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STUDIES IN RADAR CROSS SECTIONS XXXI - DIFFRACTION BY AN IMPERFECTLY CONDUCTING HALF-PLANE AT OBLIQUE INCIDENCE

by

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PREFACE

This is the thirty-first in a series of reports growing out of the study of radar cross sections at The Radiation Laboratory of The University of Michigan. Titles of the reports already published or presently in process of publication are listed on the preceding pages.

When the study was first begun, the primary aim was to show that radar cross sections can be determined theoretically, the results being in good agreement with experiment. It is believed that by and large this aim has been achieved.

In continuing this study, the objective is to determine means for computing the radar cross section of objects in a variety of different environments. This has led to an extension of the investigation to include not only the standard boundary-value problems, but also such topics as the emission and propagation of electromagnetic and acoustic waves, and phenomena connected with ionized media.

Associated with the theoretical work is an experimental program which embraces (a) measurement of antennas and radar scatterers in order to verify data determined theoretically; (b) investigation of antenna behavior and cross section problems not amenable to theoretical solution; (c) problems associated with the design and development of microwave absorbers; and (d) low and high density ionization phenomena.

K.M. Siegel

SUMMARY

The exact solution is obtained for the problem of a plane wave incident at an oblique angle on a half-plane of large but finite conductivity. The usual approximate boundary conditions are applied and these lead to coupled Wiener-Hopf integral equations from which to determine the currents excited on the surface of the sheet. The resulting expressions for the field components are found to be entirely different from those which would have been obtained by applying the technique used for the derivation of three-dimensional solutions in the case of perfectly conducting structures and, indeed, not one component is given to the required accuracy by this technique.

INTRODUCTION

In this paper attention will be confined to diffracting structures which are two-dimensional in the sense of being composed of cylinders of arbitrary cross section whose generators are all parallel to the z axis of some coordinate system.

If a structure of this type is perfectly conducting, it is possible to deduce the solution for a three-dimensional incident field from the solution for a two-dimensional field and, in particular, the solution for a plane wave at oblique incidence can be obtained from that in which the plane wave is normal to the z axis. The method is based upon the fact that any solution of the two-dimensional wave equation gives rise to a solution of the three-dimensional equation on replacing the propagation constant k by k cos β and multiplying by $e^{-ikz\sin\beta}$. If the particular solution considered represents the solution for the electromagnetic problem in which the incident field is a two-dimensional plane wave, and if it is modified in the above way and then taken to be the z component of an electric or magnetic Hertz vector whose other components are zero, a solution of the three-dimensional problem is produced. The corresponding incident field has either H_z or E_z zero (depending on whether the two-dimensional field was E or H polarized), and the two fundamental fields so generated can be combined to give the solution for any incident field. The method has been described in detail by Clemmow (Ref. 1), and was used by Senior (Ref. 2) to determine the field of a dipole in the presence of a half-plane.

When the structure is not perfectly conducting the method is no longer applicable and the question then is whether an analogous technique can be developed to treat such cases. To attempt an answer to this an obvious approach is to tackle a particular problem in the hope that its solution may indicate a general transformation and apart from the problem of a circular cylinder (for which the solution is almost trivial), one of the most simple is that of a plane wave incident at an oblique angle on a half-plane of large but finite conductivity. The present paper is devoted entirely to a consideration of this problem.

The crux of the analysis is the determination of coupled integral equations for the electric and magnetic current distributions excited on the surface of the half-plane. These are of Wiener-Hopf type and can be solved to give expressions for the currents in terms of the "split" functions which characterized the solution for normal incidence (Ref. 3). The field components are then given as integrals over the currents, and some ramifications of the results are examined.

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THE INTEGRAL EQUATIONS

Consider a thin semi-infinite sheet of large but finite conductivity occupying the half-plane $y = 0,\ x > 0$ of a rectangular Cartesian coordinate system (x,y,z). A plane wave is incident in a direction making an angle α with the positive x axis and an angle $\pi/2 - \beta$ with the z axis; $\beta = 0$ then corresponds to incidence in the plane perpendicular to the diffracting edge.

The actual form of the incident field matters little as regards the analysis, but in order to simplify the comparison of the results with those for a perfectly conducting sheet the field is taken to be a quasi three-dimensional plane wave which is E-polarized and whose components are given by

$$\underline{E}^i = (-\cos\alpha\sin\beta\cos\beta,\ -\sin\alpha\sin\beta\cos\beta,\ \cos^2\beta)e^{-ik(x\cos\alpha\cos\beta + y\sin\alpha\cos\beta + z\sin\beta)} \tag{1}$$

$$\underline{H}^i = (-Y\sin\alpha\cos\beta,\ Y\cos\alpha\cos\beta,\ 0)e^{-ik(x\cos\alpha\cos\beta + y\sin\alpha\cos\beta + z\sin\beta)} \tag{2}$$

where $Y = 1/Z$ is the intrinsic admittance of free space and a time factor $e^{-i\omega t}$ is suppressed. The above field is that which is obtained by modifying a two-dimensional E-polarized plane wave in the manner described in the previous section.

If η is the reciprocal of the complex refractive index of the material comprising the sheet, the boundary conditions to be applied can be written as

$$\underline{E} - (\underline{n}\cdot\underline{E})\underline{n} = \eta\, Z\, \underline{n}\wedge\underline{H} \ , \tag{3}$$

where $\underline{n}$ is a unit vector normal drawn outwards from the sheet. On the upper surface $\underline{n}$ is in the positive y direction, and equation (3) then gives

$$E_x = \eta\, ZH_z \ , \quad E_z = -\eta\, ZH_x$$

Similarly, on the lower surface

$$E_x = -\eta\, ZH_z \ , \quad E_z = \eta\, ZH_x$$

As a consequence of these conditions, the tangential components of both the electric and the magnetic vectors will be discontinuous on crossing the sheet, and it is convenient to regard these discontinuities as due to the presence of electric and magnetic currents in the sheet. We therefore write

$$E_z\Big|_{y=-0}^{y=+0} = I_1(x', z') \ , \qquad H_x\Big|_{y=-0}^{y=+0} = I_2(x', z')$$

$$E_x\Big|_{y=-0}^{y=+0} = I_3(x', z') \ , \qquad E_z\Big|_{y=-0}^{y=+0} = I_4(x', z')$$

where I_2 and I_4 are the electric currents and I_1 and I_3 are the magnetic ones. I_1 and I_4 are perpendicular to the diffracting edge whilst I_2 and I_3 are parallel, and when the field is normally incident ($\beta = 0$) I_3 and I_4 are identically zero.

The electric field at a point (x,y,z) can be expressed as a surface integral in the form

$$\underline{E}(x,y,z) = \frac{1}{4\pi} \int_S \left\{ ikZ(\underline{n}_\wedge \underline{H}) - (\underline{n}_\wedge \underline{E})_\wedge \nabla - (\underline{n} \cdot \underline{E}) \nabla \right\} \frac{e^{ik\rho}}{\rho} \, dS \tag{4}$$

(see, for example, Ref. 4, p. 467) where $\underline{n}$ is a unit vector normal drawn into the volume contained by the surface S. The differentiation is with respect to the coordinates of the observation point and ρ is the distance to a variable point (x',y',z') on S.

The surface S is made up of two sheets which envelop the diffracting structure, together with a cylinder of infinitely large radius centered on the z axis and meeting the two sheets at $x = \infty$. The integration over the cylindrical portion is easily shown to produce the incident field and accordingly equation (4) can be written

$$\underline{E}(x,y,z) = \underline{E}^i(x,y,z) + \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty \left\{ ikZ(\underline{n}_\wedge \underline{H}) - (\underline{n}_\wedge \underline{E})_\wedge \nabla - (\underline{n}\cdot\underline{E})\nabla \right\} \frac{e^{ik\rho}}{\rho} \Bigg|_{y=-0}^{y=+0} dx'dz' . \tag{5}$$

On the upper surface of the half-plane

$$\underline{n}_\wedge \underline{H} = (H_z, 0, -H_x)$$

from which we obtain

$$\underline{n}_\wedge \underline{H} \Bigg|_{y=-0}^{y=+0} = (I_4, 0, -I_2) .$$

Similarly,

$$(\underline{n}_\wedge \underline{E})_\wedge \nabla \frac{e^{ik\rho}}{\rho} \Bigg|_{y=-0}^{y=+0} = \left(I_3 \frac{\partial}{\partial y}, -I_3 \frac{\partial}{\partial x} - I_1 \frac{\partial}{\partial z}, I_1 \frac{\partial}{\partial y} \right) \frac{e^{ik\rho}}{\rho}$$

and

$$(\underline{n}\cdot\underline{E})\nabla \frac{e^{ik\rho}}{\rho} \Bigg|_{y=-0}^{y=+0} = i\frac{Z}{k}\left(\frac{\partial I_2}{\partial z'} - \frac{\partial I_4}{\partial x'} \right)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{e^{ik\rho}}{\rho}$$

and if these are inserted into (5), the equation becomes

$$\underline{E}(x,y,z) = \underline{E}^i(x,y,z) + \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty \Bigg\{ ikZ(I_4, 0, -I_2) - \left(I_3 \frac{\partial}{\partial y}, -I_3 \frac{\partial}{\partial x} - I_1 \frac{\partial}{\partial z}, I_1 \frac{\partial}{\partial y} \right)$$

$$- i\frac{Z}{k}\left(\frac{\partial I_2}{\partial z'} - \frac{\partial I_4}{\partial x'} \right)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \Bigg\} \frac{e^{ik\rho}}{\rho} \, dx' \, dz' . \tag{6}$$

At normal incidence the fields and currents are independent of z (and hence z') and, in addition, there is no current I_4; the terms in the third group of the above integrand then disappear.

