AIAA Guidance, Navigation, and Control Conference and Exhibit
16 - 19 August 2004, Providence, Rhode Island

AIAA 2004-5434

Applications of the Generalized Model for Solar Sails

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Abstract

In a previous paper, a generalized model for representing the propulsive force and moment from a solar sail as a function of solar illumination was presented. This generalized model can be defined with only 18 numerical coefficients for the force, and 36 for the moment, and can represent a sail of arbitrary geometry (under some mild restrictions). In this paper we revisit this general model and develop a number of applications for it that will showcase its generality and utility. Specifically, we first show that the number of coefficients needed to describe the total moment acting on the sail can be reduced to the 18 coefficients needed for the force description plus 9 additional constants, a significant reduction from the original 36 coefficients. The computation of these new constants is described. Next we present the partial derivatives of this model with respect to the sail position, attitude, and the model coefficients. Finally, we revisit some classical results, now reformulated for our new model, such as the optimal orientation to generate maximum thrust or to generate maximum energy increase in a sail trajectory.

I Introduction

A fundamental challenge for the simulation and analysis of solar sail trajectories and control laws is the specification of the sail propulsive force as a function of its sun-relative orientation. Analytically, the only known and commonly used closed-form expressions are for the "flat plate" model of a sail, which at best is only a crude approximation of a sail even if realistic optical parameters are used [1]. Realistic sails may have non-planar surfaces (or billow) and gaps where structural elements are placed. The presence of such complexity in a sail structure can have significant effects on the performance of the sail and the design of optimal control laws. The traditional approach to modeling such sails would use finite-element type models, constructing a sail out of a mesh of connected flat plate elements. For such an approach, however, every time the sail is re-oriented the entire surface must be summed over to compute the new propulsive force and new moment acting on the sail. With such a sophisticated model it is very computationally intensive to compute partial derivatives of the sail with respect to the solar orientation, these summations over the sail having to be recomputed for each new illumination geometry. Finally, as the accuracy of the model is increased, the computational burden will increase commensurately.

These issues become important as solar sail technology continues to advance and as the first practical flight of a solar sail spacecraft comes closer to becoming a reality. For solar sail spacecraft to be accepted as a viable option for space missions, it is crucial that flight tools be developed for these craft that will allow for precision navigation [2]. At the heart of precision navigation, however, is the creation of accurate and precise models of a solar sail propulsive force and moment as a function of the solar illumination. In Rios-Reyes and Scheeres (2004) this issue was addressed and a general model for a solar sail's total force and moment as an analytical function of the solar illumination geometry was proposed. This formulation introduced a series of well-defined coefficients that can be computed for any sail which can capture the effect of non-planarity and the effect of a non-uniform distribution of optical properties across a sail's surface. The coefficients are defined as the integral of higher moments of the sail surface normal vector over the sail surface. One can qualitatively think of them as "gravity coefficients" for the sail, in that they provide a complete and unique specification of the sail's properties. For notational ease, these coefficients are gathered into Cartesian tensors of rank 1, 2, and 3. For the force equation there are a total of 18 independent coefficients, while

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for the moment equation there are a total of 36 independent coefficients. That paper also gave a number of examples of these coefficients for some simple and non-simple sail shapes and geometries. These models are a real advance in our ability to model solar sails, as they provide an exact analytical formula of a sail's force and moment as a function of the solar illumination geometry, including all relevant optical and re-radiation effects. The only assumptions made in their derivation is that the sail shape does not change with the solar illumination, and that the sail does not self-shadow. Relaxation of these assumptions may be considered in the future.

In this paper, we revisit this sail model to propose a few improvements, provide a deeper discussion of the properties of the model, and give a few examples of how it can be easily used to derive some simple guidance laws for a complex sail. Specifically, this paper covers the following topics. First, we provide a concise derivation of the general model and discuss the physical meaning of some of the tensor coefficients. Next, we reconsider the moment equation with its 36 independent coefficients, and show that these can be reduced to the 18 independent coefficients of the force equation plus 9 additional independent coefficients. This is a nice result, as it reduces the total number of independent coefficients needed to specify the sail's force and moment equations to 27. We give examples of these reduced coefficients for the sail models considered in Rios-Reyes and Scheeres (2004). Following this, we provide a few practical applications of the model, beyond the direct computation of forces and moments. First we derive a series of partial derivatives of these models with respect to the solar illumination geometry, with respect to the coefficient values themselves, and with respect to the optical parameters of the model. Finally, we provide two examples of how the general model allows one to derive analytical guidance laws for the sail that maximize the change in energy of the sail or that maximize the total force produced by a complex sail.

A Concise Derivation of the Generalized Sail Model

The differential force normal and transverse to a sail element area can be expressed as:

$$d\mathbf{F}_n = -P(r)[a_1 \cos^2\alpha + a_2 \cos\alpha] dA\hat{\mathbf{n}} \tag{1}$$

$$d\mathbf{F}_t = P(r) a_3 \cos\alpha \sin\alpha dA\hat{\mathbf{t}} \tag{2}$$

where $P(r)$ is the solar radiation pressure at a distance r, $\hat{\mathbf{n}}$ is the vector normal to the differential area dA, $\hat{\mathbf{t}}$ is the vector transverse to the normal vector in the plane given by $\hat{\mathbf{n}}$ and the sail position unit vector $\hat{\mathbf{r}}$, which points into the sail, $a_1 = 1 + \rho s$, $a_2 = B_f(1-s)\rho + (1-\rho)\frac{\epsilon_f B_f - \epsilon_b B_b}{\epsilon_f + \epsilon_b}$, and $a_3 = 1 - \rho s$, ρ is the reflectivity, s is the fraction of specular reflection, ϵ is the emissivity, and B is the Lambertian coefficient with the subscripts f and b denoting the front and back surfaces, respectively. The solar radiation pressure can be calculated assuming the sun to be a point source or by taking into account the solar disk. Both of these equation are found in [1]; the latter equation is given by:

$$P(r) = \frac{2\pi I_0}{3c}\left\{1 - \left[1 - \left(\frac{R_s}{r}\right)^2\right]^{3/2}\right\} \tag{3}$$

where I_0 is the frequency integrated specific intensity, c is the speed of light, and R_s is the radius of the sun.

