388967-7-T

Senior, T.B.A.

THE UNIVERSITY OF MICHIGAN

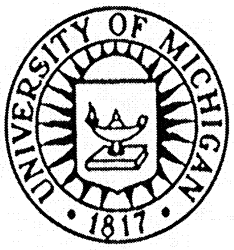
COLLEGE OF ENGINEERING

DEPARTMENT OF ELECTRICAL ENGINEERING & COMPUTER SCIENCE

Radiation Laboratory

DERIVATION AND APPLICATION OF A CLASS OF GENERALIZED BOUNDARY CONDITIONS

T.B.A. Senior
J.L. Volakis
Radiation Laboratory
Department of Electrical Engineering
and Computer Science
The University of Michigan
Ann Arbor, MI 48109



June, 1988

Ann Arbor, Michigan

Abstract

Boundary conditions involving higher order derivatives are presented for simulating surfaces whose reflection coefficients are known analytically, numerically or experimentally. Procedures for determining the coefficients of the derivatives are discussed, along with the effect of displacing the surface where the boundary conditions are applied. Provided the coefficients satisfy a duality relation, equivalent forms of the boundary conditions involving tangential field components are deduced, and these provide the natural extension to non-planar surfaces. As an illustration, the simulation of a metal-backed uniform dielectric layer is given. It is shown that fourth order conditions are capable of providing an accurate simulation for layers a quarter of a wavelength in thickness or more.

I. Introduction

The use of non-metallic materials, possibly in the form of a non-uniform or multilayer coating applied to a metallic substrate, has made necessary the development of methods for simulating material effects in scattering. This is important in the analytical treatment of simple (canonical) geometries and also for the efficient generation of numerical solutions.

A possible approach is to employ approximate boundary conditions, and this paper discusses a hierarchy of boundary conditions which allow the simulation of surfaces whose reflection coefficients are known analytically, numerically, or experimentally. The boundary conditions involve higher order normal derivatives of the normal field components on a surface to account for the angular dependence of the reflection coefficient, and, as such, constitute a generalization of the standard impedance (first order) condition. They were first considered by Karp and Karal [1] in a study of the surface waves supported by dielectric coatings. They were subsequently employed [2] to simulate a perfectly absorbing surface in a finite element analysis, and particular versions were recently developed [3, 4] to model dielectric layers with and without a metal backing.

The boundary conditions in their general form are introduced in the context of a planar surface and it is shown how they can be used to model a reflection coefficient. The required order increases with the complexity of the reflection coefficient, and the effect of displacing the surface where the boundary conditions are enforced is discussed. Provided the coefficients in the boundary conditions satisfy a duality relation, the conditions can also be expressed in terms of the tangential field

components. This shows that one effect of the higher order derivatives is to make the conditions less local in character through the inclusion of tangential field derivatives. As such they provide a natural extension to non-planar surfaces.

To illustrate the application of these conditions, the problem of a metal-backed uniform dielectric layer is examined in detail. It is found that fourth order conditions provide an excellent simulation and their accuracy is determined. The use of such higher order conditions is analytical and numerical work is also discussed.

2. Boundary Conditions for a Planar Surface

In the case of a planar surface y = 0 the proposed boundary conditions take the form

$$\prod_{m=1}^{M}\left(\frac{\partial}{\partial y}+ik\Gamma_m\right)E_y=0$$
$$\prod_{m=1}^{M'}\left(\frac{\partial}{\partial y}+ik\Gamma'_m\right)H_y=0 \qquad (1)$$

where k is the free space propagation constant and a time factor $e^{-i\omega t}$ has been assumed and suppressed. Γ_m and Γ'_m are constants which are chosen to reproduce the desired scattering properties of the surface, and since a knowledge of E_y or H_y alone is not in general sufficient to determine an electromagnetic field, the constants cannot be chosen independently of one another.

An alternative form of (1) is

$$\sum_{m=0}^{M} \frac{a_m}{(ik)^m} \frac{\partial^m}{\partial y^m} E_y = 0$$
$$\sum_{m=0}^{M'} \frac{a_m'}{(ik)^m} \frac{\partial^m}{\partial y^m} H_y = 0 \qquad (2)$$

and if, for example, M = 3,

$$a_0 = \Gamma_1 \Gamma_2 \Gamma_3 \ , \qquad a_1 = \Gamma_1 \Gamma_2 + \Gamma_2 \Gamma_3 + \Gamma_3 \Gamma_1 \ ,$$

$$a_2 = \Gamma_1 + \Gamma_2 + \Gamma_3 \ , \qquad a_3 = 1 \ .$$

When M = M' = 1 with $\Gamma_1' = 1/\Gamma_1$, the conditions are equivalent to impedance boundary conditions for a surface with normalized impedance Γ_1, and (1) can therefore be regarded as generalizations of the standard impedance boundary condition that allow the simulation of a greater variety of material properties through the inclusion of additional derivative factors.

If the plane wave

$$E_y^i = e^{-ik\,(x\cos\phi + y\sin\phi)}$$

is incident on a surface y = 0 at which the boundary conditions (1) are imposed, the reflection coefficient is

$$R(\phi) = -\prod_{m=1}^{M} \frac{\Gamma_m - \sin\phi}{\Gamma_m + \sin\phi} = -\frac{\sum_{m=0}^{M} (-1)^m a_m \sin^m \phi}{\sum_{m=0}^{M} a_m \sin^m \phi} \qquad (3)$$

with an analogous expression for the reflection coefficient R' (ϕ) associated with the component H_y. The special case $\Gamma_m = 1$, m = 1, 2, ..., M, constitutes a model of a perfectly absorbing surface [2]. We then have

$$R(\phi) = -\tan^{2M}\left(\frac{\pi}{4} - \frac{\phi}{2}\right) ,$$

and as M increases, there is an increasing range of angles about $\phi = \pi/2$ (normal incidence) where R is effectively zero.

More generally, the constants Γ_m and Γ'_m can be determined from a knowledge of the actual reflection coefficient of the surface, and some possible procedures are as follows:

(i) If analytical expressions for the reflection coefficients are available, expansion in the form (3) leads immediately to the identification of the Γ_m and Γ'_m.

(ii) The constants can be chosen to recover the poles of the analytical expressions, but this may produce a less accurate simulation if the reflection coefficient is not a ratio of polynomials in $\sin\phi$.

