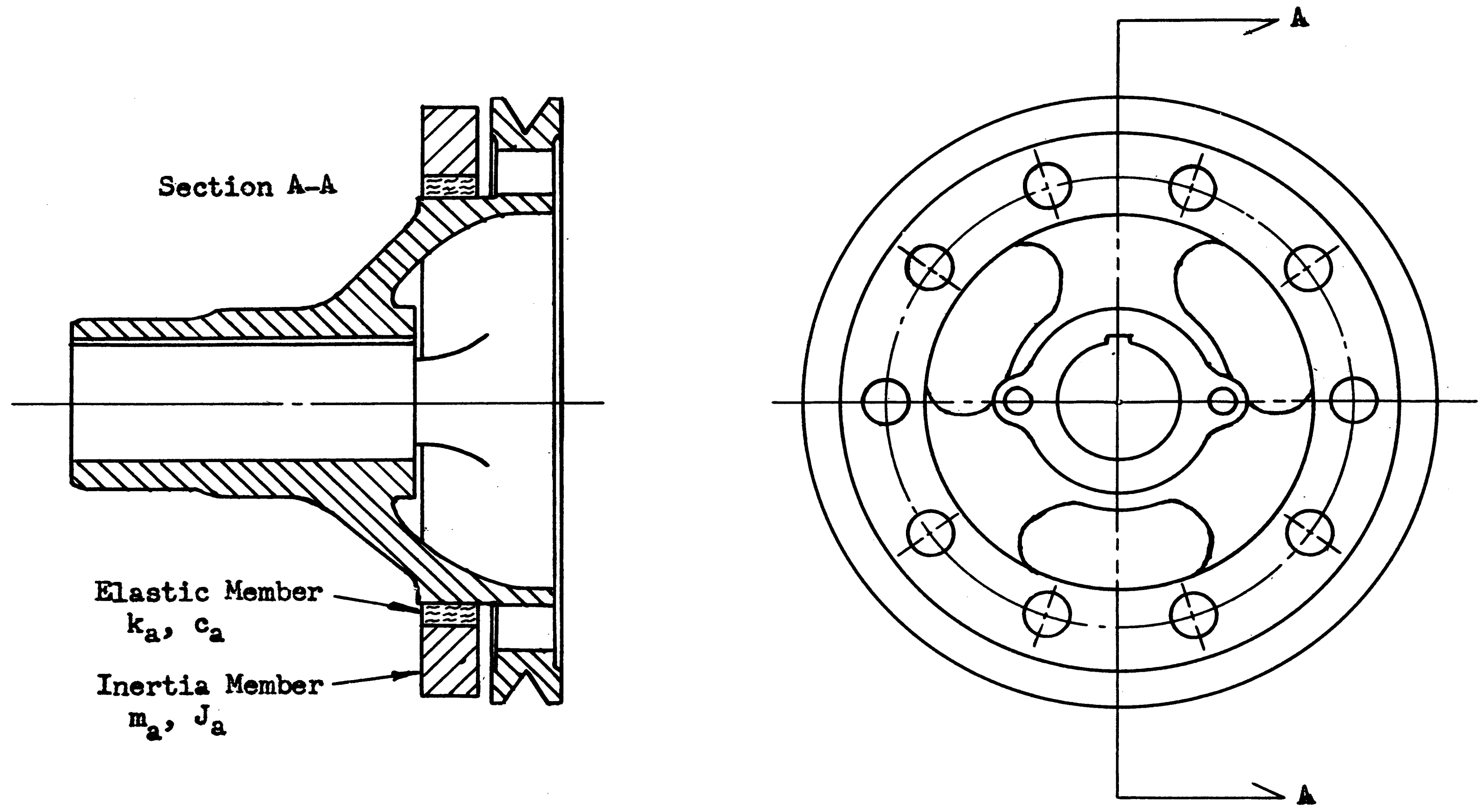

DAMPED VIBRATION ABSORBER
SIMPSON MANUFACTURING CO.
LITCHFIELD, MICHIGAN

ENGINEERING RESEARCH INSTITUTE
THE UNIVERSITY OF MICHIGAN
ANN ARBOR

Final Report

DESIGN OF DAMPED VIBRATION ABSORBERS
FOR AUTOMOTIVE ENGINES

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Project 2526

SIMPSON MANUFACTURING COMPANY
LITCHFIELD, MICHIGAN

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ABSTRACT

This report deals with the design of damped vibration absorbers used on automotive engines.

Vibration theory, which we have simplified and combined with easily obtained test data, gives a means of rapidly analyzing vibration. This is useful in initiating a new absorber design or in modifying an existing absorber for greater effectiveness.

A brief summary of torsional vibration theory and its application to simple systems is contained in the introduction, and a development of their mathematical solutions is contained in the Appendix.

The Design Sections include the development of an initial design using a torsiograph vibration curve and the relationship of absorber components to fit the design problem. This is accompanied by an illustrative example. A simple method for the modification of an absorber is also offered for adjustments after preliminary testing or modification of an existing absorber.

OBJECTIVE

The objective of this report is to develop and simplify a mathematical approach to the problem of absorber design, and to correlate the results of calculated design with those of actual tests.

NOMENCLATURE

Symbol	Quantity	Dimensions
c	Damping constant	in-lb-sec
c_c	Critical damping constant $2J\omega_n$	in-lb-sec
c/c_c	Damping factor	Dimensionless
D_x	Diameter of mass x	in.
d_x	Diameter of shaft x	in.
G	Shear modulus	$lb/in.^2$
g	Acceleration of gravity = 386 in/sec^2	in/sec^2
J_x	Moment of inertia of mass x	$lb\text{-}in\text{-}sec^2$
J_{area}	Polar moment of inertia of area	$in.^4$
k_x	Spring Constant (in rotation) of member x	lb-in/rad
L	Length	in.
M	Moment of a force	lb-in.
M_r	Absorber-engine amplitude ratio (max.)	Dimensionless
m_x	Mass (W/g) of member x	lb
R	Radius of gyration	in.
r	Radius	in.
S	Shear displacement	in.
T_o	Amplitude of torque vector	lb-in.
T_e	Torque in engine shaft at θ_e	lb-in.
t	Time	sec
V	Velocity (angular)	rad/sec
W_x	Weight of mass x	lb
w	Width	in.
θ_x	Maximum rotational displacement or maximum amplitude of mass x	rad or degrees
θ_{st}	Maximum static rotational displacement (ω=0)	rad or degrees
θ_{ae}	Maximum vibration amplitude between absorber and engine	rad
ϕ_x	Instantaneous amplitude during harmonic motion of mass x	rad
μ	Inertia ratio	Dimensionless
τ_x	Period of mass x	sec
π	3.14159	Dimensionless
ω	Circular frequency of the force	$rad\ sec^{-1}$
ω_n	Circular frequency (natural)	$rad\ sec^{-1}$
ω_{n_a}	Natural circular frequency of absorber	$rad\ sec^{-1}$
ω_{n_e}	Natural circular frequency of engine (first mode)	$rad\ sec^{-1}$

INTRODUCTION

The objectives of this investigation are: 1) to develop a mathematical approach to engineering problems involved in the design of damped vibration absorbers and 2) to determine a correlation between calculated and actual performance of absorbers in operation.

Mechanical systems, consisting of masses and elastic members, will vibrate 1) if one or more of the masses is displaced and 2) if the displaced mass causes a deformation of an elastic member. Such a vibration can take place if a mass is initially displaced and released or if a pulsating external force is applied to the system in some manner. An automobile engine, with its pistons, connecting rods, crankshaft, flywheel, camshaft, and various accessory moving parts, is a typical example of displaced masses and elastic members. Such a system presents complex vibration problems.

In many cases small vibrations in machinery are not harmful. When certain conditions prevail during mechanical operation, the vibrations may become excessive and cause serious stress, wear, and noise. Such conditions are most likely to exist when the applied external force is pulsating with the natural frequency of the machine or one of its parts. A situation of this kind is called resonance.

To develop some of these concepts in the light of absorber design a few simple torsional systems and their evaluation will be discussed. In developing equations involving the frequency of vibration of elastic members in torsion, several physical constants are required; the torsional spring constant, the angular moment of inertia, and the damping constant. Methods of determining these constants will be outlined.

TORSIONAL SPRING CONSTANT

The spring constant of a shaft in torsion is defined as the in-lb of torque required to twist the shaft one radian. This may be determined by experiment or it may be calculated from the dimensions and material properties of the shaft.

<u>Experimental method</u>.—Secure one end of the shaft against rotation. To the other end attach a lever arm of known length and support the shaft as near the arm as possible to prevent axial distortion under load. Apply a known load to the lever arm. Measure the twist or angular displacement of the shaft. Figure 1 is a test procedure suitable for shafts.

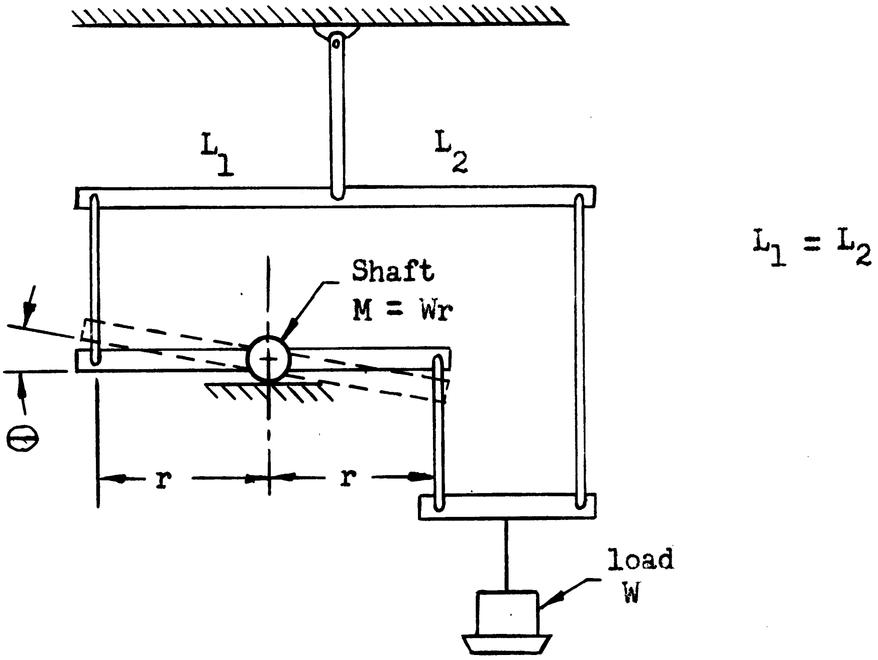


Fig. 1.

Calculation method.—The following formula for the angular displacement of a shaft in torsion may be used to calculate the spring constant.

$$M = \frac{GJ_{area}}{L} \theta$$

By definition, the spring constant

$$k = \frac{GJ_{area}}{L} .$$

The polar moment of inertia of the cross-sectional area has the dimensions of $inch^4$. If the length of the shaft L is in inches, and the modulus of rigidity G is in $lb/in.^2$ (11,500,000 for steel), k will have the dimensions of in-lb/radian. If the shaft is round,

$$J_{area} = \frac{\pi D^4}{32} .$$

For other shapes J_{area} can be found in handbooks. For a complicated shaft such as a crankshaft, k can be found by calculating the deflection of each section (i.e., bearing journals, webs, etc.) and adding these deflections under a given load.

ANGULAR MOMENT OF INERTIA

Angular moment of inertia is a measure of the resistance that a body offers to any change in its angular velocity. This may be calculated for simple, elemental geometric shapes. For more complicated bodies the moment of inertia may be calculated if the object can be divided into simple geometric shapes. The moment of inertia of any shape, simple or complex, may be determined experimentally.

Experimental method.—If a mass is suspended on a long thin vertical rod and allowed to oscillate as a torsional pendulum, the torsional spring constant of the rod will be given by

$$k = \frac{4\pi^2 J_{mass}}{\tau^2} .$$

If a disc of uniform thickness is mounted on the lower or free end of the rod and allowed to oscillate (the other end of the rod being fixed), we have

$$k = \frac{4\pi^2 J_D}{\tau_D{}^2} .$$

J_D for the disc can be calculated from

$$J_D = \frac{Wr^2}{2g} .$$

J_D is measured for the rod and disc combination. Then if a mass M, having a moment of inertia J_M, is placed on or under the disc, concentric with the rod, we have

$$k = \frac{4\pi^2(J_D + J_M)}{\tau_{D+M}^2} = \frac{4\pi^2 J_D}{\tau_D{}^2}$$

or

$$\frac{J_D + J_M}{J_D} = \left(\frac{\tau_{D+M}}{\tau_D}\right)^2$$

from which J_M can be calculated.

Calculation method.—The angular moment of inertia is defined by the formula

$$J = \frac{WR^2}{g} .$$

For a complex body, reduce it to a series of elemental geometric parts. From the dimensional data, calculate the radius of gyration R about the center of rotation for each of the parts. Determine WR^2 for each and add them together to arrive at a total. The moment of inertia for the body will then be this total divided by g. Where the dimensions are in inches and g is 386 in/sec^2, J will be in terms of in-lb-sec^2.

DAMPING CONSTANT

The frictional resistances, internal molecular and external, which oppose vibration, are termed damping forces and for most evaluations are closely approximated by values proportional to the velocity of the masses in vibration. The effects of a damping force proportional to velocity and of a restoring force proportional to displacement are to reduce both the natural frequency of vibration and the amplitudes of successive cycles. Since the amplitudes and velocities of natural torsional vibrations are small, their effect on the frequency of vibration is small. The damping force may be so large as to prevent an oscillation, in which case the damping constant is said to be larger than the critical damping constant. If the damping constant is less than critical, a free vibration, once excited, will continue to vibrate with a decreasing amplitude until it comes to rest. The rate of decay of the vibration will depend on the value of the damping constant. The damping constant is a property of the material in the system and the medium in which the oscillation takes place. It cannot be calculated but may be measured experimentally.

With a damping constant $c/c_c < 1.0$, free vibration will occur. A means of measuring c would be to measure the rate of decay of the vibration due to c (Fig. 2). The damped system is displaced and allowed to vibrate. A measure of the change in the amplitude for a number of cycles will determine c in the formula

$$\delta = \log_e \frac{X_n}{X_{n-1}} = \frac{c\tau}{2J} \ .*$$

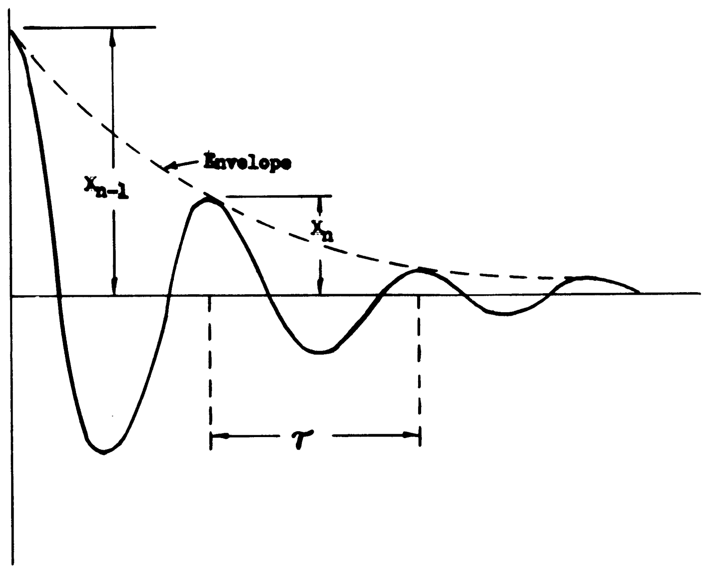


Fig. 2

The motion is a sine wave with a constant period of τ seconds. The inertia of the mass may be calculated or measured. With a measure of the amplitude of two successive cycles X_{n-1} and X_n, the damping c may be calculated. For more accurate measurement, a number of cycles may be taken, using

*The inertia J used for torsional motion is replaced by mass (m) for translatory motion.

$$\delta = \frac{1}{n - m} \log_e \frac{X_n}{X_{n-m}} = \frac{c\tau}{2J} .*$$

If the decay is rapid, the amplitude measurements should be modified to measure the amplitude at the point of tangency of the exponential envelope drawn on the decay curve. The units of c are in-lb-sec. A second method of obtaining the damping constant is explained in the Design Section of this report.

NATURAL FREQUENCY OF A SINGLE-MASS SYSTEM WITHOUT DAMPING

The simplest system that supports vibration is that of one mass and one torsional spring as shown in Fig. 3.