The total force is found by integrating and summing these expressions over the sail surface:

$$\mathbf{F} \quad = \quad \int_A (d\mathbf{F}_n + d\mathbf{F}_t) \tag{4}$$

The term $\cos\alpha$ can be obtained from:

$$\cos\alpha \quad = \quad -\hat{\mathbf{n}} \cdot \hat{\mathbf{r}} \tag{5}$$

If the normal vector at any point on the sail is defined as $\hat{\mathbf{n}} = [\hat{n}_1 \;\; \hat{n}_2 \;\; \hat{n}_3]^T$ and defining the cross product as:

$$\hat{\mathbf{n}} \times -\hat{\mathbf{r}} = -\tilde{\mathbf{n}} \cdot \hat{\mathbf{r}} \tag{6}$$

where the operator ($\tilde{}$) is defined in appendix A, the terms appearing in Eq. (2) can be expressed as:

$$\sin\alpha\hat{\mathbf{t}} = -\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \hat{\mathbf{r}}) = -\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} \cdot \hat{\mathbf{r}} \tag{7}$$

Thus, the differential force acting on an area element can be reduced to:

$$d\mathbf{F} = -P(r)[a_1(\hat{\mathbf{r}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \hat{\mathbf{r}}) - a_2(\hat{\mathbf{r}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + a_3(\hat{\mathbf{r}} \cdot \hat{\mathbf{n}})\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} \cdot \hat{\mathbf{r}}]dA \tag{8}$$

Defining the dyadic of the normal vector as in [3], $\hat{\mathbf{n}}\hat{\mathbf{n}}$, and the triadic as $\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}}$, the total force can be written as:

$$\mathbf{F} = P(r)\Big[\int_A a_2\hat{\mathbf{n}}\hat{\mathbf{n}}dA \cdot \hat{\mathbf{r}} + \hat{\mathbf{r}} \cdot \Big(-2\int_A \rho s\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}}dA - \overline{\overline{\mathbf{U}}}\int_A a_3\hat{\mathbf{n}}dA\Big) \cdot \hat{\mathbf{r}}\Big] \tag{9}$$

The integrands of all these expressions are independent of the solar incidence direction, $\hat{\mathbf{r}}$, and can be computed off-line for a given sail shape, re-used over a range of sail attitudes, and ideally can accommodate non-uniformities in the sail optical properties.

Assuming constant sail optical properties, the force can now be rewritten as:

$$\mathbf{F} = P(r)\Big[a_2\mathbf{J}^2 \cdot \hat{\mathbf{r}} - 2\rho s\hat{\mathbf{r}} \cdot \mathbf{J}^3 \cdot \hat{\mathbf{r}} - a_3(\mathbf{J}^1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\Big] \tag{10}$$

where the force surface normal distribution integrals, the cartesian tensors $\mathbf{J}^m$ of rank-m, are defined as:

$$\mathbf{J}^m = \int_A \hat{\mathbf{n}}^m dA \tag{11}$$

$$= \int_A \hat{\mathbf{n}}\hat{\mathbf{n}}\ldots\hat{\mathbf{n}}dA \tag{12}$$

The products of these tensors and the unit position vector are explained in appendix B. The force acting on a solar sail of any given shape can then be described by a set of three tensors of rank 1, rank 2 and rank 3 [4]. The assumption that the sail shape is fixed and does not change with sun-relative attitude is made.

The total moment acting on the sail can be found by integrating the expression:

$$d\mathbf{M} = \vec{\varrho} \times d\mathbf{F} = P(r)\tilde{\varrho} \cdot \Big[a_2\hat{\mathbf{n}}\hat{\mathbf{n}}dA \cdot \hat{\mathbf{r}} + \hat{\mathbf{r}} \cdot \Big(-2\rho s\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}}dA - a_3\hat{\mathbf{n}}\overline{\overline{\mathbf{U}}}dA\Big) \cdot \hat{\mathbf{r}}\Big] \tag{13}$$

over the entire sail, where $\vec{\varrho}$ is the position of the differential element dA with respect to a given reference frame on the sail. Integrating yields the total moment about the origin of the sail reference frame:

$$\begin{aligned}\mathbf{M} = P(r)\Big[&\int_A a_2\tilde{\varrho} \cdot \hat{\mathbf{n}}\hat{\mathbf{n}}dA \cdot \hat{\mathbf{r}} - 2\hat{\mathbf{r}} \cdot \int_A \rho s\tilde{\varrho} \cdot \hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}}dA \cdot \hat{\mathbf{r}} \\ &-\hat{\mathbf{r}} \cdot \int_A a_3\hat{\mathbf{n}}\tilde{\varrho} \cdot \overline{\overline{\mathbf{U}}} \cdot \hat{\mathbf{r}}dA\Big]\end{aligned} \tag{14}$$

Defining the moment surface normal distribution integrals as:

$$\mathbf{K}^m = \int_A \tilde{\varrho} \cdot \hat{\mathbf{n}}^m dA \tag{15}$$

$$\mathbf{L} = \int_A \hat{\mathbf{n}}\vec{\varrho}dA \tag{16}$$

and assuming constant optical properties, the moment can be rewritten as:

$$\mathbf{M} = P(r)\Big[a_2\mathbf{K}^2 \cdot \hat{\mathbf{r}} - 2\rho s\hat{\mathbf{r}} \cdot \mathbf{K}^3 \cdot \hat{\mathbf{r}} - a_3\hat{\mathbf{r}} \cdot \mathbf{L} \cdot \tilde{\hat{\mathbf{r}}}_0\Big] \tag{17}$$

where $\mathbf{K}^m$ and $\mathbf{L}$ are rank-m and rank-2 tensors.

B Properties of the Tensor Coefficients

The force tensor coefficients are completely symmetric in their indices, i.e., $J^m_{i_1 i_2 \ldots i_m} = J^m_{i_2 i_1 \ldots i_m}$, and so on for any two indices. Thus, for a rank-3 tensor, which could have up to 27 entries, we only need to compute 9 independent values. In general, a tensor $\mathbf{J}^m$ as defined above will only have $3m$ unique terms among its 3^m entries. Thus, the three integrals in Eq. (10) are specified by 3 + 6 + 9 = 18 numbers for the general case.

It is important to note that the force tensor coefficients are independent of the sail position and independent of the sail orientation.

Some geometric properties are embedded in the force tensor coefficients. First consider the $\mathbf{J}^1$ tensor, defining the nominal sail plane to be the $x-y$ plane, the third element of the $\mathbf{J}^1$ tensor, $\mathbf{J}^1_3$, represents the projection of the sail surface area into the sail x-y plane. The first element, $\mathbf{J}^1_1$, is the projection of the sail area onto the $y-z$ plane and the second element, $\mathbf{J}^1_2$, projects the area into the $x-z$ plane. If the sail is symmetric about the $y-z$ plane, then $\mathbf{J}^1_1$ will be zero. Similarly, if the sail is symmetric about the $x-z$ plane, $\mathbf{J}^1_2$ will be zero, since the projection onto their respective planes will be cancelled from opposite sides of the sail.

Now focus on the $\mathbf{J}^2$ tensor. The $\mathbf{J}^2_{11}$, $\mathbf{J}^2_{22}$, and $\mathbf{J}^2_{33}$ elements are expected to be non-zero even for symmetric shapes, unless the sail is completely flat, then the only non-zero element will be $\mathbf{J}^2_{33}$. If $\mathbf{J}^1_1$ is zero, then the elements $\mathbf{J}^2_{13}$ and $\mathbf{J}^2_{31}$ are zero. Also, If $\mathbf{J}^1_2$ is zero, then the elements $\mathbf{J}^2_{23}$ and $\mathbf{J}^2_{32}$ must be zero. If any of $\mathbf{J}^1_1$ or $\mathbf{J}^1_2$ are zero, for symmetric geometries, then $\mathbf{J}^2_{12}$ and $\mathbf{J}^2_{21}$ must be zero. To show this point assume that $\hat{\mathbf{n}} = \hat{\mathbf{n}}(\eta, \zeta)$, where η and ζ are cartesian variables that define the sail surface area. Now, $\mathbf{J}^1_1 = 0$ implies that $\hat{\mathbf{n}}_1(\eta, \zeta)$ is symmetric, or is an odd function, on any of these two variables, but not both. Similarly, $\mathbf{J}^1_2 = 0$ implies that $\hat{\mathbf{n}}_2(\eta, \zeta)$ is an odd function on either η or ζ, but not both and not on the same variable as $\hat{\mathbf{n}}_1(\eta, \zeta)$ is since η and ζ are mutually orthogonal. Now consider the case where $\mathbf{J}^1_1 = 0$, $\mathbf{J}^1_2 \neq 0$, and $\hat{\mathbf{n}}_1(\eta, \zeta)$ is symmetric on η, then we can write:

