroid as well as at great distances from the spheroid.

Usually one is more interested in the behavior of the scattered field at relatively great distances from the spheroid. To deduce this behavior most easily it is well to take the asymptotic form of ${}^S\underline{E}$ as $\xi \rightarrow \infty$, under the assumption that $c = 2\pi F/\lambda$ does not equal zero. To obtain the asymptotic form of equation (39) one uses equations (44) and (45):

$$
\begin{aligned}
{}^S\underline{E} = \Bigg\{ & \underline{i}_\eta \left[\sum_{n=0}^{\infty} j^{n+2} \alpha_n' \, S_{on}^{(1)}(\eta) \sin\phi \right] \\
& + \underline{i}_\phi \left[\sum_{n=0}^{\infty} j \; \alpha_n' \, \eta \, S_{on}^{(1)}(\eta) \cos\phi + j^{n+3} \beta_n' \, (1-\eta^2)^{1/2} \, S_{1n}^{(1)}(\eta) \cos\phi \right] \Bigg\} \\
& \times \left({}^I E_0 a \right) \times \frac{1}{r} \, e^{-j\frac{2\pi}{\lambda} r} .
\end{aligned}
\tag{72}
$$

Here $\alpha_n' = \alpha_n \Big/ {}^I E_0 a$ and $\beta_n' = \beta_n \Big/ {}^I E_0 a$. Observe the absence of the $\underline{i}_\xi$ component, which is the longitudinal component for large values of ξ. This, of course, is a necessary and sufficient condition that the wave be a purely transverse wave at a large distance from the spheroid, as it must be.

In order to obtain more information concerning ${}^{S}\underline{E}$ it is best to express the electric field vector in terms of components along the rectangular coordinate axes. The resulting expression for ${}^{S}\underline{E}$ is

$$
\begin{aligned}
{}^{S}\underline{E} = \Bigg\{ & \underline{i}_x \left[\sum_{n=0}^{\infty} j^{n+1} \beta_n' \sqrt{1-\eta^2}\, S_{1n}^{(1)}(\eta) \sin\phi \cos\phi \right] \\
& + \underline{i}_y \left[\sum_{n=0}^{\infty} j^{n} \alpha_n' \,\eta\, S_{on}^{(1)}(\eta) + j^{n+3} \beta_n' \sqrt{1-\eta^2}\, S_{1n}^{(1)}(\eta) \cos^2\phi \right] \\
& + \underline{i}_z \left[\sum_{n=0}^{\infty} j^{n+2} \alpha_n' \sqrt{1-\eta^2}\, S_{on}^{(1)}(\eta) \sin\phi \right] \Bigg\} \times \frac{I_{E_0} a}{r} \, e^{-j \frac{2\pi}{\lambda} r} .
\end{aligned}
\tag{73}
$$

It will be recalled that the incident wave was taken as moving along the z-axis, in the negative z-direction, with the E-vector pointing along the positive y-axis and the H-vector pointing along the positive x-axis. From equation (73) it can be seen that the back-scattered wave ($\eta = 1$) which is the wave moving along the z-axis in the positive z-direction, has a component of the E-vector along the y-direction only. Thus the back-scattered wave has the same polarization as the incident wave, a well-known property of back-scattered waves from smooth surfaces.

From equation (73) it may be seen also that the E-vector of scattered radiation being propagated in the xz-plane ($\phi = 0$ or π), has only a y-component, while the E-vector of scattered radiation moving in the yz-plane ($\phi = \pm \pi/2$), has both a y-component and a z-component, except along the y-axis ($\eta = 0$) where the y-component disappears, as it must for a transverse wave.

In order to compute the magnitude of the Poynting vector of the wave scattered by the prolate spheroid one may use equation (73) in the following form:

$$ {}^{S}\underline{E} = \left\{ \underline{i}_x \, {}^{S}T_x(\eta,\phi) + \underline{i}_y \, {}^{S}T_y(\eta,\phi) + \underline{i}_z \, {}^{S}T_z(\eta,\phi) \right\} \times \frac{{}^{I}E_0 a}{r} e^{-j\frac{2\pi}{\lambda} r} , \qquad (74) $$

where ${}^{S}T_x$, ${}^{S}T_y$, and ${}^{S}T_z$ are complex. Now let

$$ {}^{S}\underline{T} \, e^{jt} = \underline{i}_x \, {}^{S}T_x + \underline{i}_y \, {}^{S}T_y + \underline{i}_z \, {}^{S}T_z , \qquad (75) $$

and

$$ {}^{S}\underline{T} = \underline{i}_T \, {}^{S}T , \qquad (76) $$

where ${}^{S}\underline{T}$ and ${}^{S}T$ are real.

Then

$$ {}^{S}\underline{E} = \underline{i}_T \, {}^{S}T \, \frac{{}^{I}E_0 a}{r} e^{-j\left(\frac{2\pi}{\lambda} r - t\right)} . \qquad (77) $$

From the scattered field determined above, the scattering cross-section of the prolate spheroid is readily found. This scattering cross-section is defined as the interception cross-section σ of an isotropic scatterer which scatters in the direction under consideration the same power density as the prolate spheroid scatterers in this direction. The value of σ is given by

$$ \sigma = 4\pi \left({}^{S}T\right)^2 a^2 . \qquad (78) $$

The numerical determination of the back-scattering cross-section is described in the following paper.