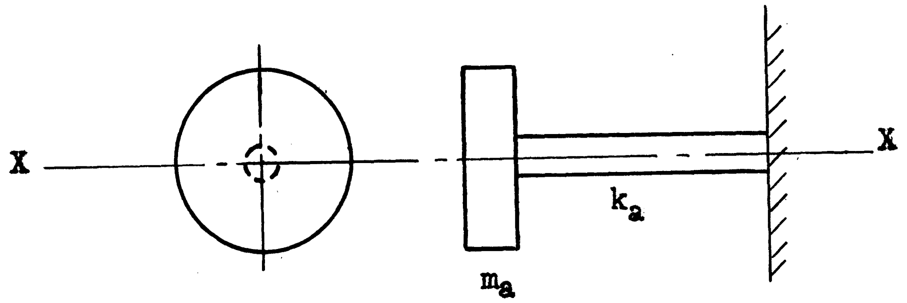


Fig. 3.

The elastic member or shaft is fastened to a rigid support on one end and a mass on the other. The amplitude of movement in rotation about the x axis at any given instant is $\phi = \Theta \cos \omega_n t$, which is derived along with an illustrative problem in Part I of the Appendix. The equation states that the instantaneous amplitude ϕ is dependent on the initial displacement Θ and varies in harmonic fashion, depending on the natural frequency ω_{n_a} where

$$\omega_{n_a} = \sqrt{\frac{ka}{Ja}} .$$

For clarity, ω_{n_a} may be changed from radians per second to cycles per second by dividing by 2π.

Increasing the stiffness of the elastic member or decreasing the moment of inertia will increase the natural frequency. A decrease in stiffness or an increase in moment of inertia will lower the natural frequency.

*The inertia J used for torsional motion is replaced by mass (m) for translatory motion.

NATURAL FREQUENCY OF A TWO-MASS SYSTEM WITHOUT DAMPING

A vibration of the same type illustrated in the single-mass system occurs in a free body consisting of two masses connected by an elastic member. Assume in the following figure (Fig. 4) that the system is free to rotate and that the shaft is supported by some means on frictionless bearings. This system results in a single natural frequency, as before, with the masses always moving 180° out of phase with each other. The natural frequency is now dependent upon both masses

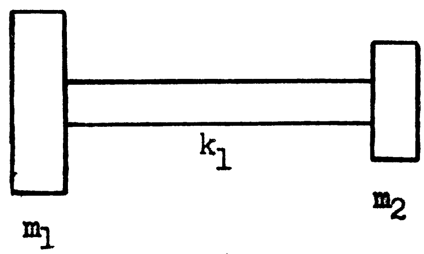


Fig. 4.

$$\omega_n = \sqrt{\frac{k_1}{J_1} + \frac{k_1}{J_2}}$$

(see Part II of the Appendix). If m_2 is made a rigid support, J_2 approaches infinity and the system becomes a single-mass system as shown in Fig. 3. The amplitudes of the masses vary inversely as the moments of inertia,

$$\frac{\theta_1}{\theta_2} = -\frac{J_2}{J_1} ,$$

the negative sign indicating the phase difference of 180°. Since the masses are moving opposite to each other at all times and connected to the same shaft twisting it as they move, a fixed point must be established in the shaft where the opposing twisting motions meet. Here no vibration occurs. This is the node point of the system. An illustration of this system may be found in Part II of the Appendix.

This example is analogous to an engine if the engine components are reduced to a simple form of two masses and a single shaft, i.e., let one mass represent the crankshaft and associated parts and the other mass represent the flywheel.

MULTIPLE MASS SYSTEMS AND FORCED VIBRATION

An absorber without damping added to the front of a crankshaft is equivalent to adding an additional mass and an elastic member to Fig. 4 of the previous discussion. This results in the three-mass system shown in Fig. 5.

The method of solution is shown in Part III of the Appendix. The solution is now two frequencies, resulting from two modes of vibration. The first mode has two adjacent masses deflecting in the same direction or phase and one end mass out of phase with the movement of the other two. The second mode has the two end masses in phase and the center mass opposed. The first mode results in a node point in one of the shafts in a position which is de-

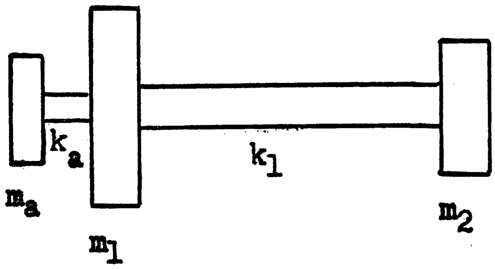


Fig. 5.

termined by the relationship between the masses and the springs in the system. The second mode has two nodes, one in each shaft, the location again depending upon the relationship of the masses and the springs in the system.

It should be noted that a system has a number of natural frequencies or modes equal to one less than the number of masses in the system. The combination shown in Fig. 6 has six natural frequencies in torsion. The first mode would have one node (with the larger mass m_7 on the right, the node will be to the right of the center of the system). As the modes increase, the nodes also increase in number up to the sixth mode which has a node in every shaft.

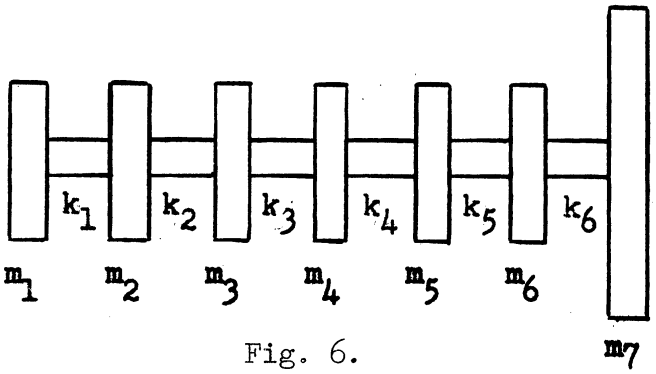


Fig. 6.

Considering such a system to represent an engine, there are six natural frequencies which could be driven into resonance and which could cause difficulty. The engine speed determines the external forced frequencies to produce resonance. In an undamped system, any force alternating at a frequency which coincides with or is a submultiple of a natural resonance will destroy the system. In a practical sense, however, all systems have some form of damping. Damping may be due to hysteresis losses in the metal, bearing friction, or air damping. Each tends to decay or reduce the amplitude of a resonating part unless an external force is applied to replace the energy losses of damping.

FORCED VIBRATION OF A SINGLE- AND TWO-MASS SYSTEM WITHOUT DAMPING

From a study of the previous examples, it is evident that the effect of an external force must be considered. When a system such as Fig. 3 is driven by a periodic external force, forced vibrations occur having the frequency of the external force and an amplitude depending on the intensity of the external force and the relation of the external force frequency to the natural frequency. The relation of the frequency and amplitude is given by the equation

$$\theta = \frac{T_o}{1 - \left(\frac{\omega}{\omega_{na}}\right)^2} .$$

(The derivation for the single-mass system can be found in Part IV of the Appendix.) An illustration of amplitude variations resulting from change in forced frequency is shown in Fig. 7. θ_{st} is the condition where the deflection θ is caused by T_o if T_o were a steady torque. At $\omega/\omega_{na} = 1$, the curve shows a discontinuity at resonance, indicating the amplitude will increase until failure takes place or some limiting action is taken to restrict the movement.

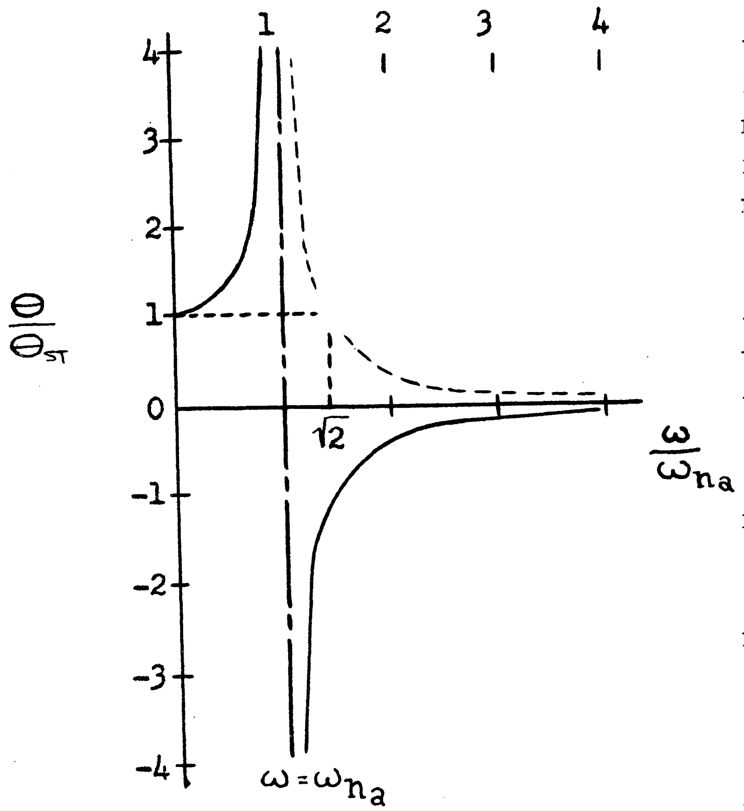


Fig. 7.

The solid lines represent the actual movement of the mass and indicate that below resonance the mass is in phase or moving with the force. Above resonance the mass is moving against the force and is shown below the line. The dotted line is merely the lower portion of the curve which is actually negative but is plotted positively according to common practice.

The two-mass system is much the same as the single-mass system, resulting in a resonant condition where the amplitudes of the masses approach infinite values 180° out of phase with each other.

FORCED VIBRATION OF A THREE-MASS SYSTEM WITH UNDAMPED ABSORBER

This three-mass system has two points of resonance where the forced frequency matches each of the two natural modes.

Figure 8 shows the relationship between the absorber and the center mass with the third mass rigidly fixed. Two discontinuities appear in the curves; one at the point where the forced frequency equals the natural frequency of the first mode of the three-mass system, and one at a point where the forced frequency equals the natural frequency of the second mode. Also note in Fig. 8 that there is a point where the value of $\omega/\omega_{na} = 1$ which results in zero amplitude for the center mass. Therefore, when the forced frequency equals the absorber natural frequency (the natural frequency of mass a and shaft a), there is no vibration in the m_1m_2 system. This is the theory of the dynamic absorber. The values given in the example in the Appendix for the single-mass spring system were chosen to be resonant at the same frequency as the natural resonance frequency of the two-mass system. Thus, when these were joined together to form Fig. 5, m_a and k_a became the absorber, "tuned" to the resonant condition of the two-mass system. The resonant point where the forced frequency pre-

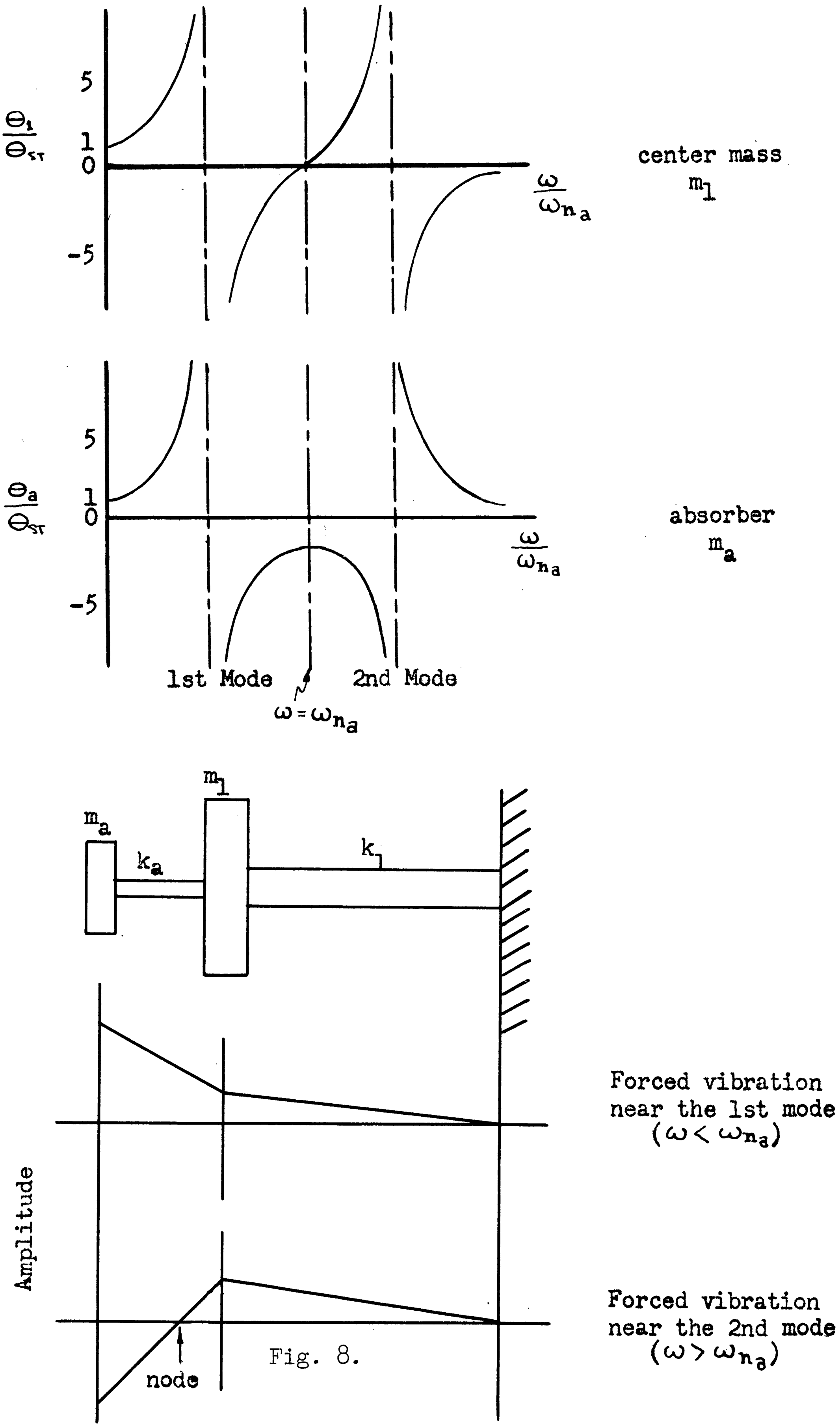

5
1
0
-5
$\frac{\Theta_1}{\Theta_{st}}$
$\frac{\omega}{\omega_{na}}$
center mass
m_1
5
1
0
-5
$\frac{\Theta_a}{\Theta_{st}}$
$\frac{\omega}{\omega_{na}}$
absorber
m_a
1st Mode
2nd Mode
$\omega = \omega_{na}$
m_a
k_a
m_1
k_1
Amplitude
Forced vibration
near the 1st mode
$(\omega < \omega_{na})$
Forced vibration
near the 2nd mode
$(\omega > \omega_{na})$
node

Fig. 8.

viously drove the two-mass system into infinite amplitude now results in zero amplitude of the two masses and a finite amplitude in the absorber. The finite amplitude of the absorber depends only on the spring value of the absorber and the external force. Since the other two masses of the original system are no longer responding to the external frequency, the absorber will "absorb" the total force by a deflection of its spring such that an equal and opposite force counteracts the external force at all times.

FORCED VIBRATION AND DAMPED ABSORBERS

If a machine has a constant or nearly constant speed, the absorber may be "tuned" to the disturbing frequency and effectively reduce the vibration. However, in an automobile engine with its large range of speeds, difficulty is encountered immediately at two resonant points in the absorber modified system, whereas in the original system, without absorber, only one resonant point may have been encountered. The solution to the new problem is damping.

With the introduction of damping we must now consider several modifications of the concepts we have described. The most noticeable change with damping present in a system is that the infinite points, where the forced frequency equals the natural frequency, now become finite and can be controlled to a great extent by the amount of damping in the system. A second change is a shifting of the resonant frequency. This effect is much less apparent and often neglected. Figure 9 shows the effect of an additional mass (damped absorber) added to a two-mass system with the end mass a fixed body.

The outer pairs of dotted lines represent the amplitude of motion of the center mass m_1 similar to that shown in the upper sketch of Fig. 8, excepting that the negative amplitudes of Fig. 8 are made positive for simplification. These curves are without damping ($c/c_c = 0$). The inner pair of dotted lines represent infinite damping ($c/c_c \to \infty$). Under these circumstances, the absorber mass m_a acts as a part of the center mass m_1, and

$$\omega_n = \sqrt{\frac{k_1}{J_a + J_1}} \quad ,$$

which is somewhat lower in natural frequency than the original system.