$$\mathbf{J}^2_{12} = \int_{-\eta_0}^{\eta_0} \int_{\zeta_1}^{\zeta_2} \hat{\mathbf{n}}_1(\eta, \zeta) \hat{\mathbf{n}}_2(\eta, \zeta) dA(\eta, \zeta) \tag{18}$$

where the limits of integration go from $-\eta_0$ to η_0, since we are assuming symmetry about η, and ζ_1 to ζ_2. Furthermore, due to this symmetry we can write:

$$\mathbf{J}^2_{12} = \int_{0}^{\eta_0} \int_{\zeta_1}^{\zeta_2} \hat{\mathbf{n}}_1(\eta, \zeta) \hat{\mathbf{n}}_2(\eta, \zeta) dA(\eta, \zeta) - \int_{0}^{\eta_0} \int_{\zeta_1}^{\zeta_2} \hat{\mathbf{n}}_1(\eta, \zeta) \hat{\mathbf{n}}_2(\eta, \zeta) dA(\eta, \zeta) = 0 \tag{19}$$

The case where $\mathbf{J}^1_1 \neq 0$, $\mathbf{J}^1_2 = 0$, and $\hat{\mathbf{n}}_2(\eta, \zeta)$ is symmetric on ζ can be shown by replacing η with ζ in the above equations. In a similar manner it can be shown that if both $\mathbf{J}^1_1$ and $\mathbf{J}^1_2$ are zero, with the symmetric assumptions previously made, then $\mathbf{J}^2_{12}$ must be zero.

For the $\mathbf{J}^3$ tensor we expect the following results. If $\mathbf{J}^1_1$ is zero, then $\mathbf{J}^3_{111}$, $\mathbf{J}^3_{221}$, $\mathbf{J}^3_{331}$, $\mathbf{J}^3_{122}$, $\mathbf{J}^3_{212}$, $\mathbf{J}^3_{133}$, and $\mathbf{J}^3_{313}$ are zero. Additionally, if $\mathbf{J}^1_2$ is zero, then $\mathbf{J}^3_{211}$, $\mathbf{J}^3_{112}$, $\mathbf{J}^3_{222}$, $\mathbf{J}^3_{332}$, $\mathbf{J}^3_{121}$, $\mathbf{J}^3_{233}$, and $\mathbf{J}^3_{323}$ will be zero. If both $\mathbf{J}^1_1$ and $\mathbf{J}^1_2$ are zero, then the elements $\mathbf{J}^3_{231}$, $\mathbf{J}^3_{321}$, $\mathbf{J}^3_{312}$, $\mathbf{J}^3_{123}$, and $\mathbf{J}^3_{213}$ will be zero. If the element $\mathbf{J}^2_{11}$ is not zero, then the elements $\mathbf{J}^3_{131}$, $\mathbf{J}^3_{311}$, and $\mathbf{J}^3_{113}$ are not zero. Finally, if the element $\mathbf{J}^2_{22}$ is not zero, then the elements $\mathbf{J}^3_{232}$, $\mathbf{J}^3_{322}$, and $\mathbf{J}^3_{223}$ are not zero. The elements $\mathbf{J}^1_3$, $\mathbf{J}^2_{33}$, and $\mathbf{J}^3_{333}$ are, in general, zero only for the trivial case when the sail area is zero.

The moment tensors $\mathbf{K}^m$ and $\mathbf{L}$ (which are rank-m and rank-2 tensors, respectively) do not have the same complete symmetry as the $\mathbf{J}^m$ do. Thus, there are more unique coefficients needed to specify them. For example, $\mathbf{L}$ has no symmetries in general and defines 9 unique coefficients. $\mathbf{K}^2$ requires 9 unique coefficients. $\mathbf{K}^3$, however, is symmetric in two of its indices, $\mathbf{K}^3_{ijk} = \mathbf{K}^3_{ikj}$, and only requires 18 entries instead of 27.

II Moment Reformulation

A Defining the Generalized Centers of Pressure

The total moment acting on a solar sail about the sail origin is defined above as:

$$
\begin{aligned}
\mathbf{M} &= P(r)\Big[\int_A a_2\tilde{\varrho}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}} - 2\hat{\mathbf{r}}\cdot\int_A \rho s\tilde{\varrho}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}} \\
&\quad -\hat{\mathbf{r}}\cdot\int_A a_3\hat{\mathbf{n}}\tilde{\varrho}\cdot\overline{\overline{\mathbf{U}}}\cdot\hat{\mathbf{r}}dA\Big] \qquad (20)
\end{aligned}
$$

Let us generalize this result to the case where the moment is taken about an arbitrary point on the sail, denoted by $\mathbf{R}$. Then for a given location in the sail body-fixed frame, $\vec{\varrho}$, the new position relative to the point defined by $\mathbf{R}$ is $\vec{\varrho}-\mathbf{R}$. Then Eq. (20) can be generalized to the moment about the point defined by $\mathbf{R}$ and be written as:

$$
\begin{aligned}
\mathbf{M}_R &= P(r)\Big[\int_A a_2(\tilde{\varrho}-\tilde{\mathbf{R}})\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}} - 2\hat{\mathbf{r}}\cdot\int_A \rho s(\tilde{\varrho}-\tilde{\mathbf{R}})\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}} \\
&\quad -\hat{\mathbf{r}}\cdot\int_A a_3\hat{\mathbf{n}}(\tilde{\varrho}-\tilde{\mathbf{R}})\cdot\overline{\overline{\mathbf{U}}}\cdot\hat{\mathbf{r}}dA\Big] \qquad (21)
\end{aligned}
$$

which can be rearranged as:

$$
\begin{aligned}
\mathbf{M}_R &= P(r)\Big[\int_A a_2\tilde{\varrho}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}} - 2\hat{\mathbf{r}}\cdot\int_A \rho s\tilde{\varrho}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}} - \hat{\mathbf{r}}\cdot\int_A a_3\hat{\mathbf{n}}\tilde{\varrho}\cdot\overline{\overline{\mathbf{U}}}\cdot\hat{\mathbf{r}}dA\Big] \\
&\quad -P(r)\Big[\int_A a_2\tilde{\mathbf{R}}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}} - 2\hat{\mathbf{r}}\cdot\int_A \rho s\tilde{\mathbf{R}}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}} - \hat{\mathbf{r}}\cdot\int_A a_3\hat{\mathbf{n}}\tilde{\mathbf{R}}\cdot\overline{\overline{\mathbf{U}}}\cdot\hat{\mathbf{r}}dA\Big] \qquad (22)
\end{aligned}
$$

