Flux Attractors and Generating Functions

by

Ross C. O'Connell

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Doctoral Committee:

Associate Professor Finn Larsen, Chair
Professor Daniel M. Burns Jr.
Professor David W. Gerdes
Associate Professor James T. Liu
Associate Professor Leopoldo A. Pando-Zayas

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ABSTRACT

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We use the flux attractor equations to study IIB supergravity compactifications with 3-form fluxes. We show that the attractor equations determine not just the values of the complex structure moduli and the axio-dilaton, but also the masses of those moduli and the gravitino. We then show that the flux attractor equations can be recast in terms of derivatives of a single generating function. A simple expression is given for this generating function in terms of the D3 tadpole and gravitino mass, with both quantities considered as functions of the fluxes. For a simple prepotential, we explicitly solve the attractor equations. We also discuss a thermodynamic interpretation of this generating function, and possible implications for the landscape.

Having solved the flux attractor equations for 3-form fluxes, we add generalized fluxes to the compactifications and study their effects. We find that when we add only geometric fluxes, the compactifications retain their no-scale structure, and minimize their scalar potential when the appropriate complex flux is imaginary self-dual (ISD). These minima are still described by a set of flux attractor equations, which can be integrated by a generating function. The expressions for the vector moduli are formally identical to the case with 3-form fluxes only, while some of the hypermoduli are determined by extremizing the generating function. We work out several orbifold examples where all vector moduli and many hypermoduli are stabilized, with VEVs given explicitly in terms of fluxes.

CHAPTER I

Introduction

String theory first received significant attention because it is a consistent quantum theory of gravity, and because it naturally incorporates gauge fields and matter as well as gravity. The development of string theory has led to significant discoveries in pure mathematics, as well as insights into the dynamics of strongly coupled gauge theories (including both QCD and condensed matter systems). While these discoveries alone ensure that string theory will continue to be an active area of research, they do not require that string theory describe fundamental interactions in our universe. In this thesis we develop machinery that may help determine whether string theory is a viable "theory of everything."

One notoriously awkward aspect of string theory is that the simplest backgrounds all have ten dimensions, six more than it seems reasonable to expect. A traditional way to solve this problem is to replace six of the extended dimensions with a compact geometry, and study when this compact space can be made small (of order the Planck length). There is then a correspondence between deformations of the compact space and massless fields in the four remaining extended dimensions, known as "moduli." Unfortunately, compact spaces which are in other ways experimentally viable tend to have hundreds of these moduli, in stark contrast to observations.

In this thesis we will investigate how the addition of background field strengths along the compact space, known as “fluxes,” to a compactification can stabilize at least some of the moduli. While the idea of using fluxes to stabilize moduli is not a new one, we will introduce a new tool, the “flux attractor equations,” which reduce the stabilization conditions to a relatively simple set of algebraic conditions. The attractor equations make the role of individual fluxes in determining the vacuum expectation values (VEVs) and masses of the moduli more transparent. They will also reveal a surprise: the moduli VEVs and mass parameters can be written as derivatives of a simple generating function, whose form is determined by the compactification geometry. This generating function also hints at a hidden set of microstates associated with individual sets of fluxes, much like the microstates that correspond to a single set of black hole charges.

1.1 Basic Ingredients

String theory has a number of different weakly-coupled descriptions. Although they are all interconnected by a series of dualities, it is usually the case that some phenomena are much more easily studied with one weakly-coupled description than with another. This is the case with flux compactifications. In this thesis, we will focus our attention on type IIA and type IIB strings, rather than heterotic strings. We prefer type II strings for two reasons. First, when compactified on Calabi-Yau manifolds they lead to a wide variety of 4D theories with $\mathcal{N} = 2$ supersymmetry. These theories have richly structured moduli spaces, and so are interesting to study, but we will find that the structure imposed by $\mathcal{N} = 2$ supersymmetry also makes the study of these theories tractable. This will be true even when we deform the Calabi-Yau, breaking some or all of the supersymmetry. Second, type II theories admit a wider variety of fluxes than heterotic theories do, making them a natural venue for the study of flux compactifications.

So long as we probe these theories at energies well below the string scale, we can study these theories using point particle actions rather than by using the full machinery of string theory. The type IIA and IIB supergravities have $\mathcal{N} = 2$ supersymmetry in 10 dimensions. Their spectra both include a graviton g_{MN}, as well as a scalar field known as the dilaton ϕ. Both theories also include an antisymmetric tensor field, B_{MN}, which can be considered as a generalization of the photon. Like the photon, it enters the action through its field strength, H_{MNP}.

Both IIA and IIB have a number of other antisymmetric tensor fields in their spectra. IIA contains fields with an odd number of indices, while IIB contains fields with an even number of indices. Like B_{MN} or the photon, they enter the action through their field strengths. Because antisymmetric tensors are naturally identified with differential forms, we will usually call the antisymmetric tensor fields "form fields" for short. All of the fields described so far as massless. Their actions are available in [3].

Both theories also contain extended, charged objects known as "D-branes." Just as the photon naturally couples to point particles, the form fields naturally couple to D-branes of the appropriate dimensionality. For example, D2 branes, which have two spatial dimensions and one time dimension, couple to a field with three indices, which in turn enters the action through a field strength with four indices. IIA therefore contains D-branes with an even number of spatial dimensions, while IIB contains D-branes with an odd number of spatial dimensions. In addition, both theories contain fundamental strings and their magnetic duals, NS5 branes.

We have described the field content of IIA and IIB strings in the supergravity limit, and in 10 flat dimensions. Let us now consider replacing six of those dimensions with a compact space. So long as the volume of the space is large, relative to the string scale, we can still describe the 10D dynamics with supergravity. So long as the volume of the space is small, relative to the energy scales of interest, we can ignore excitations on the compact space, and arrive at a 4D effective description of the theory. The 4D theory will generically include a large number of moduli. We describe the reduction in detail in chapter II, but for now we describe just the moduli arising from deformations of the compact space.

Suppose we have found a stable geometry for the compact space, identified by a metric g_{mn}. If we deform this metric by replacing $g_{mn} \to g_{mn} + \delta g_{mn}$, then the action will either increase (e.g. because δg_{mn} makes an additional contribution to the Ricci scalar) or be unchanged. If it is unchanged, we consider the metric to be a function of the modulus, $g_{mn}(z)$, such that a small change in the modulus z produces the small change in the metric δg_{mn}. Each modulus is associated with a massless field in the 4D theory.

These massless scalar fields are not consistent with observation, so we hope that there is a way to give sufficiently large masses to the moduli. Our basic strategy is to give these moduli masses by turning on the field strengths associated with form fields. The standard kinetic terms for these fields in a 10D Lagrangian are the natural generalization of the Lagrangian for electromagnetism:

$$\int F_n \wedge *_{10} F_n = \int d^{10}x \frac{\sqrt{g}}{n!} g^{M_1 N_1} \cdots g^{M_n N_n} F_{M_1 \ldots M_n} F_{N_1 \ldots N_n} \,. \tag{1.1}$$

Suppose that we turn on a background value for the field F_n. In order to preserve 4D Lorentz invariance, the indices of the background field strength must run over the compact space only, not the four extended directions. Additional restrictions are discussed in chapter II. However, once we have turned on a background value for F_n, (1.1) will become a scalar potential for the moduli, because the inverse metric used in (1.1) depends on the moduli. Although (1.1) may seem rather inscrutable, at least when considered as a scalar potential, in this thesis we will show that it can conceal a great deal of structure.

1.2 The Formal Problem of Moduli Stabilization

One can view moduli stabilization as part of the effort to construct solutions of string theory that resemble our universe. From this point of view the obvious problem with moduli is that we observe no massless scalars, and so any realistic solution of string theory must include a mechanism which gives them masses. In this thesis we take a more restrictive point of view: we will focus specifically on the stabilization of *bulk* moduli by *classical* effects. We now explain why we make these restrictions.

1.2.1 Bulk Effects and the Cosmological Constant

While string theory provides a consistent quantum theory of gravity, the fact that it naturally describes physics at the Planck scale (i.e. at energies 15 orders of magnitude above those that can be achieved at the LHC) makes it difficult to determine if it is the *correct* quantum theory of gravity. This has led many researchers to look for circumstantial evidence in favor of string theory, by trying to embed the standard model of particle physics in string theory. In particular, one would like to recover the standard model gauge groups $SU(3) \times SU(2) \times U(1)_Y$. One hopes that when fully realized, these constructions will be sufficiently restrictive to make predictions about physics above the weak scale.

Most attempts to embed the standard model in string theory involve a six-dimensional compact space, but do not require detailed knowledge of the full geometry. Instead, they only involve physics in the vicinity of some sort of defect. If we start from Type IIA or IIB string theory, these defects might be the locations of D-branes, or intersections of D-branes, since these naturally support non-abelian gauge groups. In constructions that begin from M- or F-theory (11- and 12-dimensional constructions that contain IIA and IIB strings, respectively), the gauge groups arise from geometric singularities [4–6]. Many of these models include standard model-like matter as well – F-theory models have recently been developed that have a realistic flavor hierarchy [7] – so these "local models" present a fruitful line of research. However, we will pursue a different direction in this thesis.

While local models may eventually give us important insights into both the standard model and particle physics beyond the standard model, they provide little insight into one of the great problems of contemporary physics: the unnaturally small value of the cosmological constant, which is 120 orders of magnitude smaller than would be predicted by effective field theory [8, 9]. A simple argument, due to Bousso and Polchinski [10], suggests that knowledge of the entire compactification geometry, or the "bulk physics," will be crucial in unravelling this puzzle.

The Bousso-Polchinski argument can be broken into two parts. First, they point out that scalar potentials derived from string theory tend to have a large number of discrete parameters – in the cases we study, these will be fluxes. Instantons exist that permit tunneling from one set of parameters to another. In a universe with positive cosmological constant (like ours), this is sufficient to drive eternal inflation [11–13]. That is, bubbles with different parameters, and a smaller cosmological constant, will nucleate inside of patches with larger cosmological constants. Over time, we expect all values of those discrete parameters to be explored. If the spacing between possible values of the cosmological constant is relatively uniform and sufficiently small (less than 10^{-123} in Planck units), then we expect to generate some bubbles with roughly the observed value of the cosmological constant. Given a plausible mechanism to *generate* bubbles with such a tiny value for the cosmological constant, Weinberg's anthropic arguments [14, 15] explain why we must live in such a bubble.

Bousso and Polchinski next argue that sufficiently small spacings could be achieved just by increasing the number of different cycles (roughly "handles") on the compact space, and thus the number of different fluxes. They found that the spacing between successive values of the cosmological constant scaled as

$$\delta\Lambda \sim e^{-n/2}\,, \tag{1.2}$$

where n is the number of cycles. Although it is difficult to say what a "natural" geometry for the compact space might be, many geometries that have been studied for other reasons have $n \sim \mathcal{O}\,(100)$, and so would have sufficiently fine spacing to generate the observed cosmological constant via eternal inflation.

Unfortunately, this scaling was derived by assuming that the compactification geometry, not just the compactification topology, was left unchanged by the changes in the fluxes. No known compactifications have this feature – instead, there is a strong correlation between the number of moduli, i.e. the number of ways that the compactification geometry can change, and the number of fluxes. This does not necessarily work in favor of the Bousso-Polchinski argument – in the examples considered in this thesis, the moduli always adjust so that the scalar potential is zero and the cosmological constant does not vary at all between different sets of fluxes. In other words, the effects of the bulk moduli are not negligible, and must be correctly understood in order to address the cosmological constant, at least if one is to do so in the context of eternal inflation. We remain optimistic that in the future we will be able to extend the flux attractor approach to vacua with a non-zero cosmological constant

1.2.2 Quantum Corrections and the Dine-Seiberg Problem

Our concern with moduli stabilization by classical effects may seem quaint, given that moduli are in general not protected by any symmetries and so may have masses induced entirely by quantum effects. The basic problem, originally noticed by Dine and Seiberg [16], is that many of the moduli that we study are also control parameters: a scalar known as the dilaton ϕ controls the string coupling $g_s = e^{-\phi}$, the compactification volume V_6 controls the masses of excitations on the compact space, which we have left out of our 4D theory, and many other moduli control the effects of instantons. Moreover, the regime where we have good analytic control is where all of these moduli are large.

Let us consider a single modulus, the dilaton ϕ. Because the perturbative regime is $\phi \to \infty$, quantum corrections must vanish in this limit. As we move in from $\phi \to \infty$, the potential generated by the leading correction can be either positive or negative. If it is positive, the potential pushes $\phi \to \infty$, where its mass goes to zero. If the leading term in the potential is negative, it pulls ϕ in toward zero, where perturbation theory breaks down. We might hope to construct a potential with a local minimum by balancing two or three terms at different orders against each other, but this would imply that higher order corrections were of the same size as lower order corrections – in other words, that our perturbation series had broken down. Although this argument is somewhat heuristic, we regard it as good motivation to focus on stabilization by classical effects.

1.3 Attractor Equations

Our study of moduli stabilization will make use of a new tool: the "flux attractor equations" [1, 2, 17–20]. They allow us to rewrite the stabilization conditions for the moduli as a relatively simple set of algebraic equations. These stabilization conditions are related to those that arise in the study of extremal (i.e. zero temperature) black holes in 4D, $\mathcal{N} = 2$ supergravity, as we now describe.

The supergravities of interest include three kinds of supermultiplets, whose bosonic components we now describe. Hypermultiplets include a pair of complex scalars, vector multiplets contain a vector A^i_μ and a complex scalar z^i, and the gravity multiplet contains the graviton and a single vector, A^0_μ. The index i runs from 1 to n, and we will introduce an index $I = 0, ..., n$ which runs over all of the vectors. This theory is "ungauged," so the gauge group is $U(1)^{n+1}$, and there is no potential for the scalars.

Because there is no potential for the scalars in a vacuum, there is also no potential for them infinitely far away from a black hole, so the scalars can be set to arbitrary values. However, as one moves in toward the black hole horizon, an effective potential for the scalars z^i is induced[1]. The change in the values of the scalars between spatial infinity and the horizon can be formulated as a radial *attractor flow*, with the scalars eventually taking values at the horizon (the end of the flow) that are completely independent of their values at spatial infinity. The horizon values of the scalars are determined in terms of the electric and magnetic charges of the black hole, e_I and m^I, via the *attractor equations*,

$$m^I = \mathrm{Re}\left[CZ^I\left(z^i\right)\right], \tag{1.3}$$

$$e_I = \mathrm{Re}\left[CF_I\left(z^i\right)\right]. \tag{1.4}$$

The Z^I and F_I are holomorphic functions of the scalars z^i, and will be discussed more extensively in chapter II. Together, (1.3) and (1.4) are $2(n+1)$ real equations for $n+1$ complex variables[2], the z^i and C. When the 4D theory arises from IIB strings compactified on a Calabi-Yau manifold, and the black hole is composed of wrapped D3 branes, we can write the attractor equations in a more compact form:

$$F_3 = \mathrm{Re}\left[C\Omega_3\right], \tag{1.5}$$

where Ω_3 is the holomorphic 3-form of the Calabi-Yau, and F_3 describes the world-volumes of the wrapped D3 branes, and thus the charges of the black hole.

[1]Note that the scalars are not charged under any of the $U(1)$'s. The effective potential arises because the scalars appear in the kinetic terms for the vectors, $\int \mathcal{M}_{IJ}\left(z^i\right) g^{\mu\nu} g^{\rho\sigma} F^I_{\mu\rho} F^J_{\nu\sigma}$, through the metric $\mathcal{M}_{IJ}\left(z^i\right)$.

[2]In fact, C can only be considered as an independent observable after a Kähler gauge is fixed. For more details see chapter III.

While no notion of attractor *flow* has been developed for flux compactifications, the VEVs of the moduli are determined by equations which strongly resemble the black hole attractor equations. In the simplest context there are only twice as many fluxes as would appear for a corresponding black hole, and the attractor equations are

$$
\begin{aligned}
m_h^I &= \mathrm{Re}\left[CZ^I + L^I\right], && (1.6)\\
m_f^I &= \mathrm{Re}\left[\tau CZ^I + \overline{\tau} L^I\right], && (1.7)\\
e_I^h &= \mathrm{Re}\left[CF_I + L^J F_{IJ}\right], && (1.8)\\
e_I^f &= \mathrm{Re}\left[\tau CF_I + \overline{\tau} L^J F_{IJ}\right], && (1.9)\\
0 &= L^I\left(\overline{F}_I - \overline{Z}^J F_{IJ}\right). && (1.10)
\end{aligned}
$$

Again, Z^I, F_I, and F_{IJ} are holomorphic functions of the z^i. The real black hole charges $\{m^I, e_I\}$ are replaced by the real fluxes $\left\{m_h^I, m_f^I, e_I^h, e_I^f\right\}$. An additional complex scalar, the axio-dilaton τ appears, as do additional mass parameters L^I, which can be thought of as generalizations of the parameter C. As with the black hole attractor equations, we can rewrite (1.6)-(1.10) in terms of 3-forms,

$$
\begin{aligned}
G_3 \equiv F_3 - \tau H_3 &= \overline{C\Omega}_3 + L^I \partial_I \Omega_3 \,, && (1.11)\\
0 &= L^I \int (\partial_I \Omega_3) \wedge \overline{\Omega}_3 \,, && (1.12)
\end{aligned}
$$

where the components of F_3 are the $\left\{m_f^I, e_I^f\right\}$ and the components of H_3 are $\left\{m_h^I, e_I^h\right\}$.

Altogether, (1.6)-(1.10) constitute $4n+6$ real equations for $2n+3$ complex parameters, so that in general all of the moduli and mass parameters are determined by the fluxes. Rather than dwell on the context in which these equations appear (explained in chapter II) or their derivation (performed in chapter III), for now we just point out the formal similarity between (1.3), (1.4) and (1.6)-(1.9).

Given that the attractor equations (1.3), (1.4) and (1.6)-(1.9) take similar forms, it is not entirely surprising that they have similar solutions. As originally noted by [21], solutions of the black hole attractor equations can be written in terms of a generating function,

$$CZ^I = m^I + \frac{i}{\pi}\frac{\partial}{\partial e_I} S_{\mathrm{BH}}\left(e_I, m^I\right), \tag{1.13}$$

where the generating function S_{BH} happens to be the entropy of the black hole. As we will demonstrate in chapter III, solutions of the flux attractor equations can be written as

$$CZ^I = \frac{1}{\tau - \overline{\tau}}\left[\left(m_f^I - \overline{\tau} m_h^I\right) + \left(\frac{\partial}{\partial e_I^h} + \overline{\tau}\frac{\partial}{\partial e_I^f}\right)\mathcal{G}\left(e_I^h, e_I^f, m_h^I, m_f^I\right)\right], \tag{1.14}$$

$$L^I = \frac{1}{\tau - \overline{\tau}}\left[\left(m_f^I - \tau m_h^I\right) - \left(\frac{\partial}{\partial e_I^h} + \tau\frac{\partial}{\partial e_I^f}\right)\mathcal{G}\left(e_I^h, e_I^f, m_h^I, m_f^I\right)\right]. \tag{1.15}$$

In both cases it is quite striking that the solutions can be written in terms of derivatives of a single function.

We know that the black hole entropy does more than encode the near-horizon properties of the black hole – it also counts the number of microstates that correspond to a given set of black hole charges. It is tempting to speculate that the generating function $\mathcal{G}$ also counts microstates, which now correspond to a given set of fluxes. Since string theory is replete with examples of branes "dissolving" into flux, it seems plausible that a sufficiently complicated collection of fluxes would support as many microstates as a similarly complicated collection of D-branes. While we will not pursue this line of inquiry further in this thesis, it is possible that the study of moduli stabilization will teach us something new about the degrees of freedom in string theory.

1.4 Organization of the Thesis

We begin in chapter II by providing a brief review of moduli and moduli stabilization in Type II compactifications. After providing a general discussion of the moduli that arise from form fields, we look at the $\mathcal{N} = 2$ theories that arise from compactifying IIA and IIB strings on Calabi-Yau manifolds. After examining the consistency conditions for the addition of fluxes to these compactifications, we will find that an additional ingredient is required: orientifold planes. Because of this we will compute the spectra of the $\mathcal{N} = 1$ theories that arise from compactifying IIB strings on Calabi-Yau orientifolds. We then introduce fluxes, and find that they generate a superpotential in the 4D, $\mathcal{N} = 1$ theory. The primary results of this chapter are expressions for the Kähler potential and superpotential of the 4D theory in terms of geometric objects defined on the Calabi-Yau.

In chapter III we carefully analyze the 4D, $\mathcal{N} = 1$ theories introduced in chapter II. In particular, we will derive a set of flux attractor equations that describe the stabilized values of the moduli. The basic observation is that the holomorphic 3-form Ω_3 and its derivatives provide a natural, complex basis of cohomology elements on which to expand the complex flux $G_3 \equiv F_3 - \tau H_3$. The flux attractor equations arise from imposing the stabilization conditions on G_3 when it is expanded on the complex basis. We then show that solutions to the attractor equations can be written in terms of a single generating function, as promised above, and find a general expression for the generating function:

$$\mathcal{G} = \int F_3 \wedge H_3 - 2\mathrm{Vol}^2 m_{3/2}^2 \,, \tag{1.16}$$

where the gravitino mass $m_{3/2}^2$ is considered as a function of the fluxes. After solving the attractor equations on a simple orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with eight distinct fluxes, we compute the generating function, and verify that its derivatives correctly reproduce the moduli VEVs. This chapter was originally published as [1], and is based on research conducted in collaboration with Finn Larsen.

In chapter IV we will make a first attempt at stabilizing hypermoduli, scalar fields descended from $\mathcal{N} = 2$ hypermultiplets, with fluxes. Hypermoduli are notoriously difficult to stabilize with fluxes, both in the standard GKP setup and in IIA compactifications [22]. After analyzing the scalar potential induced by a completely general set of fluxes, we find that we can add *geometric flux* to O3/O7 compactifications, stabilize some of the hypermoduli, and still have a well-formed set of attractor equations. Moreover, we find that the solutions of these attractor equations are intimately related to the solutions of flux attractor equations *without* geometric flux. This enables us to define a more general generating function that depends on the geometric fluxes in addition to the 3-form fluxes. In order to demonstrate the utility of the attractor equations, we solve a simple example with 14 distinct fluxes turned on. This chapter is based on [2], and is based on research conducted in collaboration with Finn Larsen and Daniel Robbins.

CHAPTER II

Moduli Stabilization and Fluxes

There is a large literature devoted to the study of flux compactifications, which investigates a variety of effects (classical and quantum mechanical, perturbative and non-perturbative) in a variety of contexts (IIB, IIA, or heterotic strings on geometric and non-geometric backgrounds) using a variety of techniques (worldsheet analysis, 10D supergravity, 4D effective theories). Many authors have produced useful reviews of the subject [23–29]. In this chapter we provide a very brief review of moduli stabilization by classical effects in the best studied class of compactifications: IIB strings on Calabi-Yau O3/O7 orientifolds, with 3-form flux.

We begin by reviewing the compactification of antisymmetric tensor or "form" fields, a family of fields that includes scalars, vector fields, and higher-dimensional generalizations of vector fields. Upon compactification, these fields lead to a rich spectrum of lower-dimensional fields. Compactification also allows us to add a variety of classical backgrounds, often called "fluxes," for these fields. We will show that the spectrum of moduli and allowed fluxes is essentially determined by the *cohomology* of the compact space.

Additional moduli are derived from deformations of the compact space itself. To illustrate this, we study the moduli spaces of Calabi-Yau manifolds, following the original analysis of [30]. In fact, the moduli space for Calabi-Yau manifolds factorizes into a space of Kähler deformations, and a space of complex structure deformations. Once again, the cohomology of the Calabi-Yau plays an important role, determining the dimension of these spaces. The detailed structure of each of these spaces is determined by two functions, known as prepotentials. We will illustrate how these geometric properties are reflected in the 4D effective theory by determining the spectra of the $\mathcal{N} = 2$ supergravities that result from compactifying IIA and IIB strings on a given Calabi-Yau, as well as the metric on moduli space.

While ordinary compactifications of type II strings on Calabi-Yau manifolds have many beautiful properties, they typically include a large number of unstabilized moduli. These moduli *cannot* be consistently stabilized by fluxes. In section 2.3 we will first review the construction of Giddings, Kachru, and Polchinski (GKP), who used a 10D supergravity analysis to show that one *can* consistently use fluxes to stabilize moduli when the compactification includes orientifold planes. We will then review the 4D, $\mathcal{N} = 1$ theories that are derived from Calabi-Yau orientifolds. The orientifold projection removes many moduli from the theory. The addition of fluxes then generates a superpotential that can stabilize both the remaining complex structure moduli z^i and the axio-dilaton τ.

Throughout this chapter I will assume familiarity with differential forms and the usual operators that act on them: the exterior derivative d, Hodge star $*$, and the wedge product $\wedge$. Wherever possible I will include qualitative descriptions of the results as well as explicit calculations, but differential forms are the natural language in which to discuss flux compactifications. Useful introductions to this formalism can be found in [31–33].

2.1 Compactification and Form Fields

Before considering the compactification of more complicated fields, let us consider a 10D, massless scalar field, ϕ_{10}. We assume that the 10D metric is now a product of four extended dimensions and six compact dimensions, and that the 10D Laplacian splits into a piece associated with the extended dimensions and a piece associated with the compact directions, $\nabla^2_{10} = \nabla^2_4 + \nabla^2_6$. We can then expand ϕ_{10} in terms of eigenfunctions of ∇^2_6,

$$\phi_{10} = \phi^a_4(x)\,\omega_a(y)\ , \tag{2.1}$$

$$\nabla^2_6\omega_a(y) = m^2_a\omega_a(y)\,. \tag{2.2}$$

The 10D equation of motion $\nabla^2_{10}\phi_{10} = 0$ then implies that

$$\left(\nabla^2_4 + m^2_a\right)\phi^a_4 = 0.$$

In other words, a single 10D, massless scalar gives rise to a tower of 4D scalars. The two eigenvalues of greatest interest are the lowest one, which is zero and corresponds to a constant function on the 6D space, and the second-lowest one, which is typically set by the size of the compact space[1], $m^2 \sim \ell_6^{-2}$. When the compact space is small relative to the 4D scale of interest, we can restrict our attention to the zero mode only, and ignore the effects of the higher modes altogether. The scalar field is just one example of a larger family of fields, which we will call "form fields." A similar story can be developed for this larger family, as we will now demonstrate.

2.1.1 Form Fields in Flat Space

As discussed in the introduction, both type IIA and IIB string theory have, in their 10D effective actions, terms of the form

$$\int F_n \wedge *_{10}F_n = \int d^{10}x\frac{\sqrt{g}}{n!}g^{M_1N_1}\cdots g^{M_nN_n}F_{M_1\ldots M_n}F_{N_1\ldots N_n}\,. \tag{2.3}$$

[1]In warped compactifications the effects of warping can dramatically reduce this mass scale.

These terms are generalizations of the usual kinetic term for electromagnetism. The field strength tensor for electromagnetism has two indices, and is antisymmetric in them. These $n-$form field strengths have n indices, in which they are also antisymmetric. These kinetic terms lead to the equation of motion[2]

$$d *_{10} F_n = 0\,, \tag{2.4}$$

and the Bianchi identity

$$dF_n = 0\,. \tag{2.5}$$

These are generalizations of the source-free Maxwell equations. In flat space, the solution of the Bianchi identity is

$$F_n = dC_{n-1}\,, \tag{2.6}$$

where the potential C_{n-1} is a generalization of the photon $A = A_\mu dx^\mu$. Note that the gauge transformation

$$C_{n-1} \rightarrow C_{n-1} + d\Lambda_{n-2} \tag{2.7}$$

leaves the field strength unchanged, so that potentials that differ by an exact form are physically equivalent.

2.1.2 Form Fields in Compact Spaces

Let us now consider form fields on a product of four extended dimensions and six compact dimensions. Several new features will appear, all because compact spaces typically support a number of non-trivial "harmonic" forms, i.e. forms that satisfy

$$d\omega = d *_6 \omega = 0\,, \tag{2.8}$$

[2]Note that $*_{10} F_n \sim F_{10-n}$, so that we can take either F_n or F_{10-n} to be the independent field strength. 5-form fluxes are a special case, and must satisfy the constraint $F_5 = *_{10} F_5$.

but cannot be written as $\omega = d\lambda$ or $\omega = *_6 d *_6 \lambda$. These harmonic forms are zero-modes of the 6D Laplacian (for $p-$forms), and can be thought of as forming a vector space, and the dimension of the space of harmonic $p-$forms is the pth Betti number, b_p.

The existence of these harmonic forms implies that we can modify the expansion (2.6), while still satisfying (2.4) and (2.5), by adding a harmonic term:

$$F_n = dC_{n-1} + F_n^{\text{fl}}\,. \tag{2.9}$$

The harmonic term F_n^{fl} can be expanded on a basis of harmonic forms, and we will refer to the resulting coefficients as “fluxes.” In theories with gravity one must take care when adding them, since the gravitational backreaction of the fluxes will alter the compactification geometry. We will review this backreaction in section 2.3.1. However, because the number of harmonic forms of a given degree is a topological invariant, the heuristic analysis given here correctly determines the number of different fluxes.

We also add harmonic terms to C_n. Because they are not of the form $d\Lambda$, they are not gauge transformations. We can then expand C_n much as we did ϕ. We keep only the zero modes, finding

$$C_n = c^a_{\mu\nu} dx^\mu \wedge dx^\nu \wedge h_a^{(n-2)} + c^I_\mu dx^\mu \wedge h_I^{(n-1)} + c_a h^a_{(n)}\,, \tag{2.10}$$

where the $h_a^{(n-2)}$ are a basis for the harmonic $(n-2)$-forms, the $h_I^{(n-1)}$ are a basis for the harmonic $(n-1)$-forms, and the $h^a_{(n)}$ are a basis for the harmonic n-forms. We have assumed that $n \geq 2$ – when $n = 1$ the first term does not appear. While the scalar Laplacian has a unique zero-mode, there are in general many harmonic forms, so that upon compactification C_n leads to many massless 4D fields. The c_a are 4D scalars, while the c^I_μ are 4D vectors. The $c^a_{\mu\nu}$ can be dualized to scalars: they enter the 4D Lagrangian via 3-form field strengths, which can be dualized to 1-form field strengths, which are in turn generated by scalar (0-form) fields. The 4D equations of motion for all of these fields are derived from the equation of motion for the 10D field strength, $d *_{10} F_n = 0$.

We emphasize an important distinction between fluxes and moduli. While they may seem quite similar from the 6D perspective, since they both result from expansion on harmonic forms, they play quite different roles in the 4D theory. Suppose that we multiplied the flux contribution F_n^{fl} by a function $f(x^\mu)$. The Bianchi identity $dF_n = 0$ would imply that $f(x^\mu)$ must be constant. In other words, the fluxes cannot vary as a function of the 4D coordinates, and are not associated with a dynamical 4D field. Instead, they are fixed *input* parameters. On the other hand, the moduli are dynamical, so they will roll to a minimum of the potential determined by the fluxes, even if they are initially displaced. For this reason, we consider the moduli VEVs to be *outputs*, which we compute as functions of the fluxes.

We have seen that harmonic forms play a key role in the analysis of form fields in flux compactifications. In the following the simple picture we have presented will be complicated by a number of factors: in addition to the effects of backreaction mentioned above, the 10D Lagrangians for IIA and IIB strings include Chern-Simons terms with the $p-$form potentials, and the effects of fluxes will give masses to fields which were, in this analysis, massless. All the same, the harmonic forms will continue to determine the spectrum of light fields and the fluxes that may stabilize them.

2.2 Calabi-Yau Manifolds and $\mathcal{N} = 2$ Compactifications

In addition to the form field moduli discussed in the previous section, most compactifications admit metric moduli: deformations of the compact geometry which correspond to scalar fields in the 4D theory. In this section we will introduce a relatively simple family of compactification geometries, the Calabi-Yau manifolds, and study their associated metric moduli. We will then see how they combine with the form field moduli of the IIA and IIB theories into 4D $\mathcal{N} = 2$ supermultiplets. While the number of multiplets will be determined by the number of harmonic 2- and 3-forms, we will see that the resulting moduli spaces enjoy additional structure, known as special geometry, which allows the moduli space of a Calabi-Yau manifold to be expressed in terms of two scalar functions, the prepotentials.

2.2.1 Important Properties of Calabi-Yau Manifolds

We will start from the usual definition of a Calabi-Yau manifold [34].

Definition: A Calabi-Yau n-fold is a compact, Kähler manifold with n complex dimensions, and vanishing first Chern class.

While the condition on the first Chern class is usually easiest to check when one is constructing new Calabi-Yaus, it implies several other important results. Let us review each of them in turn.

2.2.1.1 The Ricci Flat Metric

Each Calabi-Yau manifold admits a unique, Ricci flat metric. Among other things, this means that the curvature of the Calabi-Yau does not contribute to the 4D cosmological constant, since its contribution would be $\int d^6y\sqrt{g_{\mathrm{CY}}}R = 0$. Calabi-Yau manifolds are therefore naturally associated with compactifications to Minkowski space.

2.2.1.2 Reduced Holonomy and Supersymmetry

One of the most important properties of Calabi-Yau manifolds is that their holonomy group is always contained within $SU(n)$. For example, the holonomy group for a Calabi-Yau 3-fold is reduced from $SO(6) \approx SU(4)$ to $SU(3)$. This implies that spinors on the Calabi-Yau can be decomposed in representations of $SU(n)$ – in the

3-fold case, the decomposition[3] is simply $\mathbf{4} = \mathbf{3} + \mathbf{1}$. The spinor in the singlet is globally well-defined and non-vanishing. It is also covariantly constant.

When the Calabi-Yau has full $SU(n)$ holonomy, the covariantly constant spinor is unique. For compactifications from 10 to 4 dimensions, each 10D supersymmetry condition then reduces to a single 4D supersymmetry condition, so compactifying theories with 10D, $\mathcal{N} = 2$ supersymmetry (e.g. Type IIA and IIB strings in 10 flat dimensions) leads to 4D effective theories with $\mathcal{N} = 2$ supersymmetry. Note that the total amount of supersymmetry is reduced, since two 10D spinors have more degrees of freedom than two 4D spinors. Indeed, Type II string theory in 10D has 32 supersymmetries, while $\mathcal{N} = 2$ in 4D corresponds to only 8 supersymmetries.

2.2.1.3 The Hodge Diamond

In section 2.1.2 we discussed the importance of harmonic forms in analyzing general compactifications. The set of harmonic forms on a Calabi-Yau is constrained in a variety of ways. First and foremost, Calabi-Yau manifolds are by definition Kähler, and so complex. We can use the complex structure to divide the six real coordinates into three holomorphic coordinates and three anti-holomorphic coordinates. We can then choose a basis of harmonic forms that have specific numbers of holomorphic and antiholomorphic indices. For example, the harmonic 2-forms can be split into $(2, 0)$ forms (two holomorphic indices, no anti-holomorphic indices), $(1, 1)$ forms, and $(0, 2)$ forms.

We already introduced the Betti numbers, which count the number of independent harmonic $p-$forms. We can refine these to count the number of independent (p, q) forms, defining the Hodge numbers $h^{(p,q)}$. Note that one can recover the Betti numbers

[3]Calabi-Yau 3-folds whose holonomy is a proper subgroup of $SU(3)$ yield more than one spinor in a singlet, and compactifications on these 3-folds have $\mathcal{N} > 2$ supersymmetry. These Calabi-Yaus are products of K3 factors and/or tori. We consider these compactifications to be "exceptions," since there are relatively few of them compared to the number of Calabi-Yaus with full $SU(3)$ holonomy, and they will not be the primary focus of this thesis.

$$\begin{array}{ccccccc} & & & h^{(0,0)} & & & \\ & & h^{(1,0)} & & h^{(0,1)} & & \\ & h^{(2,0)} & & h^{(1,1)} & & h^{(2,0)} & \\ h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\ & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\ & & h^{(3,2)} & & h^{(2,3)} & & \\ & & & h^{(3,3)} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Figure 2.1: The Hodge diamond for a Calabi-Yau 3-fold with full $SU(n)$ holonomy.

from the Hodge numbers via $b_p = \sum_q h^{(p-q,q)}$. Calabi-Yau 3-folds with full $SU(3)$ holonomy have very few independent Hodge numbers. Complex conjugation tells us that $h^{(p,q)} = h^{(q,p)}$, while Poincaré duality implies that $h^{(p,q)} = h^{(3-q,3-p)}$. They have a unique, globally-defined, harmonic, holomorphic 3-form, which implies that $h^{(3,0)} = 1$. Full $SU(3)$ holonomy implies that $h^{(1,0)} = h^{(2,0)} = 0$. Finally, if the Calabi-Yau is connected then the volume form is unique and $h^{(3,3)} = 1$. The only free quantities are then $h^{(1,1)}$ and $h^{(2,1)}$, which will appear frequently in the following. All of these results about the Hodge numbers are neatly summarized in the "Hodge diamond," presented in Figure 2.1.

In the following we will often want to expand various fields on a basis of harmonic forms. We introduce such a basis of real forms: ω_a for the $(1,1)$-forms, $\widetilde{\omega}^a$ for the $(2,2)$-forms, and $\left\{\alpha_I, \beta^I\right\}$ for the 3-forms, with $a = 1, ..., h^{(1,1)}$ and $I = 0, ..., h^{(2,1)}$. We will assume that they satisfy the orthogonality conditions

$$\int \omega_a \wedge \widetilde{\omega}^b = \delta_a^b\,, \tag{2.11}$$

$$\int \alpha_I \wedge \beta^J = \delta_I^J\,. \tag{2.12}$$

We also define the triple intersection numbers,

$$\kappa_{abc} \equiv \int \omega_a \wedge \omega_b \wedge \omega_c \, . \tag{2.13}$$

2.2.2 Deformations and Metric Moduli

As we mentioned in the introduction, certain deformations of the compact space correspond to massless 4D scalar fields. On Calabi-Yau manifolds, we look for deformations of the metric and complex structure which leave the metric Ricci flat. We can use the complex structure to separate these deformations into two types[4], $\delta g_{m\overline{n}}$ and δg_{mn}, $\delta g_{\overline{mn}}$, with the former corresponding to deformations of the metric and the latter corresponding to deformations of the complex structure. It was demonstrated in [30] that rather than directly studying these deformations of the metric and complex structure, it is equivalent to study the deformations of two geometric objects defined on the Calabi-Yau: The Kähler (2-)form J and the holomorphic 3-form Ω_3. Metric deformations alter the Kähler form, leading to "Kähler moduli," while complex structure deformations alter Ω_3, leading to "complex structure moduli."