Two other curves are shown representing absorbers tuned to the natural frequency of the two-mass system but with different damping values. The double-peaked curve has less damping than the single-peaked curve in the center. Depending on the amount of damping, the curve may be varied within the extremes of no damping (outside dotted curves) to infinite damping (center dotted curve). Note that all curves intersect at points P and Q regardless of the damping in the system. For optimum absorber effect, the points P and Q on the curves can be adjusted to equal heights by changing the tuning of the ab-

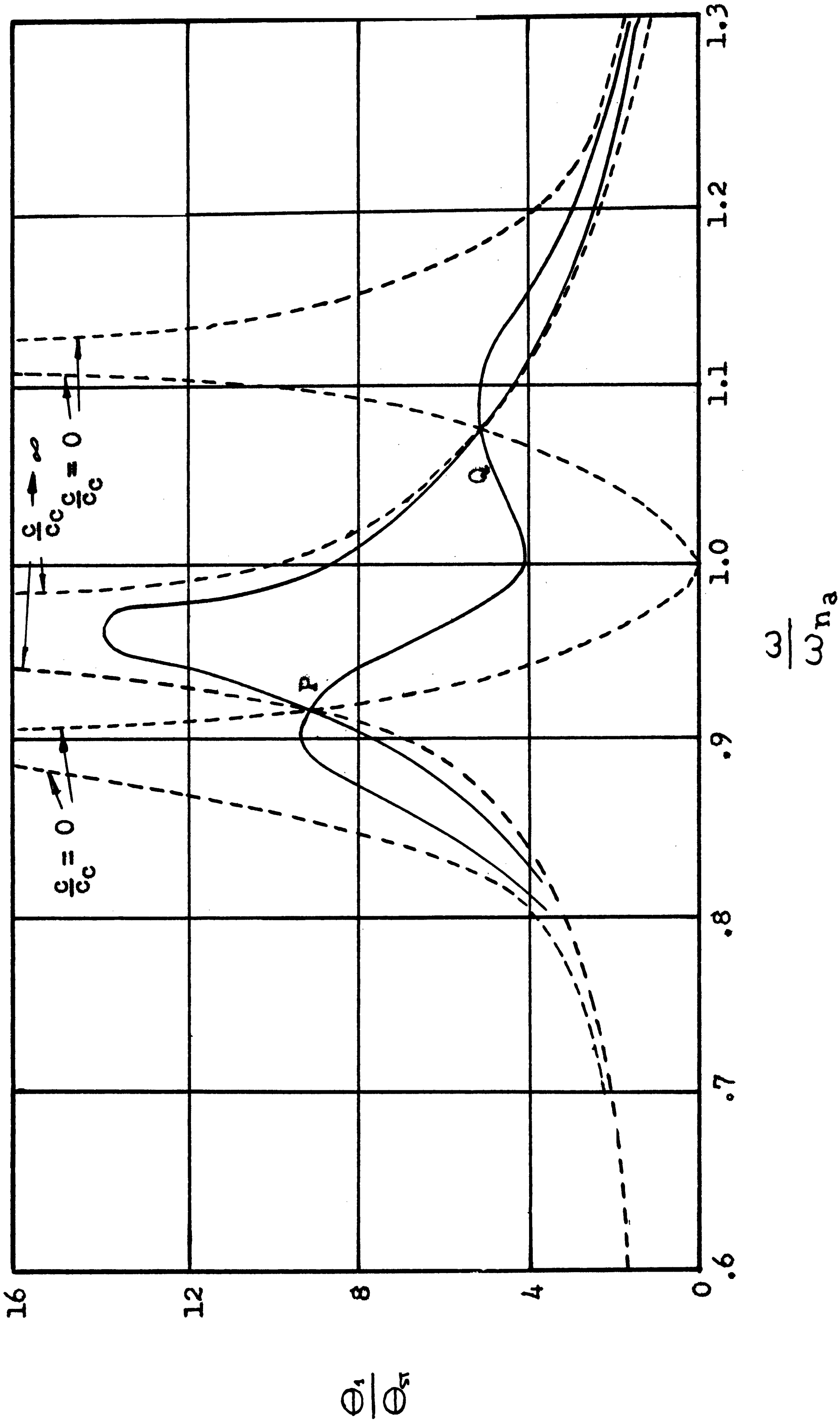

16
12
8
4
0
.6
.7
.8
.9
1.0
1.1
1.2
1.3
$\frac{\Theta_1}{\Theta_{st}}$
$\frac{\omega}{\omega_{na}}$
$\frac{c}{c_c} = 0$
$\frac{c}{c_c} = \infty$
$\frac{c}{c_c} = 0$
P
Q

Fig. 9.

sorber. For a given absorber inertia, the two outside dotted curves ($c/c_c = 0$) are fixed relative to each other. By tuning, these can be shifted relative to the center curve ($c/c_c \to \infty$). An increase in absorber inertia will separate the two outside curves, thus separating P and Q. With proper tuning this has the effect of actually lowering P and Q and of reducing the resulting maximum vibration at those points.

To find the optimum tuning point for an absorber, the formula is:

$$\omega_{n_a} = \left(\frac{1}{1 + \mu}\right) \omega_{n_1}$$

where

ω_{n_a} = natural frequency of the absorber
ω_{n_1} = natural frequency of the original system m_1k_1
μ = inertia ratio J_a/J_1

Control of the amount of damping in the system can limit the height of the curve which is a measure of the amplitude of the motion. Damping which produces a maximum at point P or Q, with P and Q equal (see Fig. 10), will give the best possible absorber for a given inertia.* In the figure, P and Q are shown of equal height and a minimum of vibration is obtained throughout the resonant range.

With modifications, the methods used to determine absorber design for a fixed torsional spring and mass system can be used to approximate multimass conditions. The more complex systems must be reduced to a resultant or equivalent inertia which will behave in a way similar to the original system. The value of the effective inertia is determined by the inertias and the amplitudes involved. The inertia of each mass and its amplitude squared is calculated and the sum of these is obtained ($\sum J\theta^2$). With the two-mass system of Fig. 4, the effective inertia $\sum J\theta^2$ would be $J_1\theta_1^2 + J_2\theta_2^2$ and the inertia ratio μ modified to $J_a/\sum J\theta^2$ when m_a is added (Fig. 5).

Considering the absorber design for engines, the inertia ratio is the absorber inertia J_a divided by the equivalent engine inertia $\sum J\theta^2$.** The equivalent engine inertia is a breakdown of the complicated spring-mass system to a system of the type shown in Fig. 11. The relationship of the amplitudes of the various masses is shown in the lower portion of this figure. If the end mass is assigned a vibration amplitude of 1.0 unit, the related movements of the other masses may be calculated. The effective inertia is then the sum of each mass times its relative amplitude squared of $\sum J\theta^2$. For ideal representation an equivalent system should have kinetic and potential energies equal to the original system. Engine resonance, shown in Fig. 12a, may be determined by

*Discussed fully in Den Hartog, Mechanical Vibrations.

**Methods for obtaining these figures are explained in Wilson, Practical Solution of Torsional Vibration Problems.

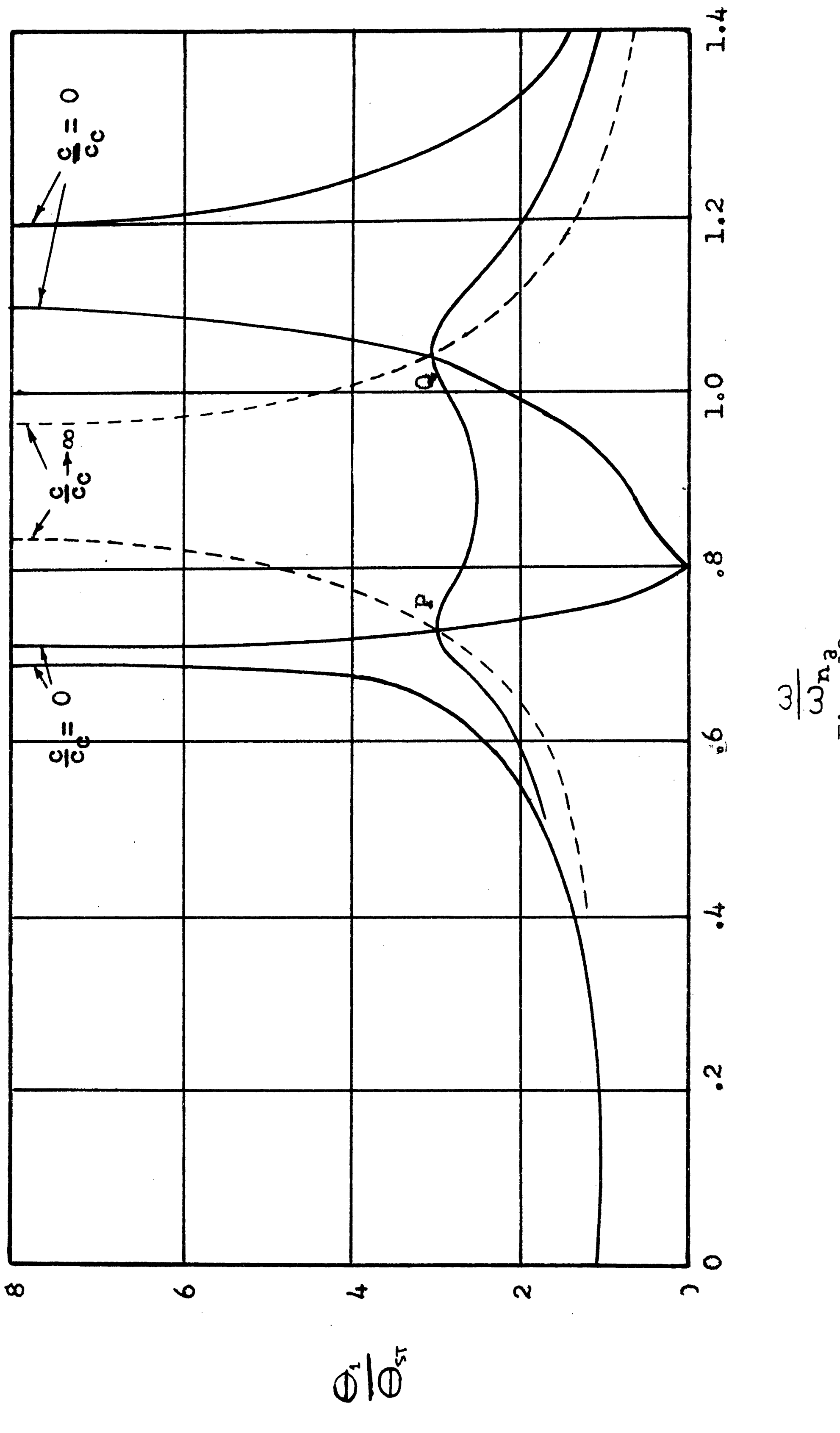
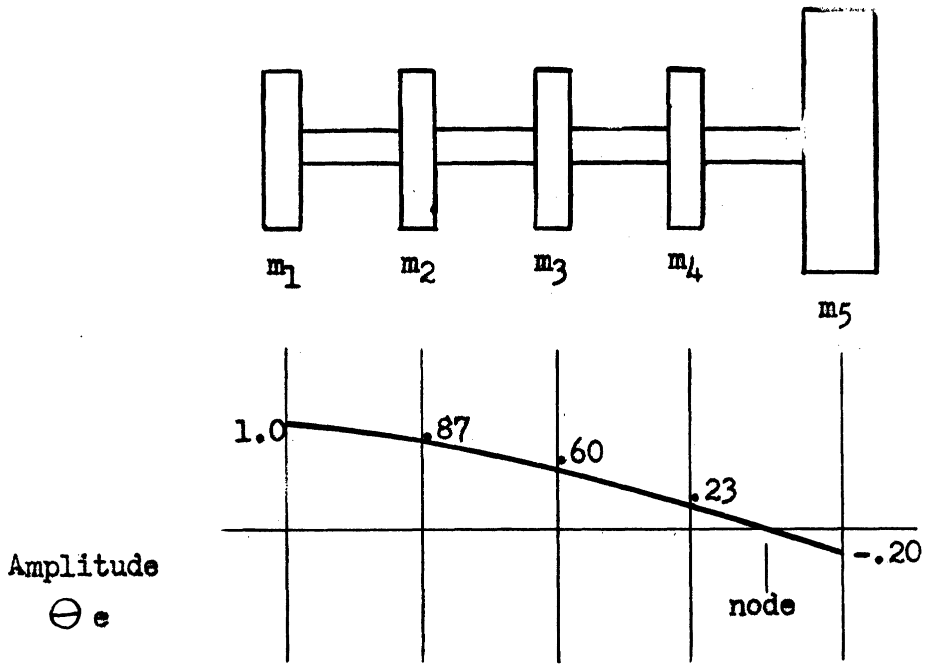


Fig. 10.



1st Mode
Engine Vibration

Fig. 11.

calculation or by test runs with measurements of engine vibration without an absorber. Several points within the speed range will show vibration amplitudes, the maximum usually being at an engine speed which excites the fundamental mode of the engine and produces a node point on the crankshaft near the flywheel.

In an automobile power train, it is possible for the engine, drive shaft, axles, and wheels to constitute a system which, in vibration, causes the engine to act much like a single mass. The predominant use of fluid-type couplings eliminates this condition to a large extent by providing effective isolation. We therefore will consider only the first mode within the engine itself.

A mode may be excited at any engine speed where the engine's pulsating torque coincides with or is a submultiple of the mode. The relationship between engine revolution and the cycle of vibration is known as the order.* For example, the engine might have a vibration frequency of the fifth order at 4000 rpm giving a resonant frequency of 5 x 4000 or 20,000 cpm or 333 cps. This same 4000 rpm resonant point may be excited at other speeds by a different order. For instance, the 333 cps or 20,000 cpm could be excited at the fourth order at 5000 rpm or at the sixth order at 3333 rpm. These excitation points depend on engine factors such as two- or four-cycle design, firing order, number of cylinders, and unbalance of moving parts.

*An order is the number of cycles of vibration per revolution of the shaft.

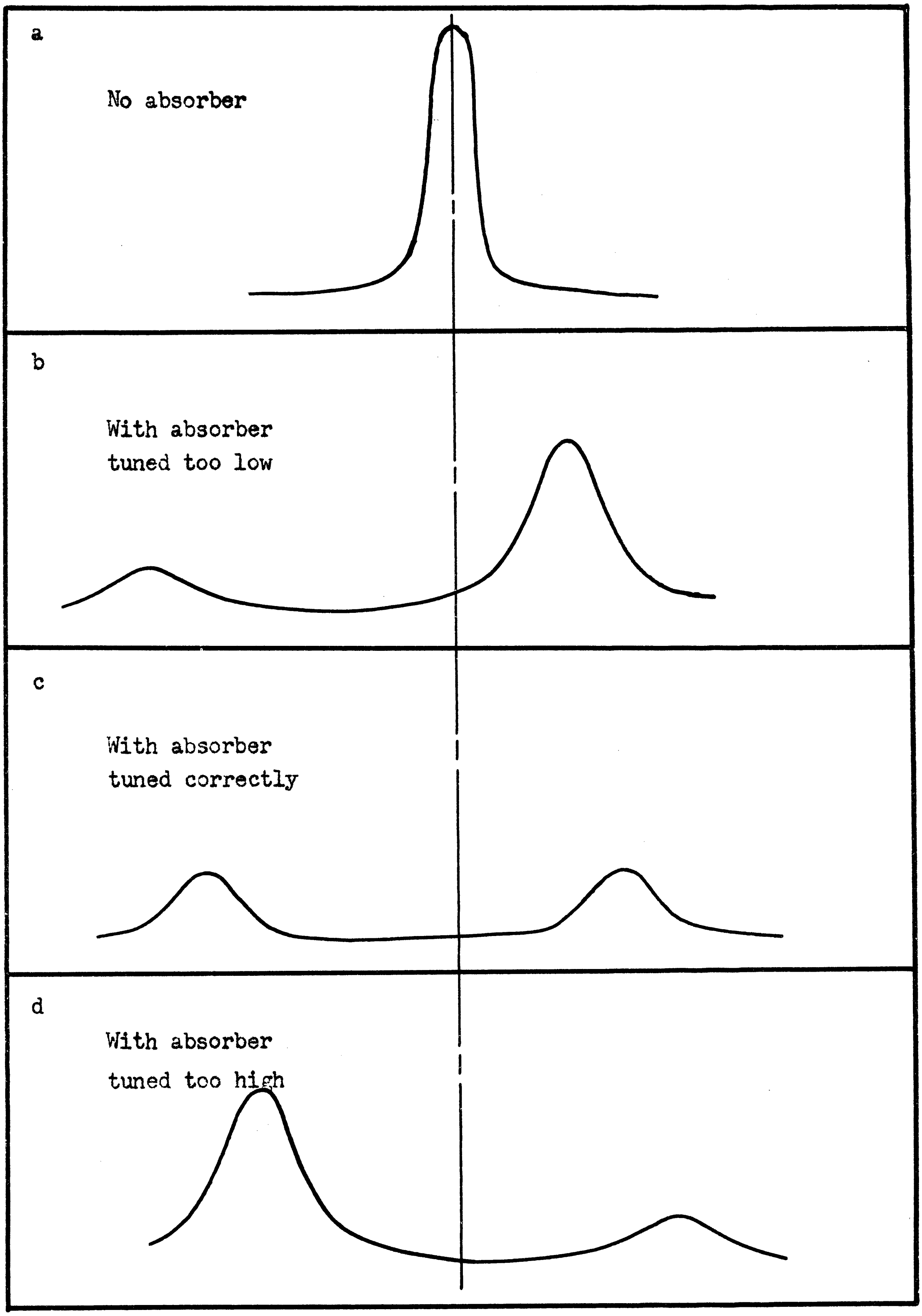

a
No absorber
b
With absorber
tuned too low
c
With absorber
tuned correctly
d
With absorber
tuned too high

Fig. 12.