As $\mathbf{R}$ denotes the position of a fixed reference point, and if the optical properties are assumed to be constant, Eq. (22) can be reduced to:

$$
\begin{aligned}
\mathbf{M}_R &= P(r)\Big[a_2\int_A \tilde{\varrho}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}} - 2\rho s\hat{\mathbf{r}}\cdot(\int_A \tilde{\varrho}\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}}dA\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} - a_3\hat{\mathbf{r}}\cdot\int_A \hat{\mathbf{n}}\tilde{\varrho}\cdot\overline{\overline{\mathbf{U}}}\cdot\hat{\mathbf{r}}dA\Big] \\
&\quad -P(r)\Big[a_2\tilde{\mathbf{R}}\cdot\mathbf{J}^2\cdot\hat{\mathbf{r}} - 2(\rho s\tilde{\mathbf{R}}\cdot\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} - a_3\hat{\mathbf{r}}\cdot\mathbf{J}^1\tilde{\mathbf{R}}\cdot\overline{\overline{\mathbf{U}}}\cdot\hat{\mathbf{r}}\Big] \qquad (23)
\end{aligned}
$$

or using the definition for the moment tensors we can write:

$$
\begin{aligned}
\mathbf{M}_R &= P(r)\Big[a_2\mathbf{K}^2\cdot\hat{\mathbf{r}} - 2\rho s\hat{\mathbf{r}}\cdot(\mathbf{K}^3\cdot\hat{\mathbf{r}}) - a_3\hat{\mathbf{r}}\cdot\mathbf{L}\cdot\tilde{\hat{\mathbf{r}}}_0\Big] - \mathbf{R}\times\mathbf{F} \\
&= \mathbf{M} - \tilde{\mathbf{R}}\cdot\mathbf{F} \qquad (24)
\end{aligned}
$$

This provide us with a general formula for the moment relative to a general point on the sail. We should note that, for any given orientation, there will be a"center of pressure" defined by $\mathbf{M}_{R_p}=0$ or location $\mathbf{R}_p$ such that:

$$
\mathbf{R}_p\times\mathbf{F} = 0 \qquad (25)
$$

In [4] a general formula for the center of pressure is found and given by:

$$
\mathbf{R}_p = \frac{1}{F^2}\mathbf{F}\times\mathbf{M} + \sigma\hat{\mathbf{F}} \qquad (26)
$$

where σ is an arbitrary distance. The center of pressure changes with $\hat{\mathbf{r}}$ for a non-flat sail shape.

Inspired by the concept of center of pressure, we can use this idea to reduce the number of independent coefficients needed to define the moment formula. In Eq. (21) we note that each integral can be written as:

$$
\int_A \hat{\mathbf{n}}(\tilde{\varrho}-\tilde{\mathbf{R}})\cdot\overline{\overline{\mathbf{U}}}dA = \mathbf{L} - \mathbf{R}\mathbf{J}^1 \qquad (27)
$$

$$\int_A (\tilde{\varrho} - \tilde{\mathbf{R}}) \cdot \hat{\mathbf{n}}\hat{\mathbf{n}} dA = \mathbf{K}^2 - \tilde{\mathbf{R}} \cdot \mathbf{J}^2 \tag{28}$$

$$\int_A (\tilde{\varrho} - \tilde{\mathbf{R}}) \cdot \hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}} dA = \mathbf{K}^3 - \tilde{\mathbf{R}} \cdot \mathbf{J}^3 \tag{29}$$

If we set each of these equations equal to zero and solve for the vector $\mathbf{R}$ that satisfies the result, we can replace the occurrence of the $\mathbf{K}^m$ and $\mathbf{L}$ tensors with a simple function of the $\mathbf{J}^m$ tensors and the appropriate $\mathbf{R}$ vector. Each equation will have a different solution, in general, exceptions occurring for sails with highly symmetric area distributions. This allows the following substitutions in Eq. (17):

$$\mathbf{L} = \mathbf{R}_L \mathbf{J}^1 \tag{30}$$
$$\mathbf{K}^2 = \tilde{\mathbf{R}}_{K^2} \cdot \mathbf{J}^2 \tag{31}$$
$$\mathbf{K}^3 = \tilde{\mathbf{R}}_{K^3} \cdot \mathbf{J}^3 \tag{32}$$

and the moment equation can be expressed as:

$$\mathbf{M} \quad = P(r)\Big[a_2(\tilde{\mathbf{R}}_{K^2} \cdot \mathbf{J}^2) \cdot \hat{\mathbf{r}} - 2\rho s \hat{\mathbf{r}} \cdot (\tilde{\mathbf{R}}_{K^3} \cdot \mathbf{J}^3) \cdot \hat{\mathbf{r}} - a_3 \hat{\mathbf{r}} \cdot (\mathbf{R}_L \mathbf{J}^1) \cdot \tilde{\hat{\mathbf{r}}}_0\Big] \tag{33}$$

The moment equation is now characterized by 9 coefficients, the vectors $\mathbf{R}_L$, $\mathbf{R}_{K^2}$, and $\mathbf{R}_{K^3}$, plus the already defined $\mathbf{J}^m$, instead of the 36 coefficients stated previously.

We call the new coefficients the "generalized centers of pressure" for the moment coefficients. Note that in order to find these generalized centers of pressure the coefficients $\mathbf{L}$, $\mathbf{K}^2$, and $\mathbf{K}^3$ must be computed first.

Eqs. (30)-(32) are "non-standard" linear equations for the generalized center of pressure vectors. Thus, we give a detailed solution of how they can be solved for.

B Solution of Generalized Centers of Pressure

The vector $\mathbf{R}_L$ can be uniquely determined if the tensor $\mathbf{J}^1$ is not identically equal to zero, which it will not be in general. If this statement holds, then Eq. (30) can be post-multiplied by $\mathbf{J}^1$ to yield:

$$\mathbf{L} \cdot \mathbf{J}^1 = (\mathbf{J}^1 \cdot \mathbf{J}^1)\mathbf{R}_L \tag{34}$$

which allows us to solve directly for $\mathbf{R}_L$:

$$\mathbf{R}_L = \frac{1}{\mathbf{J}^1 \cdot \mathbf{J}^1} \mathbf{L} \cdot \mathbf{J}^1 \tag{35}$$

Eqs. (31) and (32) cannot be solved directly, in general. We can discuss some properties of the solutions, however. First we note that the vector $\mathbf{R}_{K^m}$ is a zero left eigenvector of $\mathbf{K}^m$, or for $\mathbf{K}^2$:

$$\mathbf{R}_{K^2} \cdot \tilde{\mathbf{R}}_{K^2} \cdot \mathbf{J}^2 = \mathbf{R}_{K^2} \cdot \mathbf{K}^2 = 0 \tag{36}$$

Now, this implies that $\mathbf{K}^2$ has a zero left eigenvalue and thus will be singular. If we solve for its left eigenvector $\mathbf{R}_{K^2}$, we then arrive at the identity:

$$|\mathbf{R}_{K^2}| \tilde{\hat{\mathbf{R}}}_{K^2} \cdot \mathbf{J}^2 - \mathbf{K}^2 = 0 \tag{37}$$

which should allow us to solve for the magnitude of $\mathbf{R}_{K^2}$ by balancing terms. The same logic applied to Eq. (32) yields:

$$\mathbf{R}_{K^3} \cdot \tilde{\mathbf{R}}_{K^3} \cdot \mathbf{J}^3 = \mathbf{R}_{K^3} \cdot \mathbf{K}^3 = 0 \tag{38}$$

and

$$|\mathbf{R}_{K^3}|\tilde{\hat{\mathbf{R}}}_{K^3} \cdot \mathbf{J}^3 - \mathbf{K}^3 = 0 \tag{39}$$

It may also be possible to directly solve for the vectors, depending on the structure of the $\mathbf{J}^2$ and $\mathbf{J}^3$ tensors.