(iii) From computed or measured data for the reflection coefficients, Γ_m and Γ'_m can be obtained by curve fitting. Although this could be adequate in any given case, it would not reveal the dependence on the material parameters of the surface.

A few simple examples of a reflection coefficient R (ϕ) are sufficient to illustrate these procedures. In general, a plane wave reflection coefficient can be written as

$$R(\phi) = - \frac{\eta(\phi) - \sin\phi}{\eta(\phi) + \sin\phi} \qquad (4)$$

where η (ϕ) is an angle-dependent surface impedance. If $\eta = \eta_0$, independent of angle, the corresponding boundary condition has $\Gamma_1 = \eta_0$ with $\Gamma_m = 0$ for $m > 1$, and this is the standard impedance boundary condition [5]. Alternatively, if

$$\eta(\phi) = \eta_0 + \eta_1 \sin^2 \phi$$

then

$$\Gamma_{1,2} = \frac{1}{2\eta_1} \left\{ 1 \mp \sqrt{1 - 4\,\eta_0\,\eta_1} \right\}$$

with $\Gamma_m = 0$ for $m > 2$. As another example of procedure (i), if

$$\eta(\phi) = \cos\phi \quad ,$$

then for $\phi \leq \pi/4$ we can write $\eta(\phi) \approx 1 - 1/2 \sin^2 \phi$ without significant loss of accuracy, in which case $\Gamma_1 = 0.732$ and $\Gamma_2 = -2.732$ with $\Gamma_m = 0$ for $m > 2$. For $\phi \geq \pi/4$ we can set $\alpha = \pi/2 - \phi$ and use the same approximation for $\cos\alpha$. Alternatively, using procedure (ii), it is found that if $0 < \text{Re. } \phi < \pi/2$, $R(\phi)$ has a single pole at $\sin\phi = 0.707 = -\Gamma_1$. The resulting first order boundary condition is obviously not as accurate as the second order one given by procedure (i).

In addition to modelling a reflection coefficient, the generalized boundary conditions (1) allow some flexibility in the location of the surface where they are enforced. To show this, consider a surface $y = \tau$ at which the reflection coefficient is $R(\phi)$. The corresponding reflection coefficient at $y = 0$ is then $R(\phi) \exp(-2ik\tau \sin\phi)$, and since

$$e^{-2ik\tau \sin\phi} = \frac{1 - ik\tau \sin\phi - \frac{1}{2}(k\tau)^2 \sin^2\phi + \ldots.}{1 + ik\tau \sin\phi - \frac{1}{2}(k\tau)^2 \sin^2\phi + \ldots.} \quad , \tag{5}$$

the phase factor can be simulated using additional derivative factors in the boundary condition. In particular, if $k\tau$ is so small that terms $O\{(k\tau)^2\}$ can be neglected, one more derivative factor suffices with $\Gamma_1 = 1/(ik\tau)$, whereas to the next order in $k\tau$ two derivatives are needed with $\Gamma_{1,2} = (1 \pm i)/(ik\tau)$. This allows us to consider separately the modelling of the reflection coefficient and the phase factor, but if the location of the

simulating surface is not specified a priori, it can be chosen to minimize the order of the boundary condition for a given accuracy.

3. Equivalent Forms of the Conditions

The boundary conditions (1) are scalar conditions in as much as each involves only a single field component, but in the first order case when $M = M' = 1$ with $\Gamma_1' = 1/\Gamma_1$ it is well known [5] that they can be expressed in terms of the tangential field components. This is true for any pair of generalized boundary conditions provided the Γ_m and Γ_m' are appropriately related, and to illustrate the derivation of these equivalent boundary conditions involving tangential fields we consider the second order case.

The boundary condition for E_y is

$$\left\{\frac{\partial^2}{\partial y^2} + ik(\Gamma_1 + \Gamma_2)\frac{\partial}{\partial y} - k^2\Gamma_1\Gamma_2\right\} E_y = 0 \ ,$$

and using Maxwell's equations and the fact that $\nabla \cdot \bar{E} = 0$, this can be written as

$$\frac{\partial}{\partial x}\left\{E_x - \frac{\Gamma_1\Gamma_2 + 1}{\Gamma_1 + \Gamma_2} ZH_z + \frac{1}{ik(\Gamma_1 + \Gamma_2)}\frac{\partial E_y}{\partial x} + \frac{\partial f}{\partial z}\right\}$$

$$= -\frac{\partial}{\partial z}\left\{E_z + \frac{\Gamma_1\Gamma_2 + 1}{\Gamma_1 + \Gamma_2} ZH_x + \frac{1}{ik(\Gamma_1 + \Gamma_2)}\frac{\partial E_y}{\partial z} - \frac{\partial f}{\partial x}\right\}$$

for any function $f = f(x,z)$. Similarly, from the boundary condition for H_y,

$$\frac{\partial}{\partial x}\left\{H_x + \frac{\Gamma_1'\Gamma_2' + 1}{\Gamma_1' + \Gamma_2'} YE_z + \frac{1}{ik(\Gamma_1' + \Gamma_2')} \frac{\partial H_y}{\partial x} + \frac{\partial g}{\partial z}\right\}$$

$$= -\frac{\partial}{\partial z}\left\{H_z - \frac{\Gamma_1'\Gamma_2' + 1}{\Gamma_1' + \Gamma_2'} YE_x + \frac{1}{ik(\Gamma_1' + \Gamma_2')} \frac{\partial H_y}{\partial z} - \frac{\partial g}{\partial x}\right\}, \qquad (6)$$

and therefore

$$\frac{\partial}{\partial z}\left\{E_x - \frac{\Gamma_1' + \Gamma_2'}{\Gamma_1'\Gamma_2' + 1} ZH_z + \frac{\Gamma_1' + \Gamma_2'}{\Gamma_1'\Gamma_2' + 1} Z\frac{\partial g}{\partial x} - \frac{1}{ik(\Gamma_1'\Gamma_2' + 1)} Z\frac{\partial H_y}{\partial z}\right\}$$

$$= \frac{\partial}{\partial x}\left\{E_z + \frac{\Gamma_1' + \Gamma_2'}{\Gamma_1'\Gamma_2' + 1} ZH_x + \frac{\Gamma_1' + \Gamma_2'}{\Gamma_1'\Gamma_2' + 1} Z\frac{\partial g}{\partial z} + \frac{1}{ik(\Gamma_1'\Gamma_2' + 1)} Z\frac{\partial H_y}{\partial x}\right\} \qquad (7)$$

for any function g = g (x,z). Choose

$$f = -\frac{1}{ik(\Gamma_1'\Gamma_2' + 1)} ZH_y$$

and

$$g = \frac{\Gamma_1'\Gamma_2' + 1}{\Gamma_1' + \Gamma_2'} \quad \frac{1}{ik(\Gamma_1 + \Gamma_2)} YE_y \ .$$