The absorber will be effective at any engine speed which produces the same basic vibrating frequency. At other frequencies the absorber is ineffective. Thus, in the example above, an absorber properly tuned at 4000 rpm will work equally well at 5000 rpm and at 3333 rpm.

The tuning formula can be adapted to an engine using the effective inertia.

$$\mu = \frac{\text{absorber inertia}}{\text{effective engine inertia}}$$

$$\text{Tuning} = \frac{1}{1+\mu} \times 100\% \ .$$

This result will give the percent the absorber must be detuned from engine resonance for optimum results with a given absorber inertia. As an example of proper tuning, an engine with an equivalent inertia of .25 in-lb-sec and with an absorber with an inertia of .05 in-lb-sec would result in an absorber detuning of

$$\frac{1}{1+\frac{.05}{.25}} \times 100\% = 83.3\% \ .$$

The engine vibration amplitude at resonance equipped with an absorber is an integral part of the absorber tuning. Tuning too low will result in a high point above the original resonance and tuning too high will give a similar high point below resonance. Proper tuning will result in equal amplitudes above and below the original resonant point as explained on page 12 (see Fig. 12).

The effect of absorber inertia on the balanced resonant points appears in the following formula:

$$\frac{\theta}{\theta_{st}} = \sqrt{1+\frac{2}{\mu}} \ .$$

The ratio of θ/θ_{st} is a factor known as the dynamic magnifier and is the ratio of the resonant amplitude of either of the two resonant peaks to the deflection which would be caused by the pulsating torque if this torque were unvarying ($\omega = 0$). A greater absorber inertia results in a smaller θ/θ_{st} or more effective absorber but the rate of increase in effectiveness decreases with an increase in absorber inertia.

Another important consideration in absorber inertia is the deflection in the absorber spring (rubber movement in this case) when at resonance. The smaller the absorber, the more flexible the rubber must be to maintain a given tuned frequency by the formula

$$\omega_{n_a} = \sqrt{\frac{k_a}{J_a}} \ .$$

The results of lower spring constants means a greater deflection must take place in the rubber to produce a torque to oppose the external vibration torque.

DESIGN OF DAMPED VIBRATION ABSORBERS

The design of an absorber may be developed from a mathematical approach, a trial-and-error approach, or by methods which may include both. A successful design developed purely from the mathematics of vibration into a finished product approaches the impossible. Assumptions must be made to provide mathematical equations which are solvable. Even with these assumptions the equations become highly complex because of the inherent complexity of engines and the vibrations they represent. Much work has been done in the field of vibration to simplify this mathematical approach by reducing the engine components to simple equivalent systems which may be more easily solved. This generally involves a sacrifice in the accuracy of calculations and the final product must be tested to evaluate its worth.

The methods presented under Initial Design are simple mathematical approaches correlated with test data. Design Modification gives some suggestions on methods for modifying an existing design for optimum results.

I. INITIAL DESIGN

A. Torsiograph method.—The information necessary to design the absorber is as follows:

1. Equivalent engine inertia (with pulley if possible).

2. Torsiograph of the engine without an absorber (with pulley if possible).

This information is used in the following steps:

1. Vibration amplitude of engine with absorber.

   a. Locate resonance rpm, order, and amplitude from torsiograph curve.
   b. Locate a point 200 rpm or less from resonance on the same curve determining rpm and amplitude.
   c. Solve for dynamic magnifier and θ_{st}.
   d. Step c may be checked by taking another point and solving again.
   e. Determine absorber inertia, or select absorber inertia and determine vibration amplitude.

2. Absorber tuning

From these calculations we have these results:

1. ω_{n_a} (absorber frequency)

2. J_a (absorber inertia)

These are results using optimum damping as shown in Fig. 13.

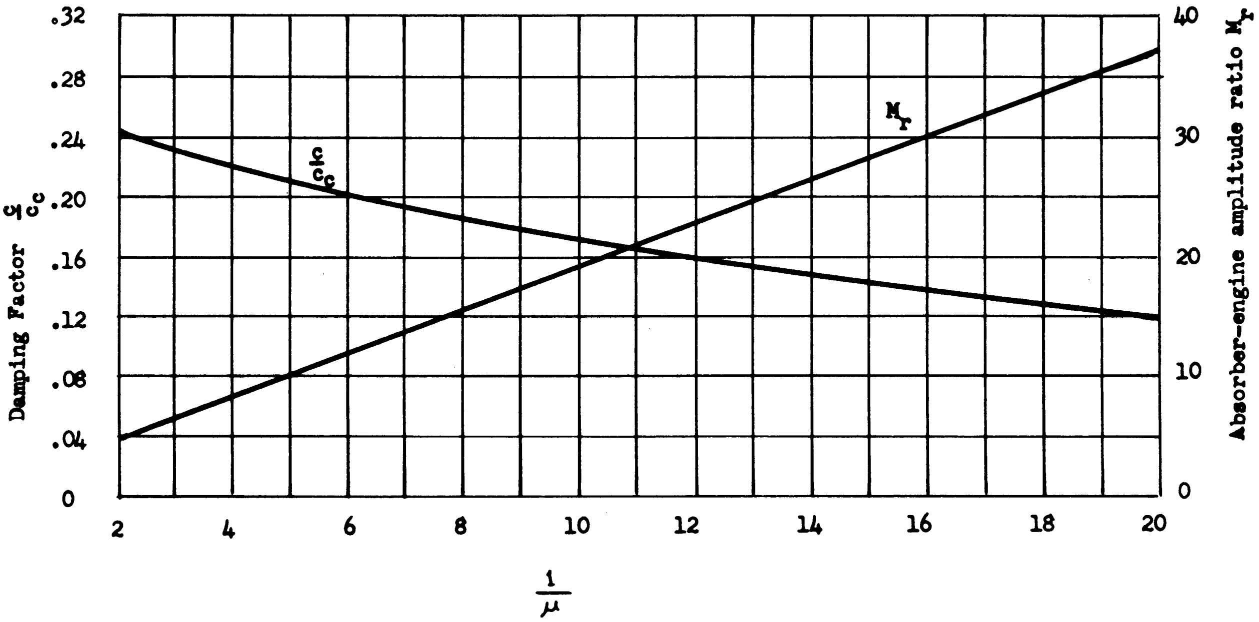


Fig. 13.

Torsiograph Method Example

The basic equation for the dynamic magnifier is used and it is a modification of previous equations to include damping.

$$\frac{\theta_e}{\theta_{st}} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_{n_e}}\right)^2\right]^2 + \left[2\frac{c}{c_c}\frac{\omega}{\omega_{n_e}}\right]^2}}$$

where

θ_e = engine amplitude in degrees double amplitude

θ_{st} = engine static amplitude at resonance in degrees double amplitude ($\omega = 0$)

ω = force frequency, cps

ω_{n_e} = natural frequency, first mode, cps, of the engine

c/c_c = damping factor of the engine

Since engine vibration is usually stated in degrees double amplitude (D.D.A.) and frequency in cps, they are used here in that manner. The following is an example of this method of development with steps similar to those on page 17.

Information

1. Equivalent engine inertia .32 lb-in-sec^2
2. Torsiograph data

rpm	D.D.A.
4400	.45
4500	.85
4600	1.40
4700	1.80 resonance, 4th order
4800	1.30
4900	.40

Steps

Step 1 a. Resonance 4700 rpm, 1.80 D.D.A. 4th order

Step 1 b. Second point 4600 rpm, 1.40 D.D.A.

Step 1 c. Solving for the dynamic magnifier

$$\omega_{n_e} = \frac{4700 \times 4}{60} = 313 \text{ cps (natural frequency)}$$

$$\omega = \frac{4700 \times 4}{60} = 313 \text{ cps (force frequency)}$$

At resonance $\omega = \omega_{n_e}$ and the equation reduces to

$$\frac{\theta_e}{\theta_{st}} = \frac{1}{2\frac{c}{c_c}} \quad \text{or} \quad \frac{c}{c_c} = \frac{\theta_{st}}{2\theta_e}, \text{ where } \theta_e = 1.8° \text{ D.D.A.}$$

$$\frac{c}{c_c} = \frac{\theta_{st}}{2(1.8)} = \frac{\theta_{st}}{3.6} \quad \text{(first equation)}$$

At the second point

$$\theta_e = 1.4° \text{ D.D.A.}$$

$$\omega = \frac{4600 \times 4}{60} = 306.5 \text{ cps (force frequency)}$$

$$\omega_{n_e} = 313 \text{ cps (natural frequency)}$$

$$\frac{\theta_e}{\theta_{st}} = \frac{1}{\sqrt{\left[1 - \left(\frac{306.5}{313}\right)^2\right]^2 + \left[2\frac{c}{c_c}\left(\frac{306.5}{313}\right)\right]^2}}$$

$$\frac{\theta_e}{\theta_{st}} = \frac{1}{\sqrt{(.042)^2 + \left(1.958\frac{c}{c_c}\right)^2}}$$

$$\frac{\theta_e^2}{\theta_{st}^2} = \frac{1}{17.64 \times 10^{-4} + 3.834\left(\frac{c}{c_c}\right)^2} \quad \text{(second equation)}$$

Substituting the first equation into the second,

$$\frac{\theta_e^2}{\theta_{st}^2} = \frac{1}{17.64 \times 10^{-4} + 3.834\left(\frac{\theta_{st}}{3.6}\right)^2}$$

with $\theta_e = 1.4°$ D.D.A.

$$\theta_e^2 = 1.96$$

$$1.96 = \frac{\theta_{st}^2}{17.64 \times 10^{-4} + 3.834\frac{\theta_{st}^2}{12.96}}$$

$$\theta_{st}^2 = 1.96\,(17.64 \times 10^{-4} + .296\,\theta_{st}^2)$$

$$\theta_{st}^2 = 34.58 \times 10^{-4} + .580\,\theta_{st}^2$$

$$.420\,\theta_{st}^2 = 34.58 \times 10^{-4}$$

$$\theta_{st} = \sqrt{\frac{34.58 \times 10^{-4}}{.420}} = \sqrt{82.3 \times 10^{-4}}$$

$$\theta_{st} = 9.07 \times 10^{-2} = .0907° \text{ D.D.A.}$$

$$\frac{\theta_e}{\theta_{st}} = \frac{1.8}{.0907} = 19.85 \text{ dynamic magnifier at resonance}$$

Step 1 d. Checking at another point

at resonance $c/c_c = \theta_{st}/3.6$ as before (first equation)

choosing 4800 rpm 1.3° D.D.A.

$$\omega = \frac{4800 \times 4}{60} = 320 \text{ cps}$$

$$\theta_e = 1.3° \text{ D.D.A.}$$

$$\frac{\theta_e}{\theta_{st}} = \frac{1}{\sqrt{\left[1 - \left(\frac{320}{313}\right)^2\right]^2 + \left[2\frac{c}{c_c}\left(\frac{320}{313}\right)\right]^2}}$$

$$\frac{\theta_e}{\theta_{st}} = \frac{1}{\sqrt{(-.045)^2 + \left(2.046\frac{c}{c_c}\right)^2}}$$

$$\frac{\theta_e^2}{\theta_{st}^2} = \frac{1}{20.25 \times 10^{-4} + 4.19\left(\frac{c}{c_c}\right)^2} \quad \text{(second equation)}$$

Substituting the first equation into the second,

$$\frac{\theta_e^2}{\theta_{st}^2} = \frac{1}{20.25 \times 10^{-4} + 4.19\left(\frac{\theta_{st}}{3.6}\right)^2}$$

with $\theta_e = 1.3°$ D.D.A.

$$\theta_e^2 = 1.69$$

$$1.69 = \frac{\theta_{st}^2}{20.25 \times 10^{-4} + 4.19\frac{\theta_{st}^2}{12.96}}$$

$$\theta_{st}^2 = 1.69(20.25 \times 10^{-4} + .323\,\theta_{st}^2)$$

$$\theta_{st}^2 = 34.2 \times 10^{-4} + .546\,\theta_{st}^2$$

$$.454\,\theta_{st}^2 = 34.2 \times 10^{-4}$$

$$\theta_{st} = \sqrt{\frac{34.2 \times 10^{-4}}{.454}} = \sqrt{75.4 \times 10^{-4}}$$

$$\theta_{st} = 8.69 \times 10^{-2} = .0869 \text{ D.D.A.}$$

$$\frac{\theta_e}{\theta_{st}} = \frac{1.8}{.0869} = 20.7 \text{ dynamic magnifier at resonance}$$

Step 1 e. From these findings we can now approximate the effect of an absorber on the system. Choosing an absorber with an inertia of .053* in-lb-sec^2, the new dynamic magnifier is:

$$\text{dynamic magnifier with absorber} = \frac{\theta_e}{\theta_{st}} = \sqrt{1 + \frac{2}{\mu}} \qquad \mu = \frac{.053}{.32} = .166$$

$$\frac{\theta_e}{\theta_{st}} = \sqrt{1 + \frac{2}{.166}} = 3.61 \text{ with effective engine inertia} = .32$$

$$\theta_e \text{ max.} = 3.61(.09) = .32^\circ \text{ D.D.A.}$$

The maximum engine vibration with absorber is estimated to be .32° D.D.A. assuming an approximate static deflection of .090 D.D.A. The vibration or harmonic torque is not known in this case. If it were, the static deflection could be calculated more accurately. The .09 is merely a value between the two static deflections originally calculated in Steps 1 c and 1 d.

Step 2. Tuning $= \frac{1}{1 + \mu} \times 100\%$ or $\omega_{n_a} = (\text{tuning } \%)\, \omega_{n_e}$

$$\frac{1}{1 + \frac{.053}{.32}} = \frac{1}{1 + .166} = .858 \times 100\% = 85.8\%$$

$$85.8\% \text{ of } 313 \text{ cps} = 269 \text{ cps - absorber tuning } (\omega_{n_a})$$

B. Harmonic torque method.—The same problem may be attacked without torsiograph information if use can be made of the following to find θ_{st} of the engine. It is advantageous to return to deflection in radians and to frequency in radians/sec.

*This can be any value determined by the vibration reduction needed as shown here, the physical size as shown on page 24, heat problems (page 28), and production costs.

$$\theta_{st} = \frac{|T_n| \sum \theta}{\omega_{ne}^2 \sum J\theta^2}$$

where

θ_{st} = static deflection in radians ($\omega = 0$)
ω_{ne} = natural engine resonance in radians/sec
$\sum J\theta^2$ = effective engine inertia without absorber
$|T_n|\sum\theta$ = nth order harmonic torque acting on engine without absorber.

$|T_n|\sum\theta$ is also given as HT_o (H = harmonic coefficient; T_o = mean torque). These values must be known to find θ_{st} and may be calculated if information is available to do so. Extensive information on this subject is available in Wilson's two volumes.[2]

From this point the problem is identical to Step 1 e of the previous method.

II. ABSORBER COMPONENTS

From the previous calculations the absorber natural frequency (ω_{na}) and inertia (J_a) are known. The damping factor (c/c_c) can be determined from Fig. 13.

These quantities are interrelated by the formula

$$\omega_{na} = \sqrt{\frac{k_a}{J_a} - \left(\frac{c}{2J_a}\right)^2} .$$

This is the basic single-mass equation

$$\omega_n = \sqrt{\frac{k}{J}} ,$$

modified to include the effects of damping. In the determination of frequency values, the damping has little effect and is neglected. However, in the calculations of amplitude of vibration, it is of great importance.

Determination of k_a in the above formula completes the minimum requirements which must be met by the absorber. It is now necessary to find methods which will enable the components to be designed to these requirements and to perform satisfactorily in service.

A. <u>Inertia component</u>.—The inertia value needed in the design has been determined. This value may be calculated or a sample measured by methods explained in the introduction. These methods are accurate and should give good results. Here is an application of the calculated method using the present type of absorber inertia member.

Calculation

Moment of inertia of the absorber using cast iron (see Fig. 14).