Deformations of the complex structure mix holomorphic and antiholomorphic coordinates, and cause the holomorphic 3-form Ω_3 to mix with the $(2,1)$ forms. This implies that there are $h^{(2,1)}$ complex structure moduli z^i, $i = 1, ..., h^{(2,1)}$,and that

$$\frac{\partial}{\partial z^i}\Omega_3 = k_i \Omega_3 + \chi_i^{(2,1)} \, . \tag{2.14}$$

The Kähler moduli t^a arise by expanding the Kähler form on the basis of $(1,1)$-

[4]While the metric can always be written in terms of real coordinates and indices g_{cd}, we can write the metric of a Kähler manifold with complex coordinates and indices, $g_{m\overline{n}}$. Since the complex structure is used to convert the real coordinates to complex ones, $g_{m\overline{n}}$ contains information about both the real metric and the complex structure. Deformations of the form $\delta g_{m\overline{n}}$ correspond to deformations of the underlying real metric, while deformations of the form δg_{mn} and $\delta g_{\overline{mn}}$ correspond to deformations of the complex structure.

forms:

$$J = v^a \omega_a \,. \tag{2.15}$$

Clearly this yields $h^{(1,1)}$ real Kähler moduli. It is customary to combine these with the fields that arise in expanding the 2-form B_2,

$$B_2 = b^0_{\mu\nu} dx^\mu \wedge dx^\nu + b^a \omega_a \,, \tag{2.16}$$

to yield $h^{(1,1)}$ complexified Kähler moduli,

$$t^a = b^a + i v^a \,. \tag{2.17}$$

The additional field $b^0_{\mu\nu}$ can be dualized to a scalar as well. Since it is clearly associated with the extended directions and not the Calabi-Yau itself, we will discuss it in the next section.

2.2.3 Form Fields and $\mathcal{N} = 2$ Multiplets

We have claimed that Calabi-Yau compactifications of IIA and IIB strings lead to 4D $\mathcal{N} = 2$ supergravities. These theories admit three kinds of supermultiplets, which we describe in terms of their bosonic components. The unique gravity multiplet contains the graviton $g_{\mu\nu}$ and a single vector, the graviphoton A^0_μ. Vector multiplets contain a vector and a single complex scalar, while hypermultiplets contain two complex scalars. In this section we will show how the form field moduli and metric moduli for both IIA and IIB compactifications can be organized into $\mathcal{N} = 2$ multiplets.

There is one subtlety in the analysis of form fields that we have not yet addressed. While it is true that in the supergravity limit IIA strings include odd form fields and IIB strings include even form fields, it is not true that all of these fields contain independent degrees of freedom. Considering once more the usual kinetic term, $\int F_n \wedge$

$*_{10}F_n$, we see that we can easily exchange the roles of F_n and $*_{10}F_n \sim F_{10-n}$, in what is known as an electric-magnetic duality. These field strengths are generated by C_n and C_{8-n}, which apparently encode the same degrees of freedom. We will treat C_n for $n \leq 4$ as independent, in order to be sure that we do not double count any moduli.

2.2.3.1 IIA Multiplets

In IIA we have two form fields to expand, C_1 and C_3. Because there are no 1-forms on a Calabi-Yau (with full $SU(3)$ holonomy), C_1 generates a single 4D vector,

$$C_1 = c^0_\mu dx^\mu \,. \tag{2.18}$$

The expansion of C_3 is more involved,

$$C_3 = c^a_\mu \omega_a \wedge dx^\mu + c^I \alpha_I - c_I \beta^I \,. \tag{2.19}$$

This contributes $h^{(1,1)}$ vectors c^a_μ and $2\left(h^{(2,1)}+1\right)$ real scalars, c^I and c_I.

It appears that c^0_μ is the graviphoton, while c^a_μ and the Kähler moduli t^a assemble into $h^{(1,1)}$ vector multiplets. If we take $h^{(2,1)}$ of the pairs $\{c^i, c_i\}$, we can group them with the complex structure moduli z^i to form $h^{(2,1)}$ hypermultiplets. The remaining pair $\{c^0, c_0\}$ combine with the dilaton ϕ and $b^0_{\mu\nu}$ to form a final hypermultiplet, for a total of $h^{(2,1)}+1$ hypermultiplets. Although $b^0_{\mu\nu}$ has two more indices than we might expect on a scalar field, it enters the 4D Lagrangian via a 3-form field strength $h^0_{\mu\nu\rho}$, which is Poincaré dual to a 1-form field strength, which in turn is just the gradient of a scalar.

2.2.3.2 IIB Multiplets

The IIB spectrum includes a scalar C_0, which leads to one 4D scalar. There is also a 2-form C_2 which is expanded as

$$C_2 = c_{\mu\nu} dx^\mu \wedge dx^\nu + c^a \omega_a \,, \tag{2.20}$$

like B_2, and the $c_{\mu\nu}$ component can be dualized to a scalar, like $b^0_{\mu\nu}$.

There is also an RR 4-form C_4, which we can expand as

$$C_4 = u^a_{\mu\nu} \omega_a \wedge dx^\mu \wedge dx^\nu + v_a \widetilde{\omega}^a + A^I_\mu \alpha_I \wedge dx^\mu - \widetilde{A}_{\mu I} \beta^I \wedge dx^\mu \,.$$

Again, the $u^a_{\mu\nu}$ can be dualized to scalars in the same way as $b^0_{\mu\nu}$ and $c_{\mu\nu}$. While this appears to give $2h^{(1,1)}$ scalars and $2\left(h^{(2,1)}+1\right)$ vectors, the field strength associated with C_4 is subject to a self-duality constraint:

$$F_5 = *F_5 \,. \tag{2.21}$$

Because of this constraint, only half of the 4D scalars $\{u^a, v_a\}$ and half of the 4D vectors $\left\{A^I_\mu, A_{\mu I}\right\}$ are independent, and C_4 contributes only $h^{(1,1)}$ scalars and $h^{(2,1)}+1$ vectors to the 4D theory.

We can combine $h^{(2,1)}$ of the vectors $\left\{A^I_\mu, A_{\mu I}\right\}$ with the complex structure moduli z^i to form $h^{(2,1)}$ $\mathcal{N}=2$ vector multiplets. The remaining vector enters the gravity multiplet. We can combine the real c^a, the surviving $h^{(1,1)}$ of $\{u^a, v_a\}$, and the complex Kähler moduli t^a to form $h^{(1,1)}$ hypermultiplets. Finally, we can combine $b_{\mu\nu}$, $c_{\mu\nu}$, C_0 and ϕ into one more hypermultiplet, for a total of $h^{(1,1)}+1$. Note that the spectrum is identical to that of a IIA compactification, except that $h^{(1,1)}$ and $h^{(2,1)}$ have been exchanged.

2.2.4 Special Geometry

So far we have determined the spectra of both IIA and IIB Calabi-Yau compactifications, and found that the topological quantities $h^{(1,1)}$ and $h^{(2,1)}$ determine them completely. However, two Calabi-Yaus can have the same Hodge numbers, yet lead to different 4D theories. This happens when the moduli spaces of the two Calabi-Yaus[5] have the same *dimension*, but different *structure.* In this section we will study the moduli space of Calabi-Yau manifolds more carefully, following [30]. We will see that the entire geometry of moduli space can be described with two holomorphic functions, known as prepotentials.

The geometry of the moduli space is encoded in its metric, so we turn out attention there. The authors of [30] demonstrated that it can be written as

$$g_{i\bar{j}} = -\frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^j}\log\left[i\int \Omega_3 \wedge \overline{\Omega}_3\right], \tag{2.22}$$

$$g_{a\bar{b}} = -\frac{\partial}{\partial t^a}\frac{\partial}{\partial \bar{t}^b}\log\left[\frac{1}{6}\int J\wedge J\wedge J\right], \tag{2.23}$$

with no mixed terms $g_{a\bar{i}}$. This indicates that the moduli space is the product of two Kähler manifolds, with Kähler potentials

$$K_z = -\log\left[i\int \Omega_3 \wedge \overline{\Omega}_3\right], \tag{2.24}$$

$$K_t = -\log\left[\int J\wedge J\wedge J\right]. \tag{2.25}$$

In fact, these Kähler potentials conceal additional structure not found in generic Kähler manifolds, in that each real Kähler potential can itself be derived from a holomorphic prepotential. Manifolds with this property are described as being *special Kähler*, or having "special geometry" [35, 36]. We first illustrate how this works for the complex structure moduli space.

[5] By the moduli space of the Calabi-Yau, we mean the space of the complexified Kähler moduli and the complex structure moduli only. This is not to be confused with the full 4D theory described in section (2.2.3), which includes additional scalars descended from the RR form fields.

We begin by expanding the holomorphic 3-form on the basis of real 3-forms,

$$\Omega_3 = Z^I \alpha_I - F_I \beta^I \,. \tag{2.26}$$

Note that there are $2\left(h^{(2,1)}+1\right)$ complex coefficients Z^I and F_I, which are holomorphic functions of the $h^{(2,1)}$ complex structure moduli z^i. An important feature of Ω_3 is that we can multiply it by any nowhere vanishing holomorphic function of the complex structure moduli and it will remain holomorphic and harmonic. Because these transformations are local on moduli space, we will often refer to them as Kähler gauge transformations. We can think of the Z^I as projective coordinates for the z^i, with $h^{(2,1)}$ combinations of them (usually Z^i/Z^0) corresponding to the z^i, and one of them (usually Z^0) determining the Kähler gauge. The F_I can then be considered as functions of the Z^I, rather than as functions of the z^i. Kähler gauge transformations send $Z^I \to e^{\lambda\left(z^i\right)} Z^I$ and $F_I \to e^{\lambda\left(z^i\right)} F_I$, indicating that the F_I are homogeneous functions of the Z^I.

We argued above that differentiating Ω_3 with respect to the complex structure moduli returns a sum of $(3,0)$ and $(2,1)$ forms. This is also true if we differentiate with respect to the Z^I. It follows that

$$0 = \int \Omega_3 \wedge \frac{\partial}{\partial Z^I} \Omega_3 \,. \tag{2.27}$$

If we write this out in terms of components, we find

$$0 = F_I - Z^J \frac{\partial}{\partial Z^I} F_J \,. \tag{2.28}$$

If we rearrange and differentiate again, we find a “curl free” condition,

$$\frac{\partial}{\partial Z^I} F_J = \frac{\partial}{\partial Z^J} F_I \,, \tag{2.29}$$

which implies that we can write the F_I themselves as the gradient of another function,

$$F_I = \frac{\partial}{\partial Z^I} F \,. \tag{2.30}$$

This new function F is the prepotential. It is holomorphic and homogeneous of degree two in the Z^I. We can contract (2.30) with Z^I and exploit the homogeneity to find an explicit expression for the prepotential,

$$F = \frac{1}{2} Z^I F_I \,. \tag{2.31}$$

As promised, we can write the Kähler potential (2.24) in terms of the prepotential as

$$K_z = -\log\left[i\left(\overline{Z}^I \frac{\partial}{\partial Z^I} F - Z^I \frac{\partial}{\partial \overline{Z}^I} \overline{F} \right) \right] . \tag{2.32}$$

Because derivatives of F with respect to the Z^I will appear frequently, we will indicate them by appending subscripts only, e.g. $F_{IJ} = \partial_I \partial_J F$.

We now wish to demonstrate that the Kähler potential K_t can be written in the same form. This is not readily apparent, given that (2.25) is not written in terms of complex fields, and does not appear to enjoy a rescaling symmetry. Let us instead begin with a proposed prepotential, holomorphic and homogeneous of degree two, and check that it reproduces (2.25). That proposed prepotential is:

$$G = -\frac{1}{6} \frac{\kappa_{abc} t^a t^b t^c}{t^0} \,, \tag{2.33}$$

where the complexified Kähler moduli t^a were introduced in (2.17). We have introduced an additional field t^0, in analogy with Z^0. After differentiating, we will set it to 1. We find

$$G_0 \equiv \frac{\partial}{\partial t^0} G = \frac{1}{6} \frac{\kappa_{abc} t^a t^b t^c}{(t^0)^2} \,, \tag{2.34}$$

$$G_a \equiv \frac{\partial}{\partial t^a} G = -\frac{1}{2} \frac{\kappa_{abc} t^b t^c}{t^0} \,. \tag{2.35}$$

A brief calculation shows that, after setting $t^0 = 1$, we have

$$\bar{t}^0 G_0 + \bar{t}^a G_a - t^0 \overline{G}_0 - t^a \overline{G}_a = \frac{4}{3} i \kappa_{abc} v^a v^b v^c = 8i \int J \wedge J \wedge J \,. \tag{2.36}$$

This demonstrates that the Kähler potential for Kähler moduli (2.25) can also be written in terms of a holomorphic prepotential, homogeneous of degree two in its arguments, just as can be done for the Kähler potential for the complex structure moduli (2.24).

2.3 Fluxes, Orientifolds, and $\mathcal{N} = 1$ Compactifications

One undesirable property of Calabi-Yau compactifications of Type II strings is that they include a preponderance of massless scalar fields, which are clearly excluded by experiment. One way to induce a potential for the moduli is to turn on n-form field strengths or "fluxes" around the compact cycles of the Calabi-Yau. Once the field strength F_n has some background value, the standard 10D kinetic term for the field strength,

$$\int F_n \wedge *_{10} F_n = \int d^{10}x \frac{\sqrt{g}}{n!} g^{m_1 p_1} \cdots g^{m_n p_n} F_{m_1 \ldots m_n} F_{p_1 \ldots p_n} \,, \tag{2.37}$$

becomes a potential for the moduli in the 4D theory, with the moduli entering through the Calabi-Yau metric. Unfortunately, fluxes alone tend to destabilize proposed compactifications by driving the compactification volume (a product of the Kähler moduli) to infinity, as we will now demonstrate. This scaling analysis does not depend on special properties of the Calabi-Yau, so we will only assume that we are compactifying on a compact six-dimensional space space X.

We would like to study this effect of the fluxes by looking at the potential in the 4D theory. However, if we start from a 10D action and simply integrate over six of the dimensions, the resulting 4D theory will have non-standard, moduli-dependent kinetic terms. For example, integrating the Einstein-Hilbert term gives

$$S_{\rm EH} = \int d^{10}x\sqrt{g^{(10)}}\frac{e^{-2\phi}}{\ell^8}R \tag{2.38}$$

$$= \int d^4x\sqrt{g_{\rm s}^{(4)}}\frac{e^{-2\phi}}{\ell^2}V_X R_{\rm s}\,, \tag{2.39}$$

where V_X is the volume of the compact space in string units, which clearly depends on the metric moduli. The metric and Ricci scalar are said to be in "string frame," since they were obtained by direct reduction of the 10D action. Analyzing the effects of the fluxes on the moduli in string frame is difficult, since both the 4D kinetic terms and the 4D scalar potential are moduli dependent.

We can recover the usual Einstein-Hilbert term from (2.39) by rescaling the metric,

$$\left(g_{\rm s}^{(4)}\right)_{\mu\nu} = \frac{e^{2\phi}}{V_X}\left(g_{\rm e}^{(4)}\right)_{\mu\nu}\,, \tag{2.40}$$

so that the kinetic term becomes

$$S_{\rm EH} = \int d^4x\sqrt{g_{\rm e}^{(4)}}\frac{1}{\ell^2}R_{\rm e}\,. \tag{2.41}$$

Since this is the standard Einstein-Hilbert term, the action is now considered to be in "Einstein frame." With the non-standard moduli dependence removed from the kinetic terms, we can now direct our attention to the scalar potential alone.

If we turn on a flux F_n on the compact space, its contribution to the Einstein frame scalar potential is

$$V_F = \frac{1}{\ell^4 V_X^2} \int_X d^6x \sqrt{g_X} g_X^{m_1 p_1} \cdots g_X^{m_n p_n} F_{m_1 \ldots m_n} F_{p_1 \ldots p_n} \,. \tag{2.42}$$

Note that we have suppressed some possible dependence on the string coupling, since we are primarily concerned with the volume modulus. Now, consider a rescaling of the metric of the compact space, $g_X \rightarrow \lambda^2 g_X$. Under such a rescaling, $\sqrt{g_X}$ and V_X scale as λ^d, and the inverse metric g_X^{mp} scales as λ^{-2}, so the scalar potential scales as λ^{-2p-6}. In other words, the scalar potential has a runaway direction, pushing the size of the compact space to infinity, spoiling any notion of "compactification." This indicates that fluxes alone cannot be used to stabilize moduli.

An intuitive understanding of this result is that increasing the volume of the Calabi-Yau reduces the energy *density* contributed by the flux, and so is favored. We might hope that we could add something else to the compactification which would favor a smaller volume. Branes are a natural choice, as their intrinsic tension might cause them to prefer small volumes. However, their Einstein frame contribution to the potential is

$$V_D = \frac{V_n}{V_X^2} T \,, \tag{2.43}$$

where T is the (positive) brane tension, and V_n is the volume of the n-cycle wrapped by the brane, again in string units. The factor of V_X^{-2} appeared during the conversion from string frame to Einstein frame. Under rescalings of the metric on the compact space, this contribution scales as λ^{n-12}. The tension of the D-brane makes the runaway less severe than it was for the fluxes, but the factor of V_X^{-2} still dominates.

We have two problems: first, none of the contributions we have considered scale with more than two powers of the volume (in string frame), and second, they are all positive. Attempts to overcome the first problem led to the no-go theorems of Maldacena and Nuñez [37]. In order to overcome the second, we must consider objects with negative tension. While these might sound problematic in pure gravity, string theory provides a consistent description of such objects, known as "orientifold planes." Their contribution to the scalar potential is essentially the same as that of a brane (2.43), but the negative tension allows us to balance them against the positive contributions from fluxes. Giddings, Kachru, and Polchinski (GKP) first demonstrated how to do this systematically [38]. We will review the details of their construction, but we can anticipate many general features from this scaling analysis. 3-form fluxes, which scale as λ^{-12}, can be balanced against O3 planes that fill the four extended dimensions and are pointlike on the Calabi-Yau, which also scale as λ^{-12}. A runaway to infinite or zero volume will only be prevented if these contributions exactly cancel one another, so stable compactifications must have $V = 0$. This will leave the volume modulus (as well as the other Kähler moduli) unstabilized, but will allow us to stabilize the complex structure moduli.

After discussing the GKP construction, which makes no detailed assumptions about the compact space, we will study O3/O7 orientifolds of Calabi-Yau compactifications. The resulting theories live in 4D Minkowski space, as is required by the GKP construction. The orientifold theory can be thought of as a projection of an $\mathcal{N} = 2$ Calabi-Yau compactification, with the orientifold reducing the amount of supersymmetry to $\mathcal{N} = 1$, as well as reducing the spectrum of massless fields. We will begin by discussing the orientifold theories without fluxes, then we will describe the superpotential that fluxes produce in the 4D $\mathcal{N} = 1$ theory.

2.3.1 GKP Analysis

The GKP construction begins with an ansatz for the 10D metric:

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}\tilde{g}_{mn}dy^m dy^n \, . \qquad (2.44)$$

Since the scale factor for the 4D Minkowski space depends on our location in the 6D space, this is referred to as a *warped product*, and $A(y)$ is referred to as the "warp factor." We make no assumptions about the metric $\tilde{g}_{mn}$ of the unwarped 6D space, but instead assume that the four extended directions have the geometry of Minkowski space, not de Sitter or anti de Sitter space.

We will assume that the 6D space is large compared to the string scale, so that the IIB supergravity action[6]

$$\begin{aligned} S_{\mathrm{IIB}} &= \frac{1}{2\ell^8}\int d^{10}x\sqrt{-g}\left\{R - \frac{\partial\tau\cdot\partial\overline{\tau}}{2\mathrm{Im}\,(\tau)^2} - \frac{G_3\cdot\overline{G}_3}{12\mathrm{Im}\,(\tau)} - \frac{F_5\cdot F_5}{4\cdot 5!}\right\} \\ &\qquad + \frac{1}{8i\ell^8}\int\frac{1}{\mathrm{Im}\,(\tau)}C_4\wedge G_3\wedge\overline{G}_3 + S_{\mathrm{loc}}\,, \end{aligned} \tag{2.45}$$

provides a reliable effective description of the 10D massless fields, as well as local objects such as branes or orientifold planes contained in S_{loc}. Many of these fields have been redefined, relative to the conventions of [3], as

$$g_{\mu\nu}^{\mathrm{here}} = e^{-\phi/2}g_{\mu\nu}^{\mathrm{there}}\,, \tag{2.46}$$

$$\tau = C_0 + ie^{-\phi}\,, \tag{2.47}$$

$$G_3 = F_3 - \tau H_3\,, \tag{2.48}$$

$$F_5^{\mathrm{here}} = F_5^{\mathrm{there}} - \frac{1}{2}C_2^{\mathrm{there}}\wedge H_3^{\mathrm{there}} + \frac{1}{2}B_2^{\mathrm{there}}\wedge F_3^{\mathrm{there}}\,. \tag{2.49}$$

The conventions used here make manifest the $SL(2,\mathbb{Z})$ action of S-duality on this

[6]Strictly speaking this is not a complete action, since it does not reproduce the constraint $F_5 = *_{10}F_5$. In fact a complete action does exist for IIB supergravity, as demonstrated in [39, 40], but the "action" presented here will be sufficient for our analysis.

action:

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \tag{2.50}$$

$$\begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_3 \\ H_3 \end{pmatrix}, \tag{2.51}$$

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \tag{2.52}$$

$$G_3 \rightarrow \frac{G_3}{c\tau + d}, \tag{2.53}$$

where $\{a, b, c, d\}$ are integers satisfying $ad - bc = 1$.

As usual, the action (2.45) must be supplemented by a constraint:

$$F_5 = *_{10} F_5 \,. \tag{2.54}$$

The only solution consistent with Poincaré invariance is

$$F_5 = (1 + *_{10})\, d\alpha\,(y) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \,, \tag{2.55}$$

for some scalar function $\alpha\,(y)$ on the compact space.

We begin by studying a subset of the equations of motion for the theory, the trace-reversed Einstein equations for the four extended dimensions:

$$R_{\mu\nu} = \ell^8 \left(T_{\mu\nu} - \frac{1}{8} g_{\mu\nu} T \right), \tag{2.56}$$

where $T_{\mu\nu}$ is the stress-tensor, and T is its trace. Both receive contributions from the usual supergravity fields, as well as from as-yet-unspecified localized sources:

$$T^{\text{loc}}_{MN} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{loc}}}{\delta g^{MN}} \,. \tag{2.57}$$

We can compute the Ricci tensor from the metric ansatz (2.44), which gives

$$R_{\mu\nu} = -\frac{1}{4}\left[\widetilde{\nabla}^2 e^{4A} - e^{-6A} g^{mn}\left(\partial_m e^{4A}\right)\left(\partial_n e^{4A}\right)\right], \tag{2.58}$$

where $\widetilde{\nabla}^2$ is the Laplace-Beltrami operator computed using the *unwarped* metric $\tilde{g}_{mn}$. Computing the stress tensor gives

$$\ell^8\left(T_{\mu\nu} - \frac{1}{8}g_{\mu\nu}T\right) = -g_{\mu\nu}\left[\frac{G_{mnp}\overline{G}^{mnp}}{48\mathrm{Im}\,(\tau)} + \frac{e^{-8A}}{4}g^{mn}\left(\partial_m\alpha\right)\left(\partial_n\alpha\right)\right] + \ell^8\left(T^{\mathrm{loc}}_{\mu\nu} - \frac{1}{8}g_{\mu\nu}T^{\mathrm{loc}}\right), \tag{2.59}$$

where the contractions of the 3-form are performed with the *warped* metric g_{mn}. Substituting (2.58) and (2.59) into (2.56) and tracing gives

$$\begin{aligned}\widetilde{\nabla}^2 e^{4A} &= e^{2A}\frac{G_{mnp}\overline{G}^{mnp}}{12\mathrm{Im}\,(\tau)} + e^{-6A}g^{mn}\left[\left(\partial_m\alpha\right)\left(\partial_n\alpha\right) + \left(\partial_m e^{4A}\right)\left(\partial_n e^{4A}\right)\right] \\ &\quad + \frac{\ell^8}{2}\left(T^m_m - T^\mu_\mu\right)_{\mathrm{loc}},\end{aligned} \tag{2.60}$$

where again all contractions on the righthand side are performed with the warped metric.

If we integrate (2.60) over the compact space the lefthand side gives zero, while the first two terms on the righthand side are positive semi-definite. We therefore conclude that in the absence of localized sources, fluxes cannot be included and the warp factor $A(y)$ must be constant. If localized sources are to be included, they must make a negative semi-definite contribution to the third term. When their contribution is negative, fluxes and warping are both permitted. As our canonical localized source, we consider a brane or orientifold plane with tension T_p, filling the four extended dimensions and wrapping a $p-3$ cycle Σ on the compact space. Its contribution to (2.60) is then

$$\left(T^m_m - T^\mu_\mu\right)_{\mathrm{loc}} = (7-p)\,T_p\delta(\sigma)\ . \tag{2.61}$$

For $p < 7$, we see that negative tension objects, such as orientifold-planes, are required. We can also freely add D7 branes and O7 planes.

Let us now examine the Bianchi identity,

$$dF_5 = H_3 \wedge F_3 + 2\ell^8 T_3 \rho_3 \,. \tag{2.62}$$

We see that the Bianchi identity is modified from the usual $dF_5 = 0$. First, the Chern-Simons term in (2.45) leads to the $H_3 \wedge F_3$ term. In the second term we include the sources of D3 charge, which include both D3 branes and O3 planes, as well as D7 branes and O7 planes, which induce D3 charge when they wrap 4-cycles. Note that ρ_3 is product of the volume form of the Calabi-Yau and delta functions which describe the location of the various sources. If we integrate (2.62) over the compact space, we find

$$\frac{1}{2\ell^8 T_3} \int H_3 \wedge F_3 = -Q_3 \,, \tag{2.63}$$

a condition known as the "tadpole constraint." This indicates that the fluxes H_3 and F_3 contribute some net D3 charge, which must be cancelled by other sources of D3 charge, most notably O3 planes.

Returning to (2.62), we can rewrite the $H_3 \wedge F_3$ in terms of G_3 and $\overline{G}_3$, evaluate dF_5 using the expression for the 5-form field strength (2.55), and find

$$\widetilde{\nabla}^2 \alpha = i e^{2A} \frac{G_{mnp} \left(*_6 \overline{G}_3 \right)^{mnp}}{12 \mathrm{Im}\,(\tau)} + 2 e^{-6A} g^{mn} \left(\partial_m \alpha \right) \left(\partial_n e^{4A} \right) + 2\ell^8 e^{2A} T_3 \rho_3 \,, \tag{2.64}$$

where we have now divided the volume form out of ρ_3, leaving only the delta functions. If we now subtract this from (2.60), we find an important constraint:

$$\begin{aligned} \widetilde{\nabla}^2 \left(e^{4A} - \alpha \right) &= \frac{e^{2A}}{6 \mathrm{Im}\,(\tau)} \left| i G_3 - *_6 G_3 \right|^2 + e^{-6A} \left| \partial \left(e^{4A} - \alpha \right) \right|^2 \\ &\quad + 2\ell^8 e^{2A} \left[\frac{1}{4} \left(T^m_m - T^\mu_\mu \right)_{\mathrm{loc}} - T_3 \rho_3 \right] . \end{aligned} \tag{2.65}$$

Once again, the lefthand side gives zero when integrated over the Calabi-Yau, and the first two terms on the righthand side are positive semi-definite. This means that when integrated, the last term can give something zero or negative, but not something positive. When the contribution of the final term is negative, i.e. when

$$\frac{1}{4}\left(T^m_m - T^\mu_\mu\right)_{\text{loc}} < T_3\rho_3\,, \tag{2.66}$$

(2.65) can be satisfied by cancellations between all three of the terms on the righthand side. This situation is rather difficult to analyze, so we will not pursue it further.

We now turn our attention to sources that satisfy

$$\frac{1}{4}\left(T^m_m - T^\mu_\mu\right)_{\text{loc}} = T_3\rho_3\,, \tag{2.67}$$

so that the contribution from the third term is zero. Such sources include D3 and D7 branes, as well as O3 and O7 planes. When only these sources are included, each term on the righthand side of (2.65) must vanish separately, i.e. we must have $\partial_m\alpha = \partial_m e^{4A}$ and G_3 must be imaginary self-dual (ISD). While the relationship between the warp factor $A(y)$ and the 5-form (determined by $\alpha(y)$) can be trivially imposed, the condition that G_3 be ISD is highly non-trivial, since the metric of the compact space enters through the Hodge star $*_6$. Indeed, the axio-dilaton τ and many of the metric moduli are stabilized by the ISD condition. Analysis of this condition will occupy a significant fraction of chapters III and IV.

2.3.2 O3/O7 Calabi-Yau Orientifolds and their Spectra

Having learned several general lessons from the GKP analysis, we now introduce a large family of compactifications that are of GKP type. These are orientifold projections of Calabi-Yau compactifications with O3 and O7 planes. The construction begins with an ordinary Calabi-Yau compactification, as described in section (2.2).

The orientifold projection introduces orientifold planes, reduces the amount of supersymmetry in the 4D theory from $\mathcal{N} = 2$ to $\mathcal{N} = 1$, and projects many of the moduli out of the theory altogether. The remaining moduli lie in $\mathcal{N} = 1$ chiral multiplets, and the moduli space still factorizes into two parts: the "vector moduli," descended from $\mathcal{N} = 2$ vector multiplets, and the "hypermoduli" descended from $\mathcal{N} = 2$ hypermultiplets. For an extensive discussion of orientifold compactifications, see [24].

We begin by studying the orientifold theories without fluxes, so that the moduli are all massless[7]. The orientifold projection removes states which are odd under the combined action of two separate involutions. The first involves worldsheet operators, always including worldsheet parity Ω_p and, in the O3/O7 case[8], also including the spacetime fermion number in the left-moving sector, $(-1)^{F_L}$. Our usual 10D fields are all eigenstates of these involutions, with eigenvalues:

Field	ϕ, $g_{\mu\nu}$	$B_{\mu\nu}$	C_0	C_1	C_2	C_3	C_4
Ω_p	+	-	-	+	+	-	-
$(-1)^{F_L}$	+	+	-	-	-	-	-

The eigenvalues of C_{n+4} are equal to those of C_n. The second factor is an involution σ of the spacetime, in this case of the Calabi-Yau only. Orientifold planes arise at fixed points of σ. In the case of O3/O7 compactifications σ is a holomorphic involution, acting as

$$\sigma^* J = J \,, \tag{2.68}$$

$$\sigma^* \Omega_3 = -\Omega_3 \,. \tag{2.69}$$

If we use local coordinates z^m, $m = 1, 2, 3$, such that $\Omega_3 = dz^1 \wedge dz^2 \wedge dz^3$, we see

[7]Note that the tadpole constraint (2.63) can be satisfied by the introduction of D3 branes, so that the theory without fluxes is well-defined.

[8]IIB strings also allow orientifolds without the factor of $(-1)^{F_L}$. These lead to compactifications with O5 and O9 planes. One can also consider orientifolds of IIA strings. These require an anti-holomorphic involution, and lead to O6 planes. We will not discuss either of these cases explicitly.

that σ must flip the sign on either one or all of the z^m, leading to O7 or O3 planes, respectively.

In order to discuss the O3/O7 spectrum we will need to refine the basis of cohomology elements introduced in 2.2.1.3, so that each element of the refined basis is either odd or even under σ. (2.68) indicates that the Calabi-Yau volume form is even under the involution, as is the constant function 1. The remaining basis elements are:

The σ-even forms: consisting of the 2-forms μ_α, their dual 4-forms $\widetilde{\mu}^\alpha$, and the 3-forms $\mathcal{A}_{\hat{I}}$ and $\mathcal{B}^{\hat{I}}$. The ranges of the indices are $\alpha = 1, ..., h_+^{(1,1)}$ and $\hat{I} = 1, ..., h_+^{(2,1)}$.

The σ-odd forms: consisting of the 2-forms ω_a, their dual 4-forms $\widetilde{\omega}^a$, and the 3-forms α_I and β^I. The ranges of the indices are $a = 1, ..., h_-^{(1,1)}$ and $I = 0, ..., h_-^{(2,1)}$. Note that by (2.69), the $(3,0)$ form is σ-odd.

Because the volume form is even, many intersections products of these forms (e.g. $\int \mathcal{A}_{\hat{I}} \wedge \beta^J$) automatically vanish. The non-zero intersections are

$$\int \mathcal{A}_{\hat{I}} \wedge \mathcal{B}^{\hat{J}} = \delta_{\hat{I}}^{\hat{J}}\,, \tag{2.70}$$

$$\int \alpha_I \wedge \beta^J = \delta_I^J\,, \tag{2.71}$$

$$\int \mu_\alpha \wedge \widetilde{\mu}^\beta = \delta_\alpha^\beta\,, \tag{2.72}$$

$$\int \omega_a \wedge \widetilde{\omega}^b = \delta_a^b\,, \tag{2.73}$$

and

$$\int \mu_\alpha \wedge \mu_\beta \wedge \mu_\gamma = \kappa_{\alpha\beta\gamma}\,, \tag{2.74}$$

$$\int \mu_\alpha \wedge \omega_b \wedge \omega_c = \widehat{\kappa}_{\alpha bc}\,. \tag{2.75}$$

Let us now compute the spectrum of the 4D theory.

The 10D scalars C_0 and ϕ are both even under $(-1)^{F_L}\,\Omega_p$, and so survive the orientifold projection. Since the holomorphic 3-form Ω_3 is σ-odd, only $h_-^{(2,1)}$ complex structure moduli z^i survive the orientifold projection. The Kähler form is σ-even, and so it must be expanded on even 2-forms,

$$J = t^\alpha \mu_\alpha \,, \tag{2.76}$$

giving rise to $h_+^{(1,1)}$ real scalars. The RR form field C_2 is odd under $(-1)^{F_L}\,\Omega_p$, and so must be expanded on σ-odd forms only:

$$C_2 = c^a \omega_a \,, \tag{2.77}$$

ans so provides $h_-^{(1,1)}$ real scalars. The NS 2-form B_2 is also odd under $(-1)^{F_L}\,\Omega_p$, so its expansion is similar to that of C_2 :

$$B_2 = u^a \omega_a \,. \tag{2.78}$$

C_4, on the other hand, is even under $(-1)^{F_L}\,\Omega_p$, so it must be expanded on σ-even forms:

$$C_4 = \rho^\alpha_{\mu\nu}\mu_\alpha dx^\mu \wedge dx^\nu + \rho_\alpha \widetilde{\mu}^\alpha + A^{\hat{I}}_\mu \mathcal{A}_{\hat{I}} \wedge dx^\mu - A_{\mu\hat{I}} \mathcal{B}^{\hat{I}} \wedge dx^\mu \,. \tag{2.79}$$

Once again, the constraint $F_5 = *_{10} F_5$ halves the number of independent components of C_4, and we will take $\rho^\alpha_{\mu\nu}$ and $A_{\mu\hat{I}}$ to be functions of ρ_α and $A^{\hat{I}}_\mu$. These contribute $h_+^{(1,1)}$ real scalars, and $h_+^{(2,1)}$ vectors, respectively. We see that the $h^{(2,1)}$ $\mathcal{N}=2$ vector multiplets split into $h_+^{(2,1)}$ $\mathcal{N}=1$ vector multiplets, with the vectors provided by $A^{\hat{I}}_\mu$, and $h_-^{(2,1)}$ $\mathcal{N}=1$ chiral multiplets, with the scalars provided by the z^i. The $h^{(1,1)}+1$ $\mathcal{N}=2$ hypermultiplets are reduced to $h^{(1,1)}+1$ chiral multiplets, with the scalars provided by ρ_α, t^α, u^a, c^a, C_0, and ϕ.

Having determined the number of light fields, we will describe the correct Kähler coordinates and Kähler potentials for these compactifications [24]. This is straightforward for the vector moduli, since their $\mathcal{N}=2$ moduli space was special Kähler. The z^i are already good Kähler coordinates, and the Kähler potential is just what we might have guessed from the analysis in section (2.2.2):

$$K_{\mathrm{V}} = -\log\left[-i\int_{\mathrm{CY}}\Omega_3\wedge\overline{\Omega}_3\right], \tag{2.80}$$

where Ω_3 is expanded only on the $\sigma-$odd 3-cycles.

The analysis of the hypermoduli is more delicate, since the orientifold picks out a Kähler submanifold of the $\mathcal{N}=2$ moduli space, which is quaternionic Kähler. We simply quote the results of [41], that the proper Kähler coordinates are

$$\tau = C_0 + ie^{-\phi}, \tag{2.81}$$

$$G^a = c^a - \tau u^a, \tag{2.82}$$

$$T_\alpha = \rho_\alpha - \frac{i}{2}e^{-\phi}\kappa_{\alpha\beta\gamma}v^\beta v^\gamma - \widehat{\kappa}_{\alpha bc}c^b u^c + \frac{1}{2}\tau\widehat{\kappa}_{\alpha bc}u^b u^c, \tag{2.83}$$

and the Kähler potential is

$$K_{\mathrm{H}} = -\log\left[-i\left(\tau-\overline{\tau}\right)\right] - 2\log\left[\frac{1}{6}e^{-3\phi/2}\int J\wedge J\wedge J\right]. \tag{2.84}$$

The dependence of the Kähler potential on the Kähler coordinates $\{\tau, G^a, T_\alpha\}$ is implicitly determined by (2.81)-(2.83), but cannot in general be directly determined. The Kähler metric and its inverse *can* be explicitly determined, but since the result is quite involved, and will not be needed until our analysis of compactifications of geometric fluxes, we postpone their discussion until chapter IV.

The introduction of 3-form fluxes along the Calabi-Yau induces a superpotential

[42, 43] in the 4D, $\mathcal{N} = 1$ effective theory. It takes the quite simple form

$$W = \int G_3 \wedge \Omega_3 \,, \tag{2.85}$$

where G_3 is the usual complex flux $F_3 - \tau H_3$. In addition to τ, this superpotential is also a function of the complex structure moduli z^i, which enter through the holomorphic 3-form. The remaining hypermoduli G^a and T_α do not appear in the superpotential, and cannot be stabilized (at the classical level) by 3-form fluxes.

While (2.85) can be derived directly by dimensional reduction, it was originally derived [42] by relating the fluxes F_3 and H_3 to the wrapped D5 and NS5 branes that are their magnetic sources. In this construction, the branes wrap 3-cycles on the Calabi-Yau, and form domain walls in the four extended dimensions. For BPS configurations, the tension of the domain wall is equal to the superpotential. This derivation was extended to compactifications that are "non-geometric," in the sense of having no 2-cycles in [44]. Recently, this superpotential was derived using worldsheet methods only [45]. Rather then review each of these derivations, we will demonstrate in chapter III that this superpotential correctly reproduces the ISD condition on the flux G_3, as discussed in section 2.3.1.