Then if

$$\frac{\Gamma_1' + \Gamma_2'}{\Gamma_1'\Gamma_2' + 1} = \frac{\Gamma_1\Gamma_2 + 1}{\Gamma_1 + \Gamma_2} , \tag{8}$$

(6) and (7) imply

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\left\{E_x - \frac{\Gamma_1\Gamma_2 + 1}{\Gamma_1 + \Gamma_2} ZH_z + \frac{1}{ik(\Gamma_1 + \Gamma_2)} \frac{\partial E_y}{\partial x} - \frac{1}{ik(\Gamma_1' + \Gamma_2')} Z\frac{\partial H_y}{\partial z}\right\} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\left\{E_z + \frac{\Gamma_1\Gamma_2 + 1}{\Gamma_1 + \Gamma_2} ZH_x + \frac{1}{ik(\Gamma_1 + \Gamma_2)} \frac{\partial E_y}{\partial z} + \frac{1}{ik(\Gamma_1' + \Gamma_2')} Z\frac{\partial H_y}{\partial x}\right\} = 0 ,$$

and as shown in [6], the only allowed solutions of these equations are zero.

Hence

$$E_x = \frac{\Gamma_1\Gamma_2 + 1}{\Gamma_1 + \Gamma_2} ZH_z - \frac{1}{ik(\Gamma_1 + \Gamma_2)} \frac{\partial E_y}{\partial x} + \frac{1}{ik(\Gamma_1' + \Gamma_2')} Z\frac{\partial H_y}{\partial z} \tag{9}$$

$$E_z = -\frac{\Gamma_1\Gamma_2 + 1}{\Gamma_1 + \Gamma_2} ZH_x - \frac{1}{ik(\Gamma_1 + \Gamma_2)} \frac{\partial E_y}{\partial z} - \frac{1}{ik(\Gamma_1' + \Gamma_2')} Z\frac{\partial H_y}{\partial x} \tag{10}$$

on $y = 0$. We note that the relation (8) connecting the Γ_m and Γ_m' is equivalent to

$$\frac{a_1'}{a_2' + a_0'} = \frac{a_2 + a_0}{a_1} , \tag{11}$$

and is a consequence of duality.

The vector forms involving the tangential field components are the obvious vehicle for extending the generalized boundary conditions (1) to a non-planar surface. In the third order case the resulting boundary conditions can be expressed as

$$\hat{n} \times \left(\hat{n} \times \left\{ \overline{E} + \frac{1}{ik(\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 + 1)} \nabla \left[\left(\Gamma_1 + \Gamma_2 + \Gamma_3 \right) \hat{n}.\overline{E} - \frac{1}{ik} \nabla_s.\overline{E} \right] \right\} \right)$$

$$= - \frac{\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_1\Gamma_2\Gamma_3}{\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 + 1} Z \hat{n} \times \left\{ \overline{H} + \frac{1}{ik(\Gamma_1'\Gamma_2' + \Gamma_2'\Gamma_3' + \Gamma_3'\Gamma_1' + 1)} \nabla \left[\left(\Gamma_1' + \Gamma_2' + \Gamma_3' \right) \hat{n}.\overline{H} - \frac{1}{ik} \nabla_s.\overline{H} \right] \right\} \tag{12}$$

provided

$$\frac{\Gamma_1' + \Gamma_2' + \Gamma_3' + \Gamma_1'\Gamma_2'\Gamma_3'}{\Gamma_1'\Gamma_2' + \Gamma_2'\Gamma_3' + \Gamma_3'\Gamma_1' + 1} = \frac{\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 + 1}{\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_1\Gamma_2\Gamma_3} \tag{13}$$

or, equivalently,

$$\frac{a_3' + a_1'}{a_2' + a_0'} = \frac{a_2 + a_0}{a_3 + a_1} \tag{14}$$

where $\hat{n}$ is a unit vector outward normal to the surface and ∇_s is the surface divergence.

The extension of (13) and (14) to higher order boundary conditions is obvious. As is evident from (1) we can recover the second order relations (8) and (11) from (13) and (14) by letting Γ_3 and Γ_3' become infinite, and similarly for the first order relations. In comparison with the first order (standard impedance) boundary conditions, one effect of going to a higher order is to make the conditions less local through the inclusion of tangential derivatives of the fields.

4. Metal-Backed Layer

A geometry of practical interest is a uniform dielectric layer of thickness τ backed by a metal (see Figure 1), and we now seek a simulation using generalized boundary conditions of the form (1) or (2). In doing so, particular attention is given to layers of reasonable thickness since, as discussed below, approximate boundary conditions are already available for very thin coatings. For this geometry the exact reflection coefficients are known, and we can therefore derive the conditions using procedure (i). These can be referred either to the surface $y = \tau+$ of the dielectric layer or, by using the expansion (5), to the surface $y = 0+$ of the metal backing.

For an H-polarized plane wave having $E_y \neq 0$ the reflection coefficient is

$$R(\phi) = - \frac{\sqrt{N^2 - \cos^2\phi}\ \tan\left(k\tau\sqrt{N^2 - \cos^2\phi}\right) - i\varepsilon \sin\phi}{\sqrt{N^2 - \cos^2\phi}\ \tan\left(k\tau\sqrt{N^2 - \cos^2\phi}\right) + i\varepsilon \sin\phi} \qquad (15)$$

where $N = \sqrt{\varepsilon\mu}$ is the complex refractive index, ε is the relative permittivity and μ is the relative permeability of the layer. The corresponding reflection coefficient for E-polarization is

$$R'(\phi) = - \frac{\sqrt{N^2 - \cos^2\phi}\ \cot\left(k\tau\sqrt{N^2 - \cos^2\phi}\right) + i\mu \sin\phi}{\sqrt{N^2 - \cos^2\phi}\ \cot\left(k\tau\sqrt{N^2 - \cos^2\phi}\right) - i\mu \sin\phi} \qquad (16)$$

where $R(\phi)$ and $R'(\phi)$ are both referred to $y = \tau+$. Expansion of the tangent and cotangent in powers of $\sin\phi$ leads immediately to the identification of the constants a_m and a'_m, but before doing this, it is of interest to examine two boundary conditions already available in the literature.