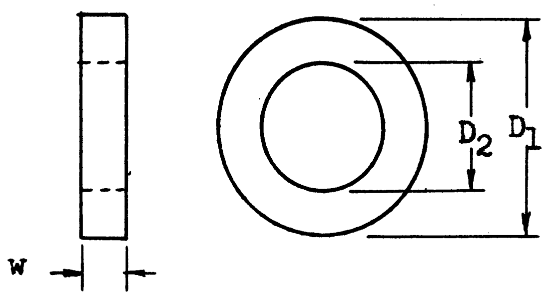


Fig. 14.

$$WR^2 = \frac{w(D_1^4 - D_2^4)}{39.2} .$$

Assume:

D_2 = 4.68 (depending on pulley diameter and gap width)
w = .5625 (depending on space, etc.)
J_a = .053 in-lb-sec^2 (page 22)

Unknown: D_1
Known: $WR^2 = J_a g$
$g = 386 \text{ in/sec}^2$
$WR^2 = 386\ J_a$
$WR^2 = 386\ (.053)$
$WR^2 = 20.45$

Substituting in equation for WR^2

$$20.45 = \frac{.5625[D_1^4 - (4.68)^4]}{39.2}$$

$$D_1^4 - 479.7 = 1425$$

$$D_1^4 = 1904.7$$

$$D_1^2 = 43.64$$

$$D_1 = 6.60 \text{ in.}$$

A factor which may be important for some designs is the inertia effect of the spring component. All elastic members throughout the discussion have been considered as weightless. This will lead to error if the spring has significant inertia of its own when compared to the inertia member itself. Methods of calculation can be found in any vibration text, but in view of the difficulty of determining the spring constant of rubber products used in these designs they do not seem important enough to develop here.

B. Elastic component.—The spring constant and the damping constant are much more difficult to determine. Testing the absorber on a frequency machine with a known inertia member will give an effective spring constant from

$$\omega_{n_a} = \sqrt{\frac{k_a}{J_a}} .$$

Once this has been established, the rubber in the absorber may be used as a basis for whatever design changes it may be necessary to make.

Properties of Rubber as an Elastic Component

The stress-strain properties of rubber are not linear. The shear-strain diagrams for rubber show a sharp increase in shear stress for an increase in shear strain as the rubber continues to be deformed. For small deflections, in the range with which we are dealing, most rubber will approximate linear characteristics. Here are a set of calculations which give an approximate indication of the rubber needed for a specific absorber design (see Fig. 15). A comparison of these calculations with those of an actual absorber under test should give a good indication of modifications which may be made on the design or the rubber composition.

No consideration has been made for effects of temperature, rubber compression, or rubber overhang (extension of the rubber beyond the inertia-member width).

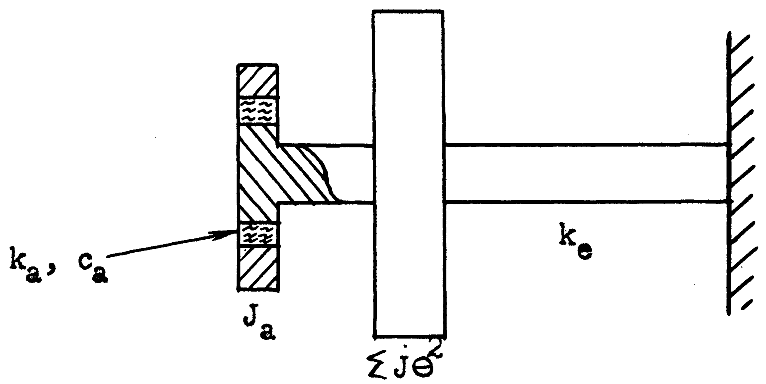


Fig. 15.

ω_{na} = 269 (2π) = 1690 rad/sec (page 22)

J_a = .053 in-lb-sec (page 22)

μ = .166 (page 22)

To obtain the absorber spring constant k_a,

$$\omega_{n_a} = \sqrt{\frac{k_a}{J_a}}$$

$$k_a = \omega_{n_a}^{\ 2} J_a$$

$$k_a = (1690)^2 (.053)$$

$$k_a = 1.51 \times 10^5 \text{ lb-in/rad.}$$

Let $\theta_{ae} = \phi_{ae}$ max. = maximum relative amplitude value between absorber and shaft

where $\phi_{ae} = \phi_a - \phi_e$ (the difference in the instantaneous values of the absorber and engine amplitudes)

Also $M_r = \theta_{ae}/\theta_{st}$ θ_{st} = engine static amplitude without absorber, radians

M_r = absorber-engine amplitude ratio

$$\theta_{st} = .09^\circ \text{ D.D.A.}$$

$$\theta_{st} = \frac{.09}{2} \times \frac{\pi}{180} = .785 \times 10^{-3} \text{ radians}$$

with $\mu = .166$

$$\frac{1}{\mu} = \frac{1}{.166} = 6.02$$

From Fig. 13,

$$M_r = 12.0$$

$$\therefore \theta_{ae} = M_r \theta_{st} = 12.0 (.785 \times 10^{-3}) = 9.42 \times 10^{-3} \text{ rad.}$$

$T_a = k_a \theta_{ae}$ where T_a = maximum vibration torque

$$T_a = (1.51 \times 10^5)(9.42 \times 10^{-3}) = 1424 \text{ in-lb}$$

We now have the information necessary to find the spring characteristics of the rubber. Using Figs. 16 and 17, we can solve for the shear modulus.

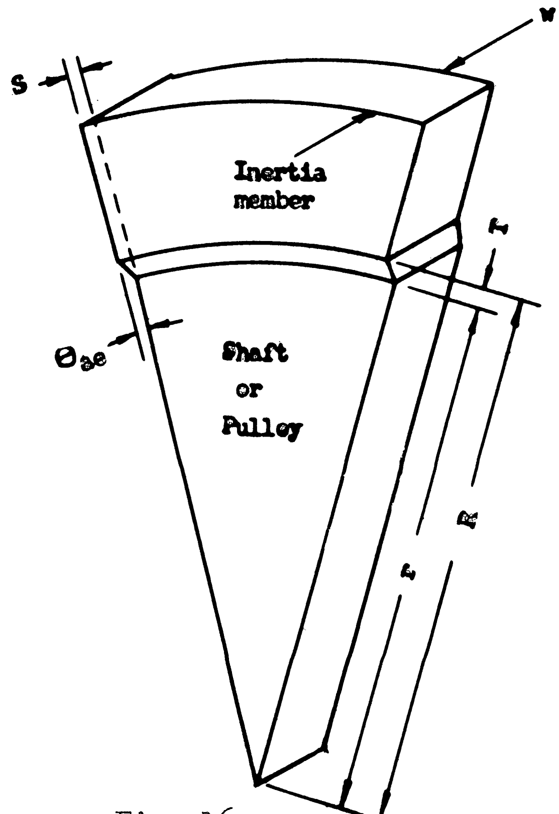


Fig. 16.

$2R$ = D_2 (Fig. 14), in.

w = Absorber width (Fig. 14), in.

S = Shear deformation, in.

T = Gap width, in.

θ_{ae} = Max. relative amplitude, radians

r = Pulley radius, in.

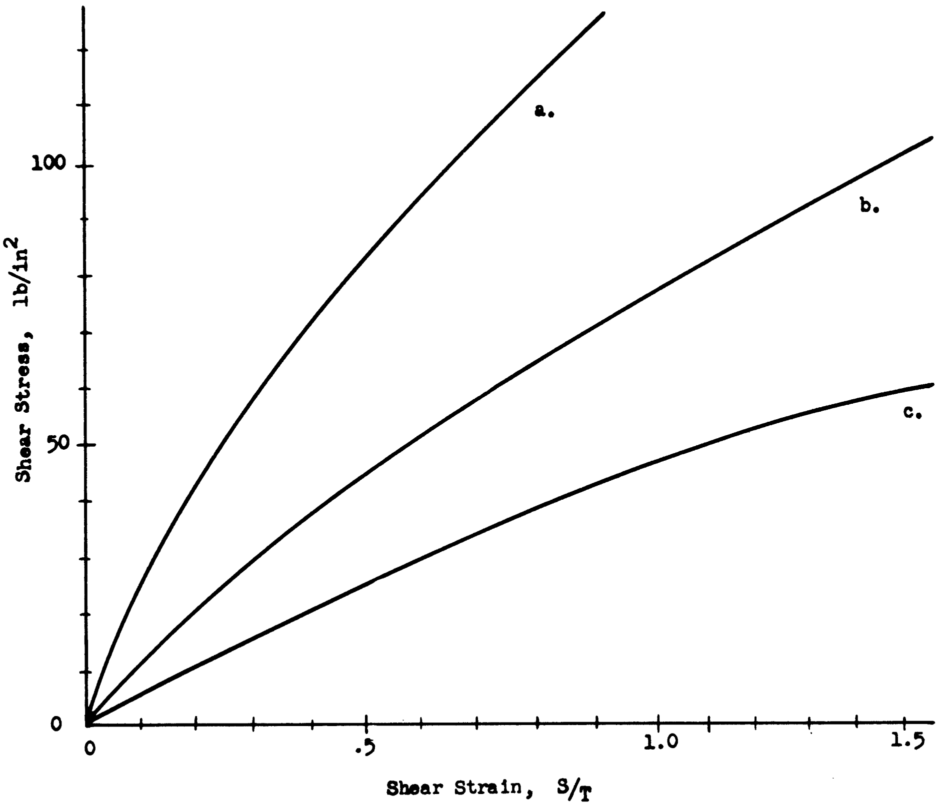


Fig. 17.

$$G \text{ (shear modulus)} = \frac{T_a\left(1 - \frac{r^2}{R^2}\right)}{4\pi r^2 w \theta_{ae}}$$

S = shear deformation = $\theta_{ae}R$
T_a = 1424 in-lb
T = .105 in. (gap)
D_2 = 4.68 in.
R = $D_2/2$ = 2.34 in.
r = $R - T$ = 2.34 - .105 = 2.24 in.
θ_{ae} = 9.42×10^{-3} radians
w = .562 in.

$$G = \frac{1424\left[1 - \left(\frac{2.24}{2.34}\right)^2\right]}{4\pi(2.24)^2(.562)(9.42 \times 10^{-3})}$$

$$S = (9.42 \times 10^{-3})(2.34) = .022 \text{ in.}$$

$$G = 273 \text{ lb/in.}^2 \text{ with } S/T = .022/.105 = .209$$

This shear modulus is correct only with small deflections because of the non-linear characteristics of rubber.

Figure 17 shows some typical curves. Curve (a) has the highest hardness (natural rubber, 60 durometer). These curves are at constant temperature and repeat stress-strain values obtained at a low loading rate. These must now be compared with the necessary point we have obtained at high-compression and high-frequency values.

To obtain these values for comparison, a rubber with a known curve (such as those on Fig. 17) should be employed in an absorber, and its natural frequency measured. Working through these calculations, all factors are known and a point can be established on the graph (Fig. 17) for comparison of the curve with no compression and low loading rate. Several other points may be found by changing the inertia member, diameters, and the effect of gap clearance and the resulting rubber compression. This will then give usable design data (i.e., shear moduli) from available rubber engineering data.

Dynamic Fatigue Life

The life of rubber (in service under absorber conditions) is for the most part determined by the fatigue characteristics of the rubber. Extensive data on natural rubber and its engineering properties including fatigue have been published by the U. S. Rubber Company.[4] Factors that affect rubber life significantly are:

a. Static load
b. Amplitude of vibration
c. Foreign elements (oil, gases) and the influence of light
d. Temperature

In our application there is no static load in shear. The U. S. Rubber Company has conducted tests which conclusively prove that static preloading is an important factor of fatigue life. Their results seem to indicate that the load, caused by compressing the rubber into the present gap spacing, will give a greater life than a nonloaded type of assembly.

Heat Problems

The temperature of the rubber is determined by the outdoor air temperature, the heat of the engine, and the internal heat created within the rubber itself through damping (hysteresis). The absorber design dictates the presence of damping and the resulting heat is unavoidable. There are several ways to relieve this problem. One is to have a large volume of rubber in which to store the heat generated. This is an advantage when the absorber is in action periodically as it would be in driving at varying speeds. A second way would be to have the heat dissipated from the rubber as quickly as possible. This can be done by an adequate air flow and by large adjoining metal surfaces on the absorber and on the pulley. Third, increasing the size of the absorber causes a decrease in the oscillation and the damping needed in the absorber for it to perform properly. Fourth, heat dissipation is determined by the conductivity of the rubber, and any alteration in the rubber compound which increases its conductivity will in turn increase its life.

To obtain an idea of the heat generation, we can continue with our design problem. From Fig. 13, with an inertia ratio $\mu = .166$ or $1/\mu = 6.02$ the damping factor $c/c_c = .2$, where $c_c = 2J_a\omega_{n_e}$, or

$$c = .2(2J_a\omega_{n_e}) = .2(2)(.053)(313)(2\pi)$$

$c = 41.6$ in-lb/rad/sec [unit-absorber damping torque (in-lb-sec), i.e., value of the damping torque in in-lb for each unit-velocity (rad/sec) difference between absorber and the mass to which it is attached.]

$$\theta_{ae} = 9.42 \times 10^{-3} \text{ radians}$$

$$\omega_{n_e} = 313 \text{ cps or } 313(2\pi) = 1970 \text{ rad/sec}$$

With θ_{ae} the maximum difference in amplitude between absorber and engine, then the instantaneous difference can be shown as

$$\phi = \theta_{ae} \cos \omega_{n_e} t \quad ;$$

differentiating $d\phi/dt = -\theta_{ae}\omega_{n_e} \sin\omega_{n_e} t$; this is the instantaneous velocity difference on which the damping coefficient (c) acts.

$$\text{Instant. velocity} = -\theta_{ae}\, \omega_{n_e} \sin \omega_{n_e} t$$

$$\text{Maximum velocity} = \theta_{ae}\, \omega_{n_e}$$

The effective velocity $= .707$ maximum $= .707\, \theta_{ae}\omega_{n_e}$

$$V_{eff.} = .707\ (9.42 \times 10^{-3})(1970) = 13.1 \text{ rad/sec}$$

The effective work from internal friction or hysteresis is $c(V_{eff.})$.

$$c\, V_{eff.} = 41.6(13.1) = 545 \text{ in-lb/sec.}$$

Since 1 Btu = 9336 in-lb.

$$\frac{545}{9336} = .05838 \text{ Btu/sec.}$$

$$1 \text{ Btu} = 252 \text{ cal}$$

or

$$14.71 \text{ cal/sec.}$$

We have then a total of 14.71 cal/sec generated in the absorber. This is the same figure regardless of the volume of rubber or any other characteristic as long as this absorber is in action and is properly damped. Improper damping or tuning will reduce this figure, but the result is increased engine vibration.

It is imperative that the rubber be able to withstand the temperature build-up that will result from this heat. This, along with conductivity, is one of the major properties an elastic product must possess, and any investigation of elastic products should be conducted with this in mind.

Specific recommendations in regard to heat dissipation other than conductivity within the heat generator (rubber) might include increasing the contact surfaces (i.e., absorber surface and pulley surface), and to reduce the temperature of the materials the heat is being dissipated into. If ventilation is a problem, it may be possible to provide air movement by some flanges in the pulley on which the absorber is mounted or an additional stamped piece added. The casting may be modified at the three spokes so that they are cantered to perform as a fan. A second method might be to add a thin stamping or to vane the casting on the pulley which would force large quantities of air through the present drilled holes which lead directly to the inner rubber surface. A modified washer could also fit on the crankshaft in front of the pulley which is bladed to act as a centrifugal fan. Another method, if the production problems could be solved, would be to have a metal cylinder imbedded into the rubber and extending out of the rubber as a fin for the transfer of heat to the air. If this cylinder were stamped from sheet metal and used as the knife edge for assembling purposes, and then perhaps the exposed fin cut and bent similar to a turbine blade, conductivity could be greatly increased at very low cost.

A combination of this last suggestion along with casting vanes or scoops on the front of the pulley at the holes would provide air cooling which may be adequate to solve the problem.

With surrounding ambient temperatures in the vicinity of 150°F, and the narrow limit of temperature differential above this because of the maximum-temperature rubber limits, efficient cooling must be realized.

Comparison Tests for Rubber

Because of the many available rubber compounds and the difficulty of determining their usefulness, some method is needed to indicate their possible behavior as absorber components. Some simple and rapid methods are available to compare characteristics of an unknown compound with a known rubber. Both damping constants and spring constants can be compared by the methods shown in Fig. 18.