Focusing on Eq. (31) first, if $\mathbf{J}^2$ is invertible we easily find the solution:

$$\tilde{\mathbf{R}}_{K^2} = \mathbf{K}^2 \cdot (\mathbf{J}^2)^{-1} \tag{40}$$

from which $\mathbf{R}_{K^2}$ can be directly solved.

For finding $\mathbf{R}_{K^3}$, $\mathbf{J^3}$ can be transformed into a $9\times$ matrix with the structure:

$$\mathbf{J}^{3M} = \begin{bmatrix} \mathbf{J}^3_{ij1} & 0 & 0 \\ 0 & \mathbf{J}^3_{ij3} & 0 \\ 0 & 0 & \mathbf{J}^3_{ij3} \end{bmatrix} \tag{41}$$

the generalized center of pressure $\mathbf{R}_{K^3}$ must also be modified to:

$$\tilde{\mathbf{R}}^M_{K^3} = \begin{bmatrix} \tilde{\mathbf{R}}_{K^3} & 0 & 0 \\ 0 & \tilde{\mathbf{R}}_{K^3} & 0 \\ 0 & 0 & \tilde{\mathbf{R}}_{K^3} \end{bmatrix} \tag{42}$$

Then, Eq. (32) can be written in the form:

$$\mathbf{K}^{3M} = \tilde{\mathbf{R}}^{3M}_{K} \cdot \mathbf{J}^{3M} \tag{43}$$

If each of the matrices $\mathbf{J}^3_{ij1}$, $\mathbf{J}^3_{ij2}$, and $\mathbf{J}^3_{ij3}$ are invertible, then $\mathbf{J}^{3M}$ is invertible and we can solve for:

$$\tilde{\mathbf{R}}^M_{K^3} = \mathbf{K}^{3M} \cdot (\mathbf{J}^{3M})^{-1} \tag{44}$$

A final approach to solving Eqs. (31) and (32) is to realize that these equations can be thought of as an overdetermined linear system of equations. Eq. (31) would be a linear system of 9 equations with 3 unknowns while Eq. (32) has 18 equations with three unknowns. Expanding Eq. (31), the corresponding equations obtained are:

$$\begin{align}
\mathbf{J}^2_{13}\mathbf{R}_{K^2y} - \mathbf{J}^2_{12}\mathbf{R}_{K^2z} &= \mathbf{K}^2_{11} \tag{45}\\
\mathbf{J}^2_{23}\mathbf{R}_{K^2y} - \mathbf{J}^2_{22}\mathbf{R}_{K^2z} &= \mathbf{K}^2_{12} \tag{46}\\
\mathbf{J}^2_{33}\mathbf{R}_{K^2y} - \mathbf{J}^2_{23}\mathbf{R}_{K^2z} &= \mathbf{K}^2_{13} \tag{47}\\
-\mathbf{J}^2_{13}\mathbf{R}_{K^2x} + \mathbf{J}^2_{11}\mathbf{R}_{K^2z} &= \mathbf{K}^2_{21} \tag{48}\\
-\mathbf{J}^2_{23}\mathbf{R}_{K^2x} + \mathbf{J}^2_{12}\mathbf{R}_{K^2z} &= \mathbf{K}^2_{22} \tag{49}\\
-\mathbf{J}^2_{33}\mathbf{R}_{K^2x} + \mathbf{J}^2_{13}\mathbf{R}_{K^2z} &= \mathbf{K}^2_{23} \tag{50}\\
\mathbf{J}^2_{12}\mathbf{R}_{K^2x} - \mathbf{J}^2_{11}\mathbf{R}_{K^2y} &= \mathbf{K}^2_{31} \tag{51}\\
\mathbf{J}^2_{22}\mathbf{R}_{K^2x} - \mathbf{J}^2_{12}\mathbf{R}_{K^2y} &= \mathbf{K}^2_{32} \tag{52}\\
\mathbf{J}^2_{23}\mathbf{R}_{K^2x} - \mathbf{J}^2_{13}\mathbf{R}_{K^2y} &= \mathbf{K}^2_{33} \tag{53}\\
& \tag{54}
\end{align}$$

which can be rearranged as:

$$
\begin{bmatrix}
0 & \mathbf{J}^2_{13} & -\mathbf{J}^2_{12} \\
0 & \mathbf{J}^2_{23} & -\mathbf{J}^2_{22} \\
0 & \mathbf{J}^2_{33} & -\mathbf{J}^2_{23} \\
-\mathbf{J}^2_{13} & 0 & \mathbf{J}^2_{11} \\
-\mathbf{J}^2_{23} & 0 & \mathbf{J}^2_{12} \\
-\mathbf{J}^2_{33} & 0 & \mathbf{J}^2_{13} \\
\mathbf{J}^2_{12} & -\mathbf{J}^2_{11} & 0 \\
\mathbf{J}^2_{22} & -\mathbf{J}^2_{12} & 0 \\
\mathbf{J}^2_{23} & -\mathbf{J}^2_{13} & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
\mathbf{R}_{K^2x} \\
\mathbf{R}_{K^2y} \\
\mathbf{R}_{K^2z}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{K}^2_{11} \\
\mathbf{K}^2_{12} \\
\mathbf{K}^2_{13} \\
\mathbf{K}^2_{21} \\
\mathbf{K}^2_{22} \\
\mathbf{K}^2_{23} \\
\mathbf{K}^2_{31} \\
\mathbf{K}^2_{32} \\
\mathbf{K}^2_{33}
\end{bmatrix}
\tag{55}
$$