CHAPTER III

GKP Flux Attractors

3.1 Introduction

The compactification of string theory from 10 to 4 dimensions is a subject of both formal and phenomenological interest. Many methods of compactification result in moduli: massless 4D scalar fields which correspond to deformations of the compactification geometry. Given the observed absence of massless scalars, these moduli are phenomenologically undesirable. As a result, much attention has been focused on the question of how moduli can be *stabilized*, i.e. how features can be added to a simple compactification so that most or all of the 4D scalar fields become massive. We can consider this question in three different levels of detail:

1. Is the proposed stabilization method consistent? That is, does the stabilized compactification still solve the 10D equations of motion?

2. Which moduli are stabilized, and what are their VEVs?

3. What are the masses of the moduli?

In this chapter we study compactifications of IIB string theory on Calabi-Yau orientifolds, with RR and NS 3-form flux in the compact directions. The *flux attractor equations* [17] describing the stabilization of the moduli strongly resemble black hole attractor equations, and we will exploit this similarity to address the questions above.

We will focus our attention on one of the 10D equations of motion. If the (real) 3-form RR flux is F_3, the (real) 3-form NS flux is H_3, and the complex axio-dilaton is τ, we define the complex 3-form flux

$$G_3 \equiv F_3 - \tau H_3 \,. \tag{3.1}$$

For large classes of compactifications to 4D Minkowski space, the 10D equations of motion require [38, 46] that G_3 be imaginary self dual (ISD):

$$*_6 G_3 = i G_3 \,. \tag{3.2}$$

Because $*_6$ involves the metric, a non-zero G_3 stabilizes some or all of the complex structure moduli and τ. Specifically, the complex structure of the Calabi-Yau is fixed so that G_3 has only $(0,3)$ and/or $(2,1)$ components. If no such combination of complex structure and τ exists, the choice of F_3 and H_3 is not consistent with compactification to Minkowski space. In order to analyze (3.2) in detail we may expand G_3 and the holomorphic 3-form, Ω_3, on a judiciously chosen basis of 3-cycles. This procedure results in the flux attractor equations, as we review in section 3.2.

The resulting algebraic equations suffer an apparent inconsistency, in that there are many more equations than moduli. If $n = b_3/2 - 1$ is the number of $(2,1)$ cycles on the Calabi-Yau, we will find $4n+4$ different (real) equations and only $2n+2$ (real) moduli. While this mismatch suggests that the system of equations is overconstrained, we will show that this is not the case. In section 3.3 we will show that the $4n+4$ attractor equations determine both the VEVs of the moduli *and* the independent parameters of their mass matrix, as well as the gravitino mass. All of these outputs together constitute $4n+4$ parameters, the same as the number of input fluxes.

Having established that the flux attractor equations determine both the moduli VEVs *and* certain mass parameters, in section 3.4 we develop an algorithm to find them. We take inspiration from OSV [21], who solved the black hole attractor equations by introducing a mixed ensemble. Accordingly, we first solve the "magnetic" half of the attractor equations, writing our $4n+4$ parameters in terms of the $2n+2$ magnetic fluxes and $2n+2$ as-yet-undetermined electric potentials. We then show that the "electric" attractor equations can be rewritten in terms of a generating function, and that they can be formally solved by a simple Legendre transform.

The existence of the generating function $\mathcal{G}$ is the principal result of this chapter. If one can determine it as a function of arbitrary fluxes, its derivatives will give back the moduli VEVs and the mass parameters. Thus $\mathcal{G}$ provides a compact summary of the flux attractor behavior, and this suggests that we study the properties of $\mathcal{G}$ directly. We initiate such a study in section 3.5, where we find a general formula for $\mathcal{G}$:

$$\mathcal{G} = \int F_3 \wedge H_3 - 2\text{Vol}^2 m_{3/2}^2. \tag{3.3}$$

Here the gravitino mass is considered as a function of arbitrary fluxes.

We proceed in section 3.6 by considering an explicit example. We use the prepotential $F = Z^1 Z^2 Z^3 / Z^0$, a setting with sixteen distinct fluxes. For a reduced set of eight of these fluxes we are able to completely solve the flux attractor equations. We then argue that the general case can be solved as well, by appealing to duality transformations.

For the sake of simplicity, we will discuss many of our results in the context of large-volume, unwarped compactifications. These lead to relatively well-understood 4D theories, and we can easily translate our findings about the 10D geometry into statements about 4D physics. However, our 10D reasoning applies equally well to strongly-warped compactifications and some non-geometric compactifications [44]. Since we are analyzing the ISD condition, which is quite robust, we expect our qualitative understanding of the flux attractor behavior, such as the existence of a generating function, to be similarly robust. On the other hand the detailed mass spectrum depends on the Kähler potential, and is therefore less robust.

As we have mentioned above, the solution of the flux attractor equations is controlled by a single generating function, which depends on the fluxes alone. In the case of the black hole attractor, the analogous function turned out to be the equilibrium value of the black hole entropy. It is tempting to speculate that the flux attractor equations also describe a thermodynamic system. Ultimately, the underlying statistical system may be related to a classical measure on this patch of the string theory landscape. We conclude in section 3.7 by summarizing the issues that must be resolved in order to make this interpretation sound.

3.2 From the ISD Condition to Attractor Equations

In this section we review some basic aspects of special geometry and flux compactifications. We then provide a simple derivation of the flux attractor equations.

3.2.1 Special Geometry

Most of the objects we are interested in, including F_3, H_3, and Ω_3, are 3-forms on the compact space. It is useful to expand these 3-forms on a real basis $\left\{\alpha_I, \beta^I\right\}$, $I = 0, ..., n$, satisfying

$$\int \alpha_I \wedge \beta^J = \delta_I^J\,, \tag{3.4}$$

$$\int \alpha_I \wedge \alpha_J = \int \beta^I \wedge \beta^J = 0\,. \tag{3.5}$$

We specify the NS fluxes H_3 and RR fluxes F_3 with respect to this basis as

$$H_3 = m_h^I \alpha_I - e_I^h \beta^I\,, \tag{3.6}$$

$$F_3 = m_f^I \alpha_I - e_I^f \beta^I\,. \tag{3.7}$$

There is an $Sp(2n+2,\mathbb{R})$ symmetry[1] that corresponds to a change in the basis $\{\alpha_I, \beta^I\}$. The fluxes $\{m_h^I, e_I^h\}$ and $\{m_f^I, e_I^f\}$ transform in the fundamental of $Sp(2n+2,\mathbb{R})$, and objects with an index $I, J, K...$ transform in the fundamental of $SO(n+1,\mathbb{R}) \subset Sp(2n+2,\mathbb{R})$.

We can also expand the holomorphic 3-form with respect to the real basis,

$$\Omega_3 = Z^I \alpha_I - F_I \beta^I \,. \tag{3.8}$$

The combination $\{Z^I, F_I\}$ is called a *symplectic section* [35], and also transforms in the fundamental of $Sp(2n+2,\mathbb{R})$. While the fluxes $e_I^{h,f}$ and $m_{h,f}^I$ were all independent parameters, the F_I and Z^I are holomorphic functions of the complex structure moduli. For our purposes, it is sufficient to treat the F_I as functions that are holomorphic and homogeneous of degree 1 in the Z^I. The functional form of the F_I is the only information about the Calabi-Yau geometry that we will use.

The holomorphic 3-form is only defined up to a holomorphic rescaling,

$$\Omega_3 \to f\left(Z^I\right)\Omega_3 \,. \tag{3.9}$$

These are the *Kähler transformations.* If, under Kähler transformations, an operator is simply multiplied by h powers of $f\left(Z^I\right)$ and $\overline{h}$ powers of $\overline{f\left(Z^I\right)}$, we will say that it is Kähler covariant with weight $\left(h, \overline{h}\right)$. For example, Ω_3 has weight $(1,0)$.

Physical moduli must be invariant under Kähler transformations. For example, on a patch where $Z^0 \neq 0$ we may use the ratios

$$z^i \equiv \frac{Z^i}{Z^0}\,, \tag{3.10}$$

where $i = 1, ..., n$. The z^i are clearly Kähler invariant. Unfortunately, this breaks

[1]Dirac quantization conditions require the magnetic fluxes $m_{h,f}^I$ and electric fluxes $e_I^{h,f}$ to take integer values, breaking $Sp(2n+2,\mathbb{R})$ to a discrete subgroup.

the $SO(n+1)$ symmetry enjoyed by the Z^I, so we will sometimes use an alternative approach to formulating Kähler invariant quantities. We will utilize a coefficient C which has weight $(-1,0)$, so that the products CZ^I are Kähler invariant.

Because Kähler transformations are local, ordinary derivatives of Kähler covariant functions do not give new Kähler covariant functions. We introduce the Kähler potential

$$K_z = -\log i \int \Omega_3 \wedge \overline{\Omega}_3 \,, \tag{3.11}$$

which generates the metric on moduli space,

$$g_{i\bar{j}} \;=\; \partial_i \overline{\partial}_{\bar{j}} K_z \,. \tag{3.12}$$

By construction, e^{K_z} has weight $(-1,-1)$. This motivates the definition of the Kähler covariant derivative of an operator of weight $\left(h,\overline{h}\right)$,

$$D_i \mathcal{O}^{\left(h,\overline{h}\right)} \equiv e^{-hK_z}\partial_i\left(e^{hK_z}\mathcal{O}^{\left(h\overline{h}\right)}\right) = \partial_i\mathcal{O}\left(h,\overline{h}\right) + h\mathcal{O}^{\left(h,\overline{h}\right)}\partial_i K_z \,. \tag{3.13}$$

We note that since the Kähler potential is real, the Kähler covariant derivative of a holomorphic object is not itself holomorphic.

It is especially interesting to consider derivatives of the holomorphic 3-form. An ordinary derivative with respect to the complex structure moduli gives a sum of $(3,0)$ and $(2,1)$ forms,

$$\partial_i \Omega_3 = k_i \Omega_3 + \chi_i \,. \tag{3.14}$$

If we instead use a Kähler covariant derivative, the Kähler potential is constructed so that the $(3,0)$ piece cancels and we are left with only a $(2,1)$ form,

$$D_i \Omega_3 = \chi_i \,. \tag{3.15}$$

This establishes a convenient *complex* basis for 3-forms on the Calabi-Yau, $\{\Omega_3, D_i\Omega_3, \overline{D_i\Omega_3}, \overline{\Omega}_3\}$ [30]. The intimate connection between the complex structure of a Calabi-Yau and its cohomology will be the primary tool that we use to analyze the ISD condition (3.2).

3.2.2 S-Duality

In addition to Kähler transformations, S-duality helps organize the flux attractor equations. Type IIB supergravity has an $SL(2,\mathbb{R})$ symmetry[2], under which

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{3.16}$$

$$\begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_3 \\ H_3 \end{pmatrix}, \tag{3.17}$$

with the constraint

$$ad - bc = 1\,. \tag{3.18}$$

The transformation of the complex flux G_3 under S-duality can be deduced from the transformations of F_3, H_3, and τ :

$$G_3 \to \frac{G_3}{c\tau + d}\,. \tag{3.19}$$

We will frequently encounter $\mathrm{Im}\,(\tau)$, which transforms as

$$\mathrm{Im}\,(\tau) \to \frac{\mathrm{Im}\,(\tau)}{|c\tau + d|^2}\,. \tag{3.20}$$

[2]Quantum effects break this to $SL(2,\mathbb{Z})$, but the distinction between the two groups will not be relevant to our analysis.

3.2.3 4D Physics of Large Volume Compactifications

The flux attractor equations are simply a rephrasing of the ISD condition (3.2). We could discuss the ISD condition entirely from the 10D point of view, but we find it useful to make reference to the resulting 4D effective theory. As long as the volume of the Calabi-Yau is large relative to the string scale, and regions of strong warping are all string scale, the result is a 4D, $\mathcal{N} = 1$ theory with the GVW superpotential [42, 43],

$$W = \int_{CY} G_3 \wedge \Omega_3 \,, \tag{3.21}$$

and Kähler potential[3],

$$K = K_z + K_\tau + K_t \tag{3.22}$$

$$= -\log\left[i \int_{CY} \Omega_3 \wedge \overline{\Omega}_3\right] - \log\left[2\mathrm{Im}\,(\tau)\right] - 2\log\left[\mathrm{Vol}\right] \,. \tag{3.23}$$

These compactifications are reviewed in e.g. [23, 25, 27, 29]. While both the superpotential and Kähler potential receive a variety of phenomenologically interesting corrections [47–49], we will not consider their effects here. Note that the 4D Kähler potential contains the Kähler potential (3.11) that we introduced earlier for the Calabi-Yau. This relationship between the 4D kinetic terms and the Calabi-Yau geometry is a special characteristic of the large-volume limit, and breaks down in the presence of significant warping (see e.g. [50–55]).

In addition to the complex structure moduli z^i and τ, the 4D theory also contains a number of Kähler moduli t^a. Rather than depending on the holomorphic volumes of three-cycles, these measure the actual volumes of two- and four-cycles. Since the

[3]The volume of the Calabi-Yau is determined by the Kähler moduli, which are not stabilized by 3-form fluxes. We have little to say about the factors of the volume that appear, but include them for completeness.

Kähler moduli do not appear in the superpotential, their F-terms are just

$$F_a = D_a W = W \partial_a K_t \,. \tag{3.24}$$

When summed up they give $\sum_a |F_a|^2 = 3\,|W|^2$, so the standard expression for the scalar potential simplifies to

$$\begin{aligned} V &= e^K \left[\sum_{A=i,\tau,a} |D_A W|^2 - 3\,|W|^2 \right] \qquad (3.25)\\ &= e^K \left(\sum_i |D_i W|^2 + |D_\tau W|^2 \right) . \qquad (3.26) \end{aligned}$$

When $W \neq 0$ the F-terms for the Kähler moduli (3.24) are non-vanishing, so SUSY is broken. However, the potential (3.25) is positive definite and has a global minimum when $F_i = D_i W = 0$ and $F_\tau = D_\tau W = 0$. Because of this, we require that $F_i = F_\tau = 0$, regardless of whether SUSY is broken.

The simple form of the Kähler potential gives the $F_i = F_\tau = 0$ conditions simple geometric interpretations. For the complex structure moduli we find

$$D_i W = \int_{CY} G_3 \wedge D_i \Omega_3 = \int_{CY} G_3 \wedge \chi_i \,, \tag{3.27}$$

so that setting $F_i = 0$ is equivalent to requiring that G_3 have no $(1,2)$ component. In addition one can verify that

$$D_\tau \int G_3 \wedge \Omega_3 = -\frac{1}{\tau - \overline{\tau}} \int \overline{G}_3 \wedge \Omega_3 \,, \tag{3.28}$$

so setting $F_\tau = 0$ is equivalent to requiring that G_3 have no $(3,0)$ component. Thus we have found that minimizing the potential (3.25) is *equivalent* to imposing the ISD condition (3.2). This is one of the reasons that the GVW superpotential is believed to accurately describe large-volume compactifications.

3.2.4 Flux Attractor Equations

The flux attractor equations were originally derived in [17] by considering F-theory compactified on $\mathrm{CY}_3 \times T^2$. For the sake of variety, we present a slightly different derivation which does not involve an explicit embedding in F-theory.

Our goal is to make the implications of the ISD condition (3.2) more explicit. Since an ISD 3-form can have only $(0,3)$ and $(2,1)$ pieces, we can expand it with respect to the complex basis introduced at the end of section (3.2.1) as:

$$G_3 = -i\mathrm{Im}\,(\tau)\left[\overline{C\Omega}_3 + C^i D_i \Omega_3\right] . \tag{3.29}$$

The overall factor of $-i\mathrm{Im}\,(\tau)$ is included for convenience. Note that C and C^i both have weight $(-1,0)$ under Kähler transformations, and transform under S-duality as

$$C \quad\rightarrow\quad (c\tau + d)\, C\,, \tag{3.30}$$

$$C^i \quad\rightarrow\quad (c\overline{\tau} + d)\, C^i\,. \tag{3.31}$$

In order to make (3.29) completely explicit we must specify the symplectic section $\left\{Z^I, F_I\right\}$, as this determines how Ω_3 depends on the complex structure moduli. We can then compute the Kähler covariant derivatives $D_i\Omega_3$, so that (3.29) becomes an *algebraic* equation for the complex structure moduli and the axio-dilaton.

One undesirable aspect of (3.29) is that the LHS contains both the real fluxes F_3 and H_3, which we think of us "inputs," and the axio-dilaton τ, which we think of as an "output." This is rectified by writing

$$\begin{pmatrix} G_3 \\ \overline{G}_3 \end{pmatrix} = \begin{pmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{pmatrix} \begin{pmatrix} F_3 \\ H_3 \end{pmatrix} = -i\mathrm{Im}\,(\tau) \begin{pmatrix} \overline{C\Omega}_3 + C^i D_i \Omega_3 \\ -C\Omega_3 - \overline{C}^i \overline{D_i \Omega}_3 \end{pmatrix}, \tag{3.32}$$

which we can easily invert:

$$\begin{pmatrix} F_3 \\ H_3 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -\overline{\tau} & \tau \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \overline{C\Omega}_3 + C^i D_i \Omega_3 \\ -C\Omega_3 - \overline{C}^i \overline{D_i \Omega}_3 \end{pmatrix} \tag{3.33}$$

$$= \begin{pmatrix} \mathrm{Re}\left[\tau\left(C\Omega_3 + \overline{C}^i \overline{D_i \Omega_3}\right)\right] \\ \mathrm{Re}\left[C\Omega_3 + \overline{C}^i \overline{D_i \Omega_3}\right] \end{pmatrix}. \tag{3.34}$$

Now the LHS of the attractor equations consists entirely of quantities that define the vacuum (fluxes), while the RHS depends on the moduli and the symplectic section (choice of $\{Z^I, F_I\}$).

The equations in (3.34) are equations for 3-forms, rather than for ordinary numbers. While this makes their geometric implications clear, if we want to actually solve the equations it will be helpful to integrate them against a real basis of 3-forms. We have already introduced the required notation in (3.6)-(3.8), so we simply quote the result,

$$m_f^I = \mathrm{Re}\left[\tau\left(CZ^I + \overline{C}^i \overline{D_i Z^I}\right)\right], \tag{3.35}$$

$$m_h^I = \mathrm{Re}\left[CZ^I + \overline{C}^i \overline{D_i Z^I}\right], \tag{3.36}$$

$$e_I^f = \mathrm{Re}\left[\tau\left(CF_I + \overline{C}^i \overline{D_i F_I}\right)\right], \tag{3.37}$$

$$e_I^h = \mathrm{Re}\left[CF_I + \overline{C}^i \overline{D_i F_I}\right]. \tag{3.38}$$

One benefit to writing the attractor equations in this form is that there is manifestly one real equation for each real flux, for a total of $4n + 4$ real equations. We will compare this to the number of moduli and other parameters quite carefully in the next section.

One may wonder to what extent it makes sense to call (3.35)-(3.38) "attractor equations." The word "attractor" implies some sort of flow along which all information about a set of initial conditions is lost, but we have not introduced any notion of attractor flow. We note that in the study of extremal black holes, there is a useful distinction between the entire attractor *flow*, which takes place between spatial infinity and the horizon, and the attractor *equations*, which describe how the moduli are stabilized at the horizon. Because (3.35)-(3.38) are closely analogous to the black hole attractor equations, we consider calling them "attractor equations" to be only a minor abuse of the term.

3.3 Attractor Equations and Mass Matrices

In expanding out the flux attractor equations, we found $4n + 4$ real equations[4] (3.35)-(3.38). This is many more than the $2n + 2$ real moduli VEVs we want to fix, the z^i and τ. The origin of this mismatch is that there are additional "outputs" of the attractor equations, namely the coefficients C and C^i. Including these outputs gives $4n + 4$ real variables, equal to the number of attractor equations. We will see that these coefficients determine the mass spectrum of the 4D theory.

3.3.1 Black Hole Attractor Equations and the Entropy

While the C^i are a new feature of the flux attractor equations, the coefficient C also appears in the more familiar context of BPS black hole attractor equations. We begin by discussing the role it plays there. Suppose we have constructed a 4D BPS Reissner-Nordström black hole by wrapping D3 branes on the 3-cycles of a Calabi-Yau manifold. The charges of the black hole can be described by a 3-form, F_3. We

[4] $n = b_3/2 - 1$, so that $n + 1$ is the number of $\mathcal{N} = 1$ vector multiplets in the 4D theory.

can expand a general real 3-form either against a real basis, or against the complex basis introduced in section 3.2.1:

$$F_3 = p^I \alpha_I - q_I \beta^I \tag{3.39}$$

$$= \mathrm{Re}\left[C\Omega_3 + C^i D_i \Omega_3\right] . \tag{3.40}$$

The expression for the spacetime central charge of the black hole is

$$W_{BH} = \int F_3 \wedge \Omega_3 \, , \tag{3.41}$$

and the BPS conditions are $D_i W_{BH} = 0$. Since F_3 does not depend on the moduli, the BPS conditions reduce to

$$\int F_3 \wedge D_i \Omega_3 = 0 \, , \tag{3.42}$$

i.e. they require that the $(1,2)$ piece of F vanishes. This simplifies the general expansion (3.40) to

$$F_3 = 2\mathrm{Re}\left[C\Omega_3\right] . \tag{3.43}$$

This is the standard black hole attractor equation, originally derived in [56–58] and reviewed in [18, 59, 60].

If we expand (3.43) on the real basis $\left\{\alpha_I, \beta^I\right\}$ we will find a counting problem. Although there are $2n+2$ real equations, there are only $2n$ real physical moduli, the z^i. In order to understand the mismatch, we first note that the righthand side of (3.43) contains $2n+4$ real parameters, $\left\{C, Z^I\right\}$. Since both C and Z^I transform under Kähler transformations we can eliminate one complex parameter, leaving $2n+2$ Kähler invariant parameters. For example, if we assume that $Z^0 \neq 0$, we can take the Kähler invariant parameters to be $\{CZ^0, z^i = Z^i/Z^0\}$. More generally, the number of Kähler invariant parameters is equal to the number of attractor equations. The non-trivial feature is that, in addition to determining the values of the moduli z^i, the black hole attractor equations fix the Kähler invariant quantity CZ^0.

It is natural to ask what the physical significance of the additional parameter is. One important place where it appears is in the black hole entropy,

$$\frac{S}{\pi} = e^{K_z} |W_{BH}|^2 \tag{3.44}$$

$$= \frac{e^{-K_z}}{|Z^0|^2} \cdot \left|CZ^0\right|^2 , \tag{3.45}$$

since (3.41) and (3.43) imply that $W_{BH} = -i\overline{C}e^{-K_z}$. In the final expression we have written the black hole entropy as the product of two Kähler-invariant factors, with the first factor depending only on the moduli z^i. We see that a change in CZ^0 leads to a change in the entropy, with the moduli held fixed.

It is sometimes stated that solving the attractor equations is equivalent to minimizing an effective potential. Our analysis shows that, in fact, the attractor equations simultaneously determine both the values of the moduli *and* the value of the effective potential. Simply minimizing the effective potential with respect to the moduli would have given us $2n$ real equations, rather than $2n+2$, and we would have had to insert the solutions for the moduli back into the effective potential to find its value at the minimum.

3.3.2 Fermion Masses

Let us now return to the flux attractor equations. (3.35)-(3.38) constitute $4n+4$ real equations, while the moduli z^i and τ constitute $2n+2$ real parameters. Our analysis of the black hole attractor equations revealed that CZ^0 contributes two more real independent parameters, but we are still left with $2n$ more equations than parameters. The new features in the flux attractor are the coefficients C^i, first introduced in (3.29). Including these in our set of Kähler-invariant parameters as $\{\tau, z^i, CZ^0, C^iZ^0\}$, we have accounted for everything that appears on the righthand side of (3.34), for a grand total of $4n+4$ parameters. Just as in the black hole case we found that different choices of charges could lead to the same moduli but different entropies, here different choices of the fluxes can lead to the same moduli, but different values of CZ^0 *and* C^iZ^0.

In large-volume compactifications, the role of the black hole entropy is played by the gravitino mass:

$$m_{3/2}^2 = e^K |W|^2 , \tag{3.46}$$

Indeed, if we substitute in the expressions (3.21) for the superpotential and (3.23) for the Kähler potential, we find

$$m_{3/2}^2 = \frac{e^{-K_z}\mathrm{Im}\,(\tau)}{2\,|Z^0|^2\,\mathrm{Vol}^2} \cdot \left|CZ^0\right|^2 . \tag{3.47}$$

Just as CZ^0 determined the entropy of the black hole attractor, it determines the gravitino mass for the flux attractor.

While we understand well enough what it means to solve for the VEVs of z^i and τ, and we know that C is related to the gravitino mass, we need to develop a physical interpretation of the C^i. We'll first observe that the C^i appear when we consider the second derivatives of the superpotential:

$$\begin{aligned} D_iD_jW &= \int G_3 \wedge D_iD_j\Omega_3 && (3.48)\\ &= \int G_3 \wedge \left(\mathcal{F}_{ijk}\overline{\chi}^k\right) && (3.49)\\ &= \mathrm{Im}(\tau)e^{-K_z}\mathcal{F}_{ijk}C^k\,, && (3.50) \end{aligned}$$

where [61]

$$\mathcal{F}_{ijk} = ie^{K_z}\int \Omega_3 \wedge \partial_i\partial_j\partial_k\Omega_3 \tag{3.51}$$

depends on both the moduli and the symplectic section[5]. We also need the mixed derivatives,

$$
\begin{aligned}
D_\tau D_i W &= -\frac{\int \overline{G}_3 \wedge \chi_i}{\tau - \overline{\tau}} && (3.52)\\
&= -\frac{1}{2}\int \left(C\Omega_3 + \overline{C^j \chi_j} \right) \wedge \chi_i && (3.53)\\
&= \frac{i}{2}\overline{C}^{\overline{j}} g_{i\overline{j}} e^{-K_z} \,. && (3.54)
\end{aligned}
$$

Here we used (3.27) and (3.28). Also, in the last step we used the relationship between the metric on complex structure moduli space (3.12) and the $(2,1)$ forms (3.14),

$$
g_{i\overline{j}} = -\frac{\int \chi_i \wedge \overline{\chi}_{\overline{j}}}{\int \Omega_3 \wedge \overline{\Omega}_3} \,. \tag{3.55}
$$

The remaining second derivative vanishes,

$$
\begin{aligned}
D_\tau D_\tau W &= \frac{2}{(\tau - \overline{\tau})^2} \int \overline{G}_3 \wedge \Omega_3 && (3.56)\\
&= 0 \,, && (3.57)
\end{aligned}
$$

since $\overline{G}_3$ has no $(0,3)$ piece.

The second derivatives of the superpotential generically determine the masses of the components of chiral multiplets. The standard expression [62] for the spinor mass matrix in 4D $\mathcal{N}=1$ supergravity is

$$
m_{\alpha\beta} = \left(D_\alpha D_\beta W - \frac{2}{3} \left(D_\alpha W \right) \left(D_\beta W \right) - \Gamma^c_{\alpha\beta} D_c W \right) \frac{m_{3/2}}{W} \,. \tag{3.58}
$$

Since the Kähler moduli are not stabilized, we will only consider $\alpha = i, \tau$. The moduli space factorizes, so the connection Γ^A_{BC} will have no mixed components, $\Gamma^a_{\alpha\beta} = 0$. Imposing the global minimum condition $D_i W = D_\tau W = 0$ reduces the mass matrix

[5] For cubic prepotentials and physical moduli $z^i = Z^i / Z^0$, $\int \Omega_3 \wedge \partial_i \partial_j \partial_k \Omega_3 = \left(Z^0\right)^2 C_{ijk}$.

to

$$m_{\alpha\beta} = e^{K/2}\sqrt{\frac{\overline{W}}{W}}D_\alpha D_\beta W\,. \tag{3.59}$$

Note that the overall phase $\sqrt{\overline{W}/W}$ could be absorbed into the definition of the fermions, though we will not do so here. Substituting in the second derivatives computed above, the fermion mass matrix simplifies to

$$\begin{pmatrix} m_{ij} & m_{i\tau} \\ m_{\tau i} & m_{\tau\tau} \end{pmatrix} = \frac{m_{3/2}}{\overline{C}} \begin{pmatrix} \mathcal{F}_{ijk}C^k & -\frac{1}{2i\mathrm{Im}(\tau)}g_{i\bar{j}}\overline{C}^{\bar{j}} \\ -\frac{1}{2i\mathrm{Im}(\tau)}g_{i\bar{j}}\overline{C}^{\bar{j}} & 0 \end{pmatrix}. \tag{3.60}$$

Here we used (3.59) and (3.46), substituted in the second derivatives (3.50), (3.54), and (3.57), then simplified using (3.11), (3.21), (3.29), and (3.55). This demonstrates how, in the large volume scenario, the C^i determine the structure of the fermion mass matrix. These masses remain finite even in the limit $m_{3/2} \sim |C| \to 0$, since the ratio $m_{3/2}/\overline{C}$ approaches a finite value.

A few comments are in order. First, the fermion mass matrix has $2n+2$ real eigenvalues, two more than there are parameters C^i. This indicates that we cannot independently determine the masses of *all* of the moduli – for example, we could consider choosing the masses of the z^i, but then the mass of τ would be determined. It is also interesting that the form of m_{ij} suggests a generalized Higgs mechanism. If we think of the $\mathcal{F}_{ijk}$ as Yukawa couplings, than C^k appears to play the role of a Higgs vacuum expectation value. While the C^k do not correspond to the expectation values of any dynamical scalars, it is possible that they can be interpreted as the expectation values of auxiliary fields. Finally, if we can make $\mathrm{Im}\,(\tau) = 1/g_s$ large, then the smallest fermion mass will be roughly $m_{3/2}g_s^2$. It would be interesting to see if such a light mode is of phenomenological interest, perhaps at an intermediate scale.

3.3.3 Scalar Masses

In supersymmetric vacua, the masses of scalar fields should match the masses of their fermionic partners. However, the no-scale vacua that we consider generically break supersymmetry. While the F-terms for the complex structure moduli and axio-dilaton vanish, $D_i W = D_\tau W = 0$, the F-terms for the Kähler moduli only vanish when $W = 0$, as shown in (3.24). In this case, the scalar mass-squared matrix takes the following form:

$$\mathcal{M}^2 = \begin{pmatrix} M_{\alpha\beta} & M_{\alpha\overline{\beta}} \\ \overline{M}_{\overline{\alpha}\beta} & \overline{M}_{\overline{\alpha\beta}} \end{pmatrix}, \tag{3.61}$$

$$M^2_{\alpha\beta} = e^K \overline{W} \left(D_\alpha D_\beta W + D_\beta D_\alpha W \right), \tag{3.62}$$

$$M^2_{\alpha\overline{\beta}} = e^K \left[g^{\gamma\overline{\delta}} D_\alpha D_\gamma W \overline{D_\beta D_\delta W} + |W|^2 g_{\alpha\overline{\beta}} \right]. \tag{3.63}$$

While (3.62) and (3.63) would be standard expressions for a theory with *only* the complex structure moduli and axio-dilaton, we verify in appendix A that they also hold when Kähler moduli are included, and supersymmetry is broken in that sector. Note that when $W = 0$, i.e. when supersymmetry is preserved, $M^2_{\alpha\beta}$ vanishes and $M^2_{\alpha\overline{\beta}} = g^{\gamma\overline{\delta}} m_{\alpha\gamma} \overline{m}_{\overline{\delta\beta}}$, as expected. When $W \neq 0$, the scalar masses are lifted above the fermion masses, and the splitting of the masses-squared is of order $m^2_{3/2} = e^K |W|^2 \sim |CZ^0|^2$.

3.4 A Generating Function for the Flux Attractor Equations

In this section we develop an algorithm which, in principle, solves the flux attractor equations. To do so we adapt the OSV solution of the black hole attractor equations [21]. We begin with a change of variables designed to automatically solve the magnetic half of the attractor equations. Next, we rewrite the electric half of the attractor equations as derivatives of a generating function. Finally, a Legendre transform provides a formal solution of the attractor equations.

The generating function itself is quite interesting. In [21], the generating function governing the black hole attractor turned out to be the free energy of the black hole. Our interest in the generating function is not restricted to this section, rather we will discuss some of its general properties in section 3.5.

3.4.1 An Alternative Formulation of the Attractor Equations

The flux attractor equations (3.35)-(3.38) contain Kähler covariant derivatives, which we find much less convenient than ordinary derivatives. We therefore consider a modified version of (3.29) that does not have this problem:

$$G_3 = -i\mathrm{Im}\,(\tau)\left[\overline{C\Omega}_3 + L^I \partial_I \Omega_3\right], \tag{3.64}$$

where $\overline{C}$ and the L^I are coefficients. Note that we differentiate with respect to the Z^I, not the z^i.

The ISD condition (3.2) allows only $(2,1)$ and $(0,3)$ pieces in the complex flux G_3. While the *ansatz* (3.64) does not contain a $(1,2)$ piece, equation (3.14) shows that the $\partial_I \Omega_3$ term includes a $(3,0)$ piece. Since the ISD condition (3.2) forbids such a term, we must choose the L^I so that it is projected out. The appropriate condition on the L^I is

$$L^I \partial_I K_z = 0\,. \tag{3.65}$$

After imposing this condition, the resulting G_3 has only $(0,3)$ and $(2,1)$ pieces. We thus conclude that (3.64) and (3.65) together are equivalent to (3.29), with

$$C^i = \frac{\partial z^i}{\partial Z^I} L^I\,. \tag{3.66}$$

If we think of the C^i as given, then this fixes n of the $n+1$ components of L^I, and

(3.65) fixes the final component.

As in section 3.2.4, we can expand (3.64) and find a set of real attractor equations. This is equivalent to replacing $C^i D_i \to L^I \partial_I$ in (3.35)-(3.38) and adding the constraint (3.65). The resulting attractor equations are:

$$m_h^I = \mathrm{Re}\left[CZ^I + L^I\right], \tag{3.67}$$

$$m_f^I = \mathrm{Re}\left[\tau CZ^I + \overline{\tau} L^I\right], \tag{3.68}$$

$$e_I^h = \mathrm{Re}\left[CF_I + L^J F_{IJ}\right], \tag{3.69}$$

$$e_I^f = \mathrm{Re}\left[\tau CF_I + \overline{\tau} L^J F_{IJ}\right], \tag{3.70}$$

$$0 = L^I \left(\overline{F}_I - \overline{Z}^J F_{IJ}\right), \tag{3.71}$$

where we have introduced $F_{IJ} \equiv \partial_I F_J$, and used (3.8) to make the constraint (3.65) more explicit. The magnetic attractor equations (3.67) and (3.68) are simpler than their counterparts (3.35) and (3.36), in that the $C^i D_i Z^I$ term reduces to L^I. Similarly, the electric attractor equations (3.69) and (3.70) are simpler than (3.37) and (3.38) since the Kähler covariant derivatives have been replaced with ordinary derivatives.

Another benefit of these reformulated attractor equations is that the L^I transform in the $n+1$ of $SO(n+1)$, just like the Z^I and the fluxes, and in contrast to the C^i. This suggests solving (3.67)-(3.70) for CZ^I and L^I, treating the L^I on an equal footing with the CZ^I, then solving (3.71) for τ. This procedure is more practical than solving (3.35)-(3.38) for the $n+1$ vector CZ^I, n vector C^i, and scalar τ, even though the results are equivalent. We will demonstrate this by completely solving an explicit example in section 3.6.

3.4.2 Magnetic Attractor Equations and the Mixed Ensemble

We now solve the flux attractor equations by adapting the OSV procedure for solving the black hole attractor equations [21]. We treat τ as a fixed variable while solving

(3.67)-(3.70), then determine it at the very end by solving (3.71). The two sets of variables we have seen so far, $\left\{CZ^I, L^I, \tau\right\}$ and $\left\{m_h^I, m_f^I, e_I^h, e_I^f, \tau\right\}$, describe two different ensembles. Following OSV, we introduce a "mixed ensemble," $\left\{m_h^I, m_f^I, \phi_h^I, \phi_f^I, \tau\right\}$, where $\phi_{h,f}^I$ are potentials conjugate to the electric fluxes. When introducing these potentials, we require that:

1. The expressions for CZ^I and L^I in terms of $\left\{m_h^I, m_f^I, \phi_h^I, \phi_f^I, \tau\right\}$ automatically solve the "magnetic" attractor equations, (3.67) and (3.68).

2. The potentials $\left\{\phi_h^I, \phi_f^I\right\}$ transform like $\left\{m_h^I, m_f^I\right\}$ under S-duality.

3. The relationship between $\left\{CZ^I, L^I, \tau\right\}$ and $\left\{m_h^I, m_f^I, \phi_h^I, \phi_f^I, \tau\right\}$ is covariant under S-duality.

These conditions determine the relationship between $\left\{CZ^I, L^I, \tau\right\}$ and $\left\{m_h^I, m_f^I, \phi_h^I, \phi_f^I, \tau\right\}$ to be

$$CZ^I = \frac{1}{\tau-\bar{\tau}}\left(m_f^I - \bar{\tau} m_h^I\right) + \frac{1}{\tau-\bar{\tau}}\left(\phi_f^I - \bar{\tau}\phi_h^I\right), \tag{3.72}$$

$$L^I = -\frac{1}{\tau-\bar{\tau}}\left(m_f^I - \tau m_h^I\right) + \frac{1}{\tau-\bar{\tau}}\left(\phi_f^I - \tau\phi_h^I\right). \tag{3.73}$$

We will also want to know how derivatives with respect to Z^I and L^I are mapped into derivatives with respect to fluxes and the potentials. Here it is important to note that both sets of variables we are considering, $\left\{CZ^I, L^I, \tau\right\}$ and $\left\{m_h^I, m_f^I, \phi_h^I, \phi_f^I, \tau\right\}$, include τ as an *independent* variable. The derivatives are therefore related by

$$\frac{1}{C}\frac{\partial}{\partial Z^I} = \frac{1}{2}\left[\left(\frac{\partial}{\partial m_h^I} + \tau\frac{\partial}{\partial m_f^I}\right) + \left(\frac{\partial}{\partial \phi_h^I} + \tau\frac{\partial}{\partial \phi_f^I}\right)\right], \tag{3.74}$$

$$\frac{\partial}{\partial L^I} = \frac{1}{2}\left[\left(\frac{\partial}{\partial m_h^I} + \bar{\tau}\frac{\partial}{\partial m_f^I}\right) - \left(\frac{\partial}{\partial \phi_h^I} + \bar{\tau}\frac{\partial}{\partial \phi_f^I}\right)\right], \tag{3.75}$$

where all derivatives are taken with τ held fixed.