The most commonly-used boundary condition for simulating a metal-backed layer is the standard impedance condition [7]. This is a first order one which can be derived from (15) and (16) by writing $\sqrt{N^2 - \cos^2\phi} \approx N$, giving

$$\Gamma_1 = \frac{1}{\Gamma'_1} = -i\frac{N}{\varepsilon}\tan(Nk\tau) \qquad (17)$$

with all other $\Gamma_m = 0$. For $k\tau < 0.52$ the conditions can be transferred to the surface $y = 0+$ with a maximum phase error of 10 degrees by employing the first two terms in (5). The resulting boundary conditions are then second order ones in which Γ_1 and Γ'_1 remain the same and Γ_2 and $\Gamma'_2 = (ik\tau)^{-1}$. These have been used [3] to simulate a metal-backed dielectric half-plane and since they are valid only if $|N| >> 1$, they will be referred to as high contrast conditions. Low contrast conditions can be derived by introducing the approximations $\tan x \approx x$ and $\cot x \approx 1/x$ in (15) and (16), giving

$$R(\phi) \approx -\frac{k\tau\sin^2\phi - i\varepsilon\sin\phi + k\tau(N^2 - 1)}{k\tau\sin^2\phi + i\varepsilon\sin\phi + k\tau(N^2 - 1)}$$

and

$$R'(\phi) \approx \frac{ik\tau\,\mu\sin\phi + 1}{ik\tau\,\mu\sin\phi - 1} \quad ,$$

from which the constants a_m and a'_m are easily found. The corresponding boundary conditions can be transferred to the surface y = 0+ using the first two terms in the expansion (5), and when only the terms of leading order in $k\tau$ are retained, the constants are

$$\begin{aligned} a_0 &= k\tau\,(N^2 - 1) & a'_0 &= 1 \\ a_1 &= i\varepsilon & a'_1 &= -ik\tau\,(\mu - 1) \\ a_2 &= -k\tau\,(\varepsilon - 1) & a'_2 &= 0 \quad . \end{aligned} \tag{18}$$

These satisfy (11) and the resulting boundary conditions are identical to those derived by Weinstein [8] by expanding the fields in the dielectric as Taylor series in y.

The high and low contrast boundary conditions provide an accurate simulation of the coating only for limited ranges of |N| and $k\tau$, but by going to a higher order condition, it is possible to produce a simulation that is valid for all |N| and a wider range of $k\tau$. To this end we write

$$\sqrt{N^2 - \cos^2\phi} \approx N - \frac{1}{2N} + \frac{\sin^2\phi}{2N} \quad .$$

By also employing the approximation

$$\tan\left(\frac{k\tau}{2N}\sin^2\phi\right) \approx \frac{k\tau}{2N}\sin^2\phi \quad ,$$

$R(\phi)$ can be expressed as

$$R(\phi) = -\frac{\sum_{m=0}^{4}(-1)^m a_m \sin^m\phi}{\sum_{m=0}^{4} a_m \sin^m\phi} \tag{19}$$

where

$$a_0 = (N - \frac{1}{2N})\left[\tan(k\tau N) - \tan(\frac{k\tau}{2N})\right]$$

$$a_1 = i\varepsilon\left[1 + \tan(k\tau N)\tan(\frac{k\tau}{2N})\right]$$

$$a_2 = \frac{1}{2N}\left\{\tan(k\tau N) - \tan(\frac{k\tau}{2N}) + k\tau(N - \frac{1}{2N})\left[1 + \tan(k\tau N)\tan(\frac{k\tau}{2N})\right]\right\} \tag{20}$$

$$a_3 = -\frac{ik\tau\varepsilon}{2N}\left[\tan(k\tau N) - \tan(\frac{k\tau}{2N})\right]$$

$$a_4 = \frac{k\tau}{4N^2}\left[1 + \tan(k\tau N)\tan(\frac{k\tau}{2N})\right] \quad .$$

Similarly, for E-polarization,

$$a_0' = (2N^2 - 1)\left[1 + \cot(k\tau N)\cot(\frac{k\tau}{2N})\right]$$

$$a_1' = i\,2N\mu\left[\cot(k\tau N) - \cot(\frac{k\tau}{2N})\right]$$

$$a_2' = 1 + \cot(k\tau N)\cot\left(\frac{k\tau}{2N}\right) + k\tau\left(N - \frac{1}{2N}\right)\left[\cot(k\tau N) - \cot\left(\frac{k\tau}{2N}\right)\right] \qquad (21)$$

$$a_3' = -i\,k\tau\mu\left[1 + \cot(k\tau N)\cot\left(\frac{k\tau}{2N}\right)\right]$$

$$a_4' = \frac{k\tau}{2N}\left[\cot(k\tau N) - \cot\left(\frac{k\tau}{2N}\right)\right] .$$

The boundary conditions implied by (20) and (21) are referred to the surface $y = \tau+$ and are fourth order ones which satisfy duality. As expected, their accuracy improves with increasing $|N|/(k\tau)$ and in Figure 2 it is shown that they predict the correct reflection coefficient for $\varepsilon = 4$, $\mu = 1$ and $\tau = 0.1\lambda$, i.e., $Nk\tau = 1.26$. When we set $a_3 = a_4 = 0$ and $a_3' = a_4' = 0$ the resulting boundary conditions are second order ones. As evident from Figure 2, their accuracy is substantially less, and the standard impedance (first order) conditions are valid only for incidence close to grazing or normal.

The accuracy of the fourth order boundary conditions is quite remarkable even for fairly thick coatings. Since the conditions correctly predict the dominant surface wave modes (see comment below), their accuracy is greatest near grazing incidence, but even at other angles the phase error is less than 2 degrees with coatings at least $\lambda/4$ thick regardless of the material properties and polarization. This is illustrated in Figures 3 and 4 where, for $\varepsilon = 2$ and 7 with μ real, the maximum layer thickness for a 2-degree phase error is shown as a function of $|N|$ for $\phi = 90$ and 45 degrees, respectively. In all cases the amplitude error is less than two percent, and if ε

and/or μ is complex, the accuracy is even better. This is expected since any loss diminishes the effect of the deeper parts of the layer. For the standard impedance boundary condition, results analogous to those in Figure 4 are shown in Figure 5. Even with the allowed phase error increased to 10 degrees, the maximum layer thickness is substantially less except at normal incidence, particularly if |N| is small. In addition, the accuracy of this boundary condition decreases as grazing incidence is approached.