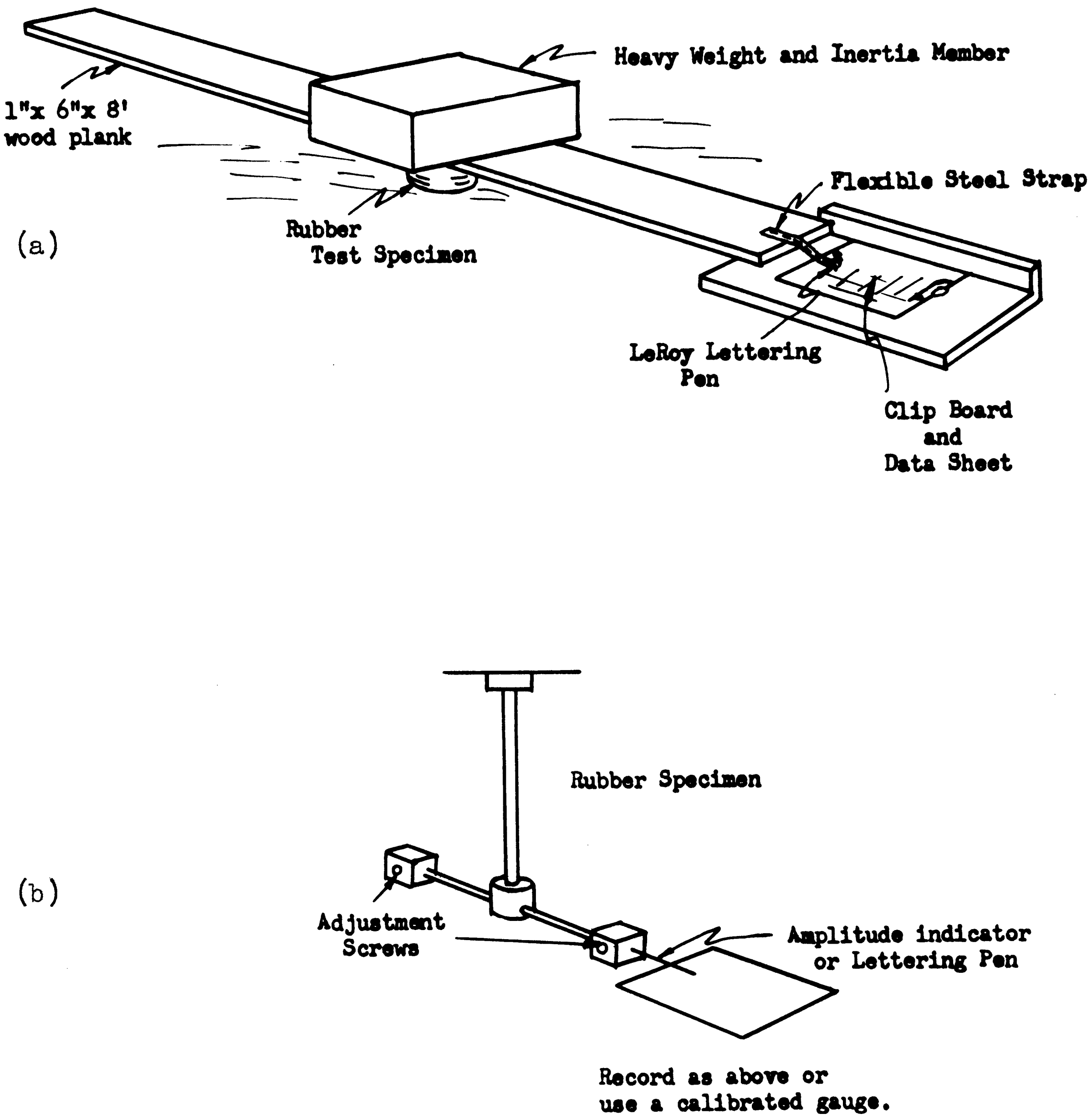


Fig. 18.

The instrumentation shown in Fig. 18a works very well for the present rubber discs used in absorbers. The discs are left stacked as they were cut, using enough of them to make a stack of about 1-1/2 in. in height. A heavy weight (1000 lb) is used to prevent movement between the discs or adjoining surfaces. A total inertia, including the extension board, of 108 in-lb-sec^2 with the present rubber used in manufacturing stacked to 1.46 in. (before weighted) gave a period τ of 71×10^{-4} seconds per cycle. Because of the high decay rate, the system has to be forced to obtain a reading of cycles vs time. This is done by applying force (with the hands) in rhythm with the oscillations and recording the time for 50 cycles to occur. This gives a relative k value for the spring constant

$$\frac{1}{\tau} = \text{cps} \quad \text{or} \quad \frac{2\pi}{\tau} = \text{rad/sec} = \omega_n \text{ (approximately)}$$

$$\omega_n = \sqrt{\frac{k}{j}}$$

$$\frac{1}{71 \times 10^{-4}} = \sqrt{\frac{k}{108}}$$

$$k = .846 \times 10^8 \text{ in-lb/rad} .$$

For an indication of the damping, a decay curve can be recorded that will be similar to Fig. 2. The recording paper may be drawn manually as the oscillations decay. With some care the period of oscillation that is recorded may be made uniform. Uniformity, however, is only to establish the position of the exponential envelope for measuring the amplitude of vibration and the period (τ) used for the calculation taken from the resonant count made earlier. For greater sensitivity the extension arm of the pen may be increased which in turn will increase the pen movement.

Figure 18b shows a method of testing small samples of rubber with the same principles involved as in the previous method. Several items should be kept in mind with either system.

1. Conditions under which comparative samples are tested must be the same.
   a. Inertia of the mass and their weight
   b. Sample size and configuration
   c. Temperature (may be varied to test effects)

2. These tests will only indicate trends in performance.

3. The frequencies used here are much lower than those used in an absorber.

4. Endurance life is not known.

III. DESIGN MODIFICATION

After preliminary resonance tests with a frequency machine, some further modifications may be necessary in the absorber design. Methods suggested under Absorber Components may be used for this purpose. The absorber is then correct as far as calculations will permit. An engine test is now in order. Short period tests will indicate any unsatisfactory tuning or inadequate reduction in amplitude.

A simple method of accurately correcting the tuning can be accomplished by the use of torsiograph curves. It is important that the data obtained be accurate and care should be taken in measurement. Two torsiograph curves are necessary, one with the absorber as normally used, and another with the absorber fixed or rigid. The curves taken with the absorber rigid could be accomplished by substitution of a mass of equal inertia in the form of some type of disc. Any other equipment which is normally rotating with the engine should also be included. This is important because the absorber tuning depends on the inertias in vibration, and the absence of any vibrating part will alter the curves. Conversely, recording instruments should not add to the inertia for the same reasons. Electronic and magnetic devices are ideal in this respect. They are capable of great accuracy with little or no inertia addition.

The data for each condition are plotted on a single sheet of graph paper, forming two curves similar to the solid curves in Fig. 9. The torsiograph curve differs from Fig. 9 in the vertical or ordinate values, which are amplitudes divided by static torque, whereas the torsiograph is amplitude or double amplitude only. This is of no importance in the application which we are using.

The curves will intersect at two points P and Q similar to Fig. 9. Choosing one of the two points, say Q, we can evaluate to find the error in tuning. Using the basic equations for a system as shown in Fig. 15, we have Equation 1 (with absorber).

$$\frac{\theta_e}{\theta_{st}} = \sqrt{\frac{KX + (X - Y)^2}{KX(X + \mu X - 1)^2 + [\mu XY - (X - 1)(X - Y)]^2}} \qquad (1)$$

μ = absorber-engine inertia ratio

$K = (2\frac{c}{c_c})^2$ where c/c_c is the absorber damping factor

$X = (\frac{\omega}{\omega_{n_e}})^2$

and $Y = (\frac{\omega_{n_a}}{\omega_{n_e}})^2$

$\frac{\theta_e}{\theta_{st}}$ = dynamic magnifier

The θ_e used here is then the value of the selected point Q on the data. θ_{st} is an unknown in this application. The damping factor c/c_c of the absorber can be any value of these particular points (P and Q), so we can select c/c_c = 0 to simplify the equation. The force frequency can be taken from the curve at the point Q and ω_{n_e} from the maximum point of the engine without absorber curve.

Since $c/c_c = 0$, then $K = 0$ and Equation 1 reduces to

$$\frac{\theta_e}{\theta_{st}} = \frac{(X - Y)}{\mu XY - (X - 1)(X - Y)} \quad .$$

From page 18 we have Equation 2 (without absorber) suitable for the single-mass system with damping (engine curve, fixed absorber).

$$\frac{\theta_e}{\theta_{st}} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_{n_e}}\right)^2\right]^2 + \left(2 \frac{c}{c_c} \frac{\omega}{\omega_{n_e}}\right)^2}}$$

Again select $c/c_c = 0$ and the equation reduces to

$$\frac{\theta_e}{\theta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_{n_e}}\right)^2}$$

Using the convention of Equation 1,

$$\left(\frac{\omega}{\omega_{n_e}}\right)^2 = X ,$$

then

$$\frac{\theta_e}{\theta_{st}} = \frac{1}{1 - X} \quad .$$

Since Equations 1 and 2 represent the same point Q

$$\frac{\theta_e}{\theta_{st}} \text{ (Equation 1)} = \frac{\theta_e}{\theta_{st}} \text{ (Equation 2)}$$

Then

$$\frac{1}{1 - X} = \frac{X - Y}{\mu XY - (X - 1)(X - Y)}$$

X,Y = known values (ω_{n_a} measured for resonance on an absorber test machine)

μ = unknown or known which is to be modified.

Solving for μ we have the re-evaluated ratio and thus can retune the absorber by $1/(1 + \mu)$, the original formula used for tuning. The care with which the absorber resonance and the graph data are measured will be reflected in the results. Damping in the engine and absorber is not important. Both factors can give considerable trouble if they have to be measured or calculated.

Once the tuning is corrected, it is a simple matter to check on the absorber damping. Engine amplitude values with the correctly tuned absorber should be near the values calculated on page 22 or recalculated with the new μ.

The two maximum peaks with values much over those calculated would indicate insufficient damping. A curve which gives only one peak close to that of the original resonance (slightly shifted to a lower frequency) indicates an excess of damping. This effect is explained on page 10.

The absorber at this stage is properly designed for maximum effectiveness. There is no indication, however, that its life in service is sufficient to satisfy the need. It remains now to examine the elastic member in the light of fatigue, aging, heat resistance, and stability with changes in temperature which will require a knowledge of the properties of the elastic member.

As work progresses on the development of new rubber-type compounds for this use, the evaluation of the compound will necessarily include the above items as well as its creep due to compression load, cost, and ease of assembly.

CONCLUSIONS

It is not only possible but advisable to begin the initial design of an absorber by using some simplified mathematical approach. The simplification of the system and the simplification of equations will necessarily result in errors. Therefore, any absorber design must be carefully tested to determine its degree of success in achieving the desired performance. Two accurate ways to test an absorber are:

1. as a separate unit on a frequency testing machine; and

2. as an integral unit on the type of engine with which it is to be used.

Fatigue life may be greatly affected by the design, i.e., preloading, amplitudes, temperature.

RECOMMENDATIONS

The methods above should be used to test the absorber for short-time performance. For an indication of its expected life, controlled runs on an engine which will duplicate anticipated operating conditions are needed, as are temperature and amplitude measurements of the absorber.

A thorough investigation should be made of any new material proposed for use as the elastic member with particular regard for conductivity, aging, fatigue, and changes due to temperature.

To determine the spring constants of possible rubber compounds, a test program should be developed which will correlate simple tests such as those explained in this report with results obtained on a frequency-measuring machine or an engine.

A similar correlation should be developed for the damping constant.

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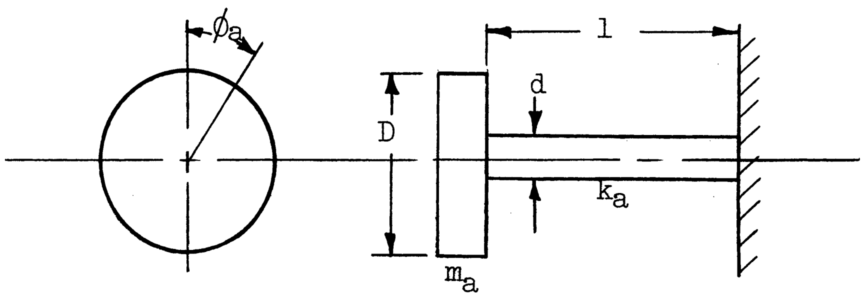
APPENDIX

PART I

NATURAL FREQUENCY OF A SINGLE-MASS SYSTEM WITHOUT DAMPING



D = diameter of inertia member (in.)
W = weight of inertia member (lb)
m_a = mass of inertia member in slugs (lb-sec^2/in.)
= W_a/g
d = diameter of shaft (in.)
l = length of shaft (in.)
k_a = spring constant of the shaft in torsion (lb-in.)
ϕ_a = instantaneous angular displacement of inertia during a cycle of harmonic motion
$\frac{d\phi}{dt} = \dot{\phi}_a$ = angular velocity at time of displacement
$\frac{d^2\phi}{dt^2} = \ddot{\phi}_a$ = angular acceleration at time of displacement
θ_a = maximum angular displacement
J_a = mass moment of inertia of m_a

There are two forces acting on the mass—the inertia force caused by its rotation and the torsion-spring force caused by the twist of the shaft due to the rotation of the mass. The inertia torque $T = J_a \ddot{\phi}_a$, which is equivalent to translatory motion where $F = ma$. Dynamic balance of forces requires that the inertia torque T has an equal but opposite torque acting on it at all times, namely, $k_a\phi_a$. Thus, equating the forces

$$J_a\ddot{\phi}_a = - k_a\phi_a \quad (1)$$

or

$$J_a\ddot{\phi}_a + k_a\phi_a = 0 \quad (1a)$$

$$\ddot{\phi}_a + \frac{k_a}{J_a}\phi_a = 0 \qquad (2)$$

Solving this differential equation we have the root r

$$r^2 + \frac{k_a}{J_a} = 0$$

$$r = \pm i\sqrt{\frac{k_a}{J_a}}$$

Therefore

$$\phi_a = A_{cos}\sqrt{\frac{k_a}{J_a}}\,t + B_{sin}\sqrt{\frac{k_a}{J_a}}\,t \qquad (3)$$

Let $\omega_n = \sqrt{k_a/J_a}$ (natural circular frequency)

$$\phi_a = A_{cos}\,\omega_n t + B_{sin}\,\omega_n t \qquad (4)$$

This is the general solution to Equation 1 with A and B constants which depend on initial conditions and determine the amplitude of motion. Selecting the starting position t = 0 when the displacement is maximum, $\phi_a = \theta_a$, i.e., the angle of rotation would be at its maximum. Therefore, the parameters are arbitrarily fixed at maximum displacement at the beginning of each cycle.

The displacement is

$$\phi_a = A_{cos}\,\omega_n t + B_{sin}\,\omega_n t \quad \text{(Equation 4)}$$

Differentiating, we obtain the velocity

$$\frac{d\phi_a}{dt} = \dot{\phi}_a = -A\,\omega_n \sin\omega_n t + B\,\omega_n \cos\omega_n t \qquad (5)$$

Differentiating again, we obtain the acceleration

$$\frac{d^2\phi_a}{dt^2} = \frac{d\dot{\phi}_a}{dt} = \ddot{\phi}_a = -A\,\omega_n^2 \cos\omega_n t - B\,\omega_n^2 \sin\omega_n t \qquad (6)$$

At t = 0 we have chosen $\phi_a = \theta_a$, as the start of the cycle. At this point the velocity is zero or $\dot{\phi}_a = 0$ and the $\cos\omega_n t = 1$ and $\sin\omega_n t = 0$.

Substituting in Equation 4,

$$\Theta_a = A \qquad (4a)$$

Substituting in Equation 5,

$$\dot{\phi}_a = B\,\omega_n \quad \text{or} \quad B = \frac{\dot{\phi}_a}{\omega_n} \qquad (5a)$$

Substituting 4a and 5a back into 4 results in

$$\phi_a = \Theta_a \cos \omega_n t + \frac{\dot{\phi}_a}{\omega_n} \sin \omega_n t \qquad (4b)$$

and with $t = 0$, $\dot{\phi}_a = 0$

$$\phi_a = \Theta_a \cos \omega_n t$$

The following is an example of this condition. Using the figure above:

Inertia member

D = 4 in.
weight = 19.3 lb

Torsion spring

shaft = k_a
length = 4 in.
diameter = .434 in.
material = cold-rolled steel

The equation for motion is $\phi_a = \Theta_a \cos \omega_n t$ where $\omega_n = \sqrt{k/J}$. Thus the natural frequency can be found if we find the value of k_a and J_a.