An assumption is now made concerning the dependence of the field upon the coordinate z. The diffracting structure itself is two-dimensional and since all the components of the incident field involve z only through a factor $e^{-ikz\sin\beta}$, it seems reasonable to assume that this dependence will also apply to the total field. A consequence of this is that

$$I_p(x', z') = e^{-ikz'\sin\beta} \, I_p(x')$$

for p = 1, 2, 3, 4, which makes it a trivial matter to carry out the integration with respect to z' in equation (6).

To evaluate the z' integral it is sufficient to consider

$$\int_{-\infty}^{\infty} e^{-ikz'\sin\beta} \frac{e^{ik\rho}}{\rho} dz' = e^{-ikz\sin\beta} \int_{-\infty}^{\infty} e^{-ik(z'-z)\sin\beta} \frac{e^{ik\rho}}{\rho} dz'$$

where, of course, $\rho = \sqrt{(x'-x)^2 + y^2 + (z'-z)^2}$, and if we put $z'-z = Q\sinh\gamma$ with $Q = \sqrt{(x'-x)^2 + y^2}$, the integral becomes

$$e^{-ikz\sin\beta} \int_{-\infty}^{\infty} e^{ikQ(\cosh\gamma - \sin\beta\sinh\gamma)} d\gamma$$

$$= e^{-ikz\sin\beta} \int_{-\infty}^{\infty} e^{ikQ\cos\beta\cosh(\gamma + \tanh^{-1}\sin\beta)} d\gamma = \pi i e^{-ikz\sin\beta} H_0^{(1)}(kQ\cos\beta) .$$

The nature of this result shows that the assumption about the z dependence is self-consistent and allows us to suppress the coordinate z throughout the subsequent analysis. Equation (6) can then be written as

$$\underline{E}(x,y)=\underline{E}^{i}(x,y)-\frac{1}{4}\int_0^{\infty}\left\{kZ(I_4,0,-I_2)+i\left(I_3\frac{\partial}{\partial y},-I_3\frac{\partial}{\partial x}+ik\sin\beta I_1,I_1\frac{\partial}{\partial y}\right)\right.$$

$$\left.+\frac{Z}{k}\left(ik\sin\beta I_2+\frac{\partial I_4}{\partial x'}\right)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},-ik\sin\beta\right)\right\}H_0^{(1)}(kQ\cos\beta)\,dx' \quad . \tag{7}$$

The above integrand involves $\frac{\partial I_4}{\partial x'}$, and if I_4 were zero at the edge of the half-plane, integration by parts would enable the derivative to be replaced by $I_4\frac{\partial}{\partial x}$. It is known that in the case of an H-polarized plane wave at normal incidence the magnetic current perpendicular to the edge vanishes at the edge (Ref. 3), and for simplicity it will be assumed that the same is true at oblique incidence. If this behaviour is not postulated at the outset, the solution of the integral equations becomes more complicated, although the analysis can still be carried through. The details are given in Appendix I and from these results it follows that $I_4(0)$ must be identically zero.

Equation (7) now gives rise to three scalar equations for the currents:

$$E_x(x,y)-E_x^{i}(x,y)=-\frac{1}{4}\int_0^{\infty}\left\{kZ\,I_4\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right)+i\,I_3\frac{\partial}{\partial y}+i\,Z\sin\beta\,I_2\frac{\partial}{\partial x}\right\}H_0^{(1)}(kQ\cos\beta)dx' \tag{8}$$

$$E_y(x,y)-E_y^i(x,y)=\frac{1}{4}\int_0^\infty\left\{i\,I_3\frac{\partial}{\partial x}+k\sin\beta\, I_1-i\,Z\sin\beta\, I_2\frac{\partial}{\partial y}-\frac{Z}{k}I_4\frac{\partial^2}{\partial x\partial y}\right\}H_0^{(1)}(kQ\cos\beta)dx' \tag{9}$$

$$E_z(x,y)-E_z^i(x,y)=\frac{1}{4}\int_0^\infty\left\{kZ\cos^2\beta\, I_2-i\,I_1\frac{\partial}{\partial y}+i\,Z\sin\beta\, I_4\frac{\partial}{\partial x}\right\}H_0^{(1)}(kQ\cos\beta)\,dx'\ . \tag{10}$$

Integral equations for I_2 and I_4 can be obtained from these by letting the field point approach the half-plane successively from above and from below. By taking the limits of equations (8) and (10) as $y \to \pm 0$ and using the fact that

$$\left(\lim_{y\to+0}+\lim_{y\to-0}\right)\int_0^\infty I_p(x')\,\frac{\partial}{\partial y}\,H_0^{(1)}(kQ\cos\beta)\,dx'=0,$$

we have

$$E_x(x,+0)+E_x(x,-0)-2E_x^i(x,0)=-\frac{1}{2}\int_0^\infty\left\{kZ\,I_4\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right)+\right.$$

$$\left.+\,i\,Z\sin\beta\, I_2\frac{\partial}{\partial x}\right\}H_0^{(1)}(k|x'-x|\cos\beta)\,dx'\ ,$$

$$E_z(x,+0)+E_z(x,-0)-2E_z^i(x,0)=\frac{1}{2}\int_0^\infty\left\{kZ\cos^2\beta\, I_2\ +\right.$$

$$\left.+\,i\,Z\sin\beta\, I_4\frac{\partial}{\partial x}\right\}H_0^{(1)}(k|x'-x|\cos\beta)\,dx'\ ,$$

and since

$$E_x(x,+0)+E_x(x,-0)=\eta\, Z\, I_4$$

$$E_z(x,+0)+E_z(x,-0)=-\eta\, Z\, I_2\ ,$$

it follows that

$$\eta\, I_4(x) + 2Y\cos\alpha\sin\beta\cos\beta\, e^{-ikx\cos\alpha\cos\beta} = -\frac{1}{2}\int_0^{\infty}\left\{k I_4\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right) + \right.$$

$$\left. + i\sin\beta\, I_2\frac{\partial}{\partial x}\right\} H_0^{(1)}(k\,|x'-x|\cos\beta)\, dx' \tag{11}$$

$$\eta\, I_2(x) + 2Y\cos^2\beta\, e^{-ikx\cos\alpha\cos\beta} = -\frac{1}{2}\int_0^{\infty}\left\{k\cos^2\beta\, I_2 + \right.$$

$$\left. + i\sin\beta\, I_4\frac{\partial}{\partial x}\right\} H_0^{(1)}(k\,|x'-x|\cos\beta)\, dx' \quad , \tag{12}$$

which are coupled Wiener-Hopf equations for the determination of the electric currents I_2 and I_4 .

To obtain the integral equations for I_1 and I_3, either of two procedures can be followed. The first of these has its origin in the observation that the currents I_2 and I_4 whose equations have already been found are both electric currents, and the starting point for the derivation of equations (11) and (12) was the expression of the electric field at a point (x,y,z) in terms of a surface integral over the currents. Consequently, it is to be expected that if the magnetic field were written as a surface integral analogous to that in equation (4), the same analysis as the above would lead to equations for I_1 and I_3.

The second method is equivalent to this, but avoids the necessity of going back to the expression of the magnetic field as a surface integral over the field components. If equations (8) and (10) are differentiated with respect to y before the field point is allowed to approach the surface of the sheet, equation (9) can then be used to determine H_x and H_z.