3.4.3 Electric Attractor Equations and the Generating Function

In the previous section we solved the magnetic attractor equations, (3.67) and (3.68). We now introduce an auxiliary function,

$$\mathcal{V} = 2\mathrm{Im}\,(\tau)\, C F_I L^I \,, \tag{3.76}$$

that simplifies the electric attractor equations, (3.69) and (3.70). This new function plays a role analogous to that of the prepotential in the solution of the black hole attractor equations. It enjoys the following properties:

1. Derivatives of $\mathcal{V}$ with respect to L^I give CF_I, one of the terms that appears in the electric attractor equations:

$$\frac{1}{2\mathrm{Im}\,(\tau)} \frac{\partial \mathcal{V}}{\partial L^I} = C F_I \,. \tag{3.77}$$

2. Derivatives with respect to Z^I give $L^J F_{IJ}$, the other term that appears in the electric attractor equations:

$$\frac{1}{2C\mathrm{Im}\,(\tau)} \frac{\partial \mathcal{V}}{\partial Z^I} = L^J F_{IJ} \,. \tag{3.78}$$

3. The factor of C in (3.76)makes $\mathcal{V}$ invariant under Kähler transformations.

4. By (3.30), (3.31), and (3.20), the factor of $\mathrm{Im}\,(\tau)$ in (3.76) makes $\mathcal{V}$ invariant under S-duality.

5. $\mathcal{V}$ is holomorphic in L^I and Z^I.

The first two properties will allow us to replace the F_I and $L^J F_{IJ}$ terms in the electric attractor equations, (3.69) and (3.70), with derivatives of $\mathcal{V}$. This is analogous to the role played by the prepotential in the solution of the electric black hole equations.

The invariance of $\mathcal{V}$ under Kähler transformations and S-duality (properties 3 and 4) will allow us to interpret it in terms of a physical quantity. Finally, we will make extensive use of holomorphy in the following manipulations.

As described above, we can rewrite the electric attractor equations (3.69) and (3.70) in terms of derivatives of $\mathcal{V}$,

$$e_I^h = \frac{1}{2\mathrm{Im}(\tau)}\mathrm{Re}\left[\left.\frac{\partial\mathcal{V}}{\partial L^I}\right|_{Z^J,L^{J\neq I},\tau} + \frac{1}{C}\left.\frac{\partial\mathcal{V}}{\partial Z^I}\right|_{Z^{J\neq I},L^J,\tau}\right], \tag{3.79}$$

$$e_I^f = \frac{1}{2\mathrm{Im}(\tau)}\mathrm{Re}\left[\tau\left.\frac{\partial\mathcal{V}}{\partial L^I}\right|_{Z^J,L^{J\neq I},\tau} + \overline{\tau}\frac{1}{C}\left.\frac{\partial\mathcal{V}}{\partial Z^I}\right|_{Z^{J\neq I},L^J,\tau}\right]. \tag{3.80}$$

We then use holomorphy of $\mathcal{V}$ to find

$$e_I^h = \frac{i}{2\mathrm{Im}(\tau)}\left(\frac{\partial}{\partial L^I} + \frac{1}{C}\cdot\frac{\partial}{\partial Z^I} - \frac{\partial}{\partial \overline{L}^I} - \frac{1}{\overline{C}}\frac{\partial}{\partial \overline{Z}^I}\right)\mathrm{Im}(\mathcal{V}), \tag{3.81}$$

$$e_I^f = \frac{i}{2\mathrm{Im}(\tau)}\left\{\tau\left(\frac{\partial}{\partial L^I} - \frac{1}{\overline{C}}\cdot\frac{\partial}{\partial \overline{Z}^I}\right) - \overline{\tau}\left(\frac{\partial}{\partial \overline{L}^I} - \frac{1}{C}\frac{\partial}{\partial Z^I}\right)\right\}\mathrm{Im}(\mathcal{V}). \tag{3.82}$$

Finally, we introduce derivatives with respect to the potentials using (3.74) and (3.75),

$$e_I^h = -\left[\frac{\partial}{\partial\phi_f^I}\mathrm{Im}(\mathcal{V})\right]_{\phi_h^{J\neq I},\phi_f^J,m_h^J,m_f^J,\tau}, \tag{3.83}$$

$$e_I^f = \left[\frac{\partial}{\partial\phi_h^I}\mathrm{Im}(\mathcal{V})\right]_{\phi_h^J,\phi_f^{J\neq I},m_h^J,m_f^J,\tau}, \tag{3.84}$$

Though we initially defined $\mathcal{V}$ in terms of L^I and Z^I, in this last step we simply substitute in (3.72) and (3.73) to make it a function of the magnetic fluxes and electric potentials.

It is remarkable that the electric attractor equations, which appear rather complex, reduce to derivatives of a single generating function! This is one of the principal results of this chapter.

Since we have made a rather long chain of substitutions and redefinitions, we briefly summarize our procedure for solving the flux attractor equations:

1. Take as inputs the fluxes $\left\{m_f^I, m_h^I, e_I^f, e_I^h\right\}$ and the symplectic section $\left\{Z^I, F_I\right\}$.

2. Insert the expressions for the F_I as functions of the Z^I into (3.76), giving $\mathcal{V}\left(L^I, CZ^I, \tau\right)$.

3. Substitute the expressions (3.72) and (3.73) into $\mathcal{V}\left(L^I, CZ^I, \tau\right)$ to get $\mathcal{V}\left(\phi_h^I, \phi_f^I, m_h^I, m_f^I, \tau\right)$.

4. Invert (3.83) and (3.84) to get expressions for ϕ_f^I and ϕ_h^I in terms of m_h^I, m_f^I, e_I^h, e_I^f, and τ.

5. Rewrite the constraint (3.71) in terms of m_h^I, m_f^I, e_I^h, e_I^f, and τ. Do this by substituting (3.72) and (3.73) into (3.71), then inserting the solutions for ϕ_f^I and ϕ_h^I in terms of m_h^I, m_f^I, e_I^h, e_I^f, and τ.

6. Solve the constraint (3.71) for τ as a function of the fluxes only. Substitute this back into the expressions for $\phi_{f,h}^I$ to get expressions for the potentials in terms of the fluxes only, and then insert τ and the potentials into the expressions (3.72) and (3.73) to get expressions for CZ^I and L^I in terms of the fluxes only.

The most difficult part of this procedure is step 4, which requires that we invert a system of $2n+2$ equations. Even in simple cases, these result in polynomials of impractically high order.

The electric attractor equations (3.83) and (3.84) take the form of thermodynamic relations, indicating that the potentials $\phi_{f,h}^I$ are *conjugate* to the fluxes $e_I^{f,h}$. This suggests the Legendre transformation

$$\mathcal{G} = \mathrm{Im}\left(\mathcal{V}\right) + e_I^h \phi_f^I - e_I^f \phi_h^I \,, \tag{3.85}$$

so that the electric attractor equations become

$$\phi_h^I = -\left[\frac{\partial \mathcal{G}}{\partial e_I^f}\right]_{e_J^h, e_{J\neq I}^f, m_h^J, m_f^J, \tau}, \tag{3.86}$$

$$\phi_f^I = \left[\frac{\partial \mathcal{G}}{\partial e_I^h}\right]_{e_{J\neq I}^h, e_J^f, m_h^J, m_f^J, \tau}. \tag{3.87}$$

This means that we only need to know a single function, $\mathcal{G}$, which is in principle determined by steps 1-4 above.

In practice, this may not be the best way to proceed. The analogue of $\mathcal{G}$ for the black hole attractor equations is the entropy S, which can be computed by many different methods. For example, the requirement that S be invariant under duality transformations severely constrains, and sometimes completely determines, its functional form [63].

3.4.4 The Constraint and the Generating Function

So far, we have demonstrated that the electric attractor equations (3.69) and (3.70) can be recast in terms of derivatives of a generating function. Indeed, we designed the generating function $\mathcal{G}$ specifically for this purpose. Next, we demonstrate a more surprising result: the constraint (3.71) can *also* be written in terms of derivatives of the same generating function.

We first compute $\tau-$derivatives of CZ^I and L^I in the $\left\{m_h^I, m_f^I, \phi_h^I, \phi_f^I, \tau\right\}$ ensemble, using (3.72) and (3.75):

$$\left.\frac{\partial Z^I}{\partial \tau}\right|_{m_h^I, m_f^I, \phi_h^I, \phi_f^I} = -\frac{Z^I}{\tau - \overline{\tau}}, \qquad \left.\frac{\partial \overline{Z}^I}{\partial \tau}\right|_{m_h^I, m_f^I, \phi_h^I, \phi_f^I} = \frac{Z^I}{\tau - \overline{\tau}}, \tag{3.88}$$

$$\left.\frac{\partial L^I}{\partial \tau}\right|_{m_h^I, m_f^I, \phi_h^I, \phi_f^I} = \frac{\overline{L}^I}{\tau - \overline{\tau}}, \qquad \left.\frac{\partial \overline{L}^I}{\partial \tau}\right|_{m_h^I, m_f^I, \phi_h^I, \phi_f^I} = -\frac{\overline{L}^I}{\tau - \overline{\tau}}. \tag{3.89}$$

Using these preliminary results, we find:

$$\frac{\partial}{\partial\tau}\left[\mathrm{Im}\left(\mathcal{V}\right)\right]_{m,\phi} = \frac{\partial}{\partial\tau}\left[2\mathrm{Im}\left(\tau\right)\mathrm{Im}\left(L^I F_I\right)\right]_{m,\phi} \tag{3.90}$$

$$\begin{aligned} &= -i\mathrm{Im}\left(L^I F_I\right) - i\mathrm{Im}\left(\tau\right)\left[\frac{\overline{L}^I}{\tau-\overline{\tau}}F_I + \frac{\overline{L}^I}{\tau-\overline{\tau}}\overline{F}_I\right] \\ &\quad -i\mathrm{Im}\left(\tau\right)\left[-L^I F_{IJ}\frac{Z^J}{\tau-\overline{\tau}} - \overline{L}^I\overline{F}_{IJ}\frac{Z^J}{\tau-\overline{\tau}}\right] \end{aligned} \tag{3.91}$$

$$= \frac{1}{2}\left[-L^I F_I - \overline{L}^I F_I + L^I F_{IJ}Z^J + \overline{L}^J\overline{F}_{IJ}Z^J\right] \tag{3.92}$$

$$= -\frac{1}{2}\overline{L}^I\left[F_I - \overline{F}_{IJ}Z^J\right], \tag{3.93}$$

using the homogeneity property $F_{IJ}Z^J = F_I$. The last line is proportional to the complex conjugate of the constraint (3.71). Setting $\partial\mathrm{Im}\left(\mathcal{V}\right)/\partial\tau = 0$ is thus equivalent to imposing (3.71). Notice that the overall factor of $\mathrm{Im}\left(\tau\right)$ included in $\mathcal{V}$, originally introduced to make $\mathcal{V}$ invariant under S-duality, is exactly what is required to recover the constraint (3.71) from $\partial\mathrm{Im}\left(\mathcal{V}\right)/\partial\tau$.

The Legendre transform that takes us from the $\left\{\phi_h^I, \phi_f^I, m_h^I, m_f^I, \tau\right\}$ ensemble to the $\left\{e_I^h, e_I^f, m_h^I, m_f^I, \tau\right\}$ ensemble does not change the equilibrium condition associated with τ. In the latter ensemble, the constraint (3.71) is equivalent to

$$\left.\frac{\partial\mathcal{G}}{\partial\tau}\right|_{e_I^h, e_I^f, m_h^I, m_f^I} = 0. \tag{3.94}$$

This completes our demonstration that the flux attractor equations can be interpreted as equilibrium conditions for a thermodynamic system. From the thermodynamic point of view, (3.93) indicates that τ is *conjugate* to the constraint (3.71).

While studying $\mathcal{G}$ in the $\left\{e_I^h, e_I^f, m_h^I, m_f^I, \tau\right\}$ ensemble may be conceptually clearer, there is a useful consequence of (3.94). Suppose we take derivatives of $\mathcal{G}$ *without* holding τ fixed. The result is:

$$\left.\frac{\partial \mathcal{G}}{\partial e_I^h}\right|_{e^h_{J\neq I},e^f_J,m_h^J,m_f^J} = \left.\frac{\partial \mathcal{G}}{\partial e_I^h}\right|_{e^h_{J\neq I},e^f_J,m_h^J,m_f^J,\tau} + \left.\frac{\partial \mathcal{G}}{\partial \tau}\right|_{e^h_I,e^f_I,m_h^I,m_f^I} \left.\frac{\partial \tau}{\partial e_I^h}\right|_{e^h_{J\neq I},e^f_J,m_h^J,m_f^J} \tag{3.95}$$

$$= \left.\frac{\partial \mathcal{G}}{\partial e_I^h}\right|_{e^h_{J\neq I},e^f_J,m_h^J,m_f^J,\tau}. \tag{3.96}$$

In other words, if we substitute the attractor value for τ into $\mathcal{G}$ we can simplify (3.86) and (3.87) to:

$$\phi_h^I = -\left[\frac{\partial \mathcal{G}}{\partial e_I^f}\right]_{e^h_J,e^f_{J\neq I},m_h^J,m_f^J}, \tag{3.97}$$

$$\phi_f^I = \left[\frac{\partial \mathcal{G}}{\partial e_I^h}\right]_{e^h_{J\neq I},e^f_J,m_h^J,m_f^J}. \tag{3.98}$$

If one can determine $\mathcal{G}$ as a function of arbitrary fluxes, then (3.97) and (3.98) determine the potentials $\phi^I_{h,f}$, (3.72) and (3.73) then determine the moduli Z^I and mass parameters L^I, and finally (3.71) determines the axio-dilaton τ. In this way the single function $\mathcal{G}$ determines the vacuum expectation values and masses of the moduli.

3.5 General Properties of the Generating Function

The generating function $\mathcal{G}$ introduced in (3.85) is the function that controls the flux attractor, giving attractor values for scalars and other physical quantities upon differentiation. In this section we initiate a general study of the generating function by demonstrating a simple relationship between $\mathcal{G}$ and the gravitino mass:

$$\mathcal{G} = \int F_3 \wedge H_3 - 2\mathrm{Vol}^2 m_{3/2}^2\,. \tag{3.99}$$

Note that the gravitino mass is to be considered a function of arbitrary fluxes. We first introduce a condensed, complex notation for the fluxes and potentials. We then exploit the homogeneity properties of $\mathcal{G}$ to prove the relationship (3.99).

3.5.1 Complex Fluxes and Potentials

One of the results of section 3.4 is that we can solve the electric and magnetic attractor equations (3.67)-(3.70) treating τ as a constant, then determine τ by solving (3.71). This justifies the introduction of the following complex fluxes and potentials:

$$m^I \equiv m_f^I - \tau m_h^I \,, \tag{3.100}$$

$$e_I \equiv e_I^f - \tau e_I^h \,, \tag{3.101}$$

$$\varphi^I \equiv \phi_f^I - \tau \phi_h^I \,. \tag{3.102}$$

We can then use (3.100) and (3.102) to rewrite (3.72) and (3.73) as

$$CZ^I = \frac{1}{2i\mathrm{Im}\,(\tau)} \left[\overline{m}^I + \overline{\varphi}^I\right] , \tag{3.103}$$

$$L^I = \frac{1}{2i\mathrm{Im}\,(\tau)} \left[-m^I + \varphi^I\right] . \tag{3.104}$$

We also define derivatives with respect to the complex electric fluxes as

$$\frac{\partial}{\partial e_I} \equiv \frac{i}{2\mathrm{Im}\,(\tau)} \left(\frac{\partial}{\partial e_I^h} + \overline{\tau} \frac{\partial}{\partial e_I^f} \right) , \tag{3.105}$$

where the normalization is chosen so that $\partial e_I / \partial e_J = \delta_I^J$. Definitions for $\partial / \partial m^I$ and $\partial / \partial \varphi^I$ are completely analogous. We can then rewrite the electric attractor equations (3.97) and (3.98) as

$$\varphi^I = 2i\mathrm{Im}\,(\tau) \frac{\partial \mathcal{G}}{\partial \overline{e}^I} \,, \tag{3.106}$$

and the expressions for CZ^I and L^I as

$$CZ^I = \frac{1}{2i\mathrm{Im}\,(\tau)}\left[\overline{m}^I - 2i\mathrm{Im}\,(\tau)\,\frac{\partial}{\partial e_I}\mathcal{G}\right], \tag{3.107}$$

$$L^I = \frac{1}{2i\mathrm{Im}\,(\tau)}\left[-m^I + 2i\mathrm{Im}\,(\tau)\,\frac{\partial}{\partial \overline{e}_I}\mathcal{G}\right]. \tag{3.108}$$

While (3.107) and (3.108) present a fairly compact version of the results of section (3.4), they treat the electric and magnetic fluxes quite differently. The generating function $\mathcal{G}$ is not homogeneous in either the electric or the magnetic fluxes *alone*, so a symplectic invariant version of (3.107) and (3.108) will be helpful. We formulate this by first introducing a new operator:

$$\partial \equiv \alpha_I \frac{\partial}{\partial e_I} + \beta^I \frac{\partial}{\partial m^I}, \tag{3.109}$$

which maps scalar functions of the fluxes to 3-forms. We then examine (3.67)-(3.70), and see that symplectic invariance requires

$$C\Omega_3 = \frac{1}{2i\mathrm{Im}\,(\tau)}\left[\overline{G}_3 - 2i\mathrm{Im}\,(\tau)\,\partial\mathcal{G}\right], \tag{3.110}$$

$$L^I\partial_I\Omega_3 = \frac{1}{2i\mathrm{Im}\,(\tau)}\left[-G_3 + 2i\mathrm{Im}\,(\tau)\,\overline{\partial}\mathcal{G}\right]. \tag{3.111}$$

These are equivalent to the electric attractor equations, so they must be supplemented by the constraint (3.71). This amounts to some flexibility in our treatment of $\mathcal{G}$. We can either use $\mathcal{G}\left(e_I, m^I, \tau\right)$ and take all derivatives with τ held fixed, as in (3.86) and (3.87), or substitute in the attractor value of τ to find $\mathcal{G}\left(e_I, m^I\right)$ and differentiate as in (3.97) and (3.98).

3.5.2 General Expression for the Generating Function

We now show that the relationship between the generating function $\mathcal{G}$ and the gravitino mass (3.99) holds for general compactifications. Our argument turns on a homogeneity property of the attractor equations that is evident from examining (3.67)-(3.71). These attractor equations are invariant under a uniform rescaling of the fluxes,

$$m^I_{h,f} \;\to\; e^\lambda m^I_{h,f}\,, \tag{3.112}$$

$$e_I^{h,f} \;\to\; e^\lambda e_I^{h,f}\,, \tag{3.113}$$

provided that we simultaneously rescale

$$CZ^I \;\to\; e^\lambda CZ^I\,, \tag{3.114}$$

$$L^I \;\to\; e^\lambda L^I\,. \tag{3.115}$$

If we then turn our attention to the expressions for the CZ^I and L^I in terms of fluxes and potentials,(3.72) and (3.73), we see that the potentials must transform as

$$\phi^I_{h,f} \to e^\lambda \phi^I_{h,f}\,. \tag{3.116}$$

Equations (3.97) and (3.98) then indicate that if the potentials are to be homogeneous of degree one in the fluxes, then $\mathcal{G}$ must be homogeneous of degree two in the fluxes. If we use the complex fluxes introduced in (3.100) and (3.101), we find that $\mathcal{G}$ is homogeneous of degree one in the complex fluxes and degree one in their conjugates. This homogeneity implies that

$$\int G_3 \wedge \partial\mathcal{G} = \left[m^I \frac{\partial}{\partial m^I} + e_I \frac{\partial}{\partial e_I}\right]\mathcal{G} = \mathcal{G}, \tag{3.117}$$

where we used the orthogonality relations (3.4) and (3.5) and expansions (3.6) and (3.7) to compute the integral. We will now use this result to compute the superpotential and Kähler potential at the attractor point, and finally the gravitino mass.

We begin with the superpotential (3.21), then substitute in (3.110):

$$\begin{aligned} CW &= \int G_3 \wedge C\Omega_3 && (3.118)\\ &= \frac{1}{2i\mathrm{Im}\,(\tau)} \left[\int G_3 \wedge \overline{G}_3 - 2i\mathrm{Im}\,(\tau) \int G_3 \wedge \partial \mathcal{G}\right] && (3.119)\\ &= \int F_3 \wedge H_3 - \mathcal{G}. && (3.120) \end{aligned}$$

In order to determine the Kähler potential we need to compute

$$\begin{aligned} |C|^2 \int \Omega_3 \wedge \overline{\Omega}_3 &= \frac{1}{4\mathrm{Im}\,(\tau)^2} \int \left(\overline{G}_3 - 2i\mathrm{Im}\,(\tau)\,\partial\mathcal{G}\right) \wedge \left(G_3 + 2i\mathrm{Im}\,(\tau)\,\overline{\partial}\mathcal{G}\right) && (3.121)\\ &= \frac{1}{4\mathrm{Im}\,(\tau)^2} \left[- \int G_3 \wedge \overline{G}_3 + 2i\mathrm{Im}\,(\tau) \left(\int G_3 \wedge \partial\mathcal{G} + \int \overline{G}_3 \wedge \overline{\partial}\mathcal{G}\right) \right.\\ &\quad \left. + 4\mathrm{Im}\,(\tau)^2 \int \partial\mathcal{G} \wedge \overline{\partial}\mathcal{G} \right] && (3.122)\\ &= -\frac{i}{\mathrm{Im}\,(\tau)} \left[\int F_3 \wedge H_3 - \mathcal{G} \right]. && (3.123) \end{aligned}$$

In the last step we used (3.117) and

$$4\mathrm{Im}\,(\tau)^2 \int \partial\mathcal{G} \wedge \overline{\partial}\mathcal{G} = - \int G_3 \wedge \overline{G}_3\,, \qquad (3.124)$$

which we prove as follows. $L^I \partial_I \Omega_3$ contains only $(3,0)$ and $(2,1)$ pieces, so if we integrate it against Ω_3 the result must vanish:

$$\begin{aligned}
0 &= \int C\Omega_3 \wedge L^I \partial_I \Omega_3 && (3.125)\\
&= -\frac{1}{4\mathrm{Im}\,(\tau)^2}\int \left(\overline{G}_3 - 2i\mathrm{Im}\,(\tau)\,\partial\mathcal{G}\right)\wedge\left(-G_3 + 2i\mathrm{Im}\,(\tau)\,\overline{\partial}\mathcal{G}\right) && (3.126)\\
&= -\frac{1}{4\mathrm{Im}\,(\tau)^2}\left[\int G_3\wedge\overline{G}_3 + 2i\mathrm{Im}\,(\tau)\left(\int \overline{G}_3\wedge\overline{\partial}\mathcal{G} - \int G_3\wedge\partial\mathcal{G}\right)\right. \\
&\quad \left. +4\mathrm{Im}\,(\tau)^2\int \partial\mathcal{G}\wedge\overline{\partial}\mathcal{G}\right] && (3.127)\\
&= -\frac{1}{4\mathrm{Im}\,(\tau)^2}\left[\int G_3\wedge\overline{G}_3 + 4\mathrm{Im}\,(\tau)^2\int \partial\mathcal{G}\wedge\overline{\partial}\mathcal{G}\right], && (3.128)
\end{aligned}$$

which implies (3.124).

We now write out the gravitino mass (3.46) with the full Kähler potential (3.23):

$$\begin{aligned}
\mathrm{Vol}^2 m_{3/2}^2 &= \frac{|CW|^2}{2i\mathrm{Im}\,(\tau)\,|C|^2\int\Omega_3\wedge\overline{\Omega}_3} && (3.129)\\
&= \frac{1}{2}\left[\int F_3\wedge H_3 - \mathcal{G}\right]. && (3.130)
\end{aligned}$$

Reorganizing this we find the generating function,

$$\mathcal{G} = \int F_3\wedge H_3 - 2\mathrm{Vol}^2 m_{3/2}^2\,, \tag{3.131}$$

as we wanted to show. We also point out a curious relationship:

$$\mathrm{Vol}^2 m_{3/2}^2 = \frac{1}{2}CW\,, \tag{3.132}$$

where both quantities are evaluated at the attractor point. One could have imagined that other duality-invariant quantities, e.g. eigenvalues of the mass matrix, would appear in one or more of these expressions, but they do not. We also point out that the combination $\mathrm{Vol}^2 m_{3/2}^2$ is independent of the Kähler moduli, which cannot be stabilized by turning on 3-form fluxes.

As a side product of our derivation, we find another interesting identity. While one combination of (3.110) and (3.111) gives (3.29), another combination appears more novel:

$$\overline{\partial}\mathcal{G} = \frac{1}{2}\left[L^I\partial_I\Omega_3 - \overline{C\Omega}_3\right]. \tag{3.133}$$

The operator introduced in (3.109) is nilpotent,

$$\int \partial \wedge \partial = \frac{\partial}{\partial e_I}\frac{\partial}{\partial m^I} - \frac{\partial}{\partial m^I}\frac{\partial}{\partial e_I} = 0\,, \tag{3.134}$$

so we find that

$$\int \overline{\partial} \wedge \left[L^I\partial_I\Omega_3 - \overline{C\Omega}_3\right] = 0\,, \tag{3.135}$$

in other words $L^I\partial_I\Omega_3 - \overline{C\Omega}_3$ is $\overline{\partial}-$closed. Indeed, according to (3.133) it is $\overline{\partial}-$exact. This observation may motivate the introduction of the generating function $\mathcal{G}$ even in cases where the F_I are not globally well-defined.

3.6 An Explicit Solution of the Attractor Equations

In this section we find an explicit solution to the attractor equations for a particular prepotential:

$$F = \frac{Z^1Z^2Z^3}{Z^0}\,. \tag{3.136}$$

This prepotential appears frequently in the supergravity literature as the STU model [64–67], while in the flux compactification literature it appears as the untwisted sector of a $T^6/\mathbb{Z}^2 \times \mathbb{Z}^2 \approx T^2 \times T^2 \times T^2$ orbifold [26, 68]. Because it is a truncation of $\mathcal{N} = 8$ supergravity it has a number of useful symmetries. On the other hand, it shares many features with more generic prepotentials, and so is of broader interest than the pure $\mathcal{N} = 8$ model.

We first write down the attractor equations explicitly for an arbitrary set of fluxes. For a *subset* of all possible fluxes, we are able to solve the attractor equations, finding explicit expressions for the complex structure moduli and τ. We then compute the generating function $\mathcal{G}$ and the gravitino mass, and verify that the proposed relationship between them (3.99) holds in this case. We conclude with a discussion of the U-duality group for this model, and describe how to generalize the solution for our subset of fluxes to a solution for general fluxes.

3.6.1 Symplectic Section and Electric Attractor Equations

In order to make the attractor equations (3.67)-(3.71) completely explicit, we need to specify the symplectic section $\left\{Z^I, F_I\right\}$. In the present case the F_I are just derivatives of the prepotential (3.136):

$$F_I = \frac{\partial F}{\partial Z^I}\,, \tag{3.137}$$

with $I = 0, 1, 2, 3$. We substitute (3.137) into (3.76) to find the generating function in the mixed ensemble:

$$\mathcal{V}\left(m^I, \varphi^I, \tau\right) = 2\mathrm{Im}\left(\tau\right) C\left[-L^0 \frac{Z^1 Z^2 Z^3}{\left(Z^0\right)^2} + L^1 \frac{Z^2 Z^3}{Z^0} + L^2 \frac{Z^3 Z^1}{Z^0} + L^3 \frac{Z^1 Z^2}{Z^0}\right]. \tag{3.138}$$

$$\begin{aligned} &= \frac{1}{2\mathrm{Im}\left(\tau\right)\left(\overline{m}^0 + \overline{\varphi}^0\right)} \left\{ \frac{-m^0 + \varphi^0}{\overline{m}^0 + \overline{\varphi}^0} \left[\overline{m}^1 + \overline{\varphi}^1\right] \left[\overline{m}^2 + \overline{\varphi}^2\right] \left[\overline{m}^3 + \overline{\varphi}^3\right] \right. \\ &\quad - \left[-m^1 + \varphi^1\right] \left[\overline{m}^2 + \overline{\varphi}^2\right] \left[\overline{m}^3 + \overline{\varphi}^3\right] - \left[\overline{m}^1 + \overline{\varphi}^1\right] \left[-m^2 + \varphi^2\right] \left[\overline{m}^3 + \overline{\varphi}^3\right] \\ &\quad \left. - \left[\overline{m}^1 + \overline{\varphi}^1\right] \left[\overline{m}^2 + \overline{\varphi}^2\right] \left[-m^3 + \varphi^3\right] \right\}. \end{aligned} \tag{3.139}$$

Since $\mathcal{V}$ is a function of magnetic charges and electric potentials, we substituted in (3.103) and (3.104) for the Z^I and L^I. The electric attractor equations (3.83) and

(3.84) require that we differentiate[6] Im $(\mathcal{V})$:

$$\overline{e}_0 = -2i\mathrm{Im}\,(\tau)\,\frac{\partial}{\partial\varphi^0}\frac{\mathcal{V}-\overline{\mathcal{V}}}{2i} \tag{3.140}$$

$$\begin{aligned} &= -\frac{1}{2\left(\overline{m}^0+\overline{\varphi}^0\right)^2}\left[\overline{m}^1+\overline{\varphi}^1\right]\left[\overline{m}^2+\overline{\varphi}^2\right]\left[\overline{m}^3+\overline{\varphi}^3\right] \\ &\quad -\frac{1}{2\left(m^0+\varphi^0\right)^2}\left\{2\frac{-\overline{m}^0+\overline{\varphi}^0}{m^0+\varphi^0}\left[m^1+\varphi^1\right]\left[m^2+\varphi^2\right]\left[m^3+\varphi^3\right]\right. \\ &\quad -\left[-\overline{m}^1+\overline{\varphi}^1\right]\left[m^2+\varphi^2\right]\left[m^3+\varphi^3\right]-\left[m^1+\varphi^1\right]\left[-\overline{m}^2+\overline{\varphi}^2\right]\left[m^3+\varphi^3\right] \\ &\quad \left.-\left[m^1+\varphi^1\right]\left[m^2+\varphi^2\right]\left[-\overline{m}^3+\overline{\varphi}^3\right]\right\}, \end{aligned} \tag{3.141}$$

$$\overline{e}_1 = -2i\mathrm{Im}\,(\tau)\,\frac{\partial}{\partial\varphi^1}\frac{\mathcal{V}-\overline{\mathcal{V}}}{2i} \tag{3.142}$$

$$\begin{aligned} &= \frac{1}{2\left(\overline{m}^0+\overline{\varphi}^0\right)}\left[\overline{m}^2+\overline{\varphi}^2\right]\left[\overline{m}^3+\overline{\varphi}^3\right] \\ &\quad +\frac{1}{2\left(m^0+\varphi^0\right)}\left\{\frac{-\overline{m}^0+\overline{\varphi}^0}{m^0+\varphi^0}\left[m^2+\varphi^2\right]\left[m^3+\varphi^3\right]\right. \\ &\quad \left.-\left[-\overline{m}^2+\overline{\varphi}^2\right]\left[m^3+\varphi^3\right]-\left[m^2+\varphi^2\right]\left[-\overline{m}^3+\overline{\varphi}^3\right]\right\}, \end{aligned} \tag{3.143}$$

where the φ^I-derivatives are defined analogous to e_I-derivatives (3.105). The equations for $\overline{e}_2$ and $\overline{e}_3$ are cyclic permutations of (3.143), so we have a system of four complex equations.

We also need to make the constraint (3.71) explicit. For the prepotential (3.136),

[6] We could also have substituted our F_I directly into the electric attractor equations (3.69) and (3.70), then made the change of variables (3.72) and (3.73). This gives an identical result, indicating that our Im $(\mathcal{V})$ correctly generates the electric attractor equations.

it reduces to

$$
\begin{aligned}
0 &= L^I\overline{F}_I - L^I F_{IJ}\overline{Z}^J \qquad (3.144)\\
&= -L^0\frac{\overline{Z}^1\overline{Z}^2\overline{Z}^3}{\left(\overline{Z}^0\right)^2} + \left[L^1\frac{\overline{Z}^2\overline{Z}^3}{\overline{Z}^0} + \text{cyc.}\right] - 2L^0\frac{Z^1Z^2Z^3}{(Z^0)^3}\overline{Z}^0 + \left[L^0\frac{Z^1Z^2}{(Z^0)^2}\overline{Z}^3 + \text{cyc.}\right]\\
&\quad + \left[L^1\frac{Z^2Z^3}{(Z^0)^2}\overline{Z}^0 + \text{cyc.}\right] - \left[L^1\frac{Z^2}{Z^0}\overline{Z}^3 + L^1\frac{Z^3}{Z^0}\overline{Z}^2 + \text{cyc.}\right]. \qquad (3.145)
\end{aligned}
$$

After we substitute in (3.72) and (3.73) this expands out to

$$
\begin{aligned}
0 &= -\left(-m^0+\varphi^0\right)\frac{(m^1+\varphi^1)(m^2+\varphi^2)(m^3+\varphi^3)}{(m^0+\varphi^0)^2}\\
&\quad + \left[\left(-m^1+\varphi^1\right)\frac{(m^2+\varphi^2)(m^3+\varphi^3)}{m^0+\varphi^0} + \text{cyc.}\right]\\
&\quad -2\left(-m^0+\varphi^0\right)\frac{(\overline{m}^1+\overline{\varphi}^1)(\overline{m}^2+\overline{\varphi}^2)(\overline{m}^3+\overline{\varphi}^3)}{(\overline{m}^0+\overline{\varphi}^0)^3}\left(m^0+\varphi^0\right)\\
&\quad + \left[\left(-m^0+\varphi^0\right)\frac{(\overline{m}^1+\overline{\varphi}^1)(\overline{m}^2+\overline{\varphi}^2)}{(\overline{m}^0+\overline{\varphi}^0)^2}\left(m^3+\varphi^3\right) + \text{cyc.}\right]\\
&\quad + \left[\left(-m^1+\varphi^1\right)\frac{(\overline{m}^2+\overline{\varphi}^2)(\overline{m}^3+\overline{\varphi}^3)}{(\overline{m}^0+\overline{\varphi}^0)^2}\left(m^0+\varphi^0\right) + \text{cyc.}\right] \qquad (3.146)\\
&\quad - \left[\left(-m^1+\varphi^1\right)\frac{\overline{m}^2+\overline{\varphi}^2}{\overline{m}^0+\overline{\varphi}^0}\left(m^3+\varphi^3\right) + \left(-m^1+\varphi^1\right)\frac{\overline{m}^3+\overline{\varphi}^3}{\overline{m}^0+\overline{\varphi}^0}\left(m^2+\varphi^2\right) + \text{cyc.}\right].
\end{aligned}
$$

This appears to be another high-order polynomial equation in many variables.

We need to invert (3.141), (3.143), and (3.146) and find both the electric potentials φ^I and τ as functions of the electric and magnetic fluxes. Doing this by brute force would be quite challenging, as each equation is at least cubic in the potentials. Although we have written the attractor equations in terms of complex potentials and fluxes they are clearly not holomorphic in the potentials, so even counting the number of distinct solutions (sometimes called "area codes" [59, 69–72]) for general fluxes appears difficult. In the following we will find a solution to these equations using the ideas developed in section (3.4).

3.6.2 Reduction to Eight Fluxes

Much of the difficulty in solving (3.141), (3.143), and (3.146) arises from their dependence on both m^I, φ^I, *and* $\overline{m}^I$, $\overline{\varphi}^I$. Things simplify quite a bit if we set $m_h^0 = m_f^i = e_0^f = e_i^h = 0$, and make the *ansatz* that $\mathrm{Re}\,(\tau) = \phi_h^0 = \phi_f^I = 0$, so that the complex fluxes and potentials become:

$$\begin{aligned} m^0 &= m_f^0\,, && (3.147)\\ m^i &= -i\mathrm{Im}\,(\tau)\, m_h^i\,, && (3.148)\\ e_0 &= -i\mathrm{Im}\,(\tau)\, e_0^h\,, && (3.149)\\ e_i &= e_i^f\,, && (3.150)\\ \varphi^0 &= \phi_f^0\,, && (3.151)\\ \varphi^i &= -i\mathrm{Im}\,(\tau)\, \phi_h^i\,. && (3.152) \end{aligned}$$

This makes it easy to take the complex conjugate of a flux or potential: $\overline{m}^0 = m^0$, $\overline{e}_i = e_i$, $\overline{\varphi}^0 = \varphi^0$, $\overline{m}^i = -m^i$, $\overline{e}_0 = -e_0$, and $\overline{\varphi}^i = -\varphi^i$.

If we apply these restrictions to (3.141), (3.143), and (3.146) we find:

$$\begin{aligned} e_0 &= -\frac{(m^1+\varphi^1)(m^2+\varphi^2)(m^3+\varphi^3)}{2\,(m^0+\varphi^0)^2}\left\{1 - 2\frac{-m^0+\varphi^0}{m^0+\varphi^0} - \frac{-m^1+\varphi^1}{m^1+\varphi^1}\right.\\ &\quad \left. -\frac{-m^2+\varphi^2}{m^2+\varphi^2} - \frac{-m^3+\varphi^3}{m^3+\varphi^3}\right\}, && (3.153)\\ e_1 &= \frac{(m^2+\varphi^2)(m^3+\varphi^3)}{2\,(m^0+\varphi^0)}\left\{1 + \frac{-m^0+\varphi^0}{m^0+\varphi^0} + \frac{-m^2+\varphi^2}{m^2+\varphi^2} + \frac{-m^3+\varphi^3}{m^3+\varphi^3}\right\}, && (3.154)\\ 0 &= \frac{(m^1+\varphi^1)(m^2+\varphi^2)(m^3+\varphi^3)}{2\,(m^0+\varphi^0)^2}\left[\frac{-m^0+\varphi^0}{m^0+\varphi^0} + \frac{-m^1+\varphi^1}{m^1+\varphi^1}\right.\\ &\quad \left. +\frac{-m^2+\varphi^2}{m^2+\varphi^2} + \frac{-m^3+\varphi^3}{m^3+\varphi^3}\right]. && (3.155) \end{aligned}$$

Note that the same prefactor appears in (3.153) and (3.155). So long as $e_0 \neq 0$, we conclude that the factor in square brackets in (3.155) must vanish. We can apply this to (3.153) and (3.154) to arrive at a simpler set of equations:

$$e_0 = -\frac{(m^1+\varphi^1)(m^2+\varphi^2)(m^3+\varphi^3)}{(m^0+\varphi^0)^3}m^0\,, \tag{3.156}$$

$$e_1 = \frac{(m^2+\varphi^2)(m^3+\varphi^3)}{(m^0+\varphi^0)(m^1+\varphi^1)}m^1\,, \tag{3.157}$$

$$0 = \frac{-m^0+\varphi^0}{m^0+\varphi^0}+\frac{-m^1+\varphi^1}{m^1+\varphi^1}+\frac{-m^2+\varphi^2}{m^2+\varphi^2}+\frac{-m^3+\varphi^3}{m^3+\varphi^3}\,. \tag{3.158}$$

As usual, expressions for e_2 and e_3 arise from cyclic permutations of (3.157). In the next section we will explicitly invert these equations.