Similarly, it is possible to solve for $\mathbf{R}_{K^3}$. Eq. (32) would yield 27 equations, from which 18 are non-repeated, with 3 unknowns. Using the definition of the dot product of a rank-2 $\mathbf{T}^2$ and a rank-3 $\mathbf{T}^3$ tensor presented in appendix A, the system of 18 equations can be obtained and expressed as:

$$
\begin{bmatrix}
0 & \mathbf{J}^3_{131} & -\mathbf{J}^3_{121} \\
0 & \mathbf{J}^3_{231} & -\mathbf{J}^3_{221} \\
0 & \mathbf{J}^3_{331} & -\mathbf{J}^3_{231} \\
-\mathbf{J}^3_{131} & 0 & \mathbf{J}^3_{111} \\
-\mathbf{J}^3_{231} & 0 & \mathbf{J}^3_{121} \\
-\mathbf{J}^3_{331} & 0 & \mathbf{J}^3_{131} \\
\mathbf{J}^3_{121} & -\mathbf{J}^3_{111} & 0 \\
\mathbf{J}^3_{221} & -\mathbf{J}^3_{121} & 0 \\
\mathbf{J}^3_{231} & -\mathbf{J}^3_{131} & 0 \\
0 & \mathbf{J}^3_{232} & -\mathbf{J}^3_{222} \\
\mathbf{J}^3_{332} & 0 & -\mathbf{J}^3_{232} \\
-\mathbf{J}^3_{232} & 0 & \mathbf{J}^3_{221} \\
-\mathbf{J}^3_{332} & 0 & \mathbf{J}^3_{231} \\
\mathbf{J}^3_{222} & 0 & -\mathbf{J}^3_{221} \\
\mathbf{J}^3_{232} & -\mathbf{J}^3_{231} & 0 \\
0 & \mathbf{J}^3_{333} & -\mathbf{J}^3_{332} \\
-\mathbf{J}^3_{333} & 0 & \mathbf{J}^3_{331} \\
\mathbf{J}^3_{332} & -\mathbf{J}^3_{331} & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
\mathbf{R}_{K^3x} \\
\mathbf{R}_{K^3y} \\
\mathbf{R}_{K^3z}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{K}^3_{111} \\
\mathbf{K}^3_{121} \\
\mathbf{K}^3_{131} \\
\mathbf{K}^3_{211} \\
\mathbf{K}^3_{221} \\
\mathbf{K}^3_{231} \\
\mathbf{K}^3_{311} \\
\mathbf{K}^3_{321} \\
\mathbf{K}^3_{331} \\
\mathbf{K}^3_{122} \\
\mathbf{K}^3_{132} \\
\mathbf{K}^3_{222} \\
\mathbf{K}^3_{232} \\
\mathbf{K}^3_{322} \\
\mathbf{K}^3_{332} \\
\mathbf{K}^3_{133} \\
\mathbf{K}^3_{233} \\
\mathbf{K}^3_{333}
\end{bmatrix}
\tag{56}
$$

Eqs. (55) and (56) can be solved using a pseudo-inverse method. Due to numerical errors in the computation of the $\mathbf{K}^m$ tensors these errors will propagate into the solution of the generalized center of pressure. One solution can be obtained by minimizing the error of the solution. Both of these equations have the general form $Ax = y$, and the solution's error can be defined as $e = y - Ax$. A solution that minimizes the error is given by [5]:

$$
x = (A^T A)^{-1} A^T y \tag{57}
$$

Eqs. (57) would fail if the matrix $A^T A$ is singular. If we consider the case for a flat plate, however, we do find that $A^T A$ is singular. If Eq. (55) is multiplied out, non-trivial terms do exist and equal:

$$
\mathbf{J}^2_{33}\mathbf{R}_{K^2y} = \mathbf{K}^2_{13} \tag{58}
$$

$$
-\mathbf{J}^2_{33}\mathbf{R}_{K^2x} = \mathbf{K}^2_{23} \tag{59}
$$

(60)

and the non-trivial elements of Eq. (56) are:

$$
\mathbf{J}^3_{333}\mathbf{R}_{K^3y} = \mathbf{K}^3_{133} \tag{61}
$$

$$
-\mathbf{J}^3_{333}\mathbf{R}_{K^3x} = \mathbf{K}^3_{233} \tag{62}
$$

The elements $\mathbf{J}^2_{33}$ and $\mathbf{J}^3_{333}$ can only be zero if the sail area is zero. Thus, it is guaranteed that in this degenerate case the above equations have a solution and the generalized center of pressure can be found.

C Computations of Generalized Centers of Pressure

We now find the generalized centers of pressure for a few different sail models of interest. Let us first use a flat sail model, taking as a reference point one of the sail corners and setting the sail in the first quadrant of a coordinate frame. For a flat sail the normal vector is $\hat{\mathbf{n}} = [0, 0, 1]^T$, so the only non-zero elements of the force tensors are $\mathbf{J}^1_3$, $\mathbf{J}^2_{33}$, and $\mathbf{J}^3_{333}$ with a value equal to the sail area A. The moment tensors can be found in Appendix B. Solving for the generalized centers of pressure we find:

$$\mathbf{R}_L = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{63}$$

$$\mathbf{R}_{K^2} = \frac{l}{2} \begin{bmatrix} 1 \\ 1 \\ \sigma_1 \end{bmatrix} \tag{64}$$

$$\mathbf{R}_{K^3} = \frac{l}{2} \begin{bmatrix} 1 \\ 1 \\ \sigma_2 \end{bmatrix} \tag{65}$$

where σ_i is an arbitrary constant. Using these definitions we can verify that the formulas for the $\mathbf{K}^m$ and $\mathbf{L}$ given in Appendix B are valid.

Let's now consider the circular sail model described in Appendix B. For the circular sail, $\mathbf{J}^2$ is invertible. Solving directly for $\mathbf{R}_{K^2}$ we obtain:

$$\mathbf{R}_{K^2} = \begin{bmatrix} 0 \\ 0 \\ \frac{R_0(6-5\alpha^2_{max}-\sqrt{1+\alpha^2_{max}}(6-8\alpha^2_{max}+\alpha^4_{max}))}{5\alpha_{max}(2+(\alpha^2_{max}-2)\sqrt{1+\alpha^2_{max}})} \end{bmatrix} \tag{66}$$

and for $\mathbf{R}_{K^3}$

$$\mathbf{R}_{K^3} = \begin{bmatrix} 0 \\ 0 \\ \frac{R_0}{4\alpha_{max}}\left(2\alpha^2_{max} - 2 - \frac{\alpha^4_{max}}{\alpha^2_{max}-\log(1+\alpha^2_{max})}\right) \end{bmatrix} \tag{67}$$

and solving for $\mathbf{R}_L$:

$$\mathbf{R}_L = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{4}R_0\alpha_{max} \end{bmatrix} \tag{68}$$

Note that all of these vectors point in the same direction, but have different magnitudes.

Finally, the generalized centers of pressure can be found for a square solar sail comprised of four triangular segments. A square solar sail with billow can be modeled by four quadrants with an oblique cone section [4]. Applying the model in [4] to a square solar sail of 100 m of length and 4% billow the force and moment tensors can be found and are presented in Appendix B. The terms with values less than 1×10^{-14} can be ignored since they are due to numeric errors. For this case, $\mathbf{J}^2$ is again invertible and the solution for $\mathbf{R}_{K^2}$ is found from Eq. (44):

$$\mathbf{R}_{K^2} = \begin{bmatrix} 0 & 8.8537e+001 & 0 \\ -8.8537e+001 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{69}$$

which implies that:

$$\mathbf{R}_{K^2} = \begin{bmatrix} 0 \\ 0 \\ -88.54 \end{bmatrix} \tag{70}$$

$\mathbf{R}_L$ can be found from Eq. (35):

$$\mathbf{R}_L = \begin{bmatrix} 0 \\ 0 \\ -1.7801 \end{bmatrix} \tag{71}$$

The result for $\mathbf{R}_{K^3}$ is found to be:

$$\mathbf{R}_{K^3} = \begin{bmatrix} 0 \\ 0 \\ 89.15 \end{bmatrix} \tag{72}$$

As in the previous case, these vectors point in the same direction with different magnitudes.