If this second method is adopted, the fact that

$$H_x = i\,\frac{Y}{k}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right)$$

and

$$H_z = i\,\frac{Y}{k}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

gives, by suitable differentiation of equations (8), (9) and (10),

$$H_x(x,y) - H_x^i(x,y) = \frac{1}{4}\int_0^\infty \left\{ kY\, I_1 \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right) - i\, I_2 \frac{\partial}{\partial y} + \right.$$

$$\left. + \, i\, Y \sin\beta\, I_3 \frac{\partial}{\partial x} \right\} H_0^{(1)}(kQ\cos\beta)\,dx' \quad , \tag{13}$$

$$H_z(x,y) - H_z^i(x,y) = \frac{1}{4}\int_0^\infty \left\{ kY \cos^2\beta\, I_3 - i\, I_4 \frac{\partial}{\partial y} - i\, Y \sin\beta\, I_1 \frac{\partial}{\partial x} \right\} H_0^{(1)}(kQ\cos\beta)\,dx' \; . \tag{14}$$

Comparison with the equations for E_x and E_z reveals the expected symmetry between the electric and magnetic fields and currents. Indeed, equations (13) and (14) correspond to (8) and (10) under the transformation

$$\underline{E} \rightarrow \underline{H} \qquad Z\underline{H} \rightarrow -Y\underline{E}$$

$$I_1 \rightarrow I_4 \qquad ZI_2 \rightarrow -YI_3$$

$$I_3 \rightarrow I_2 \qquad ZI_4 \rightarrow -YI_1$$

and this duality enables us to write down immediately the integral equations for I_1 and I_3 by reference to the equations for I_2 and I_4. The results obtained in this manner are identical to those which would have been found if the first of the above methods had been employed, and are

$$\frac{1}{\eta} I_1(x) - 2\sin\alpha\cos\beta\, e^{-ikx\cos\alpha\cos\beta} = -\frac{1}{2}\int_0^\infty \left\{ kI_1\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right) + \right.$$

$$\left. + i\sin\beta\, I_3 \frac{\partial}{\partial x}\right\} H_0^{(1)}(k\,|x'-x|\cos\beta)\, dx' \qquad (15)$$

$$\frac{1}{\eta} I_3(x) = -\frac{1}{2}\int_0^\infty \left\{ k\cos^2\beta\, I_3 + i\sin\beta\, I_1 \frac{\partial}{\partial x}\right\} H_0^{(1)}(k\,|x'-x|\cos\beta)\, dx' \qquad (16)$$

(c.f. equations 11 and 12). To complete the duality between the electric and magnetic quantities, it will be observed that the additional transformation $\eta \rightarrow \frac{1}{\eta}$ is required. The absence of the inhomogenous term in equation (16) is caused by the fact that H_z^i is zero.

3

THE SOLUTION OF THE EQUATIONS

In view of the similarity between the equations for I_1, I_2, I_3 and I_4 it is sufficient to restrict the analysis to just one of the coupled pairs. Let us therefore consider equations (11) and (12) and attempt to cast them into forms suitable for solution by the Wiener-Hopf method.

Take first equation (11). This holds only for $x > 0$, but if a function $\phi_4(x)$ is introduced such that $\phi_4(x)$ is zero for $x > 0$ and is equal to the right hand side of (11) for $x < 0$, the equation can be written as

$$\eta I_4(x) + \phi_4(x) + 2Y \cos\alpha \sin\beta \cos\beta \, \Psi(x) = -\frac{1}{2} \int_{-\infty}^{\infty} \left\{ k I_4 \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial x^2}\right) + \right.$$

$$\left. + i \sin\beta \, I_2 \frac{\partial}{\partial x} \right\} H_0^{(1)}(k|x'-x|\cos\beta)\, dx' \tag{17}$$

where I_2 and I_4 are defined to be zero for $x < 0$ and

$$\Psi(x) = \begin{cases} e^{-ikx\cos\alpha\cos\beta} & \text{for } x > 0 \\ 0 & \text{for } x < 0. \end{cases}$$

In this form the equation is valid for all x.

The application of a Fourier transform to equation (17) can be justified by a study of the growth orders of the functions $\Psi(x)$, $I_2(x)$, $I_4(x)$ and $\phi_4(x)$ for large $|x|$.

The Fourier transform of, for example, $\psi(x)$ is

$$\overline{\psi}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta x} \psi(x)dx$$

and from the definition of $\psi(x)$ it is immediately obvious that $\overline{\psi}(\zeta)$ is regular in a lower half-plane of the transform variable ζ. For the currents $I_p(x)$, $p = 1,2,3,4$, the assumption that

$$I_p(x) \sim e^{-ik\, x \cos\alpha \cos\beta}$$

as $x \rightarrow \infty$ implies that the transforms $\overline{I}_p(\zeta)$ are regular in the same region and, finally, by using the asymptotic expansion for the Hankel function when x is large and negative we have that $\overline{\phi}_p(\zeta)$ is regular in an upper half-plane. All these regions of regularity overlap and within the common strip it is permissible to apply a Fourier transform to equation (17).

By inserting the Fourier integral representation of the Hankel function, the right hand side of equation (17) becomes

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ kI_4(x') \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial x^2}\right) + i \sin\beta\, I_2(x') \frac{\partial}{\partial x} \right\} \int_{\mathcal{C}} \frac{e^{i\zeta(x-x')}}{\sqrt{k^2 \cos^2\beta - \zeta^2}} d\zeta \, dx'$$

which can be written alternatively as

$$-\frac{1}{\sqrt{2\pi}}\int_{\mathcal{C}}\left\{\frac{1}{k}(k^2-\zeta^2)\,\bar{I}_4(\zeta)-\zeta\sin\beta\,\bar{I}_2(\zeta)\right\}\frac{e^{i\zeta x}}{\Gamma}\,d\zeta$$

where $\Gamma=\sqrt{k^2\cos^2\beta-\zeta^2}$ and $\mathcal{C}$ is a straight line path from $-\infty$ to ∞ lying within the strip of regularity of the integrand. Application of a Fourier transform now gives

$$-\bar{\phi}_4(\zeta)=\sqrt{\frac{2}{\pi}}\,\frac{Y\cos\alpha\sin\beta\cos\beta}{i(\zeta+k\cos\alpha\cos\beta)}+\left(\eta+\frac{k^2-\zeta^2}{k\Gamma}\right)\bar{I}_4(\zeta)-\frac{\zeta\sin\beta}{\Gamma}\,\bar{I}_2(\zeta) \tag{18}$$

and similarly, from equation (12),

$$-\bar{\phi}_2(\zeta)=\sqrt{\frac{2}{\pi}}\,\frac{Y\cos^2\beta}{i(\zeta+k\cos\alpha\cos\beta)}-\frac{\zeta\sin\beta}{\Gamma}\,\bar{I}_4(\zeta)+\left(\eta+\frac{k\cos^2\beta}{\Gamma}\right)\bar{I}_2(\zeta)\,. \tag{19}$$

These are sufficient to specify $\bar{I}_2(\zeta)$ and $\bar{I}_4(\zeta)$.

The obvious way in which to attempt the solution is to eliminate $\bar{I}_2(\zeta)$ and $\bar{I}_4(\zeta)$ in turn and thereby obtain equations for each current separately. Unfortunately, however, the resulting equations are not capable of being treated by the Wiener-Hopf technique. An essential feature of the technique is the separation of the terms into two groups having overlapping regions of regularity, and in the process of eliminating a current it is necessary to multiply one or other of the functions $\bar{\phi}_2(\zeta)$ or $\bar{\phi}_4(\zeta)$ by a quantity which destroys its regularity in the upper half-plane. A consequence of this is that coupled equations of the above type are, in general, not capable of being solved, and it is only by virtue of a particular symmetry existing between equations (18) and (19) that $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ can be determined.

The clue to the method of solution came from an attempt to guess a relationship between the currents. If it is assumed that

$$\overline{I}_2(\zeta) = f(\zeta)\,\overline{I}_4(\zeta),$$

then $f(\zeta)$ must be such as to reduce (18) and (19) to essentially the same equation. Since both $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ are regular in a lower half-plane, $f(\zeta)$ must also be regular in this region, and from the way in which η enters into the factors multiplying the currents it is clear that $f(\zeta)$ must be chosen to satisfy

$$(k^2 - \zeta^2)f - k\,\zeta \sin\beta = (k^2 \cos^2\beta - k\,\zeta\, f \sin\beta)\, f \quad .$$

The resulting values of $f(\zeta)$ are

$$\frac{\zeta}{k \sin\beta} \quad \text{and} \quad -\frac{k \sin\beta}{\zeta} \quad ,$$

but neither of these is sufficient to bring into agreement the remaining terms in equations (18) and (19).

The failure of the method is not very surprising since it presupposes an extremely close connection between the currents. On the other hand, the method does reveal a simple relation between the coefficients of $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ in equations (18) and (19), and suggests that a profitable approach would be to use the values for $f(\zeta)$ to derive equations for certain linear combinations of the currents.

If (18) and (19) are multiplied by k sin β and ζ respectively, and then subtracted, we obtain

$$\left\{\zeta \bar{I}_2(\zeta) - k\sin\beta \bar{I}_4(\zeta)\right\}\left(\gamma + \frac{k}{\Gamma}\right) + \sqrt{\frac{2}{\pi}}\, Y \frac{(\zeta\cos\beta - k\cos\alpha\sin^2\beta)\cos\beta}{i(\zeta + k\cos\alpha\cos\beta)} = k\sin\beta\bar{\phi}_4(\zeta) - \zeta\bar{\phi}_2(\zeta), \tag{20}$$

and since the manipulation has not affected the regions of regularity, this is an equation for $\zeta \bar{I}_2(\zeta) - k\sin\beta\, \bar{I}_4(\zeta)$ which can be solved by the Wiener-Hopf technique. For this purpose, let

$$\gamma\cos\beta + \frac{k\cos\beta}{\Gamma} = \frac{K_-(\zeta)}{K_+(\zeta)},$$

where $K_+(\zeta)$ and $K_-(\zeta)$ are regular in upper and lower half-planes respectively. Bearing in mind the definition of Γ, it is apparent that $K_+(\zeta)$ and $K_-(\zeta)$ differ from the "split" functions given by Senior (Ref. 3) only in having k replaced by k cos β, and accordingly their analytical expressions can be obtained from that paper.