3.6.3 Moduli, Potentials, and Mass Parameters (Reduced Fluxes)

We begin by solving for the physical complex structure moduli,

$$z^i \equiv \frac{Z^i}{Z^0} = \frac{\overline{m}^i+\overline{\varphi}^i}{\overline{m}^0+\overline{\varphi}^0} = -\frac{m^i+\varphi^i}{m^0+\varphi^0}\,. \tag{3.159}$$

The ratio of (3.156) and (3.157) can be solved for the z^i :

$$\frac{e_i}{e_0} = -\left(\frac{m^0+\varphi^0}{m^i+\varphi^i}\right)^2\frac{m^i}{m^0} = -\frac{1}{(z^i)^2}\frac{m^i}{m^0}\,. \tag{3.160}$$

In order to avoid awkward branch cuts when we take the square root, we will carefully analyze the signs on the charges. If we insert the real charges and potentials into the previous expression,

$$\frac{e_i^f}{e_0^h} = -\frac{m_h^i}{m_f^0}\left(\frac{m_f^0+\phi_f^0}{m_h^i+\phi_h^i}\right)^2, \tag{3.161}$$

we find that $e_i^f m_f^0/e_0^h m_h^i < 0$, and thus that $e_i m^0/e_0 m^i > 0$. We must also consider the Kähler potential (3.11) with the prepotential (3.136). Evaluating it, we find

$$K_z = -\log\left|Z^0\right|^2 - \log\left[-8\mathrm{Im}\left(z^1\right)\mathrm{Im}\left(z^2\right)\mathrm{Im}\left(z^3\right)\right]. \tag{3.162}$$

The condition that the volume of each of the underlying T^2's is positive requires $\text{Im}\left(z^i\right) < 0$, which in turn implies that K_z is real. This determines the expression for the modulus:

$$z^i = -i\sqrt{\frac{e_0 m^i}{m^0 e_i}} = -i\text{Im}\left(\tau\right)\sqrt{-\frac{e_0^h m_h^i}{m_f^0 e_i^f}}\,. \tag{3.163}$$

In order to make this completely explicit we must solve for $\text{Im}\left(\tau\right)$, so we will do that next.

We can use (3.163) to simplify (3.156):

$$e_0 = z^1 z^2 z^3 m^0 = i\sqrt{\frac{\left(e_0\right)^3 m^1 m^2 m^3}{\left(m^0\right)^3 e_1 e_2 e_3}}\, m^0\,. \tag{3.164}$$

All dependence on the potentials has been eliminated, so this is a single equation that determines $\text{Im}\left(\tau\right)$. Substituting in real quantities, we find

$$1 \;=\; -\text{sgn}\left(m_f^0 e_0^h\right)\sqrt{-\text{Im}\left(\tau\right)^4\,\frac{e_0^h m_h^1 m_h^2 m_h^3}{m_f^0 e_1^f e_2^f e_3^f}}\,. \tag{3.165}$$

Note that the $\text{sgn}\left(m_f^0 e_0^h\right)$ appeared when we pulled the factor of m_f^0/e_0^h under the square root. We now find that

$$\text{Im}\left(\tau\right) \;=\; \left(-\frac{m_f^0 e_1^f e_2^f e_3^f}{e_0^h m_h^1 m_h^2 m_h^3}\right)^{1/4}, \tag{3.166}$$

where the physical condition $\text{Im}\left(\tau\right) = e^{-\phi}$ dictates that we use the real, positive branch, and implies that K_τ (3.23) is real[7].

Equation (3.165) also implies that $\text{sgn}\left(m_f^0 e_0^h\right) = -1$. We can combine this with our

[7] It is somewhat awkward that our Kähler potential requires $\text{Im}\left(\tau\right) > 0$ but $\text{Im}\left(z^i\right) < 0$, especially if we want to consider this model as a compactification of F-theory. On the other hand, our conventions are self-consistent, and chosen to agree with the bulk of the literature on flux compactifications.

earlier result that $\mathrm{sgn}\left(m_f^0 e_0^h m_h^i e_i^f\right) = -1$ to find a complete set of sign restrictions:

$$-\mathrm{sgn}\left(m_f^0 e_0^h\right) = \mathrm{sgn}\left(m_h^1 e_1^f\right) = \mathrm{sgn}\left(m_h^2 e_2^f\right) = \mathrm{sgn}\left(m_h^3 e_3^f\right) = +1. \tag{3.167}$$

Only 1/16 of the possible fluxes satisfy the physical conditions we have imposed. It is interesting to consider what might happen if we relaxed these sign restrictions. Suppose we chose signs that violated some of the conditions in (3.167), but satisfied the *product* of those conditions. The Kähler potential (3.23) would still be real, so we would still have solutions to the ISD condition, at least formally. The caveat is that the complex structures of some of the T^2's would no longer be in the upper half-plane and/or the sign of the string coupling would be negative. At a minimum, then, we would have to give up the conventional geometrical interpretation of the moduli. Going even further, we can consider signs such that the *product* of the conditions in (3.167) are violated. Then the Kähler potential (3.23) would not be real and it is not clear that the proposed solution would, in fact, be a solution. Indeed, for such flux assignments there may not be any solutions to the ISD conditions at all. In the following we will analyze only the clearly physical solutions that satisfy (3.167).

We can compare our restrictions with a more familiar one [27]. If we assume that the attractor equations can be satisfied, i.e. (3.29), then

$$\begin{aligned} \int F_3 \wedge H_3 &= \frac{1}{2i\mathrm{Im}\left(\tau\right)} \int G_3 \wedge \overline{G}_3 & (3.168) \\ &= \frac{e^{-K_z}}{2\mathrm{Im}\left(\tau\right)} \left[\left|C\right|^2 + \left|C^i\right|^2 \right], & (3.169) \end{aligned}$$

and thus $\int F_3 \wedge H_3$ is positive. The sign restrictions (3.167) are consistent with this, but stronger. If we evaluate $\int F_3 \wedge H_3$ for our reduced fluxes,

$$\int F_3 \wedge H_3 = -e_0^h m_f^0 + e_i^f m_h^i \,, \tag{3.170}$$

we see that the sign restrictions require that *each term* be positive.

Having determined $\mathrm{Im}(\tau)$ and the sign restrictions on the various fluxes, (3.163) gives explicit expressions for the complex structure moduli:

$$z^1 = -i\left[\left(-\frac{e^h_0}{m^0_f}\right)\left(\frac{m^1_h}{e^f_1}\right)\left(\frac{e^f_2}{m^2_h}\right)\left(\frac{e^f_3}{m^3_h}\right)\right]^{1/4}, \tag{3.171}$$

and cyclic permutations. These explicit expressions for the physical moduli, along with the dilaton (3.166) and the restrictions on the fluxes (3.167), are some of the principal results of this example.

Up to this point we have solved for the moduli and derived a set of restrictions on the fluxes, but we haven't yet solved for the potentials. The only equation that we haven't solved is the constraint (3.158), so let's turn our attention there. We can rewrite that equation as

$$m^0+\varphi^0 = \frac{m^0}{2}\left\{1+\frac{m^1}{m^0}\frac{m^0+\varphi^0}{m^1+\varphi^1}+\frac{m^2}{m^0}\frac{m^0+\varphi^0}{m^2+\varphi^2}+\frac{m^3}{m^0}\frac{m^0+\varphi^0}{m^3+\varphi^3}\right\}. \tag{3.172}$$

Combining (3.159) and (3.163), we find

$$m^0+\varphi^0 = \frac{m^0}{2}\left\{1-i\frac{m^1}{m^0}\sqrt{\frac{m^0e_1}{e_0m^1}}-i\frac{m^2}{m^0}\sqrt{\frac{m^0e_2}{e_0m^2}}-i\frac{m^3}{m^0}\sqrt{\frac{m^0e_3}{e_0m^3}}\right\}. \tag{3.173}$$

We now rewrite this in terms of real quantities:

$$\begin{aligned}\phi^0_f &= \frac{m^0_f}{2}\left\{-1-\mathrm{sgn}\left(m^0_fm^1_h\right)\sqrt{-\frac{m^1_he^f_1}{e^h_0m^0_f}}-\mathrm{sgn}\left(m^0_fm^2_h\right)\sqrt{-\frac{m^2_he^f_2}{e^h_0m^0_f}}\right.\\ &\qquad\left.-\mathrm{sgn}\left(m^0_fm^3_h\right)\sqrt{-\frac{m^3_he^f_3}{e^h_0m^0_f}}\right\}.\end{aligned} \tag{3.174}$$

If we again use the relation between $m^0+\varphi^0$ and $m^1+\varphi^1$, (3.159), we find the following expression for ϕ_h^1 :

$$\begin{aligned}\phi_h^1 &= \frac{m_h^1}{2}\left\{-1 - \mathrm{sgn}\left(m_h^1 m_f^0\right)\sqrt{-\frac{m_f^0 e_0^h}{m_h^1 e_1^f}} + \mathrm{sgn}\left(m_h^1 m_h^2\right)\sqrt{\frac{m_h^2 e_2^f}{m_h^1 e_1^f}}\right.\\ &\quad \left. + \mathrm{sgn}\left(m_h^1 m_h^3\right)\sqrt{\frac{m_h^3 e_3^f}{m_h^1 e_1^f}}\right\}. \qquad (3.175)\end{aligned}$$

This completes our inversion of (3.156), (3.157), and (3.158).

We emphasized earlier in this chapter that the attractor equations include the mass parameters C^i on equal terms with the moduli z^i. With (3.174) and (3.175) in hand, it is straightforward to compute the C^i. We first insert our $z^i = Z^i/Z^0$ into (3.66) to make the relationship between the C^i and L^I explicit:

$$C^i Z^0 = -z^i L^0 + L^i \,. \qquad (3.176)$$

Note that the combination $C^i Z^0$ is Kähler-invariant, while C^i alone is not. If we substitute (3.73) and (3.163) into (3.176), we find

$$
\begin{aligned}
C^1 Z^0 &= \sqrt{-\frac{e_0^h m_h^1}{m_f^0 e_1^f}}\frac{1}{2}\left(-m_f^0+\phi_f^0\right)-\frac{1}{2}\left(-m_h^1+\phi_h^1\right) && (3.177)\\
&= \frac{1}{2}\left[\mathrm{sgn}\left(m_f^0 m_h^1\right)\frac{m_h^1}{m_f^0}\sqrt{-\frac{m_f^0 e_0^h}{m_h^1 e_1^f}}\left(-m_f^0+\phi_f^0\right)-\left(-m_h^1+\phi_h^1\right)\right] && (3.178)\\
&= \frac{m_h^1}{4}\mathrm{sgn}\left(m_f^0 m_h^1\right)\sqrt{-\frac{m_f^0 e_0^h}{m_h^1 e_1^f}}\left[-3-\sum_{i=1}^{3}\mathrm{sgn}\left(m_f^0 m_h^i\right)\sqrt{-\frac{m_h^i e_i^f}{e_0^h m_f^0}}\right]\\
&\quad -\frac{m_h^1}{4}\left[-3-\mathrm{sgn}\left(m_h^1 m_f^0\right)\sqrt{-\frac{m_f^0 e_0^h}{m_h^1 e_1^f}}+\mathrm{sgn}\left(m_h^1 m_h^2\right)\sqrt{\frac{m_h^2 e_2^f}{m_h^1 e_1^f}}+\mathrm{sgn}\left(m_h^1 m_h^3\right)\sqrt{\frac{m_h^3 e_3^f}{m_h^1 e_1^f}}\right]\\
&= \frac{m_h^1}{4}\left[-3\mathrm{sgn}\left(m_f^0 m_h^1\right)\sqrt{-\frac{m_f^0 e_0^h}{m_h^1 e_1^f}}-1-\mathrm{sgn}\left(m_h^1 m_h^2\right)\sqrt{\frac{m_h^2 e_2^f}{m_h^1 e_1^f}}-\mathrm{sgn}\left(m_h^1 m_h^3\right)\sqrt{\frac{m_h^3 e_3^f}{m_h^1 e_1^f}}\right]\\
&\quad -\frac{m_h^1}{4}\left[-3-\mathrm{sgn}\left(m_h^1 m_f^0\right)\sqrt{-\frac{m_f^0 e_0^h}{m_h^1 e_1^f}}+\mathrm{sgn}\left(m_h^1 m_h^2\right)\sqrt{\frac{m_h^2 e_2^f}{m_h^1 e_1^f}}+\mathrm{sgn}\left(m_h^1 m_h^3\right)\sqrt{\frac{m_h^3 e_3^f}{m_h^1 e_1^f}}\right]\\
&= \frac{m_h^1}{2}\left[1-\mathrm{sgn}\left(m_f^0 m_h^1\right)\sqrt{-\frac{m_f^0 e_0^h}{m_h^1 e_1^f}}-\mathrm{sgn}\left(m_h^1 m_h^2\right)\sqrt{\frac{m_h^2 e_2^f}{m_h^1 e_1^f}}-\mathrm{sgn}\left(m_h^1 m_h^3\right)\sqrt{\frac{m_h^3 e_3^f}{m_h^1 e_1^f}}\right] && (3.179)
\end{aligned}
$$

If one wishes to compute the fermion and scalar mass matrices explicitly, these expressions can be substituted into (3.60), (3.62), and (3.63).

3.6.4 Generating Functions (Reduced Fluxes)

One of the principal results of this chapter is that the attractor behavior of these flux compactifications is governed by a single function $\mathcal{G}$. In this section we compute this function for our reduced fluxes. We will then verify the simple relationship between $\mathcal{G}$ and the gravitino mass.

We begin with $\mathrm{Im}\left(\mathcal{V}\right)$. If we substitute our F_I into (3.76), we find

$$\begin{aligned}\mathrm{Im}\,(\mathcal{V}) &= 2\mathrm{Im}\,(\tau)\,\mathrm{Im}\left\{C\frac{Z^1Z^2Z^3}{Z^0}\left[-\frac{L^0}{Z^0}+\frac{L^1}{Z^1}+\frac{L^2}{Z^2}+\frac{L^3}{Z^3}\right]\right\} \qquad (3.180)\\ &= 2\mathrm{Im}\,(\tau)\,\mathrm{Im}\left\{-C^2\frac{Z^1Z^2Z^3}{Z^0}\left[\frac{-m^0+\varphi^0}{m^0+\varphi^0}+\frac{-m^1+\varphi^1}{m^1+\varphi^1}\right.\right.\\ &\quad\left.\left.+\frac{-m^2+\varphi^2}{m^2+\varphi^2}+\frac{-m^3+\varphi^3}{m^3+\varphi^3}\right]\right\}. \qquad (3.181)\end{aligned}$$

The term in square brackets is just the constraint (3.158) so $\mathrm{Im}\,(\mathcal{V})=0$. If we substitute this into (3.85), we find for our reduced fluxes

$$\mathcal{G} = e^h_0\phi^0_f - e^f_i\phi^i_h\,. \qquad (3.182)$$

We compute each term separately:

$$\begin{aligned}e^h_0\phi^0_f &= \frac{1}{2}\left\{-e^h_0m^0_f+\mathrm{sgn}\left(m^0_fm^1_h\right)\sqrt{-e^h_0m^0_f}\sqrt{m^1_he^f_1}\right.\\ &\quad\left.+\mathrm{sgn}\left(m^0_fm^2_h\right)\sqrt{-e^h_0m^0_f}\sqrt{m^2_he^f_2}+\mathrm{sgn}\left(m^0_fm^3_h\right)\sqrt{-e^h_0m^0_f}\sqrt{m^3_he^f_3}\right\} \qquad (3.183)\\ e^f_1\phi^1_h &= \frac{1}{2}\left\{-e^f_1m^1_h-\mathrm{sgn}\left(m^0_fm^1_h\right)\sqrt{-e^h_0m^0_f}\sqrt{m^1_he^f_1}\right. \qquad (3.184)\\ &\quad\left.+\mathrm{sgn}\left(m^1_hm^2_h\right)\sqrt{m^1_he^f_1}\sqrt{m^2_he^f_2}+\mathrm{sgn}\left(m^1_hm^3_h\right)\sqrt{m^1_he^f_1}\sqrt{m^3_he^f_3}\right\}. \qquad (3.185)\end{aligned}$$

Putting this together yields

$$\begin{aligned}\mathcal{G} &= \frac{1}{2}\left[-e^h_0m^0_f+e^f_im^i_h\right]+\mathrm{sgn}\left(m^0_fm^1_h\right)\sqrt{-e^h_0m^0_f}\sqrt{m^1_he^f_1}+\mathrm{sgn}\left(m^0_fm^2_h\right)\sqrt{-e^h_0m^0_f}\sqrt{m^2_he^f_2}\\ &\quad+\mathrm{sgn}\left(m^0_fm^3_h\right)\sqrt{-e^h_0m^0_f}\sqrt{m^3_he^f_3}-\mathrm{sgn}\left(m^1_hm^2_h\right)\sqrt{m^1_he^f_1}\sqrt{m^2_he^f_2}\\ &\quad-\mathrm{sgn}\left(m^1_hm^3_h\right)\sqrt{m^1_he^f_1}\sqrt{m^3_he^f_3}-\mathrm{sgn}\left(m^2_hm^3_h\right)\sqrt{m^2_he^f_2}\sqrt{m^3_he^f_3}\,. \qquad (3.186)\end{aligned}$$

The term in square brackets is just $\int F_3\wedge H_3$ (3.170), while the remainder is less familiar. It is precisely what is required so that $\partial\mathcal{G}/\partial e^h_0=\phi^0_f$ and $\partial\mathcal{G}/\partial e^f_i=-\phi^i_h$, as can be readily verified. It is also closely related to the gravitino mass, as we will now see.

In order to compute the gravitino mass we substitute (3.162), (3.166), (3.171), and (3.174) into (3.47) and simplify

$$
\begin{aligned}
\mathrm{Vol}^2 m_{3/2}^2 &= -\frac{8\mathrm{Im}(\tau)\,\mathrm{Im}(z^1)\,\mathrm{Im}(z^2)\,\mathrm{Im}(z^3)}{2}\left(\frac{1}{2\mathrm{Im}(\tau)}\right)^2 \left(m_f^0+\phi_f^0\right)^2 &(3.187)\\
&= -\frac{e_0^h m_f^0}{4}\left\{1-\mathrm{sgn}\left(m_f^0 m_h^1\right)\sqrt{-\frac{m_h^1 e_1^f}{e_0^h m_f^0}}-\mathrm{sgn}\left(m_f^0 m_h^2\right)\sqrt{-\frac{m_h^2 e_2^f}{e_0^h m_f^0}}\right.\\
&\quad \left.-\mathrm{sgn}\left(m_f^0 m_h^3\right)\sqrt{-\frac{m_h^3 e_3^f}{e_0^h m_f^0}}\right\}^2 &(3.188)\\
&= \frac{1}{2}\left\{\frac{1}{2}\left[-e_0^h m_f^0+e_i^f m_h^i\right]-\mathrm{sgn}\left(m_f^0 m_h^1\right)\sqrt{-e_0^h m_f^0}\sqrt{m_h^1 e_1^f}\right.\\
&\quad -\mathrm{sgn}\left(m_f^0 m_h^2\right)\sqrt{-e_0^h m_f^0}\sqrt{m_h^2 e_2^f}-\mathrm{sgn}\left(m_f^0 m_h^3\right)\sqrt{-e_0^h m_f^0}\sqrt{m_h^3 e_3^f}\\
&\quad +\mathrm{sgn}\left(m_h^1 m_h^2\right)\sqrt{m_h^1 e_1^f}\sqrt{m_h^2 e_2^f}+\mathrm{sgn}\left(m_h^1 m_h^3\right)\sqrt{m_h^1 e_1^f}\sqrt{m_h^3 e_3^f}\\
&\quad \left.+\mathrm{sgn}\left(m_h^2 m_h^3\right)\sqrt{m_h^2 e_2^f}\sqrt{m_h^3 e_3^f}\right\}, &(3.189)
\end{aligned}
$$

If we compare this with our expression for $\mathcal{G}$ (3.186), we see that they are related by

$$
\mathcal{G}=\int F_3\wedge H_3-2\mathrm{Vol}^2 m_{3/2}^2\,, \tag{3.190}
$$

in accord with the general relationship (3.99).

3.6.5 $U-$Invariants for $F=Z^1Z^2Z^3/Z^0$

The model we are considering enjoys a large set of duality symmetries. We have not made explicit use of these dualities so far, but in this section we will show how they may be used to generalize our solution with only eight fluxes to a solution for the full set of sixteen fluxes. We take inspiration here from the STU black hole, where consideration of duality-invariant combinations of the black hole charges led to a simple expression for the generating function of the potentials [63, 65].

One part of the duality group is easily identified if we think of our prepotential as arising from compactification on $T^2 \times T^2 \times T^2$. We can interpret each z^i as the modular parameter of the ith torus, and consider modular transformations on each torus. Since the tori and their associated modular transformations factorize, their contribution to the U-duality group is just $SL(2)^3$. This is the symmetry group of the STU black hole [65], whose charges transform[8] in the $(2, 2, 2)$ of $SL(2)^3$.

IIB theories also enjoy an $SL(2)$ S-duality, independent of the $SL(2)^3$ that we have already discussed. This does not factor into discussions of the STU black hole in the IIB picture[9], as the D3-branes that one uses to construct the black hole (see section (3.3.1)) are invariant under S-duality. The fluxes H_3 and F_3, however, transform under S-duality, so we must consider the larger duality group $SL(2)^4$, under which our fluxes transform as $(2, 2, 2, 2)$.

The discussion of STU black holes in terms of $SL(2)^3$ invariants is relatively straightforward because there is a single $SL(2)^3$-invariant that one can construct from the charges [63]. This essentially determines the black hole entropy, which in turn is the generating function for the electric and magnetic potentials. On the other hand, one can construct *four* invariants[10] from the $(2, 2, 2, 2)$ of $SL(2)^4$ [74]. The quadratic $I_2 = \int F_3 \wedge H_3$ appears in most studies of IIB flux compactifications, while the other three are less familiar. Considered as polynomials in the fluxes, there are also two quartics, $I_4^{(1)}$ and $I_4^{(2)}$, and a sextic, I_6.

[8] For details of the action of $SL(2)^3$ on the charges, see e.g. [73].

[9] One can also discuss this entirely in the language of $\mathcal{N} = 2$ supergravity. In the STU black hole all of the hypermultiplets, including the universal hypermultiplet, decouple from the attractor flow. On the other hand the axio-dilaton, which descends from the universal hypermultiplet, does *not* decouple from the flux attractor.

[10] More precisely, one can construct exactly four invariants from the $(2, 2, 2, 2)$ of $SL(2, \mathbb{C})^4$. These are also invariants of $SL(2, \mathbb{R})^4$ but additional invariants might arise when we restrict to the subgroup. Possible examples include $\text{sgn}\left(m_f^0 m_h^i\right)$. We also expect some number of discrete invariants to appear upon further restriction to $SL(2, \mathbb{Z})^4$.

In section 3.6.2 we chose a reduced set of fluxes that allowed us to explicitly solve the attractor equations. One of our motivations in choosing these particular fluxes was to choose a combination that left all four $SL(2)^4$ invariants non-zero and independent. While the general expressions for these invariants are quite complicated (see [74] for details), they simplify considerably for our reduced fluxes:

$$
\begin{aligned}
I_2 &= \int F_3 \wedge H_3 = \left(-m_f^0 e_0^h\right) + \left(e_1^f m_h^1\right) + \left(e_2^f m_h^2\right) + \left(e_3^f m_h^3\right) , \qquad (3.191)\\
I_4^{(1)} &= -\left(-m_f^0 e_0^h\right)\left(e_1^f m_h^1\right) + \left(-m_f^0 e_0^h\right)\left(e_2^f m_h^2\right) + \left(e_1^f m_h^1\right)\left(e_3^f m_h^3\right)\\
&\quad - \left(e_2^f m_h^2\right)\left(e_3^f m_h^3\right) , \qquad (3.192)\\
I_4^{(2)} &= -\left(-m_f^0 e_0^h\right)\left(e_2^f m_h^2\right) + \left(-m_f^0 e_0^h\right)\left(e_3^f m_h^3\right) + \left(e_1^f m_h^1\right)\left(e_2^f m_h^2\right)\\
&\quad - \left(e_1^f m_h^1\right)\left(e_3^f m_h^3\right) , \qquad (3.193)
\end{aligned}
$$

$$
\begin{aligned}
I_6 = &\left(-m_f^0 e_0^h\right)^2\left(e_1^f m_h^1\right) + \left(-m_f^0 e_0^h\right)\left(e_1^f m_h^1\right)^2 + \left(e_2^f m_h^2\right)^2\left(e_3^f m_h^3\right) + \left(e_2^f m_h^2\right)\left(e_3^f m_h^3\right)^2\\
&-4\left(e_1^f m_h^1\right)^2\left(e_2^f m_h^2\right) - 4\left(e_1^f m_h^1\right)\left(e_2^f m_h^2\right)^2 - 4\left(-m_f^0 e_0^h\right)^2\left(e_3^f m_h^3\right) - 4\left(-m_f^0 e_0^h\right)\left(e_3^f m_h^3\right)^2\\
&+3\left(-m_f^0 e_0^h\right)\left(e_1^f m_h^1\right)\left(e_2^f m_h^2\right) + 3\left(-m_f^0 e_0^h\right)\left(e_1^f m_h^1\right)\left(e_3^f m_h^3\right)\\
&+3\left(-m_f^0 e_0^h\right)\left(e_2^f m_h^2\right)\left(e_3^f m_h^3\right) + 3\left(e_1^f m_h^1\right)\left(e_2^f m_h^2\right)\left(e_3^f m_h^3\right) . \qquad (3.194)
\end{aligned}
$$

Note that given the sign restrictions in (3.167), each term in parentheses is positive-definite. Also, note that exactly four distinct *products* of pairs of fluxes appear in the expressions for the invariants. Duality orbits of our reduced fluxes therefore sweep out a codimension 0 volume in the full space of fluxes. It is more difficult to say whether pairs of fluxes satisfying the sign constraints (3.167) span the physically allowed values of the invariants (3.191)-(3.194).

The explicit form (3.186) of the generating function $\mathcal{G}$ raises an interesting question. Three independent signs appear, $\operatorname{sgn}\left(m_f^0 m_h^1\right)$, $\operatorname{sgn}\left(m_f^0 m_h^2\right)$, and $\operatorname{sgn}\left(m_f^0 m_h^3\right)$.

One can readily verify that duality transformations that leave the subspace of reduced fluxes invariant also leave these signs, and only these signs, invariant. Although we are not certain that these signs lift to invariants of the full $SL(2)^4$, it is possible that they label different octants of the full space of fluxes, with distinct expressions for e.g. the gravitino mass in each octant.

We can use these facts to generalize our solution of the $F = Z^1Z^2Z^3/Z^0$ model with eight fluxes to a solution with all sixteen fluxes. We propose the following procedure:

1. Consider (3.191)-(3.194) to be a set of implicit functions for each pair of fluxes in terms of $I_2 = \int F_3 \wedge H_3$, $I_4^{(1,2)}$, and I_6.
2. Substitute these functions into (3.186) to get $\mathcal{G}$ as a function of the invariants.
3. Substitute the full expressions for $I_2 = \int F_3 \wedge H_3$, $I_4^{(1,2)}$, and I_6 into $\mathcal{G}$ to get an expression for $\mathcal{G}$ as a function of general fluxes.
4. Derivatives of $\mathcal{G}$ with respect to the fluxes will then give the potentials, and in turn the values of the complex structure moduli and mass parameters.
5. Solve (3.71) to determine the value of τ.

This procedure will certainly work if the eight additional fluxes are small. As they become large, global properties of the space of fluxes may present an obstruction, for example one of $\mathrm{sgn}\left(m_f^0 m_h^1\right)$, $\mathrm{sgn}\left(m_f^0 m_h^2\right)$, or $\mathrm{sgn}\left(m_f^0 m_h^3\right)$ might effectively flip. It is also possible that there are other branches of solutions that we have not identified.

Though considerations of duality-invariance have not yet led us to a complete solution of the flux attractor equations with $F = Z^1Z^2Z^3/Z^0$, we hope that future work will make our understanding of flux compactifications on this geometry as detailed as the modern understanding of the STU black hole.

3.7 Thermodynamics, Stability, and the Landscape

One of the goals of this chapter was to determine how much of the analysis of flux compactifications could be done directly on the space of input fluxes. We demonstrated that local properties of the compactification are completely determined by a single generating function $\mathcal{G}$ defined on the space of fluxes. Although we have been conservative in describing $\mathcal{G}$ as a "generating function," we hope that future analysis will reveal that it is a proper thermodynamic function, and that we can think of the fluxes themselves as the parameters of an underlying thermodynamic system. At the same time, we might worry that our success in constructing $\mathcal{G}$ hinged only on the Kähler structure of the moduli space, and that no thermodynamic interpretation exists. We now outline some of the principal challenges surrounding a thermodynamic interpretation of flux attractors.

Is $\mathcal{G}$ a Thermodynamic Function? Equations (3.97) and (3.98) look like equilibrium relations between the fluxes and their thermodynamic conjugates. In addition to equilibrium relations, thermodynamic functions also obey a set of stability conditions. For a sensible thermodynamic interpretation, we would require that stable and unstable thermodynamic equilibria correspond to stable and unstable minima of the traditional spacetime potential (3.25). Here we find an apparent mismatch between the two Hessians. While the field-theoretic mass matrix has $2n + 2$ eigenvalues, the matrix of second derivatives of $\mathcal{G}$ has $4n + 4$ eigenvalues. For guidance we might study the analogous issue in the black hole attractor. There, the Hessian of the effective potential has $2n$ eigenvalues, while the second derivatives of the entropy lead to $2n + 2$ eigenvalues.

What Kind of Thermodynamic Function is $\mathcal{G}$? In thermodynamic problems, the energy and the entropy are treated rather differently. In particular, energies are *minimized* at stable equilibria, while entropies are *maximized*. In other ensembles the energy is mapped to a free energy and the entropy to a generalized Massieu function, but free energies are still minimized and Massieu functions are still maximized. The interpretation of $\mathcal{G}$ hinges on whether it is minimized, in which case it might be interpreted as the tension of a dual domain wall [75], or maximized, in which case it could be interpreted as an entropy. Determining this requires that we fix the overall sign of $\mathcal{G}$. Doing this might be as simple as requiring that $\mathcal{G}$ be positive for stable configurations, but it could be more subtle.

What Does This Imply for the Landscape? If we can establish that $\mathcal{G}$ is an entropy, it becomes quite natural to propose $e^{\mathcal{G}}$ as a classical measure on the string theory landscape. Presumably such a measure would be related to the number of microscopic realizations of a given set of fluxes. We can go on to ask if there are any geometries for which this measure becomes strongly peaked, or whether consistency conditions (such as the tadpole constraint) require that $\mathcal{G}$ be $\mathcal{O}(1)$.

Clearly many potential obstacles lie between the generating function introduced in this chapter and a *predictive* measure on the landscape. However the prospect of such a measure is quite exciting, and so worthy of some attention.

CHAPTER IV

Flux Attractors and Geometric Fluxes

4.1 Introduction

Many closed-string backgrounds with 4D $\mathcal{N} = 1$ supersymmetry descend from backgrounds with 4D $\mathcal{N} = 2$ supersymmetry. The chiral multiplets in the $\mathcal{N} = 1$ theories then arise from projections of either $\mathcal{N} = 2$ vector multiplets or $\mathcal{N} = 2$ hypermultiplets. While it is well-understood how to use fluxes to stabilize the vector moduli in both IIB and IIA compactifications [1, 22, 38] (for review see [25, 27, 29]), it has been less clear how to stabilize the hypermoduli. In this chapter we introduce a general scheme for understanding how specific geometric and non-geometric fluxes can stabilize many more of the hypermoduli. As an example of this scheme, we will study in detail the addition of geometric fluxes to O3/O7 compactifications with 3-form flux.

Our starting point is fairly general. We make only two modest assumptions about how the hypermultiplet moduli enter into the 4D scalar potential:

- *Homogeneity*: the Kähler potential is assumed homogeneous of degree four in the imaginary parts of the hypermoduli. This is satisfied by Calabi-Yau orientifolds, as well as for more general compactifications with $SU(3) \times SU(3)$ structure.

- *Linearity*: the hypermultiplet moduli should only appear linearly in the superpotential. This is the case for compactifications with generalized fluxes, our main example.

We show that under these assumptions the scalar potential can be rewritten as a sum of universal, positive semi-definite terms, and a term governed by a metric that generally has indefinite signature. For some choices of fluxes, this final term is *also* positive semi-definite, so that we can find absolute minima of the scalar potential by setting each term separately to zero. The resulting Minkowski vacua are natural generalizations of the familiar no-scale vacua introduced by GKP [38].

Our primary example of this new class of Minkowski vacua is IIB O3/O7 compactifications with 3-form fluxes (as usual) and geometric fluxes (the new ingredient). Minimization of the scalar potential for this entire class of vacua is equivalent to an ISD condition with additional constraints. The ISD condition can be recast as a set of flux attractor equations [1, 17–20, 76] which stabilize the vector moduli z^i in the manner previously studied for O3/O7 compactifications with 3-form fluxes alone. The hypermoduli τ (the axio-dilaton) and G^a (additional 2-form moduli) enter as fixed background parameters for the purpose of vector moduli stabilization. However, as we mentioned already, the ISD condition is supplemented by additional constraints. It is those that control the stabilization of the hypermoduli.

An important ingredient in the analysis is the manner in which the vector moduli are stabilized: as shown in chapter III, solutions to the flux attractor equations can be presented as derivatives of a scalar *generating function*, whether or not there are geometric fluxes present. The hypermoduli enter this generating function as parameters that are arbitrary *a priori*. However, the constraints that control the stabilization of hypermoduli turn out to be equivalent to an *extremization principle* on the generating function over hypermoduli space. For the purpose of hypermoduli stabilization the generating function thus plays a role similar to that of a conventional potential.

In favorable circumstances the extremization over hypermoduli space may yield hypermoduli that are all stabilized at finite values. However, it may also give either runaway behavior, or flat directions. We will see choices of fluxes which realize each of these possibilities. An obvious general rule is that vacua with many fluxes turned on have fewer unstabilized moduli. More surprisingly, we find that the number of hypermoduli that can be stabilized is apparently limited by the number of vector moduli.

One of the motivations for this work is to develop the generating function formalism for flux compactifications. Certainly the generating function provides a convenient way to summarize the VEVs and the masses of the scalar fields stabilized by fluxes. Additionally, it is intriguing that the role it plays in the flux attractor equations is analogous to that played by the black hole entropy in black hole attractor equations. This analogy suggests a deep relation to counting of vacua which is obscured by the usual geometric treatment of the fluxes. It would be interesting to develop this relation further.

This chapter is organized as follows. In section 2 we review a few features of no-scale vacua, as they appear in the standard GKP context. We then generalize those constructions to the generic setup of interest here. In section 3 we provide a brief introduction to generalized fluxes and the manner in which they appear in the low energy theory. Combining with the results from section 2, we find the attractor equations for no-scale vacua with geometric fluxes. In section 4 we introduce the generating function and show that it both solves the attractor equations for vector moduli and also provides an extremization principle on hypermoduli space. In section 5 we give several explicit examples that illustrate our methods. A few technical details have been collected in appendix A.

4.2 A General Class of no-scale Vacua

In this section we seek to present the scalar potential of Type II flux compactifications as a BPS-like sum of positive semi-definite terms, thus finding Minkowski vacua when each of those terms vanish separately. The resulting minimization conditions underlie the attractor equations.

We first review the standard GKP flux vacua with just one hypermodulus, the axio-dilaton τ, and then generalize to situations with many hypermoduli. We maintain a rather general setting, albeit with assumptions on the theory motivated by subsequent applications to situations with generalized fluxes.

4.2.1 GKP Compactifications

To get started we review the simplest and most widely studied class of flux vacua [38, 46]: O3/O7 orientifold vacua in type IIB theory, with F_3 and H_3 fluxes turned on[1]. We refer to these vacua as "GKP compactifications."

In these theories the vector moduli (descended from $\mathcal{N} = 2$ vector multiplets) are the complex structure moduli z^i. The hypermoduli (descended from $\mathcal{N} = 2$ hypermultiplets) are the axio-dilaton τ and the Kähler moduli T_α.

At large volume and weak coupling, the Kähler potential factorizes into

$$K = K_z\left(z^i\right) - \log\left[-i\left(\tau - \overline{\tau}\right)\right] + K_T\left(T_\alpha\right), \tag{4.1}$$

and enjoys a homogeneity property

$$K_{\alpha\overline{\beta}}(\partial^\alpha K)(\partial^{\overline{\beta}} K) = 3\,, \tag{4.2}$$

where $K_{\alpha\overline{\beta}}$ is the T_α block of the inverse Kähler metric. This homogeneity property

[1] We also assume $h_-^{(1,1)} = 0$, since we want to describe the simplest situation.

is proven in Appendix B, as is a more general version. The 3-form RR and NS fluxes F_3 and H_3 give rise to the GVW superpotential [42, 43]

$$W = \int G_3 \wedge \Omega_3 \,, \tag{4.3}$$

where

$$G_3 \equiv F_3 - \tau H_3 \,. \tag{4.4}$$

The superpotential in this situation is linear in τ, and independent of T_α.

The scalar potential is

$$e^{-K} V = \sum_{X,Y=i,\tau,\alpha} K^{X\overline{Y}} D_X W \overline{D_Y W} - 3 \left| W \right|^2 , \tag{4.5}$$

with the Kähler derivative defined as

$$D_X W = \partial_X W + W \partial_X K \,. \tag{4.6}$$

Because the superpotential is independent of the T_α, and because the Kähler potential (4.1) factorizes, the 4D scalar potential (4.5) reduces to

$$\begin{aligned} e^{-K} V &= K^{i\overline{j}} D_i W \overline{D_j W} + K^{\tau\overline{\tau}} D_\tau W \overline{D_\tau W} + \left[K_{\alpha\overline{\beta}} \left(\partial^\alpha K \right) \left(\overline{\partial^\beta K} \right) - 3 \right] \left| W \right|^2 & (4.7) \\ &= K^{i\overline{j}} D_i W \overline{D_j W} + K^{\tau\overline{\tau}} D_\tau W \overline{D_\tau W} \,. & (4.8) \end{aligned}$$

The quantity in square brackets vanishes by virtue of the homogeneity relation (4.2). The inverse Kähler metric $K^{i\overline{j}}$ has positive eigenvalues, and $K^{\tau\overline{\tau}} = 4 \mathrm{Im} \left(\tau \right)^2$ is positive, so this potential is positive semi-definite, with an absolute minimum when $D_i W = 0$ and $D_\tau W = 0$. Since $D^\alpha W = W \partial^\alpha K$ is generically non-zero, supersymmetry is broken. The combination of supersymmetry breaking and vanishing scalar potential are the defining features of a no-scale vacuum.