Mass

$$J_a = \text{moment of inertia of } m_a = \frac{Wr^2}{2g}$$

$$J_a = \frac{19.3\,(2)^2}{386 \cdot 2} = .1 \text{ lb-in-sec}^2$$

Spring

$$k = \frac{G\,J_p}{L}$$

G = shear modulus

J_p = polar moment of area of the shaft cross section

$$J_p = \frac{\pi d^4}{32} = \frac{\pi(.434)^4}{32} = .00348 = 34.8 \times 10^{-4} \text{ in.}^4$$

$$k = \frac{(11.5 \times 10^6)(34.8 \times 10^{-4})}{4}$$

$$k = 1 \times 10^4 \text{ lb-in.}$$

Natural frequency

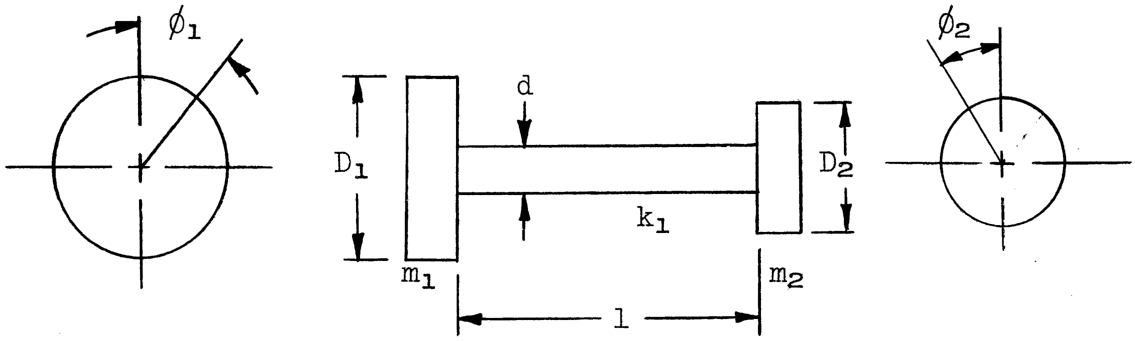
$$\omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{1 \times 10^4}{.1}} = \sqrt{10} \times 10^2 \ \frac{\text{radians}}{\text{sec}}$$

or

$$\frac{\omega_n}{2\pi} = \frac{\sqrt{10}}{2\pi} \times 10^2 = 50.3 \text{ cps}$$

PART II

NATURAL FREQUENCY OF A TWO-MASS SYSTEM WITHOUT DAMPING



D_1 = diameter of large inertia member (in.)
D_2 = diameter of small inertia member (in.)
W_1 = weight of large inertia member (lb)
W_2 = weight of small inertia member (lb)
m_1 = mass of large inertia member in slugs (lb-sec^2/in.)
m_2 = mass of small inertia member in slugs (lb-sec^2/in.)
d = diameter of shaft (in.)
l = length of shaft (in.)
k_1 = spring constant of the shaft in torsion (lb-in.)
ϕ_1 = instantaneous angular displacement of m_1
ϕ_2 = instantaneous angular displacement of m_2

As in Part I, there are two forces acting upon each mass. The spring force opposing the inertia force m_1 must also act on m_2 in the same manner to provide the twist in the shaft whenever either mass is displaced. To do this, the masses must always be opposed in their motion and their effect on the spring must be zero (at zero displacement) at the same instant.

$$J_1\ddot{\phi}_1 = -k_1(\phi_1 - \phi_2) \qquad (1)$$

$$J_2\ddot{\phi}_2 = +k_1(\phi_1 - \phi_2) \qquad (2)$$

$$J_1\ddot{\phi}_1 + k_1\phi_1 - k_1\phi_2 = 0 \qquad (1a)$$

$$J_2\ddot{\phi}_2 - k_1\phi_1 + k_1\phi_2 = 0 \qquad (2a)$$

Adding 1a and 2a,

$$J_1\ddot{\phi}_1 + J_2\ddot{\phi}_2 = 0 \tag{3a}$$

$$\ddot{\phi}_2 = -\frac{J_1}{J_2}\ddot{\phi}_1 \quad \text{or} \quad \frac{d^2\phi_2}{dt^2} = -\frac{J_1}{J_2}\frac{d^2\phi_1}{dt^2} \tag{3b}$$

$$\int d^2\phi_2 = -\frac{J_1}{J_2}\int d^2\phi_1 \tag{3c}$$

$$d\phi_2 + C_2 = -\frac{J_1}{J_2}d\phi_1 + C_1$$

Assuming t = 0 at the instant the masses are released, the velocities are zero, $\dot{\phi}_1 = d\phi_1/dt = 0$, and $\dot{\phi}_1 = d\phi_2/dt = 0$.

$$\therefore C_1 = C_2$$

This implies that the masses may be rotating at the same velocity and have no effect on the vibration. With $C_1 = C_2 = 0$, then

$$\frac{d\phi_2}{dt} = -\frac{J_1}{J_2}\frac{d\phi_1}{dt}$$

$$\int d\phi_2 = -\frac{J_1}{J_2}\int d\phi_1 \tag{3d}$$

$$\phi_2 + C_3 = -\frac{J_1}{J_2}\phi_1 + C_4 \tag{3e}$$

with C_3 and C_4 zero (values of initial displacement) and substituting 3b and 3d in 2a, we have

$$J_1\ddot{\phi}_1 + k_1\phi_1 + k_1\frac{J_1}{J_2}\phi_1 = 0 \tag{4a}$$

$$\ddot{\phi}_1 + \frac{k_1}{J_1}\phi_1 + \frac{k_1}{J_2}\phi_1 = 0 \tag{4b}$$

$$\ddot{\phi}_1 + \phi_1\left(\frac{k_1}{J_1} + \frac{k_1}{J_2}\right) = 0 \tag{4c}$$

Solving this differential equation, we have

$$r^2 + \left(\frac{k_1}{J_1} + \frac{k_1}{J_2}\right) = 0$$

$$r = \pm\sqrt{\left(-\frac{k_1}{J_1} + \frac{k_1}{J_2}\right)} = \pm i\sqrt{\left(\frac{k_1}{J_1} + \frac{k_1}{J_2}\right)}$$

$$\phi_1 = A_{\cos}\sqrt{\frac{k_1}{J_1} + \frac{k_1}{J_2}}\, t + B_{\sin}\sqrt{\frac{k_1}{J_1} + \frac{k_1}{J_2}}\, t$$

Let

$$\omega_n = \sqrt{\frac{k_1}{J_1} + \frac{k_1}{J_2}} \quad \text{(natural circular frequency) ;}$$

then

$$\phi_1 = A_{\cos}\, \omega_n t + B_{\sin}\, \omega_n t$$

This can be simplified as in Part I to

$$\phi_1 = \theta_1 \cos \omega_n t$$

The same procedure can be used to find ϕ_2

$$\phi_2 = \theta_2 \cos \omega_n t$$

The following is an example of this system.

$W_2 = 48.3$ lb $\qquad D_2 = 4$ in.
$W_1 = 48.3$ lb $\qquad D_1 = 8$ in.

shaft diameter = .718 in.

shaft length = 15 in.

$$m_2 = \frac{W_2}{g} = \frac{48.3 \text{ lb}}{386 \text{ in/sec}^2} = .125 \text{ lb sec}^2\text{/in.}$$

$$m_1 = \frac{W_1}{g} = .125 \text{ lb sec}^2\text{/in.}$$

$$J_2 = 1/8\ m_2 D_2^2 = 1/8 \times .125 \times (4)^2 = .250 \text{ lb-in-sec}^2$$

$$J_1 = 1/8\ m_1 D_1^2 = 1/8 \times .125 \times (8)^2 = 1.0 \text{ lb-in-sec}^2$$

$$k_1 = \frac{G\ J_p}{l}$$

where

$$G = 11.5 \times 10^6 \text{ lb/in.}^2 \text{ cold-rolled steel}$$

$$J_p = \frac{D^4}{32} = \frac{(.718)^4}{32} = .026$$

$$k_1 = \frac{(11.5 \times 10^6)(.026)}{15}$$

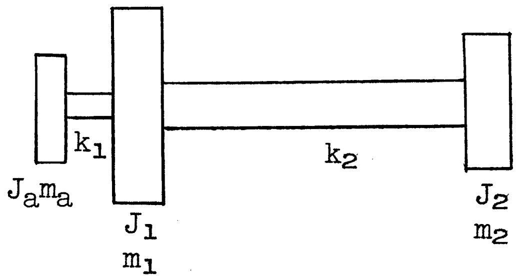
$$k_1 = 2 \times 10^4$$

$$\frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_1}{J_1} + \frac{k_1}{J_2}}$$

$$\frac{1}{2\pi}\sqrt{\frac{2 \times 10^4}{1.0} + \frac{2 \times 10^4}{.250}} = \frac{\sqrt{10} \times 10^2}{2\pi} = 50.3 \text{ cps}$$

PART III

NATURAL FREQUENCY OF A TWO-MASS SYSTEM AND VIBRATION ABSORBER WITHOUT DAMPING



$$J_a\ddot{\phi}_a = -k_a(\phi_a - \phi_1) \tag{1}$$

$$J_1\ddot{\phi}_1 = k_a(\phi_a - \phi_1) - k_1(\phi_1 - \phi_2) \tag{2}$$

$$J_2\ddot{\phi}_2 = k_1(\phi_1 - \phi_2) \tag{3}$$

$$J_a\ddot{\phi}_a + k_a\phi_a - k_a\phi_1 = 0 \tag{1a}$$

$$J_1\ddot{\phi}_1 - k_a\phi_a + k_a\phi_1 + k_1\phi_1 - k_1\phi_2 = 0 \tag{2a}$$

$$J_2\ddot{\phi}_2 - k_1\phi_1 + k_1\phi_2 = 0 \tag{3a}$$

$$\ddot{\phi}_a + \frac{k_a}{J_a}\phi_a - \frac{k_a}{J_a}\phi_1 = 0 \tag{1b}$$

$$\ddot{\phi}_1 - \frac{k_a}{J_1}\phi_a + \frac{k_a}{J_1}\phi_1 + \frac{k_1}{J_1}\phi_1 - \frac{k_1}{J_1}\phi_2 = 0 \tag{2b}$$

$$\ddot{\phi}_2 - \frac{k_1}{J_2}\phi_2 + \frac{k_1}{J_2}\phi_2 = 0 \tag{3b}$$

To simplify the solution of the above equations, we can substitute values derived previously (Appendix Parts I and II).

$W_a = 19.3$ lb $\qquad m_a = .05 \qquad J_a = .1$

$W_1 = 48.3$ lb $\qquad m_1 = .125 \qquad J_1 = 1.0$

$W_2 = 48.3$ lb $\qquad m_2 = .125 \qquad J_2 = .250$

$k_a = 1 \times 10^4 \qquad k_1 = 2 \times 10^4$

Substituting these values in 1b, 2b, and 3b,

$$\ddot{\phi}_a + \frac{1 \times 10^4}{.1}\phi_a - \frac{1 \times 10^4}{.1}\phi_1 = 0 \tag{1c}$$

$$\ddot{\phi}_1 - \frac{1 \times 10^4}{1.0}\phi_a + \frac{1 \times 10^4}{1.0}\phi_1 + \frac{2 \times 10^4}{1.0}\phi_1 - \frac{2 \times 10^4}{1.0}\phi_2 = 0 \tag{2c}$$

$$\ddot{\phi}_2 - \frac{2 \times 10^4}{.250}\phi_1 + \frac{2 \times 10^4}{.250}\phi_2 = 0 \tag{3c}$$

From previous work (see Appendix Parts I and II), solution to equations of motion of this type are $\phi = \Theta \cos \omega_n t$, we can solve Equations 1c, 2c, 3c, by substituting this solution and solving for ω_n, the natural circular frequency.

If $\phi = \Theta \cos \omega_n t$, then differentiating with respect to time gives

$$\dot{\phi} = -\Theta(\omega_n) \sin \omega_n t$$

$$\ddot{\phi} = -\Theta(\omega_n)^2 \cos \omega_n t$$

Substituting

$$-\Theta_a \omega_n^2 \cos \omega_n t + 10^5 \Theta_a \cos \omega_n t - 10^5 \Theta_1 \cos \omega_n t = 0 \tag{1d}$$

$$-\Theta_1 \omega_n^2 \cos \omega_n t - 10^4 \Theta_a \cos \omega_n t + 10^4 \Theta_1 \cos \omega_n t + 2 \times 10^4 \Theta_1 \cos \omega_n t - 2 \times 10^4 \Theta_2 \cos \omega_n t = 0 \tag{2d}$$

$$-\Theta_2 \omega_n^2 \cos \omega_n t - 8 \times 10^4 \Theta_1 \cos \omega_n t + 8 \times 10^4 \Theta_2 \cos \omega_n t = 0 \tag{3d}$$

Dividing equations by $\cos \omega_n t$

$$-\Theta_a \omega_n^2 + 10^5 \Theta_a - 10^5 \Theta_1 = 0 \tag{1e}$$

$$-\Theta_1 \omega_n^2 - 10^4 \Theta_a + 10^4 \Theta_1 + 2 \times 10^4 \Theta_1 - 2 \times 10^4 \Theta_2 = 0 \tag{2e}$$

$$- \theta_2 \omega_n^2 - 8 \times 10^4 \theta_1 + 8 \times 10^4 \theta_2 = 0 \quad (3e)$$

$$(10^5 - \omega_n^2)\theta_a + (-10^5)\theta_1 = 0 \quad (1f)$$

$$(-10^4)\theta_a + (3 \times 10^4 - \omega_n^2)\theta_1 + (-2 \times 10^4)\theta_2 = 0 \quad (2f)$$

$$(-8 \times 10^4)\theta_1 + (8 \times 10^4 - \omega_n^2)\theta_2 = 0 \quad (3f)$$

Solving by determinants

$$\begin{vmatrix} 10^5 - \omega_n^2 & -10^5 & 0 \\ -10^4 & 3 \times 10^4 - \omega_n^2 & -2 \times 10^4 \\ 0 & -8 \times 10^4 & 8 \times 10^4 - \omega_n^2 \end{vmatrix} = 0$$

$$(10^5 - \omega_n^2)(3 \times 10^4 - \omega_n^2)(8 \times 10^4 - \omega_n^2) - (-8 \times 10^4)(-2 \times 10^4)(10^5 - \omega_n^2)$$
$$- (8 \times 10^4 - \omega_n^2)(-10^4)(-10^5) = 0 \quad (4a)$$

$$24 \times 10^{13} - 104 \times 10^8 \omega_n^2 + 8 \times 10^4 \omega_n^4 - 3 \times 10^9 \omega_n^2 + 13 \times 10^4 \omega_n^4 - \omega_n^6$$
$$- 16 \times 10^{13} + 16 \times 10^8 \omega_n^2 - 8 \times 10^{13} + 10^9 \omega_n^2 = 0 \quad (4b)$$

$$\omega_n^6 - (21 \times 10^4)\omega_n^4 + (108 \times 10^8)\omega_n^2 = 0 \quad (4c)$$

Factor out ω_n^2 which is a trivial solution.

$$\omega_n^4 - (21 \times 10^4)\omega_n^2 + (108 \times 10^8) = 0$$

Solving for ω_n^2 by the Quadratic Equation

$$\omega_n^2 = \frac{21 \times 10^4 \pm \sqrt{(21 \times 10^4)^2 - 4(1)(108 \times 10^8)}}{2}$$

$$\omega_n^2 = 9 \times 10^4 \qquad \omega_n = 300 \text{ radians/sec}$$

$$\omega_n^2 = 12 \times 10^4 \qquad \omega_n = 347 \text{ radians/sec}$$

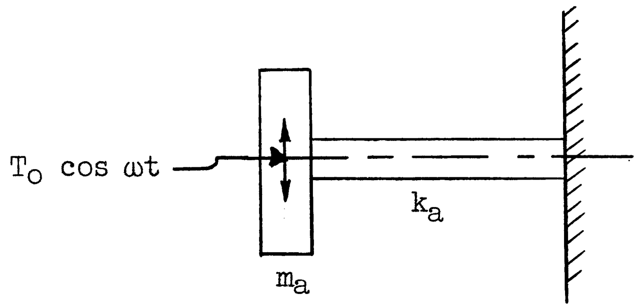
$$\frac{300}{2\pi} = 47.7 \text{ cps (first mode)}$$

$$\frac{347}{2\pi} = 55.2 \text{ cps (second mode)}$$

These are the two natural frequencies of this particular three-mass system. This method of solution is not practical for more complex systems and one of the references listed should be consulted for other means of solution. Holzer's method is one of the more widely used means of solving the more difficult problems. For our purposes, the three-mass system is all that is necessary for basic understanding, and more specifically, the system which has one mass as fixed wall as shown in Fig. 15.