Higher order boundary conditions for a metal-backed layer can also be derived using procedure (ii) of Section 2, which requires a knowledge of the complex poles of the reflection coefficients (15) and (16). The subset lying in the proper half of the complex plane are the usual surface wave poles, and the implied expansions of the reflection coefficients are

$$R(\phi) = -\sum_{m=0}^{M} \frac{\sin\theta_m - \sin\phi}{\sin\theta_m + \sin\phi}$$

$$R'(\phi) = -\sum_{m=0}^{M'} \frac{\sin\theta'_m - \sin\phi}{\sin\theta'_m + \sin\phi} \qquad (22)$$

where $f_m = -ik\sin\theta_m$ and $f'_m = -ik\sin\theta'_m$ are the propagation constants for the H-polarized (TM) and E-polarized (TE) surface waves, respectively, provided $\mathrm{Re.}\,(f_m, f'_m) > 0$.

It has been observed that the accuracy of the approximate boundary conditions deteriorates significantly as ϕ decreases unless the correct dominant surface wave fields are predicted, and the boundary conditions implied by (20) and (21) must therefore correspond to the correct surface wave poles. The number of poles (or zeros) of the reflection coefficients (15) and (16) depends on the value of Re. ($k\tau N$). In the TM case with $k\tau N$ small, only one pole exists corresponding to the lowest order surface wave mode. As $k\tau N$ increases, two additional poles appear, one of which is associated with a surface wave mode when $u > \pi$ where $u^2 = (k\tau)^2 (N^2 - 1) - \sin^2 \phi$. In the TE case, no poles exist for small $k\tau N$, but as $k\tau N$ increases, two poles appear simultaneously, and one of these is associated with the lowest order TE surface wave mode when $u > \pi/2$. An examination of the accuracy of (22) showed that they provide a good simulation of the reflection coefficients for those values of $k\tau N$ such that the corresponding poles with ($\sin \theta_m$, $\sin \theta_m'$ not much greater than unity) have just appeared; however, the accuracy decreases substantially as $k\tau N$ increases beyond the mid point between the last included pole and the next one to appear.

5. Concluding Remarks

As illustrated in the case of a metal-backed uniform dielectric layer, the generalized boundary conditions (1) provide an excellent simulation of the scattering

properties of the structure for thicknesses up to $\lambda/4$ or more for all material properties, directions of incidence and polarizations. For any other structure whose reflection coefficient is known analytically, the derivation of the appropriate boundary conditions can be carried out in a similar manner and their accuracy quantified. Alternatively, if only computed or measured data for the reflection coefficient are available, the required boundary conditions can be found by curve fitting.

For a boundary condition of any given order, the accuracy achieved depends on the location of the surface where the boundary condition is applied, and one advantage of the method we have followed is that the location is treated separately. In the example discussed, the simulating sheet was placed at the upper surface of the dielectric, and a fourth order boundary condition was found to produce excellent results. For other locations, it is necessary to expand the phase factor as indicated in (5), leading to additional derivative factors in the boundary conditions. The optimum location minimizes the complexity of the boundary conditions and/or the error in simulation, and the choice of another location could limit the accuracy achievable. If, in the above application, the simulating sheet were placed at the metal backing, the maximum layer thickness that can be reasonably handled is of order 0.1λ. With the sheet located in this manner, the result of coating the metal is simply to replace the perfectly conducting boundary condition with the appropriate generalized one. It is then only a trivial extension to simulate a metal coated on both sides.

Provided the constants Γ_m and Γ'_m (or a_m and a'_m) satisfy a duality relation (see, for example, (13) or (14)) the boundary condition can be expressed in terms of

the tangential field components. This shows that one effect of the higher order derivatives in (1) is to make the boundary conditions less local in character, and the resulting "vector" conditions provide the natural extension of (1) to non-planar surfaces.

In spite of the apparent complication of the generalized boundary conditions and their vector equivalents, it has been found possible to work with them numerically and analytically. Regardless of the order of the conditions, a sheet subject to them supports only tangential electric and magnetic currents. Thus, in a numerical solution of a scattering problem, the number of unknowns is the same as for the standard impedance (first order) boundary condition, and we have already begun to incorporate the higher order conditions into our existing sheet scattering codes employing either the moment or conjugate gradient FFT methods. Analytically, it is important to be able to determine the diffraction coefficient for the edge of a half plane or wedge subject to these boundary conditions, and thereby extend the capability of GTD scattering codes. In the case of a half plane we can use either the Maliuzhinets or Wiener-Hopf techniques and [3] is a example of the application of the former. Provided care is taken to ensure that all Fourier transforms exist in the classical sense, the Wiener-Hopf method can handle generalized boundary conditions of any order. The split functions that occur are the same as for a simple impedance boundary condition, with each derivative factor in the boundary condition giving rise to a pair of split functions. The edge diffraction coefficient then involves a product of these functions.

The proposed boundary conditions have many possible practical applications. In analytical studies of the scattering from junctions and edges, they can

be used to model single or multi-layered dielectric slabs or coatings. In numerical treatments, thicker layers can be simulated without increasing the number of unknowns at the expense of a slight increase in complexity of the integral equations. They could also be effective in modelling the dielectric layers used in printed circuit and microstrip arrays, leading to a significant simplification in the Green's functions involved, and we are now examining this approach.

References

[1] S.N. Karp and F.C. Karal, Jr., "Generalized Impedance Boundary Conditions with Applications to Surface Wave Structures," in Electromagnetic Wave Theory, Part 1, ed. J. Brown, pp. 479-483, Pergamon: New York, 1965.

[2] B. Engquist and A. Majda, "Absorbing Boundary Conditions for the Numerical Simulation of Waves," Math. Comput., vol. 31, pp. 1-24, 1977.

[3] J.L. Volakis and T.B.A. Senior, "Applications of a Class of Generalized Boundary Conditions to Scattering by a Metal-Backed Dielectric Half-Plane," submitted to Proc. IEEE.

[4] T.B.A. Senior and J.L. Volakis, "Sheet Simulation of a Thin Dielectric Layer," Radio Sci., vol. 22, pp. 1261-1272, 1987.

[5] T.B.A. Senior, "Some Problems Involving Imperfect Half-Planes," in Electromagnetic Scattering, ed. P.L.E. Uslenghi, pp. 185-219, Academic Press: New York, 1978.