The linearity of the superpotential in τ allows us to write the $D_\tau W = 0$ condition in a more illuminating manner:

$$D_\tau W = -\int H_3 \wedge \Omega_3 - \frac{1}{\tau - \overline{\tau}} \int G_3 \wedge \Omega_3 \tag{4.9}$$

$$= -\frac{1}{\tau - \overline{\tau}} \int \overline{G}_3 \wedge \Omega_3 = 0\,. \tag{4.10}$$

If we combine this with

$$D_i W = \int G_3 \wedge D_i \Omega_3 = 0\,, \tag{4.11}$$

we see that the $(3,0)$ and $(1,2)$ pieces of the complex flux G_3 must vanish. This is equivalent to the condition that G_3 must be imaginary self-dual (ISD). In the following we will find that analogues of (4.10) and (4.11), and the resulting ISD conditions, arise quite generically.

4.2.2 General Type II, $\mathcal{N} = 1$ Compactifications

The GKP compactifications are very special, but the homogeneity and linearity properties we used above apply to virtually all Type II $\mathcal{N} = 1$ flux compactifications, at least in the limit of large volume and small coupling. In the general setting we will assume that the moduli split into three groups:

Vector moduli: these are descended from $\mathcal{N} = 2$ vector multiplets, and denoted by z^i.

Stabilizable hypermoduli: these appear linearly in the superpotential, and are descended from $\mathcal{N} = 2$ hypermultiplets. We denote them by $t^{\hat{a}}$.

Unstabilizable hypermoduli: these do not appear in the superpotential, and are descended from $\mathcal{N} = 2$ hypermultiplets. We denote them $t^{\hat{\alpha}}$.

We will denote all of the hypermoduli together as t^A, with the A index running over both $\hat{a}$ and $\hat{\alpha}$. The split of the t^A into $t^{\hat{a}}$ and $t^{\hat{\alpha}}$ will depend on which fluxes we have turned on. In the simple GKP example we turned on F_3 and H_3, making the superpotential linear in τ. In this context τ is a stabilizable hypermodulus while the T_α, which did not appear in the superpotential, are unstabilizable hypermoduli. Later, we will introduce $h_-^{(1,1)}$ moduli denoted G^a which will appear linearly in the superpotential due to a coupling to geometric fluxes. Then τ, G^a will all be stabilizable hypermoduli in the terminology used here.

The most generic superpotential linear in the $t^{\hat{a}}$ and independent of the $t^{\hat{\alpha}}$ can be written as:

$$W = F(z) - t^{\hat{a}} H_{\hat{a}}(z) . \tag{4.12}$$

Both $F(z)$ and $H_{\hat{a}}(z)$ are holomorphic functions of the vector moduli z^i and independent of the hypermoduli t^A. They can be thought of as generalizations of $\int F_3 \wedge \Omega_3$ and $\int H_3 \wedge \Omega_3$.

In the general setting we assume that the Kähler potential decomposes into a term for the vector moduli z^i and another term for all the hypermoduli t^A, and enjoys the homogeneity relation:

$$K^{A\overline{B}} (\partial_A K)(\partial_{\overline{B}} K) = 4 , \tag{4.13}$$

$$K^{A\overline{B}} (\partial_{\overline{B}} K) = \bar{t}^A - t^A \equiv -2i\eta^A . \tag{4.14}$$

In the simple GKP example the Kähler potential (4.1) had independent terms for τ and for the T_α, but generally it does not decompose so neatly. In that example $K^{\tau\overline{\tau}} (\partial_\tau K)(\partial_{\overline{\tau}} K) = 1$, so (4.2) is consistent with the homogeneity (4.13). We discuss how these homogeneity relations arise in different kinds of Type II compactifications in Appendix B.

We include a brief aside on how corrections may affect our assumptions. String theory corrections are generally governed by two expansion parameters, the string coupling g_s, and the string tension α', and quantities can receive corrections both perturbative and non-perturbative in these parameters. For our models, the linearity of the superpotential in the hypermoduli will hold perturbatively to all orders, but can and will receive non-perturbative corrections (in both g_s and α') which we will neglect. The Kähler potential receives perturbative corrections in both parameters which will generically ruin our homogeneity and no-scale properties. However, if we stay at tree level in the string coupling g_s, the α' corrections to the Kähler potential still preserve the homogeneity property (4.13); for instance, the first correction, calculated in [77] simply adds a term to e^{-K} which is quartic in $\eta^\tau = \mathrm{Im}(\tau)$. The first g_s correction, however, not only ruins the homogeneity property, but in fact mixes scalars coming from hypermultiplets with those coming from vector multiplets. However, these g_s corrections are also typically accompanied by α' corrections (see for example [49]).

The no-scale cancellation in the previous subsection involved the $D^\alpha W$ terms in the scalar potential and the $-3|W|^2$ term. We are interested in similar cancellations in a more general context, so we focus our attention on the $D_A W$ and $-3|W|^2$ terms. We can use the homogeneity relations (4.13) and (4.14) to simplify them:

$$\begin{aligned} K^{A\overline{B}} D_A W \overline{D_B W} - 3|W|^2 &= K^{\hat{a}\bar{\hat{b}}} D_{\hat{a}} W \overline{D_{\hat{b}} W} + K^{\hat{\alpha}\bar{\hat{\beta}}} (\partial_{\hat{\alpha}} K) \left(\partial_{\bar{\hat{\beta}}} K\right) |W|^2 - 3|W|^2 \\ &\quad + \left[K^{\hat{a}\bar{\hat{\alpha}}} (\partial_{\bar{\hat{\alpha}}} K) \overline{W} D_{\hat{a}} W + \text{c.c.}\right] \qquad (4.15) \\ &= K^{\hat{a}\bar{\hat{b}}} \left[D_{\hat{a}} W \overline{D_{\hat{b}} W} - (\partial_{\hat{a}} K) \left(\partial_{\bar{\hat{b}}} K\right) |W|^2\right] + |W|^2 \\ &\quad + \left[K^{\hat{a}\bar{\hat{\alpha}}} (\partial_{\bar{\hat{\alpha}}} K) \overline{W} \partial_{\hat{a}} W + \text{c.c.}\right] \qquad (4.16) \\ &= K^{\hat{a}\bar{\hat{b}}} \partial_{\hat{a}} W \overline{\partial_{\hat{b}} W} + |W|^2 + \left[K^{\hat{a}\overline{A}} (\partial_{\overline{A}} K) \overline{W} \partial_{\hat{a}} W + \text{c.c.}\right] \qquad (4.17) \\ &= K^{\hat{a}\bar{\hat{b}}} \partial_{\hat{a}} W \overline{\partial_{\hat{b}} W} + |W|^2 - 2i\eta^{\hat{a}} \left[\overline{W} \partial_{\hat{a}} W - \text{c.c.}\right]. \qquad (4.18) \end{aligned}$$

In the second step we used (4.13). We then expanded out the $K^{\hat{a}\bar{\hat{b}}} D_{\hat{a}} W \overline{D_{\hat{b}} W}$ term and rearranged terms so that we could apply (4.14). Recall that $\eta^{\hat{a}}$ is just the imaginary part of $t^{\hat{a}}$.

We can evaluate the derivatives in (4.18) by virtue of the linearity of the superpotential,

$$\partial_{\hat{a}} W = -H_{\hat{a}}(z) \ . \tag{4.19}$$

The remaining terms in (4.18) simplify when written in terms of

$$\widetilde{W} \equiv F(z) - \bar{t}^{\hat{a}} H_{\hat{a}}(z) = W + 2i\eta^{\hat{a}} H_{\hat{a}}(z) \ , \tag{4.20}$$

the natural generalization of $\int \overline{G}_3 \wedge \Omega_3$ from the previous example. Adding in the remaining terms in the scalar potential, we now find

$$e^{-K} V = K^{i\bar{j}} D_i W \overline{D_j W} + \left|\widetilde{W}\right|^2 + \left[K^{\hat{a}\bar{\hat{b}}} - 4\eta^{\hat{a}}\eta^{\hat{b}}\right] H_{\hat{a}}(z) \overline{H}_{\hat{b}}(z) \ . \tag{4.21}$$

This is the natural generalization of the no-scale potential (4.8).

While the $D_i W$ and $\widetilde{W}$ terms in (4.21) are closely related to the ISD conditions in the O3/O7 example, the final set of terms is new. They will make a positive or negative contribution to the potential depending on the eigenvalues of

$$h^{\hat{a}\bar{\hat{b}}} \equiv K^{\hat{a}\bar{\hat{b}}} - 4\eta^{\hat{a}}\eta^{\hat{b}} \ . \tag{4.22}$$

The eigenvalues of $h^{\hat{a}\bar{\hat{b}}}$ are in general functions of the hypermoduli. When $h^{\hat{a}\bar{\hat{b}}}$ has one or more negative eigenvalues, the scalar potential (4.21) may admit AdS minima; we have little to say about such minima at this time. However, when the eigenvalues of

$h^{\hat{a}\bar{\hat{b}}}$ are positive semi-definite functions of the hypermoduli, Minkowski vacua arise if

$$D_i W = 0\,, \tag{4.23}$$

$$\widetilde{W} = 0\,, \tag{4.24}$$

$$H_{\underline{\hat{a}}}(z) = 0\,, \tag{4.25}$$

where the $\underline{\hat{a}}$ index runs over the non-zero eigenvalues of $h^{\hat{a}\bar{\hat{b}}}$ *only*. Whenever W is non-vanishing supersymmetry is broken because

$$D_{\underline{\hat{a}}} W = H_{\underline{\hat{a}}}(z) + (\partial_{\underline{\hat{a}}} K)\, W = (\partial_{\underline{\hat{a}}} K)\, W \neq 0. \tag{4.26}$$

Thus our solutions are generally no-scale vacua.

Before performing a detailed analysis of (4.23)-(4.25), we can ask when they are likely to have solutions. Trouble can arise if (4.23)-(4.25) together constitute more equations than we have moduli. This occurs in two cases:

- If $h^{\hat{a}\bar{\hat{b}}}$ has more positive eigenvalues than there are vector moduli z^i, we will not in general be able to solve the relevant $H_{\underline{\hat{a}}}(z) = 0$ conditions. This is because the $H_{\hat{a}}(z)$ are functions of the z^i *only*, not of the hypermoduli.

- If $h^{\hat{a}\bar{\hat{b}}}$ has *strictly* positive eigenvalues, then the number of fields z^i and $t^{\underline{\hat{a}}}$ is equal to the number of conditions in $D_i W = 0$ and $H_{\underline{\hat{a}}}(z) = 0$. Because of the additional $\widetilde{W} = 0$ condition, we do not in general expect to be able to reach the Minkowski vacuum. Instead, we expect the overall factor of e^K in the potential to lead to runaway vacua.

We therefore expect Minkowski vacua to arise when $h^{\hat{a}\bar{\hat{b}}}$ has at least one zero eigenvalue, no negative eigenvalues, and the number of positive eigenvalues is not greater than the number of z^i. The previous O3/O7 example falls into this category, since $K^{\tau\bar{\tau}} = 4\mathrm{Im}(\tau)^2$, and thus the only eigenvalue of $h^{\hat{a}\bar{\hat{b}}}$ is $h^{\tau\bar{\tau}} = 0$. When $h^{\hat{a}\bar{\hat{b}}}$ is positive semi-definite but does not satisfy these properties, we expect the overall factor of e^K in the scalar potential to lead to runaway vacua.

In order to illustrate the utility of our simplified form for the scalar potential (4.21), we will present a new set of Minkowski vacua in the next section. We will arrive at these by adding geometric flux to the O3/O7 compactifications described at the beginning of this section. The geometric flux will allow us to stabilize additional hypermoduli, which cannot be stabilized with 3-form flux alone. It also appears to lead to an infinite series of distinct vacua, as well as the ability to tune the string coupling to be arbitrarily small. We will present the full $h^{A\overline{B}}$ matrix for O3/O7 compactifications, and show that the conditions (4.23)-(4.25) can easily be converted into flux attractor equations.

4.3 Attractor Equations and Geometric Flux

The axio-dilaton τ is the only hypermodulus that enters the perturbative type IIB superpotential in the presence of RR fluxes and the 3-form NS flux H_3. There are several options for the addition of extra ingredients that give rise to dependence on more hypermoduli. Vacua with generalized NS fluxes are appealing because T-duality establishes their existence in simple cases, while mirror symmetry suggests their existence in more complicated cases. These duality considerations also largely determine how these fluxes must appear in the $\mathcal{N} = 1$ superpotential. In this section we derive stabilization conditions for the hypermoduli in this context, with emphasis on geometric fluxes (sometimes called metric fluxes).

4.3.1 Superpotential with Generalized Flux

For a long time it has been known that in a background with H-flux that lies parallel to a circle (i.e. if the circle isometry contracted with H is non-zero), a T-duality along the circle will generate a new solution in which some components of

H-flux have been exchanged for some non-constant components of the metric [78]. The effect of these new metric components can be thought of as a twisting of the circle over the rest of the geometry, encoded in the Cartan equation

$$de^i = f^i_{jk} e^j \wedge e^k \ . \tag{4.27}$$

The coefficients f^i_{jk} serve as analogs of H_{ijk}, the components of the original H-flux. Indeed, upon reduction to four dimensions, these components appear as parameters of the low-energy theory in much the same way as H_{ijk} do [79–81].

If there are more circle isometries, one might be able to perform a further T-duality, converting some of the f^i_{jk} into new objects Q^{ij}_k known as non-geometric fluxes. In the presence of non-geometric fluxes, the string background no longer has the structure of a geometric manifold, but can still be understood as torus fibers varying over a base, where the transition functions between patches include string dualities [82, 83]. From a low-energy perspective, the non-geometric nature of the background isn't relevant, and the components Q^{ij}_k appear in a natural way in the superpotential. In fact, from the low energy perspective, one is also tempted to include objects R^{ijk}, which would correspond to T-dualizing all three legs of some H-flux. From a ten-dimensional perspective, it's not clear whether these latter fluxes can in fact be constructed (indeed it is not clear whether all possible configurations of the other geometric and non-geometric fluxes can be engineered), but the manner in which they would appear in the effective theory is essentially determined by symmetry considerations. For a more detailed discussion see the review [28] and references therein.

For the purposes of studying the superpotential and the tadpole constraints, it will be useful to introduce a slightly different organizational scheme for generalized fluxes. In order to present this scheme, we must first give a basis for the cohomology of the underlying Calabi-Yau orientifold where each element has definite parity under the orientifold involution. For the remainder of this section we will specialize to O3/O7 compactifications of type IIB string theory and take the basis for even forms:

- The constant function 1 and the volume form φ, both even under the orientifold involution.
- The 2-forms μ_α and their dual 4-forms $\widetilde{\mu}^\alpha$. All are even under the orientifold (so $\alpha = 1, \ldots, h^{1,1}_+$).
- The 2-forms ω_a and their dual 4-forms $\widetilde{\omega}^a$. All are odd under the orientifold (so $a = 1, \ldots, h^{1,1}_-$).

We will also introduce symplectic bases for the 3-forms where each element has definite parity under the orientifold involution:

- $(\mathcal{A}_{\hat{I}}, \mathcal{B}^{\hat{I}})$ are even (so $\hat{I} = 1, \ldots, h^{2,1}_+$).
- (α_I, β^I) are odd (so $I = 0, \ldots, h^{2,1}_-$). The extra index value is because the $(3,0)$ and $(0,3)$ forms are odd.

Now, in compactifications with H-flux, it is often very useful to replace the local expressions H_{ijk} for the components of H_3 with a global expansion

$$H_3 = m^I_h \alpha_I - e^h_I \beta^I \ , \tag{4.28}$$

where m^I_h and e^h_I are the magnetic and electric components of the 3-form flux. To obtain the analogous expansions for the geometric and non-geometric fluxes, one should recast the H-flux not just as a 3-form, but as a linear operator which maps p-forms to $(p+3)$-forms (by wedging with H_3). The geometric fluxes f^i_{jk} similarly define a map from p-forms to $(p+1)$-forms, while the non-geometric fluxes Q^{ij}_k and R^{ijk} give maps from p-forms to $(p-1)$- and $(p-3)$-forms, respectively. Altogether, we can combine these linear maps into an operator $\mathcal{D}$ [84, 85], which we can view as an operator of odd degree on the basis forms of the underlying space. In particular we can write expansions of $\mathcal{D}$ acting on the even forms

$$-\mathcal{D}\cdot 1 = H_3 = m_h^I \alpha_I - e_I^h \beta^I \ , \tag{4.29}$$

$$-\mathcal{D}\mu_\alpha = \widehat{r}_\alpha = \widehat{r}_\alpha^{\hat{I}} \mathcal{A}_{\hat{I}} - \widehat{r}_{\alpha\hat{I}} \mathcal{B}^{\hat{I}} \ , \tag{4.30}$$

$$-\mathcal{D}\omega_a = r_a = r_a^I \alpha_I - r_{aI} \beta^I \ , \tag{4.31}$$

$$-\mathcal{D}\widetilde{\mu}^\alpha = \widehat{q}^\alpha = \widehat{q}^{\alpha I} \alpha_I - \widehat{q}_I^\alpha \beta^I \ , \tag{4.32}$$

$$-\mathcal{D}\widetilde{\omega}^a = q^a = q^{a\hat{I}} \mathcal{A}_{\hat{I}} - q_{\hat{I}}^a \mathcal{B}^{\hat{I}} \ , \tag{4.33}$$

$$-\mathcal{D}\varphi = s = s^{\hat{I}} \mathcal{A}_{\hat{I}} - s_{\hat{I}} \mathcal{B}^{\hat{I}} \ . \tag{4.34}$$

The point here is just that H_3 and Q_k^{ij} reverse the parity of forms under the orientifold projection, while f_{jk}^i and R^{ijk} preserve it. We will not need the detailed map between the component fluxes $f_{jk}^i, Q_k^{ij}, R^{ijk}$ and the 3-forms $H_3, \widehat{r}_\alpha, r_a, \widehat{q}^\alpha, q^a, s$ (given in [86]) because we will use only the latter terminology from here on. For completeness, we note that there is of course also an action analogous to (4.29)-(4.34) on the odd degree cohomology, but again we do not need the details.

Now, it turns out that the H-flux, the geometric fluxes labeled r_a, and the non-geometric fluxes labeled $\widehat{q}^\alpha$ all contribute to the superpotential, while the geometric fluxes $\widehat{r}_\alpha$ and the non-geometric fluxes q^a and s contribute to D-terms [87]. For the rest of this chapter, we will focus on only the fluxes that enter the superpotential, and set the latter group of fluxes to zero.

The operator $\mathcal{D}$ can be viewed as a generalization of the twisted exterior derivative $d_H = d - H_3\wedge$, which is the natural differential operator on forms in the presence of H_3-flux. For generalized fluxes, we would replace this with $\mathcal{D}$ and, when acting on d-closed forms, we are left with just the linear action (4.29)-(4.34) of $\mathcal{D}$. For consistency, the operator $\mathcal{D}$ must be nilpotent, $\mathcal{D}^2 = 0$, like the usual exterior derivative [84, 86].

This constraint implies that the set of 3-forms H_3, r_a, and $\widehat{q}^\alpha$ are all symplectically orthogonal[2], i.e. that [87]

$$\int H_3 \wedge r_a = \int H_3 \wedge \widehat{q}^\alpha = \int r_a \wedge r_b = \int r_a \wedge \widehat{q}^\alpha = \int \widehat{q}^\alpha \wedge \widehat{q}^\beta = 0 \ , \tag{4.35}$$

for all a, b, α, β. Another perspective on these constraints is to view them as NS source tadpole equations. For instance, $\int H_3 \wedge r_a$ contributes to the tadpole equation of NS5-branes wrapping the two-cycle labeled by a, while $\int r_a \wedge r_b$ represents KK-monopole charge, , and other combinations correspond to more exotic sources [88]. Our models will not include any of these NS sources, so the condition of symplectic orthogonality stands.

Let us now briefly describe the hypermoduli, and the manner in which they descend from $\mathcal{N} = 2$ hypermultiplets [41]. For a type IIB O3/O7 compactification they are:

- $\tau = C_0 + ie^{-\phi}$, the axio-dilaton.
- G^a, $a = 1, \ldots, h^{1,1}_-$. These arise from the complexified 2-form potential $C_2 - \tau B_2 = (c^a - \tau u^a)\omega_a = G^a \omega_a$. There is one of these for each 2-form ω_a which is odd under the orientifold involution.
- T_α, $\alpha = 1, \ldots, h^{1,1}_+$. These are obtained by expanding a certain 4-form built out of the RR potential $C_4 = \rho_\alpha \widetilde{\mu}^\alpha$ as well as the Kähler form $J = v^\alpha \mu_\alpha$.

In fact, all of these hypermoduli can be conveniently and succinctly obtained by expansion of a formal sum of even degree forms [89],

[2]In the case at hand, where only H_3, r_a, $\widehat{q}^\alpha$ are nonzero, demanding that $\mathcal{D}^2 = 0$ on the cohomology of the underlying space is equivalent to the condition that the 3-forms form a symplectically orthogonal set. However, we actually need to demand that $\mathcal{D}^2 = 0$ on locally defined closed forms, and this requirement can be slightly more stringent. In this chapter we shall only make use of the symplectic orthogonality conditions, with the understanding that our generalized fluxes may be somewhat more constrained.

$$\Phi_c = e^{-B} \wedge C_{RR} + ie^{-\phi}\left(e^{-B+iJ}\right) = \tau + G^a\omega_a + T_\alpha\widetilde{\mu}^\alpha \ , \tag{4.36}$$

where $C_{RR} = C_0 + C_2 + C_4$ is a formal sum of RR potentials.

We can now write down the perturbative superpotential in the presence of the generalized fluxes. It takes exactly the same form as the familiar GVW superpotential [42]:

$$W = \int G_3 \wedge \Omega_3 \ , \tag{4.37}$$

where

$$\Omega_3 = Z^I\alpha_I - F_I\beta^I \ , \tag{4.38}$$

is the usual holomorphic 3-form. We generalize from $G_3 = F_3 - \tau H_3$ in the GVW case to

$$G_3 = F_3 + \mathcal{D}\Phi_c = F_3 - \tau H_3 - G^a r_a - T_\alpha\widehat{q}^\alpha \ , \tag{4.39}$$

when all the hypermoduli are taken into account. It is often useful to present this complex flux in terms of components. Generalizing (4.28) we expand the complex flux on basis 3-forms,

$$G_3 = m^I\alpha_I - e_I\beta^I \ , \tag{4.40}$$

where now the complex flux components are

$$m^I = m^I_f - \tau m^I_h - G^a r^I_a - T_\alpha\hat{q}^{\alpha I} \ , \tag{4.41}$$

$$e_I = e^f_I - \tau e^h_I - G^a r_{aI} - T_\alpha\hat{q}^\alpha_I \ . \tag{4.42}$$

The complex flux G_3 is a combination of the fluxes, $F_3, H_3, r_a, \widehat{q}^\alpha$ that we consider "inputs," parts of the definition of the vacuum, and then the hypermoduli τ, G^a, T_α which constitute dynamical fields.

The superpotential in component form is

$$W = e_I Z^I - m^I F_I \ . \tag{4.43}$$

It is worth emphasizing that the superpotential depends on vector moduli (complex structure moduli) and hypermoduli (Kähler moduli) in quite different ways:

- **Vector moduli:** enter through the symplectic section (Z^I, F_I) in the familiar manner, described by special geometry and a holomorphic prepotential F with derivative F_I. The physical moduli can (in one patch) be taken as the ratios $z^i = Z^i/Z^0$, $i = 1, \ldots, h_-^{(2,1)}$.

- **Hypermoduli:** enter *linearly* through the generalized complex flux (4.39). It is this property that we assumed from the outset in the general discussion in section 4.2.2.

4.3.2 Spacetime Potential with Geometric Flux

We next compute the spacetime potential (4.21) from the superpotential (4.43). For this we need the Kähler potential for the hypermoduli which, at large volume, is essentially the volume of the compactification manifold

$$K_{\rm H} = -\log[-i(\tau - \bar{\tau})^4 \, (\mathcal{V}_6)^2] \ . \tag{4.44}$$

The Calabi-Yau volume $\mathcal{V}_6$ (equal to $(\kappa v^3)/6$ in the notation below) depends implicitly on the hypermoduli τ, G^a, T_α, so it requires some effort to carry out differentiations with respect to these scalar fields and obtain the Kähler metric. The final result for the inverse Kähler metric becomes [41]

$$
\begin{aligned}
K^{\tau\bar{\tau}} &= -(\tau-\bar{\tau})^2 \ , && (4.45)\\
K^{\tau\bar{a}} &= (\tau-\bar{\tau})^2 u^a \ , && (4.46)\\
K^{\tau}{}_{\bar{\alpha}} &= -\frac{1}{2}(\tau-\bar{\tau})^2 \left(\widehat{\kappa}u^2\right)_\alpha \ , && (4.47)\\
K^{a\bar{b}} &= (\tau-\bar{\tau})^2 \left[\frac{1}{6}\left(\kappa v^3\right)\left(\widehat{\kappa}v\right)^{-1\,ab} - u^a u^b\right] \ , && (4.48)\\
K^{a}{}_{\bar{\alpha}} &= (\tau-\bar{\tau})^2 \left[-\frac{1}{6}\left(\kappa v^3\right)\left[\left(\widehat{\kappa}u\right)\left(\widehat{\kappa}v\right)^{-1}\right]^a_\alpha + \frac{1}{2}u^a\left(\widehat{\kappa}u^2\right)_\alpha\right], && (4.49)\\
K_{\alpha\bar{\beta}} &= (\tau-\bar{\tau})^2 \left[\frac{1}{6}\left(\kappa v^3\right)\left(\kappa v\right)_{\alpha\beta} - \frac{1}{4}\left(\kappa v^2\right)_\alpha\left(\kappa v^2\right)_\beta \right. \\
&\qquad \left. + \frac{1}{6}\left(\kappa v^3\right)\left[\left(\widehat{\kappa}u\right)\left(\widehat{\kappa}v\right)^{-1}\left(\widehat{\kappa}u\right)\right]_{\alpha\beta} - \frac{1}{4}\left(\widehat{\kappa}u^2\right)_\alpha\left(\widehat{\kappa}u^2\right)_\beta\right] . && (4.50)
\end{aligned}
$$

Here we have introduced intersection numbers

$$
\int \mu_\alpha \wedge \mu_\beta \wedge \mu_\gamma = \kappa_{\alpha\beta\gamma} \ , \qquad \int \mu_\alpha \wedge \omega_a \wedge \omega_b = \widehat{\kappa}_{\alpha\, ab} \ , \tag{4.51}
$$

and used a shorthand notation for contractions

$$
\left(\kappa v^3\right) = \kappa_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma \ , \qquad \left(\widehat{\kappa}v\right)_{ab} = \widehat{\kappa}_{\alpha\, ab} v^\alpha \ , \qquad \text{etc.} \tag{4.52}
$$

The spacetime potential (4.21) depends on the matrix $h^{A\bar{B}}$ introduced in (4.22), which is essentially the the inverse Kähler metric. The fields η^A are the imaginary part of the hypermoduli, here

$$
\begin{aligned}
2i\eta^\tau &= \tau-\bar{\tau} = 2ie^{-\phi} \, , && (4.53)\\
2i\eta^a &= -(\tau-\overline{\tau})u^a \, , && (4.54)\\
2i\eta_\alpha &= \frac{\tau-\overline{\tau}}{2}\left[\left(\widehat{\kappa}u^2\right)_\alpha - \left(\kappa v^2\right)_\alpha\right] . && (4.55)
\end{aligned}
$$

With this information, we easily find the matrix (4.22):

$$h^{\tau\bar{\tau}} = 0\,, \tag{4.56}$$

$$h^{\tau\bar{a}} = 0\,, \tag{4.57}$$

$$h^{\tau}{}_{\bar{\alpha}} = 2e^{-2\phi}\left(\kappa v^2\right)_{\alpha}\,, \tag{4.58}$$

$$h^{a\bar{b}} = -\frac{2}{3}e^{-2\phi}\left(\kappa v^3\right)\left(\widehat{\kappa}v\right)^{-1\,ab}\,, \tag{4.59}$$

$$h^{a}{}_{\bar{\alpha}} = \frac{2}{3}e^{-2\phi}\left(\kappa v^3\right)\left[\left(\widehat{\kappa}u\right)\left(\widehat{\kappa}v\right)^{-1}\right]^{a}{}_{\alpha} - 2e^{-2\phi}u^a\left(\kappa v^2\right)_{\alpha}\,, \tag{4.60}$$

$$\begin{aligned} h_{\alpha\bar{\beta}} = e^{-2\phi}\Big\{ & -\frac{2}{3}\left(\kappa v^3\right)\left(\kappa v\right)_{\alpha\beta} - \frac{2}{3}\left(\kappa v^3\right)\left[\left(\widehat{\kappa}u\right)\left(\widehat{\kappa}v\right)^{-1}\left(\widehat{\kappa}u\right)\right]_{\alpha\beta} \\ & + \left(\kappa v^2\right)_{\alpha}\left(\widehat{\kappa}u^2\right)_{\beta} + \left(\widehat{\kappa}u^2\right)_{\alpha}\left(\kappa v^2\right)_{\beta}\Big\}\,. \end{aligned} \tag{4.61}$$

The vanishing of the components $h^{\tau\bar{\tau}} = h^{\tau\bar{a}} = 0$ is significant. It means that, if we consider just the τ and G^a hypermoduli then $h^{A\bar{B}}$ has one zero eigenvalue. Moreover, its remaining eigenvalues are positive, since $(\widehat{\kappa}v)_{ab}$ is a negative-definite symmetric matrix inside the Kähler cone[3]. According to the general criteria at the end of section 4.2.2 this means that all of τ and the G^a would be stabilized. We are primarily interested in this setup, and will develop it further.

With the fluxes included in (4.39), T_α is also a stabilizable modulus. However, both $h^{a}{}_{\bar{\alpha}}$ and $h_{\alpha\bar{\beta}}$ have ambiguous signs, so including all of the fluxes from (4.39) will generically lead to AdS vacua. Since we are well-equipped to study Minkowski vacua, we will set $\widehat{q}^{\alpha} = 0$ for the remainder of the chapter. This reduces the complex flux G_3 from (4.39) to

$$G_3 = F_3 - \tau H_3 - G^a r_a\,, \tag{4.62}$$

[3]This follows from the fact that the inverse Kähler metric above must be positive definite at all points in the Kähler cone, and in particular when $u^a = 0$.

reduces the components of G_3 from (4.41) and (4.42) to

$$m^I = m_f^I - \tau m_h^I - G^a r_a^I \,, \tag{4.63}$$

$$e_I = e_I^f - \tau e_I^h - G^a r_{aI} \,, \tag{4.64}$$

and renders T_α unstabilizable.

For specific orientifold examples there can be other suitable truncations which can include some of the T_α. For instance, in a background with $h_-^{1,1} = 0$ and some particular even 2-form μ_1 satisfying $\mu_1 \wedge \mu_1 = 0$ (for instance one can construct suitable examples as certain complete intersections in products of projective spaces), then $h_-^{(1,1)} = 0$ and so we could truncate to T_1 alone (no τ). However, such solutions are not generic.

4.3.3 Attractor Equations from ISD Conditions

We are now ready to derive the attractor equations that describe moduli stabilization of O3/O7 compactifications with geometric flux as well as conventional 3-form fluxes.

The starting point is a subset (4.23)-(4.24) of the conditions for Minkowski vacua

$$D_i W = \int G_3 \wedge D_i \Omega_3 = 0 \,, \tag{4.65}$$

$$\widetilde{W} = \int \overline{G}_3 \wedge \Omega_3 = 0 \,. \tag{4.66}$$

The geometric flux r_a and the hypermoduli G^a just enter through the complex flux G_3 (4.62). The form of the conditions (4.65)-(4.66) is therefore the same as when there is no geometric flux. Indeed, these equations agree with the ISD conditions (4.10) and (4.11) for O3/O7 compactifications with 3-form flux alone. As we will make explicit, this means we can proceed as if there were no geometric fluxes, and then determine the hypermoduli from the constraints (4.24) and (4.25) at the end.

In the absence of geometric fluxes, it is known that (4.65) and (4.66) are best analyzed in the complex basis $\left\{\Omega_3, D_i\Omega_3, \overline{D_i\Omega}_3, \overline{\Omega}_3\right\}$ for the 3-form cohomology. Symplectic orthogonality then determines the complex flux G_3 as

$$G_3 = \overline{C\Omega}_3 + C^i D_i\Omega_3 \,, \tag{4.67}$$

with equality in the sense of cohomology. Since the complex basis consists of eigenforms of the Hodge star ($* = +i$ on $\overline{\Omega}_3$, $D_i\Omega_3$ and $* = -i$ on Ω_3, $\overline{D_j\Omega_3}$), it is manifest that G_3 is a generic ISD flux. The expansion coefficients[4].

Fluxes can be interpreted as a twisting of the exterior derivative $d \to \mathcal{D}$, as we have reviewed in section 4.3.1. The complex basis $\left\{\Omega_3, D_i\Omega_3, \overline{D_i\Omega}_3, \overline{\Omega}_3\right\}$ is certainly a good basis for the 3-form cohomology of the underlying Calabi-Yau [30], but relatively little is known about the corresponding twisted cohomology. We can justify the continued use of the complex basis by observing that the fluxes we consider preserve $SU(3)$ structure, even though they generally spoil the $SU(3)$ holonomy. The basis elements Ω_3, $D_i\Omega_3$ transform in representations of $SU(3)$; the $SU(3)$ structure ensures that they satisfy the usual orthogonality relations, and that they retain their eigenvalues under the Hodge star [90, 91]. We can therefore apply (4.67) also after the introduction of geometric fluxes, with the equality holding up to terms that vanish in the integral.

The covariant derivative with respect to the z^i that appears in (4.67) is awkward (because it obscures symplectic invariance) and also presents challenges in practical computations (because the Kähler potential enters). It is advantageous to replace it with an ordinary derivative with respect to the Z^I, i.e.

[4] The normalization of these is changed compared with chapter III: $C_{\text{here}} = i\text{Im}\,(\tau)\, C_{\text{there}}$, $C^i_{\text{here}} = -i\text{Im}\,(\tau)\, C^i_{\text{there}}$, and $L^I_{\text{here}} = -i\text{Im}\,(\tau)\, L^I_{\text{there}}$.

$$G_3 = \overline{C\Omega_3} + L^I \partial_I \Omega_3 \,. \qquad (4.68)$$

In doing so we must be conscious of the fact that ordinary derivatives of Ω_3 contain a term proportional to Ω_3:

$$\partial_I \Omega_3 = (\partial_I K)\, \Omega_3 + \dots \,. \qquad (4.69)$$

The G_3 (4.67) cannot contain a term proportional to Ω_3 so we must impose an additional constraint:

$$L^I \partial_I K = 0 \,, \qquad (4.70)$$

on the L^I. There is indeed one more complex parameter among the L^I than there is among the C^i, which is consistent with the addition of one complex constraint. Our result (4.68) is the attractor equation, written as a relation between 3-forms.

The attractor equations are perhaps more transparent when written in terms of the real basis (α_I, β^I) of odd 3-forms introduced in section 4.3.1. Then the moduli are encoded in the symplectic section (Z^I, F_I) introduced in (4.38) and the flux components take the form (4.41)-(4.42). The component form of the attractor equation (4.68) becomes:

$$m^I = \overline{CZ}^I + L^I \,, \qquad (4.71)$$

$$e_I = \overline{CF}_I + L^J F_{IJ} \,. \qquad (4.72)$$

We consider CZ^I and L^I to be the independent variables in the attractor equations. The CZ^I determine the physical moduli z^i as well as an additional parameter, CZ^0, which only appears in the scalar mass matrix. The L^I are all mass parameters. CF_I and F_{IJ} are functions of the CZ^I – the specific functional forms are determined by the symplectic section of the Calabi-Yau. Since the number of attractor equations in (4.71),(4.72) is equal to the number of variables in CZ^I, L^I, solving (4.71) and (4.72) should give CZ^I and L^I as functions of the complex fluxes m^I and e_I. This is true whether or not there are any geometric fluxes.

Now, this type of solution does not yet determine CZ^I and L^I as functions of the real, physical fluxes, because the complex fluxes (m^I, e_I) are themselves functions of the hypermoduli. This dependence on the hypermoduli is governed by several constraints. There is both the universal constraint (4.70), written in components as

$$0 = \overline{CF}_I L^I - \overline{CZ}^I L^J F_{IJ}\,, \tag{4.73}$$

and generally also the constraints (4.25). When only geometric fluxes have been included, these latter constraints are the conditions:

$$H_a\left(z\right) = \int \Omega_3 \wedge r_a = 0\,, \tag{4.74}$$

which we can write in terms of components as

$$0 = r_{Ia} CZ^I - r_a^I CF_I\,. \tag{4.75}$$

We emphasize that we have not set $D_a W = 0$, but instead that $D_a W = W \partial_a K$ leads to supersymmetry breaking when $W \neq 0$. This stands in contrast to the flux attractor equations for $SU\left(3\right) \times SU\left(3\right)$ structure compactifications developed in [19, 76]. These attractor equations only described supersymmetric ($W = 0$) Minkowski vacua, while the attractor equations presented here describe non-supersymmetric ($W \neq 0$) Minkowski vacua as well.

Let us summarize the procedure we propose. We first solve the attractor equations (4.71) and (4.72) for CZ^I, L^I. The result will be in terms of the complex fluxes

(m^I, e_I) that depend on both τ and G^a. In the next step we use the constraints (4.73) and (4.75) together to determine τ and the G^a. The procedure is particularly simple in the standard GKP case where there is no geometric flux, and so the complex fluxes depend only on τ. Then there is just a single constraint (4.73) to solve. In the remainder of the chapter we will study the more general case including geometric fluxes.