III Partial Derivatives of the Force and Moment Equations

Since any change in sail attitude or position are directly related to changes in $\hat{\mathbf{r}}$, the partial derivative equations we define can be used for navigation or control purposes. Also, the partial derivatives help us identify how sensitive the sail is to changes or inaccuracies in parameter estimates, and can be used to improve its design.

The unit position vector can be written in terms of the sun-line angle α and the cone angle δ_f in the sail-fixed frame as:

$$\hat{\mathbf{r}} = \begin{bmatrix} -\cos\delta_f \sin\alpha \\ -\sin\delta_f \sin\alpha \\ -\cos\alpha \end{bmatrix} \tag{73}$$

where δ_f is taken in the positive sense from the x-body-fixed axis as shown in Fig. 1. Many of the partial derivatives will involve knowing the partial derivative of $\hat{\mathbf{r}}$ with respect to itself, which is given by:

$$\frac{\partial \hat{\mathbf{r}}}{\partial \hat{\mathbf{r}}} = \overline{\overline{\mathbf{U}}} - \hat{\mathbf{r}}\hat{\mathbf{r}} = \overline{\overline{\mathbf{U}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} \tag{74}$$

where $\overline{\overline{\mathbf{U}}}$ is the identity dyad. We note that any changes in $\hat{\mathbf{r}}$ will be perpendicular to its direction and cannot be along its direction, hence we see that $\overline{\overline{\mathbf{U}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} \cdot \hat{\mathbf{r}} = 0$. With this definition, the partial derivative of the force with respect to $\hat{\mathbf{r}}$ is given by:

$$\begin{aligned} \frac{\partial \mathbf{F}}{\partial \hat{\mathbf{r}}} &= P(r)\Big[a_2\mathbf{J}^2 \cdot \overline{\overline{\mathbf{U}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} - 2\rho s\hat{\mathbf{r}} \cdot (\mathbf{J}^3 \cdot \overline{\overline{\mathbf{U}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}}) - 2\rho s(\mathbf{J}^3 \cdot \hat{\mathbf{r}}) \cdot \overline{\overline{\mathbf{U}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} \\ &\quad -a_3\hat{\mathbf{r}}(\mathbf{J}^1 \cdot \overline{\overline{\mathbf{U}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}}) - a_3(\mathbf{J}^1 \cdot \hat{\mathbf{r}})\overline{\overline{\mathbf{U}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}}\Big] \cdot \\ &= P(r)\Big[a_2\mathbf{J}^2 - 4\rho s(\hat{\mathbf{r}} \cdot \mathbf{J}^3) - a_3\mathbf{J}^1\hat{\mathbf{r}} - a_3(\mathbf{J}^1 \cdot \hat{\mathbf{r}})\overline{\overline{\mathbf{U}}}\Big] \cdot \overline{\overline{\mathbf{U}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} \end{aligned} \tag{75}$$

Carrying out the same procedure for the moment equation we obtain:

$$\begin{aligned}\frac{\partial \mathbf{M}}{\partial \hat{\mathbf{r}}} &= P(r)\Big[a_2(\tilde{\mathbf{R}}_{K^2}\cdot\mathbf{J}^2)\cdot\overline{\overline{\overline{\mathbf{U}}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} - 2\rho s\hat{\mathbf{r}}\cdot(\tilde{\mathbf{R}}_{K^3}\cdot\mathbf{J}^3)^{T_{13}}\cdot\overline{\overline{\overline{\mathbf{U}}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} \\ &\quad -2\rho s\hat{\mathbf{r}}\cdot(\tilde{\mathbf{R}}_{K^3}\cdot\mathbf{J}^3)\cdot\overline{\overline{\overline{\mathbf{U}}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} - a_3(\hat{\mathbf{r}}\cdot\widetilde{\mathbf{R}_L\mathbf{J}^1})\cdot\overline{\overline{\overline{\mathbf{U}}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} + a_3\tilde{\hat{\mathbf{r}}}\cdot(\mathbf{R}_L\mathbf{J}^1)^T\cdot\overline{\overline{\overline{\mathbf{U}}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}}\Big] \end{aligned} \tag{76}$$

or:

$$\begin{aligned}\frac{\partial \mathbf{M}}{\partial \hat{\mathbf{r}}} &= P(r)\Big[a_2\tilde{\mathbf{R}}_{K^2}\cdot\mathbf{J}^2 - 2\rho s\hat{\mathbf{r}}\cdot(\tilde{\mathbf{R}}_{K^3}\cdot\mathbf{J}^3)^{T_{13}} - 2\rho s\hat{\mathbf{r}}\cdot\tilde{\mathbf{R}}_{K^3}\cdot\mathbf{J}^3 \\ &\quad -a_3(\hat{\mathbf{r}}\cdot\widetilde{\mathbf{R}_L\mathbf{J}^1}) + a_3\tilde{\hat{\mathbf{r}}}\cdot(\mathbf{R}_L\mathbf{J}^1)^T\Big]\cdot\overline{\overline{\overline{\mathbf{U}}}}_{\hat{\mathbf{r}}\hat{\mathbf{r}}} \end{aligned} \tag{77}$$

where the transpose operator T_{13} implies that the first and third indices are transposed from a rank-3 tensor with indices. Now the force partial derivatives with respect to the sun-line angle can be evaluated:

$$\frac{\partial \mathbf{F}}{\partial \alpha} = \frac{\partial \mathbf{F}}{\partial \hat{\mathbf{r}}}\cdot\frac{\partial \hat{\mathbf{r}}}{\partial \alpha} = \frac{\partial \mathbf{F}}{\partial \hat{\mathbf{r}}}\cdot\begin{bmatrix}-\cos\delta_f\cos\alpha \\ -\sin\delta_f\cos\alpha \\ \sin\alpha\end{bmatrix} \tag{78}$$

and for the cone angle:

$$\frac{\partial \mathbf{F}}{\partial \delta_f} = \frac{\partial \mathbf{F}}{\partial \hat{\mathbf{r}}}\cdot\frac{\partial \hat{\mathbf{r}}}{\partial \delta_f} = \frac{\partial \mathbf{F}}{\partial \hat{\mathbf{r}}}\cdot\begin{bmatrix}\sin\delta_f\sin\alpha \\ -\cos\delta_f\sin\alpha \\ 0\end{bmatrix} \tag{79}$$

and similarly for the moment equation.

One way to find the partial derivatives of the force with respect to the force tensors is to write the force using the summation convention as:

$$\mathbf{F}_i = P(r)\Big[a_2\mathbf{J}^2_{ij}\hat{\mathbf{r}}_j - 2\rho s\mathbf{J}^3_{kij}\hat{\mathbf{r}}_k\hat{\mathbf{r}}_j - a_3\mathbf{J}^1_j\hat{\mathbf{r}}_j\hat{\mathbf{r}}_i\Big] \tag{80}$$

Then, the force partial derivatives with respect to the force tensors can be expressed as:

$$\frac{\partial \mathbf{F}_i}{\partial \mathbf{J}^1_k} = -P(r)a_3\hat{\mathbf{r}}_k\hat{\mathbf{r}}_i \tag{81}$$

$$\frac{\partial \mathbf{F}_i}{\partial \mathbf{J}^2_{kl}} = P(r)a_2\delta_{ik}\hat{\mathbf{r}}_l \tag{82}$$

$$\frac{\partial \mathbf{F}_i}{\partial \mathbf{J}^2_{kl}} = -2P(r)\rho s\delta_{il}\hat{\mathbf{r}}_k\hat{\mathbf{r}}_m \tag{83}$$

where δ is the Kronecker delta function. The partial derivatives of the moment with respect to the moment tensors can be found in a similar manner.