PART IV

FORCED VIBRATION OF A SINGLE-MASS SYSTEM



The case of the single-mass system in Part I with a harmonic force applied to the mass gives:

$$J_a\ddot{\phi} = - k_a\phi + T_o \cos \omega t \tag{1a}$$

$$J_a\ddot{\phi} + k_a\phi = T_o \cos \omega t \tag{1b}$$

This is a linear nonhomogeneous differential equation whose general solution takes the form $\phi_g = \phi_c + \phi_p$, where ϕ_c, the complementary solution, is the solution of the equation $J_a\ddot{\phi} - k_a\phi = 0$, which is the natural frequency condition. ϕ_p is the particular solution where $T_o \cos \omega t$ is the external torque. Because of some small amount of damping in all vibrating systems, the ϕ_c factor will not be a sustained oscillation and will drop out after a period of time. Therefore, only the particular solution ϕ_p need be considered.

$$\ddot{\phi} + \frac{k_a}{J_a}\phi = \frac{T_o \cos \omega t}{J_a} \tag{1c}$$

Solving

$$r^2 + \frac{k_a}{J_a} = \frac{T_o \cos \omega t}{J_a}$$

$$r = \pm i\sqrt{k_a/J_a} \text{ roots of the left side of the equation}$$

$$r' = 0 \pm i\omega \text{ roots of the right side}$$

$$\phi_p = c_1e^o \cos \omega t + c_2e^o \sin \omega t = c_1 \cos \omega t + c_2 \sin \omega t \tag{2}$$

Since c_1 and c_2 must satisfy Equation 1c, we can solve for c_1 and c_2 by substituting in 1c.

$$\phi_p = c_1 \cos \omega t + c_2 \sin \omega t$$

Differentiating $\dot{\phi}_p = - c_1\omega \sin \omega t + c_2\omega \cos \omega t$

Second differentiation

$$\ddot{\phi}_p = - c_1\omega^2 \cos \omega t - c_2\omega^2 \sin \omega t$$

Substituting

$$- c_1\omega^2 \cos \omega t - c_2\omega^2 \sin \omega t + \frac{k_a}{J_a} (c_1 \cos \omega t + c_2 \sin \omega t) \equiv \frac{T_o}{J_a} \cos \omega t$$

$$\cos \omega t \left(-c_1\omega^2 + \frac{k_a}{J_a} c_1\right) + \sin \omega t \left(-c_2\omega^2 + \frac{k_a}{J_a} c_2\right) \equiv \frac{T_o}{J_a} \cos \omega t$$

$$- c_1\omega^2 + \frac{k_a}{J_a} c_1 = \frac{T_o}{J_a} \quad \text{and} \quad - c_2\omega^2 + \frac{k_a}{J_a} c_2 = 0$$

$$\therefore \; c_2 = 0$$

$$c_1 \left(-\omega^2 + \frac{k_a}{J_a}\right) = \frac{T_o}{J_a}$$

$$c_1 = \frac{T_o}{J_a(k/J - \omega^2)} = \frac{T_o}{k_a - J_a\omega^2} = \frac{T_o/k_a}{1 - (J_a/k_a)\omega^2}$$

With $\omega_n = \sqrt{k_a/J_a}$

$$\omega_n^2 = \frac{k_a}{J_a}$$

$$c_1 = \frac{T_o/k_a}{1 - \dfrac{\omega^2}{\omega_{na}^2}}$$

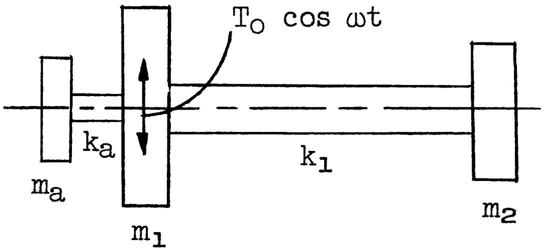
$$\phi_p = \frac{T_o/k_a}{1 - \dfrac{\omega^2}{\omega_{na}^2}} \cos \omega t$$

$$\theta = \frac{T_o/k_a}{1 - \frac{\omega^2}{\omega_{na}^2}} \qquad \text{maximum deflection}$$

Note the maximum deflection becomes infinite when the external force frequency equals the natural frequency independent of the amount of external force. Practically, the damping will limit the deflections, and the size of the external force becomes an important factor in determining the resultant amplitude.

PART V

FORCED VIBRATION OF A THREE-MASS SYSTEM



$$J_a\ddot{\phi}_a = -k_a(\phi_a - \phi_1) \tag{1}$$

$$J_1\ddot{\phi}_1 = k_a(\phi_a - \phi_1) - k_1(\phi_1 - \phi_2) + T_o \cos \omega t \tag{2}$$

$$J_2\ddot{\phi}_2 = k_1(\phi_1 - \phi_2) \tag{3}$$

$$J_a\ddot{\phi}_a = k_a\phi_a - k_a\phi_1 = 0 \tag{1a}$$

$$J_1\ddot{\phi}_1 - k_a\phi_a + k_a\phi_1 + k_1\phi_1 - k_1\phi_2 = T_o \cos \omega t \tag{2a}$$

$$J_2\ddot{\phi}_2 - k_1\phi_1 + k_1\phi_2 = 0 \tag{3a}$$

$$\ddot{\phi}_a + \frac{k_a}{J_a}\phi_a - \frac{k_a}{J_a}\phi_1 = 0 \tag{1b}$$

$$\ddot{\phi}_1 - \frac{k_a}{J_1}\phi_a + \frac{k_a}{J_1}\phi_1 + \frac{k_1}{J_1}\phi_1 - \frac{k_1}{J_1}\phi_2 = \frac{T_o \cos \omega t}{J_1} \tag{2b}$$

$$\ddot{\phi}_2 - \frac{k_1}{J_2}\phi_1 + \frac{k_1}{J_2}\phi_2 = 0 \tag{3b}$$

Using again the values chosen for Part III, all equations being identical with the exception of T_o cos ωt for Equation 2, we can advance to Equations 1f, 2f, and 3f.

$$(10^5 - \omega^2)\theta_a + (-10^5)\theta_1 = 0 \tag{1f}$$

$$(-10^4)\theta_a + (3 \times 10^4 - \omega^2)\theta_1 + (-2 \times 10^4)\theta_2 = \frac{T_o}{J_1} \tag{2f}$$

$$(-8 \times 10^4)\theta_1 + (8 \times 10^4 - \omega^2)\theta_2 = 0 \tag{3f}$$

Solving by determinants,

$$\theta_a = \frac{-(8 \times 10^4 - \omega^2)(T_o/J_1)(-10^5)}{(10^5 - \omega^2)(8 \times 10^4 - \omega^2)(3 \times 10^4 - \omega^2) - (-8 \times 10^4)(-2 \times 10^4)(10^5 - \omega^2) - (8 \times 10^4 - \omega^2)(-10^4)(-10^5)}$$

Note - The denominator of the above expression is simplified in Part III of the Appendix in terms of the natural frequency and reduces to the expression shown below.

$$[\omega^2]\ [\omega^4 - (21 \times 10^4)\omega^2 + (108 \times 10^8)]$$

$$\theta_a = \frac{-(T_o/J_1)(-10^5)(8 \times 10^4 - \omega^2)}{[\omega^2]\ [\omega^4 - (21 \times 10^4)\omega^2 + (108 \times 10^8)]}$$

$$\theta_a = \frac{-(T_o/J_1)(-10^5)(8 \times 10^4 - \omega^2)}{(\omega^2)(\omega^2 - 9 \times 10^4)(\omega^2 - 12 \times 10^4)} = \frac{-(T_o/J_1)(10^5)(\omega_n^2 - 8 \times 10^4)}{(\omega^2)(\omega^2 - 9 \times 10^4)(\omega^2 - 12 \times 10^4)}$$

θ_1 and θ_2 can be determined similarly giving the following equations:

$$\theta_1 = \frac{-(T_o/J_1)(10^5 - \omega^2)(8 \times 10^4 - \omega^2)}{(\omega^2)(\omega^2 - 9 \times 10^4)(\omega^2 - 12 \times 10^4)}$$

$$\theta_2 = \frac{-(T_o/J_1)(10^5 - \omega^2)(-8 \times 10^4)}{(\omega^2)(\omega^2 - 9 \times 10^4)(\omega^2 - 12 \times 10^4)}$$

The important point in this example is to see that all amplitudes of θ_a, θ_1, and θ_2 are dependent on the factors of their denominators, which are all identical. If any of these factors were zero, the amplitudes would go to infinity, the points of resonance. Solving for these points, we can set each factor to zero.

$\omega^2 = 0$	trivial solution	
$\omega^2 - 9 \times 10^4 = 0$	$\omega = 300$ radians/sec	First mode
$\omega^2 - 12 \times 10^4 = 0$	$\omega = 347$ radians/sec	Second mode

These values are the same as those found for the natural resonant frequencies of the system in Part III.

A comparison of the numerators gives an indication of the absorber action. θ_1 has two points where the driving frequency results in zero amplitude:

10^5 and 8×10^4. The 10^5 is the resonant point of m_a-k_a, causing Θ_1 and Θ_2 to be zero, and 8×10^4 is the resonant point of m_2-k_1, which causes Θ_1 and Θ_a to be zero. Both end masses have absorber action at their respective resonant frequencies.

Figure 8 is a set of curves similar to those which would be obtained in this example.

The methods of solution we have used are not suitable for systems of more than three masses. Holzer's method is a simple form usable for multiple-mass systems and the foregoing examples are meant only to show the mechanics involved in solving for such solutions.

PART VI

Damping in a vibrating system is most easily represented by viscous damping. For comparison with undamped systems and the differential equations used, here is a general equation for vibration with damping.

$$J\ddot{\phi} + c\dot{\phi} + k\phi = T_o \cos \omega t$$

The viscous damping factor, introduced as $c\dot{\phi}$, indicates that damping forces are a function of velocity. We will not pursue the solutions to damped vibration problems with the new factor $c\phi$ because they become far too complex for what little is gained. However, some of the texts listed in the bibliography deal with this.

For our purposes, the differences in damped and undamped vibrations may best be explained by a few vector diagrams of a single-mass system, shown on the following page.

The torque vector shown represents the force causing the vibration. The length of the vector is T_o and it is a constant magnitude rotating ω radians per second. At a given time t, the vector will have rotated ωt radians. The force applied to a mass causing torsional vibration is that portion of T_o projected down to the horizontal line, namely $T_o \cos \omega t$. Thus, the torque component will vary from a positive value of T_o through zero to a negative value and again to a positive value during one revolution (2π radians).

With the torque rotating as described, the effect on a system both with and without damping is shown below resonance, at resonance, and above resonance. The dashed lines on the diagram simply indicate that the vectors are in the same position and are separated only for the sake of clarity.

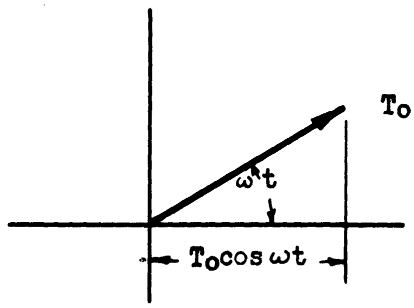
All vectors rotate at ω rad/sec such as the torque vector shown. Since there is no motion relative to one another, they can be pictured as vectors at some time t. Note that in all cases the vector sum of the spring force $k\theta$, the inertia force $J\omega^2\theta$, the damping force $c\omega\theta$ (present only in cases with damping), and the force T_o must all add up to zero (form a closed loop).

Below Resonance

Below resonance the external force T_o is mainly opposed by the spring force $k\theta$. With no damping, the movement of the mass θ is in phase with the force T_o. With damping, a portion of the torque must oppose the damping force and the amplitude is always some degree removed from the applied torque.

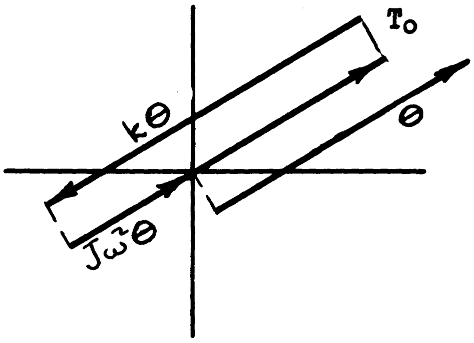
VECTOR REPRESENTATION
OF
A SINGLE-MASS SYSTEM

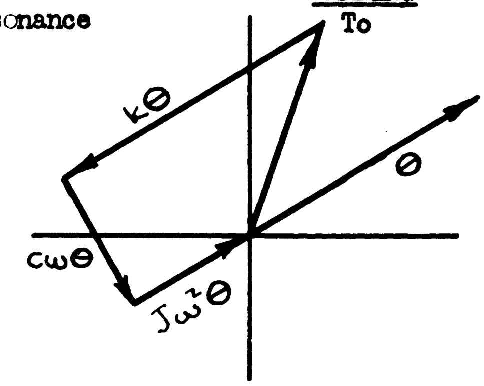
Torque



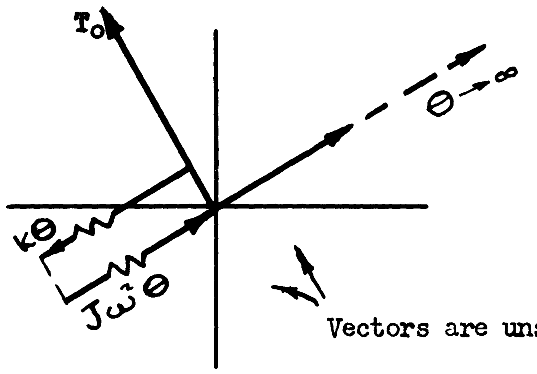
NO DAMPING | DAMPING

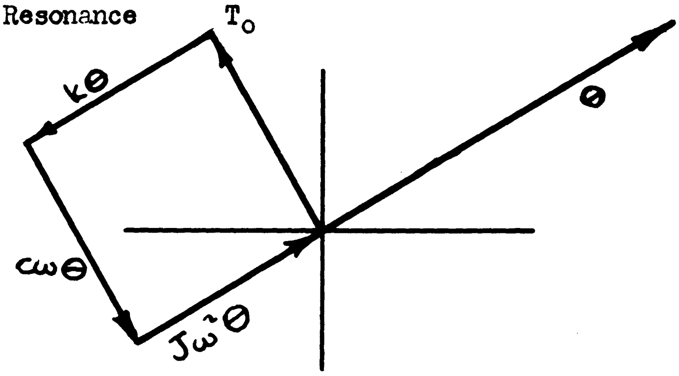
Below Resonance



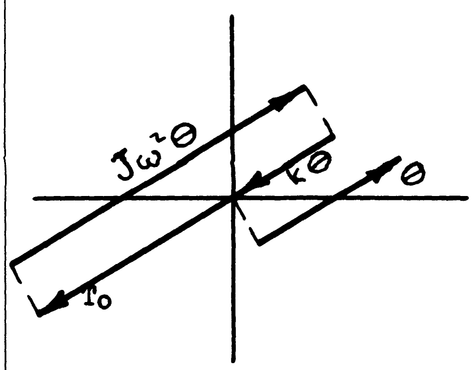


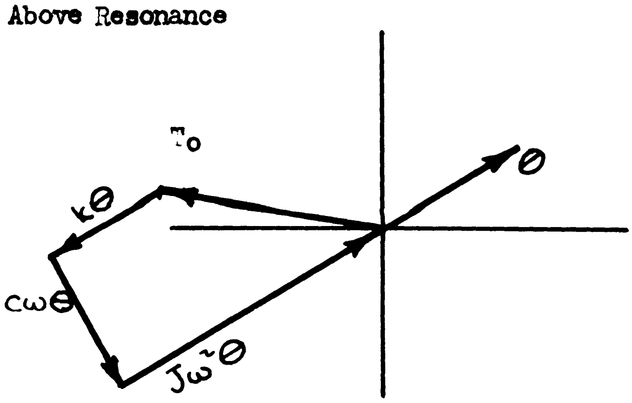
At Resonance





Above Resonance





At Resonance

At resonance, the torque is 90° removed from the amplitude and since the spring force must be out of phase and the inertia force in phase with the amplitude, the undamped system cannot be balanced and therefore results in infinite values as shown in Fig. 7 (page 8). In the damped case, the torque is counteracted only by the damping force. Since the damping force is the value of $c\omega\theta$, the amplitude at resonance must increase to balance the torque. An increase in the damping constant c in the system would serve directly to reduce the amplitude.

Above Resonance

Above resonance the external torque is mainly opposed by the inertia force and it becomes increasingly dominant as the frequency ω increases, causing a decrease in the amplitude. In the undamped case, the motion or amplitude θ is 180° out of phase with the torque (see Fig. 7).

The relationship of the vectors shown is only for three specific forced frequencies. It should be realized that for every forced frequency above and below resonance the vectors will have varied magnitudes and phase angles with each other. Some relationships, however, are always present. These are:

1. Spring force ($k\theta$) 180° opposed to amplitude (θ)
2. Inertia force ($J\omega^2\theta$) in phase with amplitude (θ)
3. Damping force 180° opposed to velocity, i.e., 90° removed from amplitude (θ)