There is one subtlety: although the constraints (4.73), (4.75) appear to determine all of τ, G^a, in fact the number of τ, G^a that we can stabilize is limited by the number $h_-^{(2,1)}$ of physical moduli $z^i = Z^i/Z^0$. If we divide (4.75) by CZ^0 and use the homogeneity properties of the F_I, we see that the hypermoduli enter into (4.75) only via the z^i, so only $h_-^{(2,1)}$ distinct combinations of the hypermoduli are constrained. When $h_-^{(1,1)} > h_-^{(2,1)}$, either $h_-^{(1,1)} - h_-^{(2,1)}$ hypermoduli will remain unstabilized, or there will be no solutions to (4.75) and we are forced into a runaway vacuum.

The situation is ameliorated somewhat by that fact that not all of the constraints (4.75) can be independent. The geometric fluxes r_a are 3-forms that must be symplectically orthogonal due to the tadpole conditions (4.35). There are at most $h_-^{(2,1)} + 1$ such three-forms, so only $h_-^{(2,1)} + 1$ of the constraints (4.75) can be independent. This is still one more than $h_-^{(2,1)}$, the number of z^i's and thus the number of independent equations we can solve, according to the argument in the previous paragraph. For generic geometric fluxes and $h_-^{(1,1)} > h_-^{(2,1)}$ we will therefore find no solutions to (4.75), but for a co-dimension one subspace of the space of possible geometric fluxes, we expect to be effective at stabilizing hypermoduli.

Summary of this section: The principal results are the attractor equations, (4.71) and (4.72), and the constraints (4.73) and (4.75). These equations illuminate how particular fluxes stabilize particular moduli. In the following sections we will show that solutions to these attractor equations can be succinctly summarized by a single generating function, as was the case without geometric fluxes. We will also solve several examples where as many moduli as possible are stabilized.

4.4 Generating Functions with Geometric Flux

While the flux attractor equations (4.71), (4.72) and constraints (4.73), (4.75), are considerably simpler than the equations that would arise from direct minimization of the potential, they cannot be solved explicitly for a generic Calabi-Yau. Nevertheless, we can establish several general properties of the solutions. First of all, the solutions for all of the moduli and mass parameters can be presented as derivatives of a single generating function. We demonstrated this in chapter III for the standard GKP setup, and here we extend the result to include geometric fluxes.

We will present two versions of the generating function, which give rise to two different stabilization procedures. The first version depends on both the complex fluxes and the hypermoduli, with the stabilized values of the hypermoduli determined by extremizing the generating function with respect to the hypermoduli. The second version employs a reduced generating function that depends on the real fluxes only. In both cases the stabilization of the vector moduli is treated separately from the stabilization of the hypermoduli.

4.4.1 Explicit Expression for the Generating Function

We begin by rewriting the electric and magnetic attractor equations, (4.71) and (4.72) as:

$$\begin{aligned} \overline{CZ}^I &= \frac{1}{2}\left(m^I + \phi^I\right), & (4.76)\\ L^I &= \frac{1}{2}\left(m^I - \phi^I\right), & (4.77)\\ \overline{CF}_I &= \frac{1}{2}\left(e_I + \theta_I\right), & (4.78)\\ L^J F_{IJ} &= \frac{1}{2}\left(e_I - \theta_I\right), & (4.79) \end{aligned}$$

where the ϕ^I and θ_I are (typically non-holomorphic) functions of the complex fluxes m^I and e_I. Although it may appear that arbitrary ϕ^I and θ_I solve (4.71) and (4.72), leading to essentially arbitrary solutions for CZ^I and L^I, the solutions for ϕ^I and θ_I are in fact related to one another in a nonlinear fashion. This is because F_I and F_{IJ} are not independent parameters, but are fixed functions of the Z^I, with the specific functional form determined by the symplectic section of the Calabi-Yau. In order to solve (4.76)-(4.79), we must substitute the expressions for CZ^I and L^I in terms of m^I and ϕ^I into (4.78) and (4.79), then solve for ϕ^I and θ_I. Doing this directly is difficult even for relatively simple Calabi-Yaus.

Considered as equations that determine the potentials ϕ^I and θ_I in terms of the complex fluxes m^I and e_I, (4.76)-(4.79) are exactly the same whether or not we have introduced geometric fluxes. We can therefore use a result proven in chapter III, namely that all solutions for the ϕ^I and θ_I can be written as derivatives of a real generating function[5] $\mathcal{G}$:

$$\begin{aligned} \phi^I &= (\tau - \overline{\tau}) \frac{\partial \mathcal{G}}{\partial \overline{e}_I} \,, &(4.80)\\ \theta_I &= -(\tau - \overline{\tau}) \frac{\partial \mathcal{G}}{\partial \overline{m}^I} \,. &(4.81) \end{aligned}$$

Although the additional minus sign in (4.81) may look awkward, it is necessary because $\left(\partial/\partial\overline{e}_I, -\partial/\partial\overline{m}^I\right)$ is a good symplectic vector, while $\left(\partial/\partial\overline{e}_I, \partial/\partial\overline{m}^I\right)$ is not. The derivatives of $\mathcal{G}$ are taken with the other *complex* fluxes, as well as τ and the G^a, held fixed. If we consider $\mathcal{G}$ as a thermodynamic function, (4.80) and (4.81) identify ϕ^I and θ_I as the potentials conjugate to $\overline{e}_I$ and $\overline{m}^I$, respectively, and so we will frequently refer to them as "the potentials."

Another result of chapter III that still holds after the introduction of geometric

[5]Strictly speaking, we can arrive at a whole family of generating functions by changing the normalizing factor of $(\tau - \overline{\tau})$ to various other functions of the hypermoduli. In section (4.4.2) we will see that the choice of $(\tau - \overline{\tau})$ is preferred, even after we have introduced geometric fluxes.

flux is that $\mathcal{G}$ is homogeneous of degree $(1,1)$ in the complex fluxes. In other words,

$$\mathcal{G}\left(\lambda m^J, \lambda e_J, \tilde{\lambda}\overline{m}^J, \tilde{\lambda}\overline{e}_J, \tau, G^a\right) = \lambda\tilde{\lambda}\mathcal{G}\left(m^J, e_J, \overline{m}^J, \overline{e}_J, \tau, G^a\right), \qquad (4.82)$$

for any $\lambda, \tilde{\lambda} \in \mathbb{C}$. This implies that the potentials ϕ^I and θ_I are homogeneous of degree $(1,0)$. It also allows us to write an explicit expression for $\mathcal{G}$:

$$\mathcal{G} = e_I\frac{\partial \mathcal{G}}{\partial e_I} + m^I\frac{\partial \mathcal{G}}{\partial m^I} \qquad (4.83)$$

$$= -\frac{1}{\tau - \overline{\tau}}\left\{e_I\overline{\phi}^I - m^I\overline{\theta}_I\right\}. \qquad (4.84)$$

The first line follows from the homogeneity of $\mathcal{G}$, while the second follows by substituting in (4.80) and (4.81). Given an explicit solution of (4.76)-(4.79), we can compute ϕ^I and θ_I, then use (4.84) to compute $\mathcal{G}$. We also see that whenever the flux attractor equations have multiple sets of solutions, each solution will correspond to a different generating function.

4.4.2 Stabilizing the Hypermoduli

Once the potentials have been determined, we have solved the attractor equations (4.71) and (4.72) for the unknowns CZ^I and L^I, with the hypermoduli treated as given parameters. To find the stabilized values of the hypermoduli we can substitute our solutions for CZ^I and L^I into the constraints (4.73) and (4.75) and solve. In this section we will present an alternate procedure: simply extremize $\mathcal{G}$ with respect to the hypermoduli.

We first present the universal constraint in a simplified form. If we substitute (4.76)-(4.79) into (4.73), we find

$$0 = \overline{CF}_I L^I - \overline{CZ}^I L^J F_{IJ} = -\frac{1}{2}\left(\phi^I e_I - \theta_I m^I\right). \qquad (4.85)$$

In order to recover the universal constraint and the constraints (4.75) from our new procedure, we need the derivatives of $\mathcal{G}$ with the real fluxes, rather than the complex fluxes, held fixed.

We begin by writing the $\tau-$derivative of $\mathcal{G}$ with the real fluxes held fixed[6] :

$$\left.\frac{\partial \mathcal{G}}{\partial \tau}\right|_{\mathbb{R}} = \left.\frac{\partial \mathcal{G}}{\partial \tau}\right|_{\mathbb{C}} + \frac{\partial \mathcal{G}}{\partial e_I}\frac{\partial e_I}{\partial \tau} + \frac{\partial \mathcal{G}}{\partial m^I}\frac{\partial m^I}{\partial \tau} \tag{4.86}$$

$$= \frac{1}{(\tau-\overline{\tau})^2}\left\{\overline{\phi}^I e_I - \overline{\theta}_I m^I\right\} + \frac{1}{\tau-\overline{\tau}}\left\{\overline{\phi}^I e_I^h - \overline{\theta}_I m_h^I\right\} . \tag{4.87}$$

In the first line we used $\mathbb{R}$ and $\mathbb{C}$ as a shorthand to indicate that the real fluxes and complex fluxes, respectively, are held fixed. The second line follows by application of (4.84), (4.80)-(4.81), (4.63), and (4.64). In the standard GKP setup, this expression reduces to

$$\left.\frac{\partial \mathcal{G}}{\partial \tau}\right|_{\mathbb{R}} = \frac{1}{(\tau-\overline{\tau})^2}\left\{\overline{\phi}^I\left[e_I^f - \tau e_I^h + (\tau-\overline{\tau})\, e_I^h\right] - \overline{\theta}^I\left[m_f^I - \tau m_h^I + (\tau-\overline{\tau})\, m_h^I\right]\right\} \tag{4.88}$$

$$= \frac{1}{(\tau-\overline{\tau})^2}\left\{\overline{\phi}^I \overline{e}_I - \overline{\theta}_I \overline{m}^I\right\} . \tag{4.89}$$

Comparing (4.89) with (4.85), we see that extremizing $\mathcal{G}$ with respect to τ, while holding the real fluxes fixed, reproduces (4.73) in the standard GKP setup.

After adding geometric fluxes, (4.87) reduces to

$$\left.\frac{\partial \mathcal{G}}{\partial \tau}\right|_{\mathbb{R}} = \frac{1}{(\tau-\overline{\tau})^2}\left\{\overline{\phi}^I\left[e_I^f - \overline{\tau} e_I^h - G^a r_{Ia}\right] - \overline{\theta}^I\left[m_f^I - \overline{\tau} m_h^I - G^a r_a^I\right]\right\} , \tag{4.90}$$

so a $\tau-$derivative alone is insufficient to reproduce (4.85). However, we can combine

[6]The specific form of $\left.\frac{\partial \mathcal{G}}{\partial \tau}\right|_{\mathbb{C}}$ was ultimately determined by the introduction of $(\tau-\overline{\tau})$, rather than some other function of the hypermoduli, in (4.80) and (4.81). Using $(\tau-\overline{\tau})$ we will find simple conditions on $\left.\frac{\partial \mathcal{G}}{\partial \tau}\right|_{\mathbb{R}}$ and $\left.\frac{\partial \mathcal{G}}{\partial G^a}\right|_{\mathbb{R}}$, while using other functions of the hypermoduli would lead to much more awkward conditions.

(4.90) with

$$\left.\frac{\partial\mathcal{G}}{\partial G^a}\right|_{\mathbb{R}} = \left.\frac{\partial\mathcal{G}}{\partial G^a}\right|_{\mathbb{C}} + \frac{\partial\mathcal{G}}{\partial e_I}\frac{\partial e_I}{\partial G^a} + \frac{\partial\mathcal{G}}{\partial m^I}\frac{\partial m^I}{\partial G^a} \tag{4.91}$$

$$= \frac{1}{\tau-\overline{\tau}}\left\{\overline{\phi}^I r_{Ia} - \overline{\theta}_I r^I_a\right\}\,, \tag{4.92}$$

to find

$$(\tau-\overline{\tau})\left.\frac{\partial\mathcal{G}}{\partial\tau}\right|_{\mathbb{R}} + \left(G^a - \overline{G}^a\right)\left.\frac{\partial\mathcal{G}}{\partial G^a}\right|_{\mathbb{R}} = \frac{1}{\tau-\overline{\tau}}\left\{\overline{\phi}^I\overline{e}_I - \overline{\theta}_I\overline{m}^I\right\}\,. \tag{4.93}$$

Setting this linear combination of derivatives of $\mathcal{G}$ to zero thus reproduces (4.73), even when geometric fluxes are included.

We also need to recover the remaining constraint (4.75) from derivatives of $\mathcal{G}$. This is straightforward, because the tadpole constraints (4.35) imply that

$$\overline{m}^I r_{Ia} - \overline{e}_I r^I_a = 0\,, \tag{4.94}$$

and so allow us to rewrite (4.92) as

$$\left.\frac{\partial\mathcal{G}}{\partial G^a}\right|_{\mathbb{R}} = \frac{1}{\tau-\overline{\tau}}\left\{\left(\overline{m}^I+\overline{\phi}^I\right)r_{Ia} - \left(\overline{e}_I+\overline{\theta}_I\right)r^I_a\right\} \tag{4.95}$$

$$= \frac{1}{\tau-\overline{\tau}}\left\{CZ^I r_{Ia} - CF_I r^I_a\right\}\,. \tag{4.96}$$

Comparing this with (4.75), we see that extremizing $\mathcal{G}$ with respect to the G^a, while holding the real fluxes fixed, reproduces the $H_a(z)=0$ attractor equations. Combining this with (4.93), we find that we must extremize over τ as well. It is somewhat surprising that the tadpole constraints play a crucial role here, given that they do not appear anywhere else in our study of the flux attractor equations.

Let us summarize our results about $\mathcal{G}$ so far. Suppose that we have somehow determined $\mathcal{G}$ as a function of the complex fluxes and the hypermoduli. (4.80) and (4.81) then determine the potentials ϕ^I and θ_I as functions of the complex fluxes and τ, and (4.76) and (4.77) in turn determine the stabilized values of the vector moduli and mass parameters. The remaining dependence of these quantities on the hypermoduli, through the complex fluxes, is fixed by extremizing $\mathcal{G}$ with respect to the hypermoduli, while holding the real fluxes fixed. Upon substituting the values of the hypermoduli into the expressions for ϕ^I and θ_I, we have determined the values of all the moduli, as well as the values of the mass parameters CZ^0 and L^I.

4.4.3 Reduced Generating Function

One peculiar aspect of the generating function described so far is that the fluxes and hypermoduli appear in $\mathcal{G}$ on roughly equal footing, but are treated very differently when we solve for the various moduli. We will now show how the moduli z^i and mass parameters CZ^0 and L^I can be determined from a reduced generating function, $\widetilde{\mathcal{G}}$, which depends on the real fluxes only. Formally, $\widetilde{\mathcal{G}}$ is constructed by substituting the stabilized values of the hypermoduli into $\mathcal{G}$.

We first address a preliminary issue concerning the map between real and complex fluxes. While we have already recorded the expressions for the complex fluxes in terms of the real fluxes (4.41)-(4.42), we will also need to know how derivatives with respect to the complex fluxes are related to derivatives with respect to the real fluxes, and this relationship is slightly subtle. When discussing the real fluxes we always explicitly include the full set $\left\{m_h^I, m_f^I, e_I^h, e_I^f, r_a^I, r_{aI}\right\}$, but when discussing the complex fluxes we tend to include only m^I, e_I, and their complex conjugates. In fact the complete set consists of $\left\{m^I, \overline{m}^I, e_I, \overline{e}_I, r_a^I, r_{aI}\right\}$. This implies that the relationship between the real and complex derivatives is:

$$\frac{\partial}{\partial m^I} = -\frac{1}{\tau-\overline{\tau}}\left(\overline{\tau}\frac{\partial}{\partial m_f^I}+\frac{\partial}{\partial m_h^I}\right), \tag{4.97}$$

$$\frac{\partial}{\partial e_I} = -\frac{1}{\tau-\overline{\tau}}\left(\overline{\tau}\frac{\partial}{\partial e_I^f}+\frac{\partial}{\partial e_I^h}\right). \tag{4.98}$$

We might have expected derivatives with respect to r_a^I or r_{aI} to appear here as well, but the complex derivatives must give zero when acting on r_a^I and r_{aI}, so such terms cannot appear.

The decomposition of (4.97) and (4.98) into derivatives with respect to real fluxes suggests that we define a set of real potentials,

$$\phi_f^I = \frac{\partial \mathcal{G}}{\partial e_I^h}\,, \qquad \phi_h^I = -\frac{\partial \mathcal{G}}{\partial e_I^f}\,, \tag{4.99}$$

$$\theta_I^f = -\frac{\partial \mathcal{G}}{\partial m_h^I}\,, \qquad \theta_I^h = \frac{\partial \mathcal{G}}{\partial m_f^I}\,. \tag{4.100}$$

related to the complex potentials via

$$\phi^I = \phi_f^I - \tau \phi_h^I\,, \tag{4.101}$$

$$\theta_I = \theta_I^f - \tau \theta_I^h\,. \tag{4.102}$$

Note that the derivatives with respect to the real fluxes are taken with the hypermoduli held fixed.

We now define $\widetilde{\mathcal{G}}$ as $\mathcal{G}$ with all hypermoduli replaced by their stabilized values, written as functions of the real fluxes. While this is the natural way to turn $\mathcal{G}$ into a function of the fluxes alone, we would like to know how $\widetilde{\mathcal{G}}$ relates to the attractor equations. A simple calculation shows that

$$\frac{\partial \widetilde{\mathcal{G}}}{\partial e_I^h} = \frac{\partial \mathcal{G}}{\partial e_I^h} + \frac{\partial \mathcal{G}}{\partial \tau}\frac{\partial \tau}{\partial e_I^h} + \frac{\partial \mathcal{G}}{\partial G^a}\frac{\partial G^a}{\partial e_I^h} \tag{4.103}$$

$$= \phi_f^I\,. \tag{4.104}$$

The second and third terms vanish because the hypermoduli are determined by extremizing $\mathcal{G}$ with respect to τ and G^a. We see that derivatives of $\widetilde{\mathcal{G}}$ return the real potentials, and therefore determine the complex potentials ϕ^I and θ_I as functions of

τ.

The procedure we follow to determine the values of the moduli and mass parameters if we know the reduced generating function $\widetilde{\mathcal{G}}$ is slightly different from the procedure we follow if we have $\mathcal{G}$. We first differentiate $\widetilde{\mathcal{G}}$ to determine the real potentials. This gives us the moduli and mass parameters as functions of the real fluxes and the hypermoduli:

$$\overline{CZ}^I = \frac{1}{2}\left[\left(m_f^I - \tau m_h^I - G^a r_a^I\right) + \left(\frac{\partial \widetilde{\mathcal{G}}}{\partial e_I^h} + \tau \frac{\partial \widetilde{\mathcal{G}}}{\partial e_I^f}\right)\right], \tag{4.105}$$

$$L^I = \frac{1}{2}\left[\left(m_f^I - \tau m_h^I - G^a r_a^I\right) - \left(\frac{\partial \widetilde{\mathcal{G}}}{\partial e_I^h} + \tau \frac{\partial \widetilde{\mathcal{G}}}{\partial e_I^f}\right)\right]. \tag{4.106}$$

We then substitute these expressions into (4.73) and (4.75) and solve to find τ and the G^a.

We believe that $\widetilde{\mathcal{G}}$ is a conceptually simpler object to study than $\mathcal{G}$ since it is a function of the fluxes alone, rather than a function of fluxes and hypermoduli. We will also see an example below where we cannot determine a closed form for $\mathcal{G}$, but are able to compute $\widetilde{\mathcal{G}}$.

4.5 Examples

In order to establish the utility of the flux attractor equations and the generating function formalism, we will now analyze two compactifications that admit both 3-form fluxes and geometric fluxes. We will solve the attractor equations (4.71) and (4.72) and the constraints (4.73) and (4.75) directly, then use the results to reconstruct the generating function.

One important input for the flux attractor equations is the prepotential, which determines the F_I and F_{IJ} via

$$F_I = \partial_I F\,, \qquad F_{IJ} = \partial_I \partial_J F\,.$$

In our first example we will study a particular $\mathbb{Z}_4$ orbifold of T^6, which gives rise to a prepotential

$$F_{T^6/\mathbb{Z}_4} = -iZ^0 Z^1\,. \tag{4.107}$$

In the second example, we will use the STU prepotential,

$$F_{\mathrm{STU}} = \frac{Z^1 Z^2 Z^3}{Z^0}\,. \tag{4.108}$$

The simplicity of the $T^6/\mathbb{Z}_4$ example makes it easy to demonstrate the logic of both the flux attractor equations and the generating function. While the STU example is more involved, we believe it is representative of what one would find when studying the large class of cubic prepotentials.

An interesting property of the attractor equations (4.71), (4.72) and constraints (4.73), (4.75), is that they do not include or require detailed information about the space of hypermoduli. While we might have imagined that e.g. the triple intersection numbers would play an important role, at least in the $H_a = 0$ equations, they do not. Rather, we only need to know $h_-^{(1,1)}$, which determines the number of different geometric fluxes r_a that can induce new F-terms, and thus stabilize additional moduli. In the following we will carefully establish that there are constructions that give rise to these prepotentials that *also* have $h_-^{(1,1)} \neq 0$.

4.5.1 $T^6/\mathbb{Z}_4$

In this example, a relatively simple prepotential will allow us to compute the generating function $\mathcal{G}$ for generic fluxes. After describing the orbifold construction that gives rise to (4.107), we will solve the attractor equations (4.71) and (4.72). This gives the potentials ϕ^I and θ_I as functions of the complex fluxes, which we will then use to write the generating function $\mathcal{G}$. We will also write the system of equations that determines the values of the stabilized moduli as functions of the real fluxes, and see that two hypermoduli can be stabilized.

4.5.1.1 Orbifold Construction

Let us consider an $\mathcal{N} = 2$ supersymmetric orbifold $T^6/\mathbb{Z}_4$, where the action of $\mathbb{Z}_4$ on the complex coordinates is generated by

$$\Theta \cdot (z_1, z_2, z_3) = (iz_1, iz_2, -z_3)\,. \tag{4.109}$$

The untwisted sector of this orbifold gives rise to 5 $(1,1)$-forms and one $(2,1)$-form. The twisted sector content depends on which T^6 lattice we are acting. To be concrete, let us pick the A_3^2 lattice (the root lattice of $SU(4) \times SU(4)$). For this choice the twisted sectors contribute 20 $(1,1)$-forms but no 3-forms (see, e.g. [92]), so the only complex structure moduli will come from the untwisted sector, and we do not need to perform any truncations when computing the prepotential or, eventually, the generating function.

Now let us construct an $\mathcal{N} = 1$ supersymmetric orientifold by combining the involution

$$\sigma \cdot (z_1, z_2, z_3) = (z_1, -z_2, z_3)\,, \tag{4.110}$$

with $\Omega(-1)^{F_L}$, where Ω here represents a worldsheet parity transformation and F_L is the left-moving fermion number on the worldsheet. The involutions σ and $\Theta^2\sigma$ each give rise to sets of untwisted sector O7-planes, while the involutions $\Theta\sigma$ and $\Theta^3\sigma$ give rise to twisted sector O7-planes which wrap exceptional divisors at the θ^2 fixed points. There are no O3-planes in this model.

Under this orientifold involution, three of the untwisted sector $(1,1)$-forms are invariant, while the other two change sign (all of the twisted sector $(1,1)$-forms are invariant), giving $h_-^{(1,1)} = 2$ and $h_+^{(1,1)} = 23$. All of the 3-forms change sign, so $h_-^{(2,1)} = 1$ and $h_+^{(2,1)} = 0$. Thus, in principle we can turn on geometric fluxes r_1 and r_2 as well as H_3 and F_3, and each of these 3-forms has four components.

For a certain choice of symplectic basis, the coefficients of the holomorphic three form (4.38) correspond to a prepotential

$$F = -iZ^0 Z^1. \tag{4.111}$$

With this information we can turn to a computation of the generating function.

4.5.1.2 Solutions for Potentials and $\mathcal{G}$

Using (4.76)-(4.79) and the prepotential (4.111), we can solve for e_I and θ_I in terms of m^I and ϕ^I.

$$\begin{aligned} e_0 &= \overline{CF_0} + L^J F_{0J} = i\left(\overline{CZ^1} - L^1\right) = i\phi^1, & (4.112)\\ e_1 &= \overline{CF_1} + L^J F_{1J} = i\left(\overline{CZ^0} - L^0\right) = i\phi^0, & (4.113)\end{aligned}$$

which inverts to

$$\phi^0 = -ie_1, \qquad \phi^1 = -ie_0. \tag{4.114}$$

Similarly, we have

$$\theta_0 = im^1, \qquad \theta_1 = im^0. \tag{4.115}$$

Inserting these results into the expression (4.84) we find

$$\mathcal{G} = -\frac{i}{\tau - \bar{\tau}}\left(e_0\overline{e_1} + e_1\overline{e_0} + m^0\overline{m^1} + m^1\overline{m^0}\right). \tag{4.116}$$

4.5.1.3 Solutions for Hypermoduli and $\widetilde{\mathcal{G}}$

We can now derive the constraints. From the complex conjugate of (4.92) we find

$$\left.\frac{\partial \mathcal{G}}{\partial \overline{G^a}}\right|_{\mathbb{R}} = \frac{i}{\tau - \bar{\tau}} \left(r_{a1} e_0 + r_{a0} e_1 + r_a^1 m^0 + r_a^0 m^1 \right) = 0 \,, \tag{4.117}$$

and after imposing this constraint we can write the complex conjugate of (4.93) as

$$\left.\frac{\partial \mathcal{G}}{\partial \bar{\tau}}\right|_{\mathbb{R}} = -\frac{2i}{(\tau - \bar{\tau})^2} \left(e_0 e_1 + m^0 m^1 \right) = 0 \,. \tag{4.118}$$

Setting these expressions to zero will stabilize some of our hypermoduli. Now all three of H_3, r_1, and r_2 must be symplectically orthogonal by the tadpole conditions (4.35), but in our model a symplectically orthogonal set of 3-forms is at most two-dimensional. Because of this, we can only hope to fix at most two linear combinations of the three moduli τ, G^1, and G^2. More explicitly, if the two independent orthogonal three-forms are denoted ξ_1 and ξ_2, and we write $H_3 = A_\tau \xi_1 + B_\tau \xi_2$, $r_a = A_a \xi_1 + B_a \xi_2$, then the complex flux is given by $G_3 = F_3 - x_1 \xi_1 - x_2 \xi_2$, where $x_1 = A_\tau \tau + A_a G^a$ and $x_2 = B_\tau \tau + B_a G^a$. Since the minimization procedure depends on the hypermoduli only via the complex flux, we can only hope to stabilize the linear combinations x_1 and x_2, leaving a third linear combination unfixed.

4.5.2 The STU Model

With this example we add geometric fluxes to a compactification with an STU prepotential. This example was studied carefully in the absence of geometric fluxes in [1, 64–67]. Substituting the symplectic section determined by (4.108) into (4.72), the electric attractor equations become

$$e_0 = -\frac{\overline{CZ}^1\overline{CZ}^2\overline{CZ}^3}{\left(\overline{CZ}^0\right)^2} + 2L^0\frac{CZ^1CZ^2CZ^3}{\left(CZ^0\right)^3} - \left(L^1\frac{CZ^2CZ^3}{\left(CZ^0\right)^2} + \text{cyc.}\right), \quad (4.119)$$

$$e_1 = -\frac{\overline{CZ}^2\overline{CZ}^3}{\overline{CZ}^0} - L^0\frac{CZ^2CZ^3}{\left(CZ^0\right)^2} + L^2\frac{CZ^3}{CZ^0} + L^3\frac{CZ^2}{CZ^0}\,. \quad (4.120)$$

Cyclic permutations of (4.120) give the remaining two electric attractor equations. Since we can use (4.71) to rewrite the L^I in terms of m^I and CZ^I, these are four complex, non-holomorphic, non-linear equations for the CZ^I. By using (4.76) we could recast (4.119) and (4.120) as equations for the ϕ^I rather than the CZ^I. Rather than solve directly for the CZ^I or the potentials, we will find it most useful to use $z^i = Z^i/Z^0$, so that (4.119) and (4.120) are considered as equations for the z^i and CZ^0.

While black hole attractor equations with the STU prepotential can be solved explicitly for arbitrary black hole charges, the attractor equations (4.119) and (4.120) do not admit an explicit solution for general fluxes. Since we are interested in finding explicit solutions that illuminate the results of sections (4.3) and (4.4), we will only turn on four components of F_3, four components of H_3, and six geometric fluxes. These 14 real flux components will allow us to explicitly stabilize the three complex vector moduli, z^i, and four complex hypermoduli, τ and three of the G^a. While it is not possible to compute explicitly the generating function $\mathcal{G}$ for these fluxes, we will compute the *reduced* generating function $\widetilde{\mathcal{G}}$, with the result given in equation (4.176).

An issue that will arise at several points in our analysis is the role of the tadpole constraints (4.35). Once we are analyzing equations involving real fluxes, imposing the tadpole constraints will consistently lead to significantly simpler expressions for the stabilized values of the moduli, the mass parameters, and the generating function. While simplifying with the tadpole constraints will not alter the algebraic relationships between these quantities, they do affect their *derivatives.* Since one of our goals is to illustrate how derivatives of the generating function reproduce the moduli and mass parameters, the primary results of sections 4.5.2.3-4.5.2.5 will be presented both with and without the tadpole constraints (4.35) imposed.

4.5.2.1 The Enriques Calabi-Yau and the STU Prepotential

We saw in section (4.3.2) that geometric fluxes could only induce new F-terms when $h_-^{(1,1)} \neq 0$. Unfortunately, the standard orbifold construction that leads to the STU prepotential, $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, has $h_-^{(1,1)} = 0$. Another construction that leads to the STU prepotential, but which has $h_-^{(1,1)} = 8$ is an orientifold of the Enriques Calabi-Yau.

The construction of the Enriques Calabi-Yau [93] begins with $K3 \times T^2$. The $K3$ factor admits the freely-acting Enriques involution, θ_1, under which the holomorphic 2-form is odd. Orbifolding $K3$ by θ_1 would give the Enriques surface, but we will instead orbifold $K3 \times T^2$ by $\theta_1\theta_2$, where θ_2 takes the torus coordinate z^3 to $-z^3$. The resulting surface is a self-mirror Calabi-Yau with $h^{(1,1)} = h^{(2,1)} = 11$. In the orbifold limit of the underlying $K3$ factor, the untwisted sector contributes $h^{(1,1)} = h^{(2,1)} = 3$, while the twisted sector contributes $h^{(1,1)} = h^{(2,1)} = 8$. The prepotential is governed by the triple intersection numbers

$$\kappa_{123} = 1\,, \tag{4.121}$$

$$\kappa_{3ab} = C_{ab}\,, \tag{4.122}$$

where C_{ab} is the Cartan matrix of E_8, and $a, b = 4, ..., 11$. Type II compactifications on the Enriques Calabi-Yau have $\mathcal{N} = 2$ supersymmetry.

The final step in the construction is the orientifold projection [94], which reduces the amount of supersymmetry to $\mathcal{N} = 1$. This employs a second involution which gives

-1 when acting on the 2-forms ω_a, $+1$ when acting on ω_1 and ω_2, and inverts the T^2. This splits the 2-form cohomology such that $h_-^{(1,1)} = 8$ and $h_+^{(1,1)} = 3$. Because the 3-forms are constructed by wedging together 2-forms on the underlying $K3$ with 1-forms on the underlying T^2, the 3-form cohomology splits with $h_-^{(2,1)} = 3$ and $h_+^{(2,1)} = 8$. The triple intersection numbers (4.121) determine that three surviving complex structure moduli will be governed by the STU prepotential.

4.5.2.2 Complex Fluxes and the Vector Moduli

Since we cannot explicitly solve the attractor equations (4.119) and (4.120) with generic fluxes, we impose the following reality conditions on the complex fluxes:

$$\overline{m}^0 = m^0\,, \tag{4.123}$$

$$\overline{m}^i = -m^i\,, \tag{4.124}$$

$$\overline{e}_0 = -e_0\,, \tag{4.125}$$

$$\overline{e}_i = e_i\,. \tag{4.126}$$

We also make a complementary ansatz for the potentials:

$$\overline{\phi}^0 = \phi^0\,, \tag{4.127}$$

$$\overline{\phi}^i = -\phi^i\,, \tag{4.128}$$

$$\overline{\theta}_0 = -\theta_0\,, \tag{4.129}$$

$$\overline{\theta}_i = \theta_i\,. \tag{4.130}$$

This reduction was previously utilized in chapter III, where it was found to be a useful compromise between completely general fluxes (where (4.119) and (4.120) cannot be solved explicitly) and solubility (since overly simple fluxes do not stabilize all of the moduli).

An important feature of the attractor equations (4.71) and (4.72) is that the fluxes enter only via the complex fluxes m^I and e_I. This means that they lead to the same solutions for the moduli and mass parameters as functions of the complex fluxes, with or without geometric fluxes. Since (4.119) and (4.120) were already solved in chapter III, we simply quote the solutions:

$$
\begin{aligned}
CZ^0 &= \frac{1}{4}\left(m^0 - i\sum_i m^i\sqrt{\frac{m^0 e_i}{e_0 m^i}}\right), &\quad (4.131)\\
z^i &= -i\sqrt{\frac{e_0 m^i}{m^0 e_i}}\,, &\quad (4.132)
\end{aligned}
$$

with no summation over i in (4.132). The requirement that the metric on moduli space remain positive, which in turn requires $\mathrm{Im}\,(z^i) < 0$ and $\mathrm{Im}\,(\tau) > 0$, implies a condition on the complex fluxes:

$$
i\frac{m^I}{e_I} > 0\,, \qquad (4.133)
$$

with no summation over I. This implies that the quantities under the square roots in (4.131) and (4.132) are real and positive, and we will ensure for the remainder of this section that only real, positive quantities appear under square roots. We will also take the positive branch of all square roots.

The universal constraint (4.73) is also written in terms of the complex fluxes alone, and so is the same with or without geometric fluxes. Generically, it takes the form of a condition that the complex fluxes must satisfy. We again quote the result from chapter III:

$$
\frac{e_0 m^1 m^2 m^3}{m^0 e_1 e_2 e_3} = -1\,. \qquad (4.134)
$$

In fact this condition was used in the derivation of (4.131) and (4.132), where it helped to find compact and explicit solutions. Because of this, (4.131) and (4.132) actually satisfy (4.119) and (4.120) only up to terms that vanish after the application of (4.134).

For completeness, we also record the potentials ϕ^I and θ_I. These are determined by substituting the solutions (4.131) and (4.132) into (4.76) and (4.78), which gives:

$$\phi^0 = -\frac{1}{2}\left(m^0 + i\sum_i m^i\sqrt{\frac{m^0 e_i}{e_0 m^i}}\right), \tag{4.135}$$

$$\phi^1 = \frac{1}{2}\left(-m^1 + i\sqrt{\frac{e_0 m^1}{m^0 e_1}}m^0 + \sqrt{\frac{e_2 m^1}{m^2 e_1}}m^2 + \sqrt{\frac{e_3 m^1}{m^3 e_1}}m^3\right), \tag{4.136}$$

$$\theta_0 = \frac{1}{2}\left(e_0 - i\frac{e_0}{m^0}\sum_i m^i\sqrt{\frac{m^0 e_i}{e_0 m^i}}\right), \tag{4.137}$$

$$\theta_1 = \frac{1}{2}\left(ie_0\sqrt{\frac{m^0 e_1}{e_0 m^1}} - e_1 + im^2\sqrt{-\frac{e_2 e_1}{m^2 m^1}} + im^3\sqrt{-\frac{e_1 e_3}{m^1 m^3}}\right). \tag{4.138}$$

The expressions for ϕ^2, ϕ^3, θ_2, and θ_3 follow from cyclic permutations of (4.136) and (4.138). This completes our discussion of the attractor equations in terms of complex fluxes. In order to proceed further, we will need to specify precisely which real fluxes we are turning on.

4.5.2.3 Stabilization of the Vector Moduli

We now choose specific real fluxes consistent with the reality conditions (4.123)-(4.126). This will allow us to compute the quantities associated with the vector moduli in terms of real fluxes alone. We will also translate the sign restrictions (4.133) into restrictions on the real fluxes.

For m^0 and e_i, which must be real, we turn on only m^0_f and e^f_i. For the purely imaginary fluxes e_0 and m^i,we turn on e^h_0 and m^i_h, as well as several geometric fluxes. In accord with the argument in section 4.2.2, we turn on only three of the eight possible r_a, since we expect that turning on more r_a would make the constraints (4.75) insoluble. We will replace the a index with $\tilde{\imath} = 1, 2, 3$, and turn on $r_{\tilde{\imath}0}$ and $r^1_{\tilde{1}}$, $r^2_{\tilde{2}}$, $r^3_{\tilde{3}}$. The six real components of the geometric fluxes are chosen so that the tadpole constraints $\int r_{\tilde{\imath}} \wedge r_{\tilde{\jmath}} = 0$ and $\int r_{\tilde{\imath}} \wedge H_3 = 0$ are automatically satisfied. The non-trivial tadpole constraints are

$$0 = \int r_{\tilde{i}} \wedge F_3 = m^0_f r_{\tilde{i}0} - e^f_i r^i_{\tilde{i}}\,, \tag{4.139}$$

and

$$n = \int F_3 \wedge H_3 = -m^0_f e^h_0 + m^i_h e^f_i\,, \tag{4.140}$$

where the integer n is determined by the number of O3 planes and D3 branes.