If equation (20) is multiplied through by $\cos\beta\, K_+(\zeta)$, it can be written as

$$\left\{\zeta \bar{I}_2(\zeta) - k\sin\beta \bar{I}_4(\zeta)\right\} K_-(\zeta) - \sqrt{\frac{2}{\pi}}\, Y k\cos\alpha\cos^2\beta \frac{K_+(-k\cos\alpha\cos\beta)}{i(\zeta + k\cos\alpha\cos\beta)}$$

$$= -\sqrt{\frac{2}{\pi}}\, Y\cos^2\beta \frac{(\zeta\cos\beta - k\cos\alpha\sin^2\beta)K_+(\zeta) + k\cos\alpha\, K_+(-k\cos\alpha\cos\beta)}{i(\zeta + k\cos\alpha\cos\beta)}$$

$$+ \left\{k\sin\beta\bar{\phi}_4(\zeta) - \zeta\bar{\phi}_2(\zeta)\right\}\cos\beta\, K_+(\zeta).$$

The left hand side is regular in a lower half-plane, whilst the right hand side is regular in an upper half-plane, and, moreover, these two regions have a common strip. It follows that each side must be equal to a function which is regular throughout the whole ζ plane and its growth order as $|\zeta| \to \infty$ then shows it to be at most a constant. Hence

$$\zeta \bar{I}_2(\zeta) - k\sin\beta\,\bar{I}_4(\zeta) = -i\sqrt{\frac{2}{\pi}}\, Y k\cos\alpha\cos^2\beta \, \frac{K_+(-k\cos\alpha\cos\beta)}{(\zeta + k\cos\alpha\cos\beta)\, K_-(\zeta)} + \frac{A'}{K_-(\zeta)} \tag{21}$$

where A' is independent of ζ .

To determine $\bar{I}_2(\zeta)$ and $\bar{I}_4(\zeta)$ individually, another linear combination is considered which introduces the second value of $f(\zeta)$. Multiplying equation (18) by ζ , equation (19) by k sin β, and then adding, we have

$$\left\{k\sin\beta\,\bar{I}_2(\zeta) + \zeta\bar{I}_4(\zeta)\right\}\left(\gamma + \frac{\Gamma}{k}\right) + \sqrt{\frac{2}{\pi}}\, Y\sin\beta\cos\beta\, \frac{k\cos\beta + \zeta\cos\alpha}{i(\zeta + k\cos\alpha\cos\beta)}$$

$$= -\zeta\bar{\phi}_4(\zeta) - k\sin\beta\,\bar{\phi}_2(\zeta), \tag{22}$$

which is an equation of the Wiener-Hopf type for $k\sin\beta\bar{I}_2(\zeta) + \zeta\bar{I}_4(\zeta)$. The factor multiplying these currents can be written

$$\frac{\gamma\,\Gamma}{k\cos\beta}\left(\frac{\cos\beta}{\gamma} + \frac{k\cos\beta}{\Gamma}\right) = \frac{\gamma\,\Gamma}{k\cos\beta}\;\frac{L_-(\zeta)}{L_+(\zeta)}$$

where $L_+(\zeta)$ and $L_-(\zeta)$ are regular in an upper and a lower half-plane respectively, and since $L_+(\zeta)$ and $L_-(\zeta)$ differ from $K_+(\zeta)$ and $K_-(\zeta)$ only in having $\eta\cos\beta$ replaced by $\frac{\cos\beta}{\eta}$, their expressions can also be deduced from the formulae given by Senior (Ref. 3).

If equation (22) is multiplied through by $\frac{k\cos\beta}{\eta}\frac{L_+(\zeta)}{\sqrt{k\cos\beta+\zeta}}$, its terms can be rearranged to give

$$\left\{k\sin\beta\bar{I}_2(\zeta)+\zeta\bar{I}_4(\zeta)\right\}\sqrt{k\cos\beta-\zeta}\,L_-(\zeta)+\sqrt{\frac{2}{\pi}}\,Y\,\frac{k^2}{\eta}\,\frac{\sin^2\alpha\sin\beta\cos^3\beta}{i(\zeta+k\cos\alpha\cos\beta)}\,\frac{L_+(-k\cos\alpha\cos\beta)}{\sqrt{k\cos\beta(1-\cos\alpha)}}$$

$$=-\sqrt{\frac{2}{\pi}}\,Y\,\frac{k}{\eta}\,\frac{\sin\beta\cos^2\beta}{i(\zeta+k\cos\alpha\cos\beta)}\left[(k\cos\beta+\zeta\cos\alpha)\cdot\frac{L_+(\zeta)}{\sqrt{k\cos\beta+\zeta}}-\right.$$

$$\left.-k\sin^2\alpha\cos\beta\,\frac{L_+(-k\cos\alpha\cos\beta)}{\sqrt{k\cos\beta(1-\cos\alpha)}}\right]-\left\{\zeta\bar{\phi}_4(\zeta)+k\sin\beta\bar{\phi}_2(\zeta)\right\}\frac{k\cos\beta}{\eta}\,\frac{L_+(\zeta)}{\sqrt{k\cos\beta+\zeta}}$$

(23)

and by the same argument as before each side of this equation must be equal to a function regular throughout the whole ζ plane. Using the fact that for large $|\zeta|$, $L_+(\zeta)$ and $L_-(\zeta)$ are O(1), the right hand side of (23) then shows that the analytic function is at most a constant, and hence,

$$k\sin\beta\bar{I}_2(\zeta)+\zeta\bar{I}_4(\zeta)=i\sqrt{\frac{2}{\pi}}\,Y\,\frac{k^2}{\eta}\,\frac{\sin^2\alpha\sin\beta\cos^3\beta}{\zeta+k\cos\alpha\cos\beta}\,\frac{L_+(-k\cos\alpha\cos\beta)}{\sqrt{k\cos\beta(1-\cos\alpha)(k\cos\beta-\zeta)}\,L_-(\zeta)}$$

$$+\frac{B'}{\sqrt{k\cos\beta-\zeta}\,L_-(\zeta)} \qquad (24)$$

where B' is independent of ζ .

Expressions for $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ can be obtained from (21) and (24) by eliminating each current in turn. The resulting formulae can be simplified to some extent by defining new constants A and B such that

$$A' = i\sqrt{\frac{2}{\pi}}\, Y \cos\beta \left\{ A + K_+(-k\cos\alpha\cos\beta)\right\} ,$$

$$B' = i\sqrt{\frac{2}{\pi}}\, \frac{Y}{\eta} \sin^2\alpha \sin\beta \cos^2\beta \sqrt{\frac{k\cos\beta}{1-\cos\alpha}}\, B ,$$

and this leads to the equations

$$(\zeta^2+k^2\sin^2\beta)\,\overline{I}_2(\zeta) = i\sqrt{\frac{2}{\pi}}\, Y\zeta\cos\beta\, \frac{\left\{\zeta K_+(-k\cos\alpha\cos\beta) + A(\zeta+k\cos\alpha\cos\beta)\right\}}{(\zeta+k\cos\alpha\cos\beta)\,K_-(\zeta)}$$

$$+\, i\sqrt{\frac{2}{\pi}}\, Y\frac{k}{\eta} \sin^2\alpha \sin^2\beta\cos^2\beta \sqrt{\frac{k\cos\beta}{(1-\cos\alpha)(k\cos\beta-\zeta)}}\, \frac{\left\{kL_+(-k\cos\alpha\cos\beta)+B(\zeta+k\cos\alpha\cos\beta)\right\}}{(\zeta+k\cos\alpha\cos\beta)\,L_-(\zeta)}$$

(25)

$$(\zeta^2+k^2\sin^2\beta)\,\overline{I}_4(\zeta) = -i\sqrt{\frac{2}{\pi}}\, Y k\sin\beta\cos\beta\, \frac{\left\{\zeta K_+(-k\cos\alpha\cos\beta)+A(\zeta+k\cos\alpha\cos\beta)\right\}}{(\zeta+k\cos\alpha\cos\beta)\,K_-(\zeta)}$$

$$+\, i\sqrt{\frac{2}{\pi}}\, Y\frac{\zeta}{\eta} \sin^2\alpha\sin\beta\cos^2\beta \sqrt{\frac{k\cos\beta}{(1-\cos\alpha)(k\cos\beta-\zeta)}}\, \frac{\left\{kL_+(-k\cos\alpha\cos\beta)+B(\zeta+k\cos\alpha\cos\beta)\right\}}{(\zeta+k\cos\alpha\cos\beta)\,L_-(\zeta)} .$$

(26)

As they stand, the above equations do not have the required regularity in that the expressions for $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ have poles at the points $\zeta = \pm ik\sin\beta$ which lie in the region where the transforms must be free of singularities. If the poles are eliminated, however, two conditions are obtained and these serve to specify A and B uniquely.