[6] T.B.A. Senior, "Impedance Boundary Conditions for Imperfectly Conducting Surfaces," Appl. Sci. Res., vol. 8(B), pp. 418-436, 1960.

[7] T.B.A. Senior, "Approximate Boundary Conditions," IEEE Trans. Antennas Propagat., vol. AP-29, pp. 826-829, 1981.

[8] A.L. Weinstein, The Theory of Diffraction and the Factorization Method, Golem Press: Boulder, CO., 1969.

Legends for Figures

Fig. 1: Metal-backed dielectric layer.

Fig. 2: Reflection coefficient phase vs ϕ for a metal-backed dielectric layer of thickness $\lambda/10$ having $\varepsilon = 4$ and $\mu = 1$. Comparison based on 4th, 2nd and 1st order boundary conditions: (a) H-polarization, (b) E-polarization.

Fig. 3: Maximum allowed thickness vs |N| for a metal-backed dielectric layer modelled using the 4th order boundary conditions at $y = \tau+$, with a 2-degree phase (and/or 2 percent amplitude) error. Curves shown are for $\varepsilon = 2$ and $\varepsilon = 7$ with $\phi = 90$ degrees. Results are indistinguishable for H- and E-polarizations.

Fig. 4: Maximum allowed thickness vs |N| for a metal-backed dielectric layer modelled using the 4th order boundary conditions at $y = \tau+$, with a 2-degree phase (and/or 2 percent amplitude) error. Curves shown are for $\varepsilon = 2$ and $\varepsilon = 7$ with $\phi = 45$ degrees: (a) H-polarization, (b) E-polarization.

Fig. 5: Maximum allowed thickness vs |N| for a metal-backed dielectric layer modelled using the first order (standard impedance) boundary conditions at $y = \tau+$ with 2-degree and 10-degree phase errors. Curves shown are for $\varepsilon = 2$ and $\varepsilon = 7$ with $\phi = 45$ degrees: (a) H-polarization, (b) E-polarization.

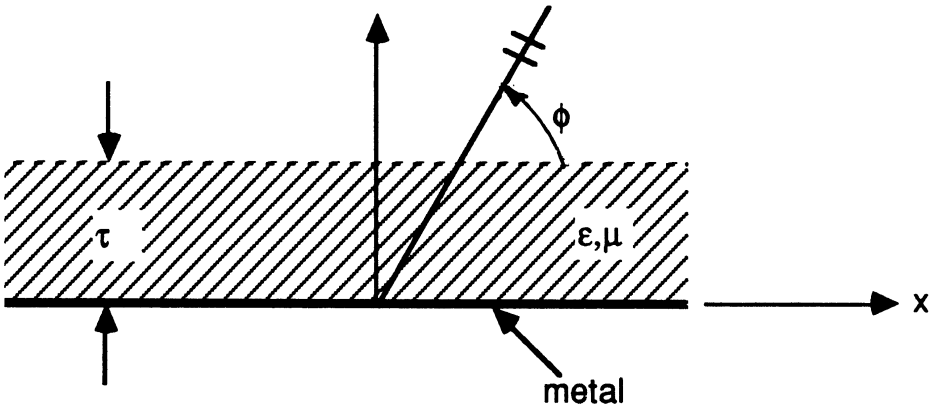
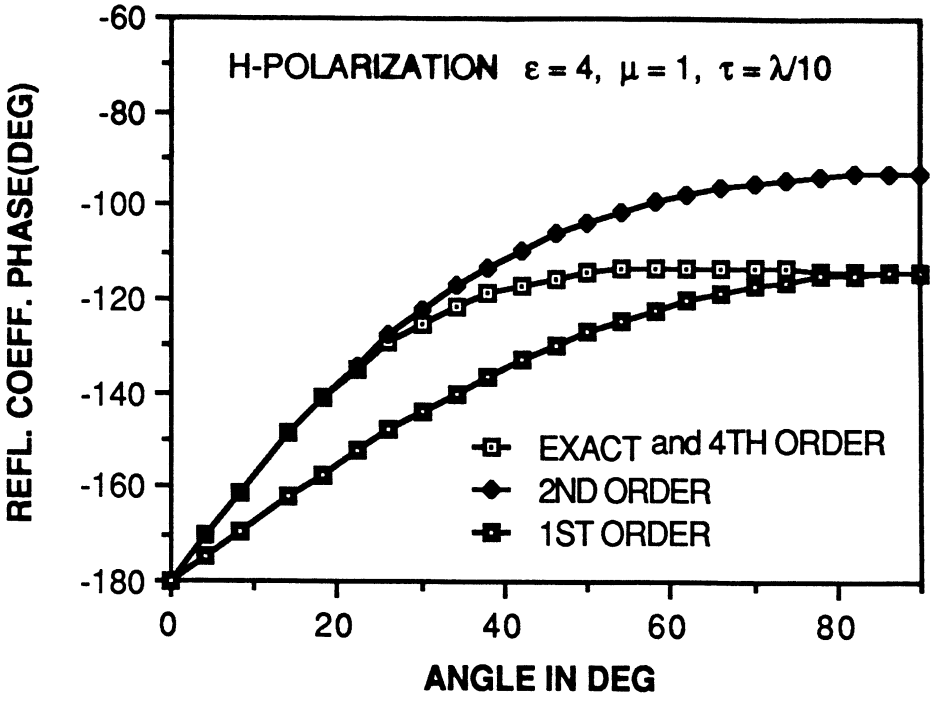
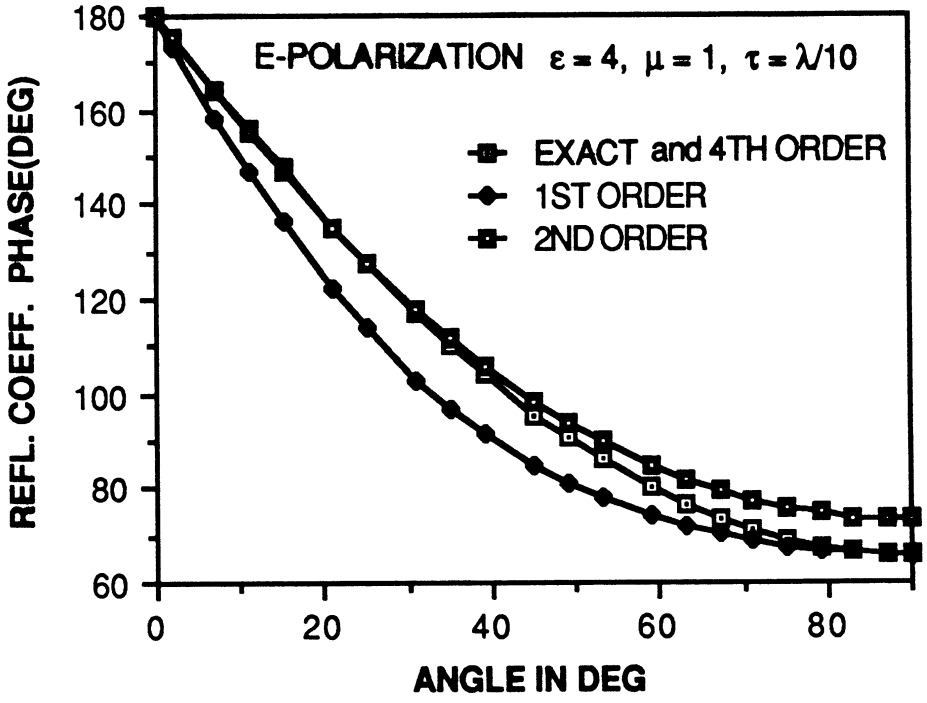