We now write out explicitly the final set of constraints (4.75):

$$0 \;=\; \int r_{\tilde{i}} \wedge \Omega_3\,. \tag{4.141}$$

For $r_{\tilde{1}}$ this reduces to

$$r_{\tilde{1}0} = r^1_{\tilde{1}} z^2 z^3\,, \tag{4.142}$$

with the other equations following by cyclic permutations. Inside the Kähler cone $\mathrm{Im}\,(z^i) < 0$, so we deduce that

$$\frac{r_{\tilde{1}0}}{r^1_{\tilde{1}}} < 0\,. \tag{4.143}$$

We can use (4.132) and (4.134) to rewrite (4.142) in terms of complex fluxes:

$$-\frac{r_{\tilde{1}0}}{r^1_{\tilde{1}}} \;=\; \sqrt{\left(\frac{e_0}{m^0}\right)^2 \frac{m^2 m^3}{e_2 e_3}} \tag{4.144}$$

$$\;=\; -i\frac{e_1}{m^1}\sqrt{-\frac{e_2 e_3}{m^2 m^3}}\,. \tag{4.145}$$

Because the hypermoduli will enter via the complex fluxes, we would like an expression with the complex fluxes isolated and linear. By combining (4.145) and its permutations, we find

$$i\frac{m^i}{e_i} = \left(\frac{r^i_{\tilde{i}}}{r_{\tilde{i}0}}\right)^2 \sqrt{-\frac{r_{\tilde{1}0} r_{\tilde{2}0} r_{\tilde{3}0}}{r^1_{\tilde{1}} r^2_{\tilde{2}} r^3_{\tilde{3}}}}\,, \tag{4.146}$$

with no summation over i. Substituting this back into (4.134), we find

$$-i\frac{e_0}{m^0} = \sqrt{-\frac{r_{\tilde{1}0}r_{\tilde{2}0}r_{\tilde{3}0}}{r^1_{\tilde{1}}r^2_{\tilde{2}}r^3_{\tilde{3}}}}\,. \tag{4.147}$$

For the set of geometric and 3-form fluxes we turn on $e_i = e^f_i$ and $m^0 = m^0_f$. We can therefore use (4.146) and (4.147) to determine the remaining complex fluxes m^i and e_0, which implicitly depend on the hypermoduli, in terms of the real fluxes alone. Then (4.131) and (4.132) give explicit expressions for the stabilized moduli and CZ^0 in terms of real fluxes:

$$\begin{aligned} CZ^0 &= \frac{1}{4}\left(m^0_f + \sum_i e^f_i \frac{r^i_{\tilde{i}}}{r_{\tilde{i}0}}\right)\,, & (4.148)\\ z^i &= i\frac{r^i_{\tilde{i}}}{r_{\tilde{i}0}}\sqrt{-\frac{r_{\tilde{1}0}r_{\tilde{2}0}r_{\tilde{3}0}}{r^1_{\tilde{1}}r^2_{\tilde{2}}r^3_{\tilde{3}}}}\,. & (4.149)\end{aligned}$$

The complex potentials (4.135)-(4.138) similarly become

$$\begin{aligned} \phi^0 &= \frac{1}{2}\left(-m^0_f + \sum_i e^f_i \frac{r^i_{\tilde{i}}}{r_{\tilde{i}0}}\right)\,, & (4.150)\\ \phi^1 &= -\frac{i}{2}\frac{r^1_{\tilde{1}}}{r_{\tilde{1}0}}\left(m^0_f - e^f_1\frac{r^1_{\tilde{1}}}{r_{\tilde{1}0}} + e^f_2\frac{r^2_{\tilde{2}}}{r_{\tilde{2}0}} + e^f_3\frac{r^3_{\tilde{3}}}{r_{\tilde{3}0}}\right)\sqrt{-\frac{r_{\tilde{1}0}r_{\tilde{2}0}r_{\tilde{3}0}}{r^1_{\tilde{1}}r^2_{\tilde{2}}r^3_{\tilde{3}}}}\,, & (4.151)\\ \theta_0 &= \frac{i}{2}\left(-m^0_f + \sum_i e^f_i \frac{r^i_{\tilde{i}}}{r_{\tilde{i}0}}\right)\sqrt{-\frac{r_{\tilde{1}0}r_{\tilde{2}0}r_{\tilde{3}0}}{r^1_{\tilde{1}}r^2_{\tilde{2}}r^3_{\tilde{3}}}}\,, & (4.152)\\ \theta_1 &= \frac{1}{2}\frac{r_{\tilde{1}0}}{r^1_{\tilde{1}}}\left(m^0_f - e^f_1\frac{r^1_{\tilde{1}}}{r_{\tilde{1}0}} + e^f_2\frac{r^2_{\tilde{2}}}{r_{\tilde{2}0}} + e^f_3\frac{r^3_{\tilde{3}}}{r_{\tilde{3}0}}\right)\,, & (4.153)\end{aligned}$$

with the other ϕ^i and θ_i given by cyclic permutations of (4.151) and (4.153).

So far we have not utilized the tadpole constraints (4.139). After imposing the tadpole constraints, we find

$$CZ^0 = m^0_f\,, \tag{4.154}$$

$$z^i = \frac{i}{e^f_i m^0_f}\sqrt{-m^0_f e^f_1 e^f_2 e^f_3}\,, \tag{4.155}$$

and

$$\phi^0 = m^0 = m^0_f\,, \tag{4.156}$$

$$\phi^i = m^i = -\frac{i}{e^f_i}\sqrt{-m^0_f e^f_1 e^f_2 e^f_3}\,, \tag{4.157}$$

$$\theta_0 = e_0 = \frac{i}{m^0_f}\sqrt{-m^0_f e^f_1 e^f_2 e^f_3}\,, \tag{4.158}$$

$$\theta_i = e_i = e^f_i\,. \tag{4.159}$$

If we compare (4.156)-(4.159) with (4.77), we find that $L^I = 0$ for this choice of fluxes, so that the only non-zero mass parameter is CZ^0. This indicates that the only mass scale is $m^2_{3/2} \sim |CZ^0|^2$.

4.5.2.4 Stabilization of the Hypermoduli

In (4.123)-(4.126) we chose e_0 and m^i to be purely imaginary. This implies that $\mathrm{Re}\,(\tau) = \mathrm{Re}\left(G^{\tilde{i}}\right) = 0$, so we will rewrite the hypermoduli as

$$\tau = i\tau_2\,, \tag{4.160}$$

$$G^{\tilde{i}} = ig^{\tilde{i}}\,, \tag{4.161}$$

where τ_2 and $g^{\tilde{i}}$ are real.

Our expressions (4.146) and (4.147) for the complex fluxes in terms of the real fluxes, along with the definitions (4.63) and (4.64), give a system of linear equations that determine the hypermoduli τ_2 and $g^{\tilde{i}}$:

$$e_0 = -i\tau_2 e_0^h - ig^{\tilde{i}} r_{\tilde{i}0} = im_f^0 \sqrt{-\frac{r_{\tilde{1}0} r_{\tilde{2}0} r_{\tilde{3}0}}{r_{\tilde{1}}^1 r_{\tilde{2}}^2 r_{\tilde{3}}^3}}\,, \tag{4.162}$$

$$m^i = -i\tau_2 m_h^i - ig^{\tilde{i}} r_{\tilde{i}}^i = -ie_i^f \left(\frac{r_{\tilde{i}}^i}{r_{\tilde{i}0}}\right)^2 \sqrt{-\frac{r_{\tilde{1}0} r_{\tilde{2}0} r_{\tilde{3}0}}{r_{\tilde{1}}^1 r_{\tilde{2}}^2 r_{\tilde{3}}^3}}\,. \tag{4.163}$$

Note that we have not yet imposed any tadpole constraints. We can rewrite this system of equations in matrix form,

$$\begin{pmatrix} e_0^h & r_{\tilde{1}0} & r_{\tilde{2}0} & r_{\tilde{3}0} \\ m_h^1 & r_{\tilde{1}}^1 & 0 & 0 \\ m_h^2 & 0 & r_{\tilde{2}}^2 & 0 \\ m_h^3 & 0 & 0 & r_{\tilde{3}}^3 \end{pmatrix} \begin{pmatrix} \tau_2 \\ g^{\tilde{1}} \\ g^{\tilde{2}} \\ g^{\tilde{3}} \end{pmatrix} = \sqrt{-\frac{r_{\tilde{1}0} r_{\tilde{2}0} r_{\tilde{3}0}}{r_{\tilde{1}}^1 r_{\tilde{2}}^2 r_{\tilde{3}}^3}} \begin{pmatrix} -m_f^0 \\ e_1^f \left(r_{\tilde{1}}^1/r_{\tilde{1}0}\right)^2 \\ e_2^f \left(r_{\tilde{2}}^2/r_{\tilde{2}0}\right)^2 \\ e_3^f \left(r_{\tilde{3}}^3/r_{\tilde{3}0}\right)^2 \end{pmatrix}. \tag{4.164}$$

We now need only invert the 4×4 matrix of NS fluxes in order to determine the hypermoduli. This can be done in general, but the result is both quite long and not particularly illuminating. We instead quote the result with the tadpole constraints (4.139) and (4.140) imposed,

$$\tau_2 = \frac{4}{n}\sqrt{-m_f^0 e_1^f e_2^f e_3^f}\,, \tag{4.165}$$

$$g^{\tilde{1}} = \frac{1}{r_{\tilde{1}}^1 e_1^f}\left(1 - \frac{4}{n} e_1^f m_h^1\right)\sqrt{-m_f^0 e_1^f e_2^f e_3^f}\,, \tag{4.166}$$

with the expressions for $g^{\tilde{2}}$ and $g^{\tilde{3}}$ analogous to (4.166).

Now that we have computed the VEVs of all the moduli, it is interesting to see what restrictions on the moduli and the fluxes are imposed by the combination of the tadpole constraints, (4.139) and (4.140), and the requirement that we stay inside the Kähler cone, i.e. that the Kähler metric remain positive. While the tadpole constraints are naturally written in terms of the fluxes alone, we can use our explicit expressions to see how the moduli are constrained. Similarly, the Kähler cone restrictions are naturally written in terms of the moduli, but the explicit solutions allow us to rewrite them as restrictions on the fluxes.

In the case without geometric fluxes, the combination of tadpole constraints and Kähler cone restrictions is quite restrictive. For example, in chapter III, where we used the same combination of F_3 and H_3 as here, but no geometric fluxes, we found that staying inside the Kähler cone required

$$e_0^h m_f^0 < 0\,, \qquad e_1^f m_h^1 > 0\,,$$
$$e_2^f m_h^2 > 0\,, \qquad e_3^f m_h^3 > 0\,.$$

When we compare these with the tadpole constraint

$$n = -e_0^h m_f^0 + e_i^f m_h^i\,,$$

we see that each term on the right-hand side is positive, so no individual flux can be larger than n. This renders the number of distinct choices of $\left\{e_0^h, m_f^0, e_i^f, m_h^i\right\}$ finite and rather small. It also keeps the string coupling $g_s = 1/\tau_2$ of order 1. We will now argue that these restrictions are far less severe when geometric fluxes are included.

The crux of our argument is that introducing geometric fluxes does not lead to additional Kähler cone restrictions. While we still need to ensure that $\mathrm{Im}\,(z^i) < 0$ and that $t > 0$, there is apparently no such restriction on the $g^{\tilde{i}}$. We have already seen the restrictions imposed on the geometric fluxes by these requirements (4.143), and can use the tadpole constraints (4.139) to find a restriction on the RR fluxes:

$$\frac{e_i^f}{m_f^0} < 0\,. \tag{4.167}$$

We do not, however, find any restriction[7] on the signs of e_0^h or m_h^i. If we choose

[7]For example, it might appear that such a restriction would arise from (4.146) or (4.147), which involve the complex fluxes e_0 and m^i and so implicitly involve e_0^h and m_h^i. If one rewrites the

fluxes such that $m_h^i/e_0^h < 0$, we can get cancellations between the terms in (4.140). These cancellations allow us to choose infinite series of fluxes that satisfy all physical constraints. In particular, we can take the RR fluxes large and, by (4.165), send the string coupling $g_s = 1/\tau_2$ to zero. It would be interesting to see how perturbative and non-perturbative corrections might modify this result.

4.5.2.5 The Generating Function

We now compute the main object of interest in this chapter, the generating function for the attractor equations. Since the result is surprisingly simple, we will first compute the numerical value of the generating function with the tadpole constraints (4.139) imposed. We will next compute the reduced generating function $\widetilde{\mathcal{G}}$ without imposing the tadpole constraints, in order to check the results of section 4.4.

With the tadpole constraints (4.139) and (4.140) imposed, we find a surprisingly simple expression for the numerical value of the generating function $\widetilde{\mathcal{G}}$. If we combine the expressions for the complex potentials in (4.156)-(4.159) with our explicit expression for the generating function (4.84), we find

$$\mathcal{G} = \frac{1}{\tau - \overline{\tau}} \left\{ m^I \overline{e}_I - e_I \overline{m}^I \right\} = \frac{1}{\tau - \overline{\tau}} \int \overline{G}_3 \wedge G_3 \,. \qquad (4.168)$$

The NS tadpole constraints (4.35) imply that the geometric fluxes make no contribution to the integral in (4.168). It therefore reduces to

$$\mathcal{G} = -\int F_3 \wedge H_3 = -n \,. \qquad (4.169)$$

An analogous generating function was derived in chapter III using an identical pre-

complex fluxes using (4.119), (4.120), (4.139), and (4.140), both (4.146) and (4.147) reduce to $\sqrt{-m_f^0 e_1^f e_2^f e_3^f} > 0$, which is automatically satisfied whenever (4.167) is satisfied.

potential and choice of F_3 and H_3, but with no geometric fluxes:

$$\mathcal{G}_{\text{there}} = n - \frac{1}{2}\left[-\text{sgn}\left(m_f^0\right)\sqrt{-e_0^h m_f^0} + \sum_i \text{sgn}\left(m_h^i\right)\sqrt{m_h^i e_i^f}\right]^2 . \quad (4.170)$$

This result, along with our expressions for the moduli (4.155) and (4.165), indicate that the solutions without geometric flux cannot be recovered from the solutions with geometric flux by formally sending the geometric fluxes to zero. Instead, this limit is discontinuous, suggesting that there is no sense in which we can add "a little" geometric flux. This is consistent with our expectation that the geometric fluxes obey a Dirac quantization condition, just as the fluxes F_3 and H_3 do.

Although the expression for the generating function in (4.169) is quite elegant, its derivatives will not reproduce the real potentials ϕ_f^0, ϕ_h^i, θ_0^h, and θ_i^f because we repeatedly used the tadpole constraints (4.139) and (4.140) to simplify the expression, and using these constraints alters the *derivatives* of the generating function. We now compute $\widetilde{\mathcal{G}}$ *without* using the tadpole constraints, and verify that its derivatives correctly reproduce the real potentials.

In order to compute both the reduced generating function and the real potentials, we need to compute τ_2 without using the tadpole constraints. If we go back to (4.164) and invert we find

$$\tau_2 = \frac{\Delta_f}{\Delta_h}\sqrt{-\frac{r_{\tilde{1}0} r_{\tilde{2}0} r_{\tilde{3}0}}{r_{\tilde{1}}^1 r_{\tilde{2}}^2 r_{\tilde{3}}^3}}\,, \quad (4.171)$$

where we introduced the combinations

$$\Delta_f \equiv m_f^0 + \sum_i \frac{r_{\tilde{i}}^i}{r_{\tilde{i}0}} e_i^f\,, \quad (4.172)$$

$$\Delta_h \equiv -e_0^h + \sum_i \frac{r_{\tilde{i}0}}{r_{\tilde{i}}^i} m_h^i\,, \quad (4.173)$$

which will appear quite frequently in the following. We now substitute the expressions for the complex potentials (4.150)-(4.153), the expressions for the complex fluxes

(4.162) and (4.163), and the value of τ_2 (4.171) into (4.84) to find the reduced generating function:

$$
\begin{aligned}
\widetilde{\mathcal{G}} &= -\frac{i}{2\tau_2}\left[m^I\overline{\theta}_I - e_I\overline{\phi}^I\right] && (4.174)\\
&= -\frac{1}{2}\frac{\Delta_h}{\Delta_f}\left[m_f^0\left(\Delta_f - 2m_f^0\right) + \sum_i\left\{e_i^f\frac{r_{\tilde{i}}^i}{r_{\tilde{i}0}}\left(\Delta_f - 2e_i^f\frac{r_{\tilde{i}}^i}{r_{\tilde{i}0}}\right)\right\}\right] && (4.175)\\
&= -\frac{1}{2}\Delta_h\left[\Delta_f - 2\frac{\left(m_f^0\right)^2 + \sum_i\left(e_i^f r_{\tilde{i}}^i/r_{\tilde{i}0}\right)^2}{\Delta_f}\right]. && (4.176)
\end{aligned}
$$

This is the principal result of this example, a single function that summarizes all aspects of the stabilized vector moduli. Comparing (4.176) with (4.170), it is interesting that (4.176) has two factors, one that is independent of F_3 and one that is independent of H_3, while each term in (4.170) mixes F_3 and H_3. Upon imposing the tadpole constraints (4.139) we recover (4.169), as expected.

We substitute (4.150)-(4.153) and (4.171) into (4.101) and (4.102) to find the real potentials:

$$
\begin{aligned}
\phi_f^0 &= \frac{1}{2}\left(\Delta_f - 2m_f^0\right), && (4.177)\\
\phi_h^i &= \frac{1}{2}\Delta_h\frac{r_{\tilde{i}}^i}{r_{\tilde{i}0}}\left[1 - 2\frac{e_i^f r_{\tilde{i}}^i}{\Delta_f r_{\tilde{i}0}}\right], && (4.178)\\
\theta_0^h &= -\frac{1}{2}\Delta_h\left[1 - 2\frac{m_f^0}{\Delta_f}\right], && (4.179)\\
\theta_i^f &= \frac{1}{2}\frac{r_{\tilde{1}0}}{r_{\tilde{1}}^1}\left(\Delta_f - 2e_i^f\frac{r_{\tilde{i}}^i}{r_{\tilde{i}0}}\right). && (4.180)
\end{aligned}
$$

These expressions agree with the derivatives (4.99) and (4.100) of the reduced generating function (4.176), up to terms that vanish when the tadpole constraints (4.139) are imposed, in accord with the arguments of section (4.4). This validates the generating function approach to flux attractor equations, even after the introduction of geometric fluxes.

APPENDICES

APPENDIX A

Scalar Mass Matrix in No-Scale Compactifications

In this appendix we present an explicit computation of the scalar mass matrix for no-scale compactifications.

We divide the scalar potential into two terms as follows:

$$\begin{aligned} V_{\rm tot} &= V + V_0 && \text{(A.1)} \\ &= e^K g^{\alpha\bar{\beta}} D_\alpha W \overline{D_\beta W} + e^K \left(g^{a\bar{b}} D_a W \overline{D_b W} - 3\,|W|^2 \right). && \text{(A.2)} \end{aligned}$$

The indices $\alpha, \beta, \gamma ...$ run over the complex structure moduli $i, j, k ...$ and axio-dilaton τ, and $a, b, ...$ run over the Kähler moduli. Because the superpotential is independent of the Kähler moduli, their F-terms are (3.24)

$$D_a W = W \partial_a K\,. \tag{A.3}$$

The inverse metric is such that

$$g^{a\bar{b}} \partial_a K \overline{\partial}_{\bar{b}} K = 3\,, \tag{A.4}$$

so that $V_0 = 0$. The remaining term V is positive semi-definite, so the absolute minima of the scalar potential all have vanishing cosmological constant. This is why these solutions are called "no-scale."

Since $V_0 = 0$, we do not expect this term to make a contribution to the mass matrix. We now show explicitly that this is the case, beginning with the contribution to $M^2_{\alpha\beta}$ from V_0 :

$$\begin{aligned}
\partial_\beta \partial_\alpha V_0 &= \partial_\beta \left\{ e^K \left[g^{a\bar{b}} \left(D_\alpha D_a W \overline{D_b W} + D_a W D_\alpha \overline{D_b W} \right) + D_a W \overline{D_b W} \partial_\alpha g^{a\bar{b}} \right] - 3\overline{W} D_\alpha W \right\} \\
&= e^K \left[g^{a\bar{b}} \left(D_\beta D_\alpha D_a W \overline{D_b W} + D_\alpha D_a W D_\beta \overline{D_b W} \right) + \left(\partial_\beta g^{a\bar{b}} \right) D_\alpha D_a W \overline{D_b W} \right. \\
&\quad + g^{a\bar{b}} \left(D_\beta D_a W D_\beta \overline{D_b W} + D_a W D_\beta D_\alpha \overline{D_b W} \right) + \left(\partial_\beta g^{a\bar{b}} \right) D_a W D_\alpha \overline{D_b W} \\
&\quad \left. + \partial_\alpha g^{a\bar{b}} \left(D_\beta D_a W \overline{D_b W} + D_a W D_\beta \overline{D_b W} \right) + D_a W \overline{D_b W} \partial_\beta \partial_\alpha g^{a\bar{b}} - 3\overline{W} D_\beta D_\alpha W \right].
\end{aligned}$$

Since the Kähler potential factorizes into $K = K_z \left(z^i, \overline{z^{\bar{i}}} \right) + K_\tau \left(\tau, \overline{\tau} \right) + K_t \left(t^a, \overline{t^{\bar{a}}} \right)$ we find that $\partial_\alpha g^{a\bar{b}} = 0$, and simplify further:

$$\begin{aligned}
\partial_\beta \partial_\alpha V_0 &= e^K \left[g^{a\bar{b}} \left(D_\beta D_\alpha D_a W \overline{D_b W} + D_\alpha D_a W D_\beta \overline{D_b W} \right) \right. \\
&\quad \left. + g^{a\bar{b}} \left(D_\beta D_a W D_\beta \overline{D_b W} + D_a W D_\beta D_\alpha \overline{D_b W} \right) - 3\overline{W} D_\beta D_\alpha W \right]. \quad \text{(A.5)}
\end{aligned}$$

Since $\partial_\alpha \partial_a K = 0$, we have

$$D_\alpha D_a W = D_\alpha \left(W \partial_a K \right) \quad \text{(A.6)}$$

$$= \left(D_\alpha W \right) \partial_a K \quad \text{(A.7)}$$

and

$$D_\alpha \overline{D_b W} = D_\alpha \left(\overline{W} \partial_b K \right) \quad \text{(A.8)}$$

$$= 0\,. \quad \text{(A.9)}$$

This, combined with (A.4), gives

$$\partial_\beta \partial_\alpha V_0 \;=\; e^K \left[\left(g^{a\bar{b}} \partial_a K \overline{\partial}_{\bar{b}} K\right) \overline{W} D_\beta D_\alpha W - 3\overline{W} D_\beta D_\alpha W\right] \tag{A.10}$$

$$\;=\; 0\,, \tag{A.11}$$

so V_0 indeed makes no contribution to $M^2_{\alpha\beta}$.

The contributions to $M^2_{\alpha\overline{\beta}}$ from V_0 simplify in a similar way:

$$\overline{\partial}_{\overline{\beta}} \partial_\alpha V_0 \;=\; \overline{\partial}_{\overline{\beta}} \left\{ e^K \left[g^{a\bar{b}} \left(D_\alpha D_a W \overline{D_b W} + D_a W D_\alpha \overline{D_b W} \right) + D_a W \overline{D_b W} \partial_\alpha g^{a\bar{b}} \right] - 3\overline{W} D_\alpha W \right\}$$

$$= e^K \Big[g^{a\bar{b}} \left(\overline{D}_{\overline{\beta}} D_\alpha D_a W \overline{D_b W} + D_\alpha D_a W \overline{D_\beta D_b W} \right)$$

$$-3 \left(\overline{W} \overline{D}_{\overline{\beta}} D_\alpha W + D_\alpha W \overline{D_\beta W} \right) \Big] \tag{A.12}$$

$$= 0\,. \tag{A.13}$$

So our expectations were correct, and V_0 makes no contribution to the scalar mass matrix.

We emphasize that in computing the contributions from V_0 to the mass matrix we have not set $D_\alpha W = 0$, we have only used the factorization of the Kähler potential. Our conclusion that V_0 makes no contribution to the scalar mass matrix thus holds for metastable local minima, where $D_\alpha W \neq 0$, as well as absolute minima, where $D_\alpha W = 0$.

Next we compute the contributions to the mass matrix from V. Since we are interested in absolute minima of the potential, we will set $D_\alpha W = 0$. We begin with contributions to $M^2_{\alpha\beta}$:

$$\partial_\beta \partial_\alpha V \;=\; \partial_\beta \left\{ e^K \left[g^{\gamma\bar{\delta}} \left(D_\alpha D_\gamma W \overline{D_\delta W} + D_\gamma W D_\alpha \overline{D_\delta W} \right) + D_\gamma W \overline{D_\delta W} \partial_\alpha g^{\gamma\bar{\delta}} \right] \right\}$$

$$= e^K g^{\gamma\bar{\delta}} \left(D_\alpha D_\gamma W D_\beta \overline{D_\delta W} + D_\beta D_\gamma W D_\alpha \overline{D_\delta W} \right) \tag{A.14}$$

We can eliminate the mixed derivatives using

$$
\begin{aligned}
D_\alpha \overline{D_\beta W} &= D_\alpha \left(\overline{\partial_\beta W} + \overline{W} \overline{\partial}_{\overline{\beta}} K\right) && \text{(A.15)} \\
&= \overline{W} \partial_\alpha \overline{\partial}_{\overline{\beta}} K && \text{(A.16)} \\
&= \overline{W} g_{\alpha\overline{\beta}} \,, && \text{(A.17)}
\end{aligned}
$$

so that (A.14) simplifies to

$$
\begin{aligned}
M^2_{\alpha\beta} = \partial_\beta \partial_\alpha V &= e^K g^{\gamma\overline{\delta}} \left[D_\alpha D_\gamma W D_\beta \overline{D_\delta W} + D_\beta D_\gamma W D_\alpha \overline{D_\delta W}\right] && \text{(A.18)} \\
&= e^K \overline{W} \left(D_\alpha D_\beta W + D_\beta D_\alpha W\right). && \text{(A.19)}
\end{aligned}
$$

We'll follow the same procedure for $M^2_{\alpha\overline{\beta}}$,

$$
\begin{aligned}
M^2_{\alpha\overline{\beta}} = \overline{\partial}_{\overline{\beta}} \partial_\alpha V &= \overline{\partial}_{\overline{\beta}} \left\{ e^K \left[g^{\gamma\overline{\delta}} \left(D_\alpha D_\gamma W \overline{D_\delta W} + D_\gamma W D_\alpha \overline{D_\delta W}\right) + D_\gamma W \overline{D_\delta W} \partial_\alpha g^{\gamma\overline{\delta}} \right] \right\} \\
&= e^K g^{\gamma\overline{\delta}} \left[D_\alpha D_\gamma W \overline{D_\beta D_\delta W} + \overline{D}_{\overline{\beta}} D_\gamma W D_\alpha \overline{D_\delta W}\right] \\
&= e^K \left[g^{\gamma\overline{\delta}} D_\alpha D_\gamma W \overline{D_\beta D_\delta W} + |W|^2 g_{\alpha\overline{\beta}} \right]. && \text{(A.20)}
\end{aligned}
$$

Our results for the scalar mass matrices, (A.19) and (A.20), agree with the standard results for $\mathcal{N} = 1$ supergravity, e.g. eq. 23.27 in [62]. We have verified that the Kähler moduli do not make any additional contributions.

We also see that when $W \neq 0$, i.e. when SUSY is broken, the scalar masses-squared are lifted above the fermion masses-squared by $\mathcal{O}\left(m^2_{3/2}\right)$.

APPENDIX B

Homogeneity Conditions

We collect here several known results about the Kähler potentials for hypermoduli in $\mathcal{N} = 1$ compactifications of Type II theories.

Homogeneity of Hypermoduli Kähler Potentials

In this section we will recall the form of the tree-level Kähler potential for hypermoduli, K, for various $\mathcal{N} = 1$ type II compactifications. For each we will demonstrate that K is independent of the real parts of the hypermoduli, and that e^{-K} is homogeneous of degree four in the imaginary parts of the hypermoduli.

IIB O3/O7

This is the case of greatest interest in this chapter. We recall that the hypermoduli (the scalar fields which descend from the $\mathcal{N} = 2$ hypermultiplets) consist of the axio-dilaton τ, a field G^a corresponding to each two-form ω_a which is odd under the orientifold involution, and a field T_α corresponding to each even four-form $\widetilde{\mu}^\alpha$. In terms of the real fields (the RR potentials C_0, $C_2 = c^a \omega_a$, and $C_4 = \rho_\alpha \widetilde{\mu}^\alpha$, the dilaton

ϕ, the B-field $B_2 = u^a \omega_a$, and the Kähler form $J = v_\alpha \mu^\alpha$), they are given by[1] :

$$\tau = C_0 + ie^{-\phi}, \tag{B.1}$$

$$G^a = c^a - \tau u^a, \tag{B.2}$$

$$T_\alpha = \rho_\alpha - \frac{i}{2} e^{-\phi} \left(\kappa v^2\right)_\alpha - \left(\widehat{\kappa} c u\right)_\alpha + \frac{1}{2} \tau \left(\widehat{\kappa} u^2\right)_\alpha, \tag{B.3}$$

as follows from (4.36). We have made use of the intersection numbers defined in (4.51) and (4.52).

Now the Kähler potential for these fields is

$$K = -4 \ln \left[-i\left(\tau - \bar{\tau}\right)\right] - 2 \ln \left[\mathcal{V}_6\right], \tag{B.4}$$

where the volume

$$\mathcal{V}_6 = \frac{1}{6} \int J^3 = \frac{1}{6} \left(\kappa v^3\right), \tag{B.5}$$

is implicitly viewed as a function of T_α, τ, and G^a. One then computes the Kähler metric by using the map (B.1)-(B.3) and the expression (B.4) to compute the derivatives of K with respect to the complex fields (which can only be written explicitly in terms of the real fields, since there are no general expressions for the v_α in terms of the complex fields). Inverting that Kähler metric then gives the expressions which appear in (4.45)-(4.50).

We would like to understand the scaling properties of the (exponential of the) Kähler potential when we scale the complex fields. Looking at (B.1)-(B.3), we see that sending $\{\tau, G^a, T_\alpha\} \to \{\lambda\tau, \lambda G^a, \lambda T_\alpha\}$ for some real λ is equivalent to an action on the real fields

[1] These conventions differ in some important ways from [41]. In particular, the definition of the v^α differs by a dilaton factor ($v^\alpha_{\text{there}} = e^{-\phi/2} v^\alpha_{\text{here}}$), essentially the difference between string frame and Einstein frame, and T_α differs by an overall numerical factor ($T^{\text{there}}_\alpha = (3i/2) T^{\text{here}}_\alpha$). They adhere more closely to [89].

$$\left\{C_0, c^a, \rho_\alpha, e^{-\phi}, u^a, v_\alpha\right\} \longrightarrow \left\{\lambda C_0, \lambda c^a, \lambda\rho_\alpha \lambda e^{-\phi}, u^a, v_\alpha\right\}, \tag{B.6}$$

i.e. everything scales with weight one except for u^a and v_α. But then it follows immediately that

$$e^{-K} = 2^4 e^{-4\phi} \mathcal{V}_6^2 , \tag{B.7}$$

is a function of the imaginary parts of the fields which is homogeneous of degree four, from the $e^{-\phi}$ dependence.

We can also consider the simpler case with $h_-^{1,1} = 0$, so there are no G^a. In this case, one can separately scale τ and the T_α, $\{\tau, T_\alpha\} \to \{\lambda\tau, \mu T_\alpha\}$. In terms of the real fields, this would be

$$\left\{C_0, \rho_\alpha, e^{-\phi}, v^\alpha\right\} \longrightarrow \left\{\lambda C_0, \mu\rho_\alpha, \lambda e^{-\phi}, \lambda^{-\frac{1}{2}}\mu^{\frac{1}{2}} v^\alpha\right\}. \tag{B.8}$$

Comparing with (B.7), we see that e^{-K} is homogeneous of degree $(1, 3)$ in the scalings of (τ, T_α). In particular, this fact can be used to show (4.2).

IIA O6

For type IIA compactifications which are orientifolds of Calabi-Yau manifolds, and which can contain O6-planes, the hypermoduli now come from the complex structure moduli of the space. Indeed, in general the orientifold involution (which, in order to preserve $\mathcal{N} = 1$ supersymmetry, must be an anti-holomorphic involution of the Calabi-Yau, and must act as minus one on the volume form of the space) can act on the holomorphic three-form as $\sigma \cdot \Omega_3 = e^{2i\theta}\overline{\Omega_3}$ for some constant phase θ. Also, given a symplectic basis a_K and b^K, we can expand

$$\Omega_3 = Z^K a_K - F_K b^K. \tag{B.9}$$

As usual, the F_K here can be derived from a holomorphic prepotential $F(Z^K)$, which depends on our choice of symplectic basis. Then [95], the hypermoduli come from expanding

$$\Omega_c = C_3 + 2i\mathrm{Re}\left(C\Omega_3\right), \tag{B.10}$$

where C is a compensator field that ensures that the expression above is invariant under Kähler transformations. If we wish to be more explicit, it is convenient to choose a symplectic basis in which the a_K are even under the orientifold involution and the b^K are odd (we can always do this since the volume form is odd) and then we can simply expand

$$\Omega_c = 2N^K a_K, \qquad N^K = \frac{1}{2}\xi^K + i\mathrm{Re}\left(CZ^K\right), \tag{B.11}$$

where we have also expanded $C_3 = \xi^K a_K$.

The Kähler potential for these fields is simply [22, 41]

$$K = -2\ln\left[2\int \mathrm{Re}\left(C\Omega_3\right)\wedge *\mathrm{Re}\left(C\Omega_3\right)\right]. \tag{B.12}$$

From this expression it is obvious that the Kähler potential depends only on the imaginary parts of the complex fields N^K, and that

$$e^{-K} = \left[2\int \mathrm{Re}\left(C\Omega_3\right)\wedge *\mathrm{Re}\left(C\Omega_3\right)\right]^2, \tag{B.13}$$

is a homogeneous function of degree four in the $\mathrm{Im}(N^K)$.

IIA and IIB, $SU(3)\times SU(3)$, $\mathcal{N}=1$

In fact, these homogeneity properties are even more general. Both of the examples above could have been formulated by saying that our complex hypermoduli fields are obtained as expansion coefficients of a formal sum of complex forms [89],

$$\Phi_c = e^{-B} C_{RR} + i\mathrm{Re}\,(\Phi)\,, \tag{B.14}$$

where $\Phi = e^{-\phi} e^{-B+iJ}$ for IIB (see (4.36)), and $\Phi = C\Omega_3$ for IIA. In both cases, the Kähler potential is given by

$$K = -2\ln\left[i\left\langle \Phi, \overline{\Phi}\right\rangle\right]\,, \tag{B.15}$$

where the pairing $\langle\cdot,\cdot\rangle$ is the Mukai pairing, defined on even and odd forms respectively as

$$\langle\varphi,\psi\rangle = \begin{cases} \int\left(\varphi_0\psi_6 - \varphi_2\wedge\psi_4 + \varphi_4\wedge\psi_2 - \varphi_6\psi_0\right), \\ \int\left(-\varphi_1\wedge\psi_5 + \varphi_3\wedge\psi_3 - \varphi_5\wedge\psi_1\right). \end{cases} \tag{B.16}$$

Again, from this formulation it is evident that K depends only on the imaginary parts of the fields, and e^{-K} is homogeneous of degree four.

This formulation is more general than the compactifications we have been considering so far. We could easily incorporate type IIB O5/O9 models, or we could include compactifications with $SU(3)\times SU(3)$-structure [89, 91, 96–100], which are in some sense the most general compactifications of type II that have $\mathcal{N}=1$ supersymmetry in four dimensions. Typically, these “spaces" are not even geometric, but nonetheless they have the structure displayed above, so that the effective $\mathcal{N}=1$ supergravity in four dimensions has a Kähler potential with the given homogeneity properties.

Identities Implied by Homogeneity

In the previous section we showed that the Kähler potentials for virtually all Type II, $\mathcal{N} = 1$ compactifications obey

$$\eta^A \frac{\partial}{\partial \eta^A} e^{-K} = 4e^{-K} \,, \tag{B.17}$$

where the index A runs over all of the hypermoduli, and η^A indicates the imaginary parts of those moduli. We also showed that the Kähler potential is independent of the real parts of the hypermoduli. We will now demonstrate how the homogeneity property (B.17) implies (4.13) and (4.14),

$$\begin{aligned} K^{A\overline{B}} \left(\partial_A K\right) \left(\partial_{\overline{B}} K\right) &= 4 \,, \\ K^{A\overline{B}} \left(\partial_{\overline{B}} K\right) &= -2i\eta^A \,, \end{aligned}$$

which played a central role in section 4.2.2.

We begin by relating complex derivatives to η^A-derivatives:

$$\partial_A = \frac{1}{2} \left(\frac{\partial}{\partial \xi^A} - i \frac{\partial}{\partial \eta^A} \right) . \tag{B.18}$$

We can use this to relate complex derivatives of the Kähler potential K to η^A derivatives of e^{-K} :

$$\partial_A K = -e^K \partial_A \left(e^{-K}\right) = \frac{i}{2} e^K \frac{\partial}{\partial \eta^A} e^{-K} \,, \tag{B.19}$$

A similar result follows for the Kähler metric $K_{A\overline{B}}$. We have

$$\partial_A \partial_{\overline{B}} e^{-K} = e^{-K} \left[\left(\partial_A K\right) \left(\partial_{\overline{B}} K\right) - \partial_A \partial_{\overline{B}} K \right] , \tag{B.20}$$

so

$$
\begin{aligned}
K_{A\overline{B}} \equiv \partial_A \partial_{\overline{B}} K &= (\partial_A K)(\partial_{\overline{B}} K) - \frac{1}{4} e^{-K} \frac{\partial}{\partial \eta^A} \frac{\partial}{\partial \eta^B} e^K & \text{(B.21)} \\
&= \frac{1}{4}\left[e^{2K} \left(\frac{\partial}{\partial \eta^A} e^{-K} \right) \left(\frac{\partial}{\partial \eta^B} e^{-K} \right) - e^{-K} \frac{\partial}{\partial \eta^A} \frac{\partial}{\partial \eta^B} e^K \right]. & \text{(B.22)}
\end{aligned}
$$

In the last step we used (B.19) to write $K_{A\overline{B}}$ in terms of η^A derivatives only. If we now contract $K_{A\overline{B}}$ with η^A, we can use (B.17):

$$
\begin{aligned}
\eta^A K_{A\overline{B}} &= \frac{1}{4}\left[4e^K \frac{\partial}{\partial \eta^B} e^{-K} - 3e^K \frac{\partial}{\partial \eta^B} e^{-K} \right] & \text{(B.23)} \\
&= \frac{1}{4} e^K \frac{\partial}{\partial \eta^B} e^{-K} & \text{(B.24)} \\
&= \frac{i}{2} \partial_{\overline{B}} K \, , & \text{(B.25)}
\end{aligned}
$$

We can now contract with the inverse metric $K^{A\overline{B}}$ to arrive at (4.14):

$$
K^{A\overline{B}} \partial_{\overline{B}} K = -2i\eta^A \, . \tag{B.26}
$$

Contracting this expression with $\partial_A K$ and using (B.17) again we find:

$$
\begin{aligned}
K^{A\overline{B}} (\partial_A K)(\partial_{\overline{B}} K) &= -2i\eta^A \partial_A K & \text{(B.27)} \\
&= e^K \eta^A \frac{\partial}{\partial \eta^A} e^{-K} & \text{(B.28)} \\
&= 4 \, . & \text{(B.29)}
\end{aligned}
$$

This is just (4.13), so we have demonstrated that (B.17) implies (4.13) and (4.14).

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