The equations for $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ each give rise to the same conditions on A and B, and it is therefore sufficient to consider only equation (26). For $\overline{I}_4(\zeta)$ to be regular at $\zeta = ik\sin\beta$, the right hand side of (26) must be zero at this point in order to balance the corresponding factor on the left, and the condition for this is

$$A = i\,e^{i\beta/2}\,\frac{1}{\eta}\sin^2\alpha\sin\beta\cos\beta\sqrt{\frac{\cos\beta}{1-\cos\alpha}}\;\frac{K_-(ik\sin\beta)}{L_-(ik\sin\beta)}\;B - i\sin\beta\,\frac{K_+(-k\cos\alpha\cos\beta)}{\cos\alpha\cos\beta+i\sin\beta}$$

$$+\; i\,e^{i\beta/2}\,\frac{1}{\eta}\sin^2\alpha\sin\beta\cos\beta\sqrt{\frac{\cos\beta}{1-\cos\alpha}}\;\frac{L_+(-k\cos\alpha\cos\beta)}{\cos\alpha\cos\beta+i\sin\beta}\;\frac{K_-(ik\sin\beta)}{L_-(ik\sin\beta)}\;. \tag{27}$$

Similar analysis applied to the point $\zeta = -ik\sin\beta$ gives

$$A = -i\,e^{-i\beta/2}\,\frac{1}{\eta}\sin^2\alpha\sin\beta\cos\beta\sqrt{\frac{\cos\beta}{1-\cos\alpha}}\;\frac{K_-(-ik\sin\beta)}{L_-(-ik\sin\beta)}\;B + i\sin\beta\,\frac{K_+(-k\cos\alpha\cos\beta)}{\cos\alpha\cos\beta-i\sin\beta}$$

$$-\; i\,e^{-i\beta/2}\,\frac{1}{\eta}\sin^2\alpha\sin\beta\cos\beta\sqrt{\frac{\cos\beta}{1-\cos\alpha}}\;\frac{L_+(-k\cos\alpha\cos\beta)}{\cos\alpha\cos\beta-i\sin\beta}\;\frac{K_-(-ik\sin\beta)}{L_-(-ik\sin\beta)} \tag{28}$$

and from (27) and (28) it is a simple matter to determine A and B. The values obtained are

$$A = \frac{\sin\beta}{(\cos^2\alpha\cos^2\beta+\sin^2\beta)P(\beta)}\left[\left\{i\cos\alpha\cos\beta\,Q(\beta) - \sin\beta\,P(\beta)\right\}K_+(-k\cos\alpha\cos\beta)\right.$$

$$\left. +\; 2\sin^2\alpha\sin\beta\cos\beta\sqrt{\frac{\cos\beta}{1-\cos\alpha}}\;L_+(-k\cos\alpha\cos\beta)\right] \tag{29}$$

$$B = \frac{1}{(\cos^2\alpha\cos^2\beta + \sin^2\beta)P(\beta)} \left[\left\{ i\sin\beta\, Q(\beta) - \cos\alpha\cos\beta\, P(\beta) \right\} L_+(-k\cos\alpha\cos\beta) + 2\eta \frac{\cos\alpha}{\sin^2\alpha} \sqrt{\frac{1-\cos\alpha}{\cos\beta}}\, K_+(-k\cos\alpha\cos\beta) \right] \tag{30}$$

where

$$P(\beta)\,,\ Q(\beta) = e^{i\beta/2}\frac{K_-(ik\sin\beta)}{L_-(ik\sin\beta)} \pm e^{-i\beta/2}\frac{K_-(-ik\sin\beta)}{L_-(-ik\sin\beta)}\,. \tag{31}$$

If the expressions for $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ are now examined, several facts are immediately apparent. Since all the split functions are O(1) for large $|\zeta|$, $\eta \neq 0$, it follows that

$$\overline{I}_2(\zeta) = O(|\zeta|^{-1})\,, \qquad \overline{I}_4(\zeta) = O(|\zeta|^{-3/2})$$

and hence

$$I_2(x) \sim \text{constant}, \qquad I_4(x) \sim x^{1/2}$$

as $x \to 0$. This shows that the electric current perpendicular to the edge is zero for x = 0 (in accordance with our previous assumption), whilst the electric current parallel to the edge is finite there. The same behaviour is found* in the case of

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* In Reference 3 it is incorrectly stated that $K_+(\zeta)$, etc. are $O(|\zeta|^{-1/2})$ for large $|\zeta|$. This does not affect the derivation of the solution given therein, but does invalidate the statements about the behaviour of the currents for small x.

normal incidence ($\beta = 0$), and it will be noted that a consequence of the finite conductivity is the removal of the current singularity at the edge.

When $\beta = 0$ equation (29) gives A = 0, and equations (25) and (26) then reduce to

$$\overline{I_2}(\zeta) = i \frac{2}{\pi} Y \frac{K_+(-k\cos\alpha)}{(\zeta + k\cos\alpha)\, K_-(\zeta)}$$

and

$$\overline{I_4}(\zeta) = 0 \quad ,$$

which agree with the known solution for normal incidence. The only other case of interest is that in which $\eta = 0$, but this is most conveniently considered in the next section.

It is now time to turn our attention to the magnetic currents $\overline{I_1}(\zeta)$ and $\overline{I_3}(\zeta)$. These have to be determined from the integral equations (15) and (16), which are of similar form to the equations (11) and (12) already discussed. Although the correspondence is not complete because of the lack of symmetry in the incident electric and magnetic fields, it is sufficiently close for us to omit the details of the solution. By the same method as was used for equations (11) and (12) it is found that

$$(\zeta^2 + k^2\sin^2\beta)\,\overline{I_1}(\zeta) = -i\sqrt{\frac{2}{\pi}}\, k\sin\alpha\sin^2\beta\cos^2\beta \left\{\frac{kL_+(-k\cos\alpha\cos\beta) + C(\zeta + k\cos\alpha\cos\beta)}{(\zeta + k\cos\alpha\cos\beta)\, L_-(\zeta)}\right\}$$

$$- i\sqrt{\frac{2}{\pi}}\,\eta\,\zeta\sin\alpha\cos\beta\sqrt{\frac{k\cos\beta}{(1-\cos\alpha)(k\cos\beta - \zeta)}}\left\{\frac{\zeta K_+(-k\cos\alpha\cos\beta) + D(\zeta + k\cos\alpha\cos\beta)}{(\zeta + k\cos\alpha\cos\beta)\, K_-(\zeta)}\right\}$$

(32)

$$(\zeta^2+k^2\sin^2\beta)\overline{I}_3(\zeta) = i\sqrt{\frac{2}{\pi}}\,\zeta\sin\alpha\sin\beta\cos^2\beta\,\frac{\left\{kL_+(-k\cos\alpha\cos\beta)+C(\zeta+k\cos\alpha\cos\beta)\right\}}{(\zeta+k\cos\alpha\cos\beta)L_-(\zeta)}$$

$$-\,i\sqrt{\frac{2}{\pi}}\,\eta k\sin\alpha\sin\beta\cos\beta\sqrt{\frac{k\cos\beta}{(1-\cos\alpha)(k\cos\beta-\zeta)}}\;\frac{\left\{\zeta K_+(-k\cos\alpha\cos\beta)+D(\zeta+k\cos\alpha\cos\beta)\right\}}{(\zeta+k\cos\alpha\cos\beta)\,K_-(\zeta)}$$

(33)

(c.f. equations 25 and 26), where the constants C and D have the values

$$C = \frac{1}{(\cos^2\alpha\cos^2\beta+\sin^2\beta)P'(\beta)}\left[\left\{i\sin\beta\,Q'(\beta) - \cos\alpha\cos\beta\,P'(\beta)\right\}L_+(-k\cos\alpha\cos\beta)\right.$$

$$\left.+\,2\eta\cos\alpha\sqrt{\frac{\cos\beta}{1-\cos\alpha}}\;K_+\,(-k\cos\alpha\cos\beta)\right]$$

(34)

$$D = \frac{\sin\beta\sqrt{\dfrac{\cos\beta}{1-\cos\alpha}}}{(\cos^2\alpha\cos^2\beta+\sin^2\beta)P'(\beta)}\left[\left\{i\cos\alpha\cos\beta\,Q'(\beta) - \sin\beta\,P'(\beta)\right\}K_+(-k\cos\alpha\cos\beta)\right.$$

$$\left.+\,2\sin\beta\cos\beta\sqrt{\frac{1-\cos\alpha}{\cos\beta}}\;L_+(-k\cos\alpha\cos\beta)\right]$$

(35)

with

$$P'(\beta)\,,\,Q'(\beta) = e^{-i\beta/2}\,\frac{K_-(ik\sin\beta)}{L_-(ik\sin\beta)} \pm e^{i\beta/2}\,\frac{K_-(-ik\sin\beta)}{L_-(-ik\sin\beta)}\,.$$

(36)

The remarks made about the electric currents $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ apply also to the magnetic currents $\overline{I}_3(\zeta)$ and $\overline{I}_1(\zeta)$.