Figure 1


H-POLARIZATION ε = 4, μ = 1, τ = λ/10
REFL. COEFF. PHASE(DEG)
-60
-80
-100
-120
-140
-160
-180
0
20
40
60
80
ANGLE IN DEG
EXACT and 4TH ORDER
2ND ORDER
1ST ORDER

(a)


E-POLARIZATION ε = 4, μ = 1, τ = λ/10
REFL. COEFF. PHASE(DEG)
180
160
140
120
100
80
60
0
20
40
60
80
ANGLE IN DEG
EXACT and 4TH ORDER
1ST ORDER
2ND ORDER

(b)

Figure 2

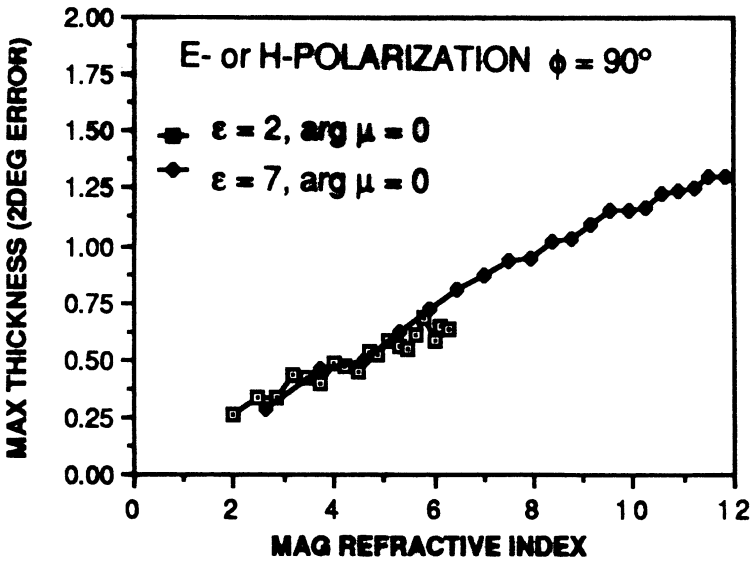
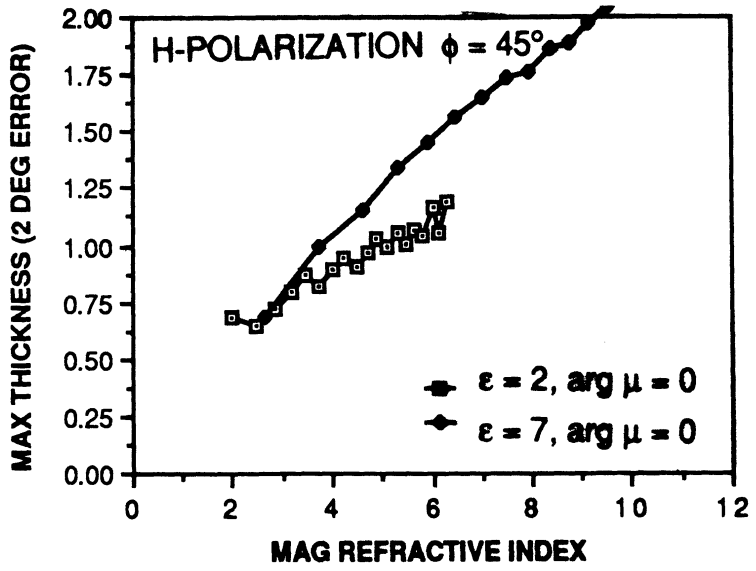
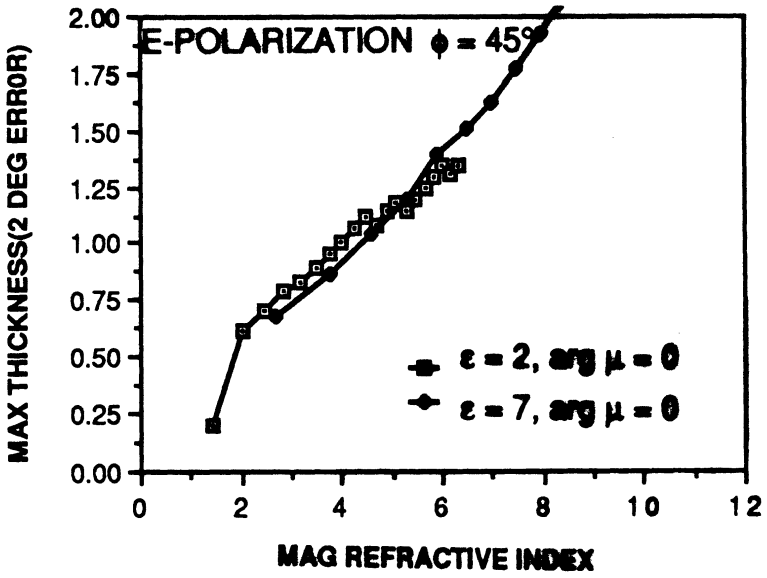

E- or H-POLARIZATION ϕ = 90°
ε = 2, arg μ = 0
ε = 7, arg μ = 0
MAX THICKNESS (2DEG ERROR)
MAG REFRACTIVE INDEX
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
2
4
6
8
10
12