4

THE FIELD COMPONENTS

Since the electric and magnetic fields can be written in terms of the transforms $\overline{I}_p(\zeta)$ alone, explicit determination of the currents $\overline{I}_p(x)$ is unnecessary. From equations (8), (9) and (10) we have, by inserting the Fourier representation of the Hankel function and restoring the z dependence,

$$E_x = E_x^i - \frac{1}{4}\sqrt{\frac{2}{\pi}}\int_{\mathcal{C}}\left\{\frac{Z}{k}\,\frac{k^2-\zeta^2}{\Gamma}\,\overline{I}_4(\zeta) - Z\,\frac{\zeta\sin\beta}{\Gamma}\,\overline{I}_2(\zeta) - \overline{I}_3(\zeta)\,\frac{y}{|y|}\right\}\phi\, d\zeta \tag{37}$$

$$E_y = E_y^i - \frac{1}{4}\sqrt{\frac{2}{\pi}}\int_{\mathcal{C}}\left\{\frac{\zeta}{\Gamma}\,I_3(\zeta) - \frac{k\sin\beta}{\Gamma}\,\overline{I}_1(\zeta) - Z\left[\sin\beta\,\overline{I}_2(\zeta) + \frac{\zeta}{k}\,\overline{I}_4(\zeta)\right]\frac{y}{|y|}\right\}\phi\, d\zeta \tag{38}$$

$$E_z = E_z^i + \frac{1}{4}\sqrt{\frac{2}{\pi}}\int_{\mathcal{C}}\left\{Z\,\frac{k\cos^2\beta}{\Gamma}\,\overline{I}_2(\zeta) - Z\,\frac{\zeta\sin\beta}{\Gamma}\,\overline{I}_4(\zeta) + \overline{I}_1(\zeta)\,\frac{y}{|y|}\right\}\phi\, d\zeta \tag{39}$$

with

$$\phi = e^{i\zeta x + i|y|\Gamma - ikz\sin\beta} ,$$

and if these are expressed in the form

$$\underline{E} = \underline{E}^i + \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\underline{\mathcal{E}}(\zeta)}{\zeta^2+k^2\sin^2\beta}\;\frac{\cos\beta}{\zeta+k\cos\alpha\cos\beta}\;\phi\, d\zeta , \tag{40}$$

the components of $\underline{\mathcal{E}}(\zeta)$ are

$$\mathcal{E}_x(\zeta) = \frac{1}{\eta}\,\frac{\zeta}{k}\,\sin\alpha\sin\beta\cos\beta\,\sqrt{k\cos\beta\,(1+\cos\alpha)(k\cos\beta+\zeta)}\;\frac{kL_+(-k\cos\alpha\cos\beta)+B(\zeta+k\cos\alpha\cos\beta)}{L_-(\zeta)}$$

$$-\frac{k^2}{\sqrt{k^2\cos^2\beta-\zeta^2}}\,\sin\beta\;\frac{\zeta K_+(-k\cos\alpha\cos\beta)+A(\zeta+k\cos\alpha\cos\beta)}{K_-(\zeta)}$$

$$- \frac{y}{|y|} \zeta \sin\alpha \sin\beta \cos\beta \; \frac{kL_+(-k\cos\alpha\cos\beta) + C(\zeta + k\cos\alpha\cos\beta)}{L_-(\zeta)}$$

$$+ \eta \frac{y}{|y|} k\sin\beta \sqrt{\frac{k\cos\beta(1+\cos\alpha)}{k\cos\beta - \zeta}} \; \frac{\zeta K_+(-k\cos\alpha\cos\beta) + D(\zeta + k\cos\alpha\cos\beta)}{K_-(\zeta)} \tag{41}$$

$$\mathcal{E}_y(\zeta) = \frac{\zeta^2 + k^2\sin^2\beta}{\sqrt{k^2\cos^2\beta - \zeta^2}} \sin\alpha\sin\beta\cos\beta \; \frac{1}{L_-(\zeta)} \Big[kL_+(-k\cos\alpha\cos\beta) + C(\zeta + k\cos\alpha\cos\beta)$$

$$- \frac{1}{\eta} \frac{y}{|y|} \sqrt{\frac{\cos\beta(1+\cos\alpha)(k\cos\beta+\zeta)}{k}} \Big\{ kL_+(-k\cos\alpha\cos\beta) + B(\zeta + k\cos\alpha\cos\beta) \Big\} \Big] \tag{42}$$

$$\mathcal{E}_z(\zeta) = -\frac{1}{\eta} \sin\alpha\sin^2\beta\cos\beta \sqrt{k\cos\beta(1+\cos\alpha)(k\cos\beta - \zeta)} \; \frac{kL_+(-k\cos\alpha\cos\beta) + B(\zeta + k\cos\alpha\cos\beta)}{L_-(\zeta)}$$

$$- \frac{k\zeta}{\sqrt{k^2\cos^2\beta - \zeta^2}} \; \frac{\zeta K_+(-k\cos\alpha\cos\beta) + A(\zeta + k\cos\alpha\cos\beta)}{K_-(\zeta)}$$

$$+ \frac{y}{|y|} k\sin\alpha\sin^2\beta\cos\beta \; \frac{kL_+(-k\cos\alpha\cos\beta) + C(\zeta + k\cos\alpha\cos\beta)}{L_-(\zeta)}$$

$$+ \eta \frac{y}{|y|} \zeta \sqrt{\frac{k\cos\beta(1+\cos\alpha)}{k\cos\beta - \zeta}} \; \frac{\zeta K_+(-k\cos\alpha\cos\beta) + D(\zeta + k\cos\alpha\cos\beta)}{K_-(\zeta)} \tag{43}$$

The corresponding equations for the magnetic field can be deduced from (37), (38) and (39) by using the duality referred to in Section 2, and if we similarly write

$$\underline{H} = \underline{H}^i + \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\mathcal{H}(\zeta)}{\zeta^2 + k^2 \sin^2\beta} \; \frac{\cos\beta}{\zeta + k\cos\alpha\cos\beta} \; \phi \, d\zeta \tag{44}$$

(c.f. equation 40), it can be shown that

$$\mathcal{H}_x(\zeta) = \eta \frac{\zeta}{k} \sqrt{k\cos\beta(1+\cos\alpha)(k\cos\beta+\zeta)} \; \frac{\zeta K_+(-k\cos\alpha\cos\beta) + D(\zeta + k\cos\alpha\cos\beta)}{K_-(\zeta)}$$

$$+ \frac{k^2}{\sqrt{k^2\cos^2\beta - \zeta^2}} \sin\alpha\sin^2\beta\cos\beta \; \frac{kL_+(-k\cos\alpha\cos\beta) + C(\zeta + k\cos\alpha\cos\beta)}{L_-(\zeta)}$$

$$- \frac{y}{|y|} \zeta \; \frac{\zeta K_+(-k\cos\alpha\cos\beta) + A(\zeta + k\cos\alpha\cos\beta)}{K_-(\zeta)}$$

$$- \frac{1}{\eta} \frac{y}{|y|} k\sin\alpha\sin^2\beta\cos\beta \sqrt{\frac{k\cos\beta(1+\cos\alpha)}{k\cos\beta - \zeta}} \; \frac{kL_+(-k\cos\alpha\cos\beta) + B(\zeta + k\cos\alpha\cos\beta)}{L_-(\zeta)} \tag{45}$$

$$\mathcal{H}_y(\zeta) = \frac{\zeta^2 + k^2\sin^2\beta}{\sqrt{k^2\cos^2\beta - \zeta^2}} \; \frac{1}{K_-(\zeta)} \Bigg[\zeta K_+(-k\cos\alpha\cos\beta) + A(\zeta + k\cos\alpha\cos\beta)$$

$$- \eta \frac{y}{|y|} \sqrt{\frac{\cos\beta(1+\cos\alpha)(k\cos\beta+\zeta)}{k}} \left\{ \zeta K_+(-k\cos\alpha\cos\beta) + D(\zeta + k\cos\alpha\cos\beta) \right\} \Bigg] \tag{46}$$

$$\mathcal{H}_z(\zeta) = -\eta \sin\beta \sqrt{k\cos\beta(1+\cos\alpha)(k\cos\beta+\zeta)} \; \frac{\zeta K_+(-k\cos\alpha\cos\beta) + D(\zeta + k\cos\alpha\cos\beta)}{K_-(\zeta)}$$

$$+ \frac{k\zeta}{\sqrt{k^2\cos^2\beta - \zeta^2}} \sin\alpha\sin\beta\cos\beta \; \frac{kL_+(-k\cos\alpha\cos\beta) + C(\zeta + k\cos\alpha\cos\beta)}{L_-(\zeta)}$$

$$+ \frac{y}{|y|} k \sin\beta \; \frac{\zeta K_+(-k\cos\alpha\cos\beta) + A(\zeta + k\cos\alpha\cos\beta)}{K_-(\zeta)}$$

$$- \frac{1}{\eta} \frac{y}{|y|} \zeta \sin\alpha \sin\beta \cos\beta \sqrt{\frac{k\cos\beta(1+\cos\alpha)}{k\cos\beta - \zeta}} \; \frac{kL_+(-k\cos\alpha\cos\beta) + B(\zeta + k\cos\alpha\cos\beta)}{L_-(\zeta)} \tag{47}$$