Figure 3

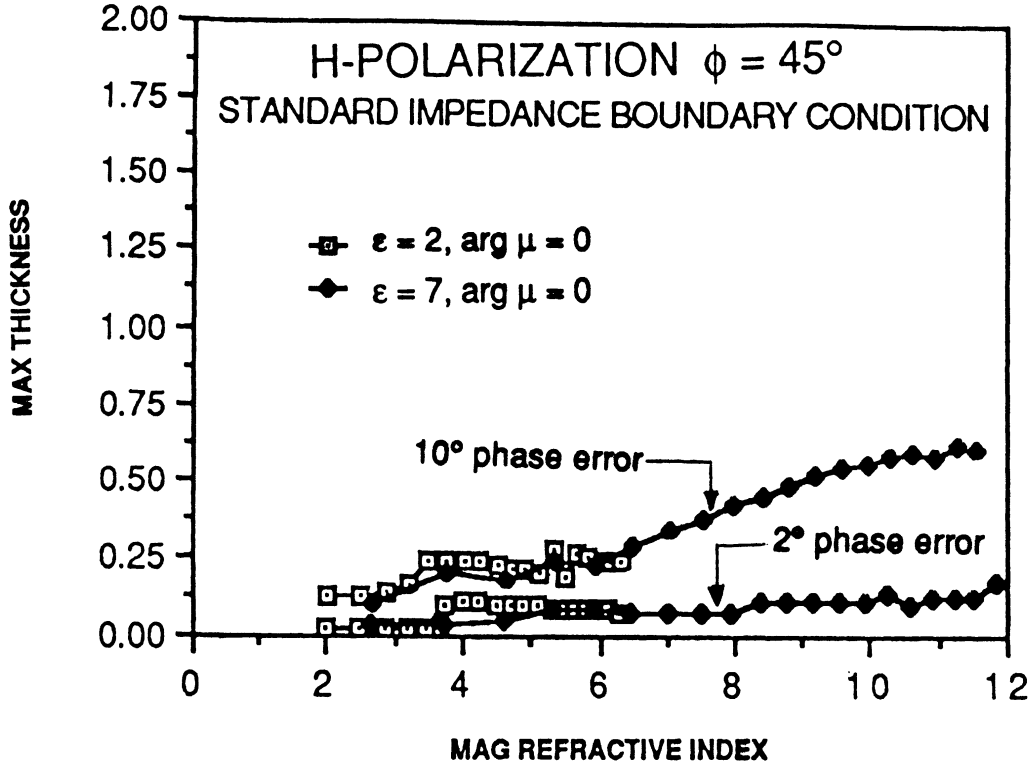

H-POLARIZATION ϕ = 45°
MAX THICKNESS (2 DEG ERROR)
MAG REFRACTIVE INDEX
ε = 2, arg μ = 0
ε = 7, arg μ = 0
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
2
4
6
8
10
12

(a)

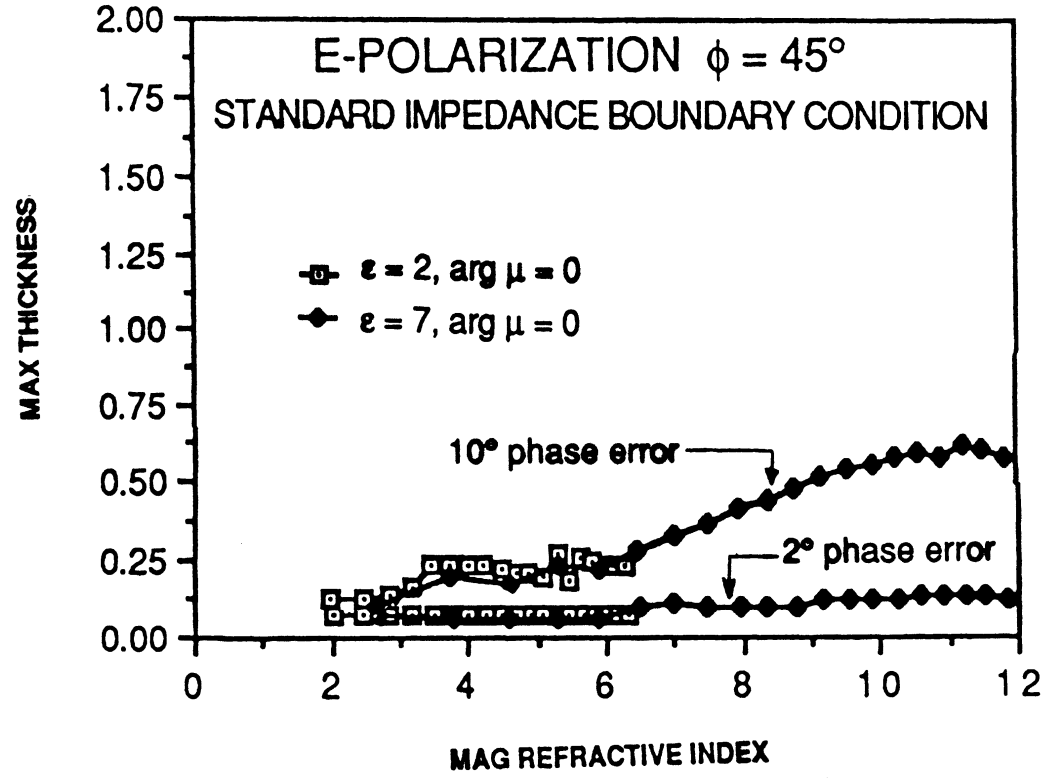

E-POLARIZATION ϕ = 45°
MAX THICKNESS(2 DEG ERR0R)
MAG REFRACTIVE INDEX
ε = 2, arg μ = 0
ε = 7, arg μ = 0
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
2
4
6
8
10
12

(b)

Figure 4


H-POLARIZATION ϕ = 45°
STANDARD IMPEDANCE BOUNDARY CONDITION
ε = 2, arg μ = 0
ε = 7, arg μ = 0
10° phase error
2° phase error
MAX THICKNESS
MAG REFRACTIVE INDEX
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
2
4
6
8
10
12

(a)


E-POLARIZATION ϕ = 45°
STANDARD IMPEDANCE BOUNDARY CONDITION
ε = 2, arg μ = 0
ε = 7, arg μ = 0
10° phase error
2° phase error
MAX THICKNESS
MAG REFRACTIVE INDEX
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
2
4
6
8
10
12

(b)

Figure 5