An exact evaluation of the integrals in (40) and (44) would appear to be impossible, but for most purposes approximate values are sufficient and these can be obtained by the method of steepest descents. The present paper, however, is more concerned with the formulas themselves; and of particular interest is the extent to which they differ from the ones predicted by the technique described in Section 1. If the technique were valid for imperfect conductivity, the field components would be given by

$$\underline{E} = \left(-\frac{i\sin\beta}{k} \frac{\partial U}{\partial x}, \; -\frac{i\sin\beta}{k} \frac{\partial U}{\partial y}, \; U\cos^2\beta \right)$$

$$\underline{H} = \frac{iY}{k} \left(-\frac{\partial U}{\partial y}, \; \frac{\partial U}{\partial x}, \; 0 \right)$$

where U is the modified two-dimensional solution, and hence

$$\underline{E} = \underline{E}^i - \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\underline{R}}{\sqrt{k^2\cos^2\beta - \zeta^2}} \; \frac{\cos\beta}{\zeta + k\cos\alpha\cos\beta} \; \phi \, d\zeta \tag{48}$$

$$\underline{H} = \underline{H}^i - \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\underline{S}}{\sqrt{k^2\cos^2\beta - \zeta^2}} \; \frac{\cos\beta}{\zeta + k\cos\alpha\cos\beta} \; \phi \, d\zeta \tag{49}$$

with

$$\underline{R} = \left(\zeta \sin\beta, \Gamma \frac{y}{|y|} \sin\beta,\ k\cos^2\beta\right) \frac{K_+(-k\cos\alpha\cos\beta)}{K_-(\zeta)} \left\{ 1 - \eta \frac{y}{|y|} \sqrt{\frac{(1+\cos\alpha)(k\cos\beta+\zeta)}{k\cos\beta}} \right\}$$

$$\underline{S} = \left(\Gamma \frac{y}{|y|}, -\zeta, 0\right) \frac{K_+(-k\cos\alpha\cos\beta)}{K_-(\zeta)} \left\{ 1 - \eta \frac{y}{|y|} \sqrt{\frac{(1+\cos\alpha)(k\cos\beta+\zeta)}{k\cos\beta}} \right\}.$$

It is immediately obvious that this differs from the true field, and with many of the components the discrepancy is at least $O(\eta)$. On the other hand, the boundary conditions themselves only reproduce the physical conditions to $O(\eta)$, and if this accuracy were achieved by even two of the components found by the above technique, they might be sufficient to describe that portion of the true solution to which a physical interpretation can be attached. To determine whether such components exist, it is necessary to expand $\underline{\mathcal{E}}(\zeta)$ and $\underline{\mathcal{H}}(\zeta)$ as series in η.

Using the expressions for the split functions (Ref. 3), we have

$$K_+(\zeta) = \sqrt{\frac{k\cos\beta+\zeta}{k\cos\beta}} \left\{ 1 - \frac{\eta}{\pi k} \sqrt{k^2\cos^2\beta - \zeta^2} \left(\frac{\pi}{2} - \sin^{-1}\frac{\zeta}{k\cos\beta} \right) + \frac{\eta\zeta}{\pi k} \left(\log\frac{\eta\cos\beta}{2} - 1 \right) + O(\eta^2 \log\eta) \right\} \tag{50}$$

$$L_+(\zeta) = \sqrt{\frac{\eta}{\cos\beta}} \left\{ 1 - \frac{\eta k}{\pi\sqrt{k^2\cos^2\beta - \zeta^2}} \left(\frac{\pi}{2} - \sin^{-1}\frac{\zeta}{k\cos\beta} \right) + O(\eta^2) \right\} \tag{51}$$

where the expansions have been carried out under the assumption that ζ is finite. Substitution into the equations for A, B, C and D gives

$$A = \frac{\eta}{\pi}\frac{\sin^2\beta}{\cos\beta}\left\{\sqrt{1-\cos\alpha}\left(\log\frac{\eta\cos\beta}{2}-1\right) - \sqrt{1+\cos\alpha}\,(\pi-\alpha) + (\eta\log\eta)\right\}$$

$$B = -\sqrt{\frac{\eta}{\cos\beta}}\;\frac{1}{(1+\cos\alpha)\cos\beta}\left\{1+\frac{\eta}{\pi\cos\beta}\left(\log\frac{\eta\cos\beta}{2}-1\right)-\frac{\eta}{\pi}(\pi-\alpha)\frac{1+\cos\alpha}{\sin\alpha\cos\beta} + O(\eta^2\log\eta)\right\}$$

$$C = -\frac{\eta}{\pi}\sqrt{\frac{\eta}{\cos\beta}}\left\{\log\frac{\eta\cos\beta}{2} - 1 - (\pi-\alpha)\cot\alpha + O(\eta\log\eta)\right\}$$

$$D = -\frac{\eta}{\pi}\frac{(\pi-\alpha)\sin^2\beta}{\sin\alpha\sqrt{\cos\beta}}\left\{1+O(\eta\log\eta)\right\},$$

and it is now a simple matter to compare the two sets of field components.

From a study of equations (40) and (44) it is seen that because of the factor $(\zeta^2+k^2\sin^2\beta)$ in the integrands, the components H_z and E_y are the only ones for which the technique is likely to succeed. The former is identically zero according to equation (49), but from equation (44) the true field has

$$\mathcal{H}_z = \eta\sin\beta\,(\zeta^2+k^2\sin^2\beta)\left\{\frac{\zeta\sin\alpha}{\sqrt{k^2\cos^2\beta-\zeta^2}} + \frac{1}{\pi}\frac{y}{|y|}\sqrt{\frac{1-\cos\alpha}{k\cos\beta(k\cos\beta-\zeta)}}\right.$$

$$\left[(\zeta+k\cos\alpha\cos\beta)\left(\log\frac{\eta\cos\beta}{2}-1\right) - k(\pi-\alpha)\cos\alpha\cos\beta\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}\right.$$

$$\left.\left. + \zeta\left(\frac{\pi}{2}+\sin^{-1}\frac{\zeta}{k\cos\beta}\right)\sqrt{\frac{k\cos\beta-\zeta}{k\cos\beta+\zeta}}\right] + O(\eta\log\eta)\right\}$$

which is of order $\eta \log \eta$. For the component E_y a similar analysis shows that the value predicted by the technique is again in error by terms $O(\eta \log \eta)$ and, in fact, this is the same with all the field components. In consequence, the technique which is so useful for treating perfectly conducting bodies fails completely when the surfaces are imperfectly conducting.

The above discussion is based upon a comparison of the expansions for $\underline{\mathcal{E}}(\zeta)$ and $\underline{\mathcal{H}}(\zeta)$ with those for

$$\frac{\zeta^2+k^2\sin^2\beta}{\sqrt{k^2\cos^2\beta-\zeta^2}}\,\underline{R} \qquad \text{and} \qquad \frac{\zeta^2+k^2\sin^2\beta}{\sqrt{k^2\cos^2\beta-\zeta^2}}\,\underline{S}\ ,$$

and the implication is that discrepancies are automatically reflected in the corresponding expansions for the fields. Since the proof is not quite straightforward, a few words of explanation are in order.

The expansions for the split functions were derived under the assumption that ζ is finite, and this restriction applies to all the subsequent formulae. In equations (40) and (44), however, the variable of integration ζ takes values from $-\infty$ to ∞ and if the expansions for $\underline{\mathcal{E}}(\zeta)$ and $\underline{\mathcal{H}}(\zeta)$ are merely integrated term by term, the expressions which are obtained are incorrect.

The difficulty can be overcome by using the method of steepest descents to approximate the integrals in (40) and (44). Almost the entire non-exponential portions of the integrands can then be removed at the saddle point $\zeta = \zeta_0$ and included in this are the functions $\underline{\mathcal{E}}(\zeta)$ and $\underline{\mathcal{H}}(\zeta)$. Since ζ_0 is, of course, finite, $\underline{\mathcal{E}}(\zeta)$ and $\underline{\mathcal{H}}(\zeta)$ can now be expanded as before and the individual terms generate corresponding terms in the expansions for $\underline{E}$ and $\underline{H}$. A direct consequence of this is that all the field components contain terms in $\eta \log \eta$ in addition to those involving powers of η alone.

5
CONCLUSIONS

In this paper the solution has been obtained for the problem of a plane wave incident at an oblique angle on a metallic half-plane. The analysis requires the solution of coupled Wiener-Hopf integral equations for the Fourier transforms of the four current distributions, and the resulting expressions for the fields are exact, subject only to the (physical) approximation implied by the impedance-type boundary conditions. The solution has applications to the coastal refraction of radio waves, but this topic is reserved for future consideration.

It has been shown that the technique commonly employed to determine oblique incidence solutions for perfectly conducting bodies cannot be used when the conductivity is finite and, indeed, all the field components produced in this way are in error by terms $O(\eta \log \eta)$. It would be extremely valuable if a technique could be developed for treating finite conductivity, but unfortunately the analysis has not suggested one. From the solution given in this paper it is clear that such a method would have to call upon the normal incidence results for both polarizations, with η replaced by $\eta \cos\beta$ in one case, and by $\eta \sec\beta$ in the other. However, the presence of the factor $\varsigma^2 + k^2 \sin^2\beta$ in the expressions for all but two components and, in addition, the occurrence of the complicated constants A, B, C and D, make the existence of a technique very unlikely.

REFERENCES

1. Clemmow, P.C. 1951 Proc. Roy. Soc. (A) 205, 286.

2. Senior, T.B.A. 1953 Quart. Journ. Mech. Appl. Math. 6, 101.

3. Senior, T.B.A. 1952 Proc. Roy. Soc. (A) 213, 436.

4. Stratton, J.A. 1941 Electromagnetic Theory. New York: McGraw-Hill.

APPENDIX: PROOF THAT $I_4(0) = 0$

If it is not assumed at the outset that $I_4(0)$ is zero, the equations from which to determine $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ are

$$-\overline{\phi}_4(\zeta) = \sqrt{\frac{2}{\pi}}\,\frac{Y\cos\alpha\sin\beta\cos\beta}{i(\zeta+k\cos\alpha\cos\beta)} - \frac{i\zeta}{k\sqrt{2\pi}}\,\frac{I_4(0)}{\Gamma} + \left(\gamma+\frac{k^2-\zeta^2}{k\,\Gamma}\right)\overline{I}_4(\zeta) - \frac{\zeta\sin\beta}{\Gamma}\,\overline{I}_2(\zeta) \tag{A1}$$

$$-\overline{\phi}_2(\zeta) = \sqrt{\frac{2}{\pi}}\,\frac{Y\cos^2\beta}{i(\zeta+k\cos\alpha\cos\beta)} - \frac{i\sin\beta}{\sqrt{2\pi}}\,\frac{I_4(0)}{\Gamma} - \frac{\zeta\sin\beta}{\Gamma}\,\overline{I}_4(\zeta) + \left(\gamma+\frac{k\cos^2\beta}{\Gamma}\right)\overline{I}_2(\zeta) \tag{A2}$$

which differ from equations (18) and (19) only by the single terms involving $I_4(0)$.

As in Section 3, the first step is to derive new integral equations for those linear combinations of $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ which are suggested by the functions $f(\zeta)$. If equations (A1) and (A2) are multiplied by $k\sin\beta$ and ζ respectively and then subtracted, a Wiener-Hopf integral equation for $\zeta\,\overline{I}_2(\zeta) - k\sin\beta\,\overline{I}_4(\zeta)$ is obtained. What is more, this process has also served to eliminate $I_4(0)$ and consequently the solution is identical to that given in equation (21).

The second equation for a linear combination of $\overline{I}_2(\zeta)$ and $\overline{I}_4(\zeta)$ is found by multiplying (A1) by ζ and (A2) by $k\sin\beta$. On adding the resulting equations, we have

$$\left\{k\sin\beta\,\overline{I}_2(\zeta) + \zeta\,\overline{I}_4(\zeta)\right\}\left(\gamma+\frac{\Gamma}{k}\right) - \frac{i\,I_4(0)}{k\sqrt{2\pi}}\,\frac{\zeta^2+k^2\sin^2\beta}{\Gamma}$$

$$+\sqrt{\frac{2}{\pi}}\,Y\sin\beta\cos\beta\,\frac{k\cos\beta+\zeta\cos\alpha}{i(\zeta+k\cos\alpha\cos\beta)} = -\zeta\,\overline{\phi}_4(\zeta) - k\sin\beta\,\overline{\phi}_2(\zeta)$$

which can be written alternatively as

$$\left[\left\{k\sin\beta\,\overline{I}_2(\zeta) + \zeta\overline{I}_4(\zeta)\right\}(k\cos\beta+\zeta) - \frac{i\,I_4(0)}{\sqrt{2\pi}}\,\frac{\zeta^2+k^2\sin^2\beta}{k\cos\beta-\zeta}\right]\sqrt{k\cos\beta-\zeta}\;L_-(\zeta)$$

$$+\frac{i\,I_4(0)}{\sqrt{2\pi}}\sqrt{\frac{k\cos\beta}{2}}\,\frac{\zeta^2+k^2\sin^2\beta}{k\cos\beta-\zeta}\,L_+(k\cos\beta)$$

$$+\sqrt{\frac{2}{\pi}}\,Y\,\frac{k^2}{\eta}\,(k\cos\beta+\zeta)\,\frac{\sin^2\alpha\sin\beta\cos^3\beta}{i(\zeta+k\cos\alpha\cos\beta)}\,\frac{L_+(-k\cos\alpha\cos\beta)}{\sqrt{k\cos\beta(1-\cos\alpha)}}$$

$$= -\sqrt{\frac{2}{\pi}}\,Y\,\frac{k}{\eta}\,(k\cos\beta+\zeta)\,\frac{\sin\beta\cos^2\beta}{i(\zeta+k\cos\alpha\cos\beta)}\left[(k\cos\beta+\zeta\cos\alpha)\,\frac{L_+(\zeta)}{\sqrt{k\cos\beta+\zeta}}\right.$$

$$\left.-k\sin^2\alpha\cos\beta\,\frac{L_+(-k\cos\alpha\cos\beta)}{\sqrt{k\cos\beta(1-\cos\alpha)}}\right]$$

$$-\frac{i\,I_4(0)}{\sqrt{2\pi}}\,k\cos\beta\,\frac{\zeta^2+k^2\sin^2\beta}{k\cos\beta-\zeta}\left\{\frac{L_+(\zeta)}{\sqrt{k\cos\beta+\zeta}}-\frac{L_+(k\cos\beta)}{\sqrt{2k\cos\beta}}\right\}$$

$$-\left\{\zeta\overline{\phi}_4(\zeta)+k\sin\beta\,\overline{\phi}_2(\zeta)\right\}\frac{k\cos\beta}{\eta}\sqrt{k\cos\beta+\zeta}\;L_+(\zeta)\;.$$

This is now in the form required for a Wiener-Hopf split. The left hand side is regular in a lower half-plane, whilst the right hand side is regular in an upper half-plane, and since the two regions overlap, each side must be equal to a function regular throughout the whole ζ plane. The growth orders of the two sides as $|\zeta| \to \infty$ then specify the function as $B'' + C''\,\zeta$, where B'' and C'' are independent of ζ, and hence

$$k\sin\beta\,\overline{I}_2(\zeta) + \zeta\,\overline{I}_4(\zeta) = i\sqrt{\frac{2}{\pi}}\,Y\,\frac{k^2}{\eta}\,\frac{\sin^2\alpha\sin\beta\cos^3\beta}{\zeta+k\cos\alpha\cos\beta}\,\frac{L_+(-k\cos\alpha\cos\beta)}{\sqrt{k\cos\beta(1-\cos\alpha)(k\cos\beta-\zeta)}\,L_-(\zeta)}$$

$$-\frac{i\,I_4(0)}{\sqrt{2\pi}}\,\frac{\zeta^2+k^2\sin^2\beta}{k^2\cos^2\beta-\zeta^2}\left\{\sqrt{\frac{k\cos\beta}{2(k\cos\beta-\zeta)}}\,\frac{L_+(k\cos\beta)}{L_-(\zeta)}-1\right\}+\frac{B''+C''\zeta}{(k\cos\beta+\zeta)\sqrt{k\cos\beta-\zeta}\,L_-(\zeta)}\,. \tag{A3}$$

From equations (21) and (A3), $\overline{I}_2(\zeta)$ can be eliminated to give an expression for $\overline{I}_4(\zeta)$ alone. The equation obtained in this manner is

$$(\zeta^2+k^2\sin^2\beta)\,\overline{I}_4(\zeta) = i\sqrt{\frac{2}{\pi}}\,Y\,k^2\cos\alpha\sin\beta\cos^2\beta\,\frac{K_+(-k\cos\alpha\cos\beta)}{(\zeta+k\cos\alpha\cos\beta)\,K_-(\zeta)}$$

$$+\,i\sqrt{\frac{2}{\pi}}\,Y\,\frac{k\zeta}{\eta}\,\sin^2\alpha\sin\beta\cos^2\beta\,\sqrt{\frac{k\cos\beta}{(1-\cos\alpha)(k\cos\beta-\zeta)}}\,\frac{L_+(-k\cos\alpha\cos\beta)}{(\zeta+k\cos\alpha\cos\beta)L_-(\zeta)}$$

$$-\frac{i\,I_4(0)}{\sqrt{2\pi}}\,\zeta\,\frac{\zeta^2+k^2\sin^2\beta}{k^2\cos^2\beta-\zeta^2}\left\{\sqrt{\frac{k\cos\beta}{2(k\cos\beta-\zeta)}}\,\frac{L_+(k\cos\beta)}{L_-(\zeta)}-1\right\}$$

$$+\,\frac{(B''+C''\zeta)\zeta}{(k\cos\beta+\zeta)\sqrt{k\cos\beta-\zeta}\;L_-(\zeta)}-\frac{A'k\sin\beta}{K_-(\zeta)}$$

from which it is seen that $\overline{I}_4(\zeta) = O(|\zeta|^{-3/2})$. The Fourier transform relationship now gives

$$I_4(x) \sim x^{1/2} \qquad \text{as } x \to 0$$

and accordingly the electric current perpendicular to the edge is zero there.