The force partial derivative with respect to distance from the sun r is given by:

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial r} &= \Big[a_2\mathbf{J}^2\cdot\hat{\mathbf{r}} - 2\rho s(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} - a_3(\mathbf{J}^1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\Big]\frac{\partial P(r)}{\partial r} \\ &= -\Big[a_2\mathbf{J}^2\cdot\hat{\mathbf{r}} - 2\rho s(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} - a_3(\mathbf{J}^1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\Big]\frac{2\pi I_0R_s^2}{cr^3}\sqrt{1-\frac{R_s^2}{r^2}} \end{aligned} \tag{84}$$

The force partial derivatives with respect to the optical parameters ρ and s are given by:

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial \rho} &= P(r)\Big[\mathbf{J}^2\cdot\hat{\mathbf{r}}\frac{\partial a_2}{\partial \rho} - 2s(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} - (\mathbf{J}^1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\frac{\partial a_3}{\partial \rho}\Big] \\ &= P(r)\Big[\mathbf{J}^2\cdot\hat{\mathbf{r}}(B_f(1-s) - \frac{\epsilon_f B_f - \epsilon_b B_b}{\epsilon_f + \epsilon_b}) - 2s(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} - (\mathbf{J}^1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}s\Big]\end{aligned} \tag{85}$$

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial s} &= P(r)\Big[\mathbf{J}^2\cdot\hat{\mathbf{r}}\frac{\partial a_2}{\partial s} - 2\rho(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} - (\mathbf{J}^1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\frac{\partial a_3}{\partial s}\Big] \\ &= P(r)\Big[-\mathbf{J}^2\cdot\hat{\mathbf{r}}B_f\rho - 2\rho(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} + (\mathbf{J}^1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\rho\Big]\end{aligned} \tag{86}$$

The previous two Eqs. can be linearly dependent for certain special cases; when $\mathbf{J}^2\cdot\hat{\mathbf{r}}$, $(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}}$, and $\hat{\mathbf{r}}$ are parallel, or antiparallel. For a flat sail, this is satisfied when the sun-sail angle is zero since the aforementioned vectors will have components only along the third direction and thus will be linearly dependent. For this to be true for a general case, each of the components of these vectors must be equal to each other. To derive these relationships, $\mathbf{J}^2\cdot\hat{\mathbf{r}}$ and $(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}}$ will be expanded. Expanding $\mathbf{J}^2\cdot\hat{\mathbf{r}}$ yields:

$$\mathbf{J}^2\cdot\hat{\mathbf{r}} = \begin{bmatrix} \mathbf{J}^2_{11}\hat{\mathbf{r}}_1 + \mathbf{J}^2_{12}\hat{\mathbf{r}}_2 + \mathbf{J}^2_{13}\hat{\mathbf{r}}_3 \\ \mathbf{J}^2_{12}\hat{\mathbf{r}}_1 + \mathbf{J}^2_{22}\hat{\mathbf{r}}_2 + \mathbf{J}^2_{23}\hat{\mathbf{r}}_3 \\ \mathbf{J}^2_{13}\hat{\mathbf{r}}_1 + \mathbf{J}^2_{13}\hat{\mathbf{r}}_2 + \mathbf{J}^2_{33}\hat{\mathbf{r}}_3 \end{bmatrix} \tag{87}$$

where the subscripts indicate the element of the corresponding tensor or vector. Performing the same procedure for $(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}}$:

$$(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} = \begin{bmatrix} \mathbf{J}^3_{111}\hat{\mathbf{r}}_1^2 + 2\mathbf{J}^3_{121}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_2 + 2\mathbf{J}^3_{131}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_3 + \mathbf{J}^3_{212}\hat{\mathbf{r}}_2^2 + 2\mathbf{J}^3_{231}\hat{\mathbf{r}}_2\hat{\mathbf{r}}_3 + \mathbf{J}^3_{331}\hat{\mathbf{r}}_3^2 \\ \mathbf{J}^3_{121}\hat{\mathbf{r}}_1^2 + 2\mathbf{J}^3_{221}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_2 + 2\mathbf{J}^3_{231}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_3 + \mathbf{J}^3_{222}\hat{\mathbf{r}}_2^2 + 2\mathbf{J}^3_{232}\hat{\mathbf{r}}_2\hat{\mathbf{r}}_3 + \mathbf{J}^3_{332}\hat{\mathbf{r}}_3^2 \\ \mathbf{J}^3_{131}\hat{\mathbf{r}}_1^2 + 2\mathbf{J}^3_{231}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_2 + 2\mathbf{J}^3_{331}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_3 + \mathbf{J}^3_{232}\hat{\mathbf{r}}_2^2 + 2\mathbf{J}^3_{332}\hat{\mathbf{r}}_2\hat{\mathbf{r}}_3 + \mathbf{J}^3_{333}\hat{\mathbf{r}}_3^2 \end{bmatrix} \tag{88}$$

finally we are left with three equations that must be satisfied for the force partial derivatives with respect to s and ρ to be linearly dependent:

$$\begin{aligned}\mathbf{J}^2_{11}\hat{\mathbf{r}}_1 + \mathbf{J}^2_{12}\hat{\mathbf{r}}_2 + \mathbf{J}^2_{13}\hat{\mathbf{r}}_3 = \mathbf{J}^3_{111}\hat{\mathbf{r}}_1^2 + 2\mathbf{J}^3_{121}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_2 + 2\mathbf{J}^3_{131}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_3 + \mathbf{J}^3_{212}\hat{\mathbf{r}}_2^2 \\ +2\mathbf{J}^3_{231}\hat{\mathbf{r}}_2\hat{\mathbf{r}}_3 + \mathbf{J}^3_{331}\hat{\mathbf{r}}_3^2 = (\mathbf{J}^1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}_1\end{aligned} \tag{89}$$

$$\begin{aligned}\mathbf{J}^2_{12}\hat{\mathbf{r}}_{01} + \mathbf{J}^2_{22}\hat{\mathbf{r}}_2 + \mathbf{J}^2_{23}\hat{\mathbf{r}}_3 = \mathbf{J}^3_{121}\hat{\mathbf{r}}_1^2 + 2\mathbf{J}^3_{221}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_2 + 2\mathbf{J}^3_{231}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_3 \\ +\mathbf{J}^3_{222}\hat{\mathbf{r}}_2^2 + 2\mathbf{J}^3_{232}\hat{\mathbf{r}}_2\hat{\mathbf{r}}_3 + \mathbf{J}^3_{332}\hat{\mathbf{r}}_3^2 = (\mathbf{J}^1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}_2\end{aligned} \tag{90}$$

$$\begin{aligned}\mathbf{J}^2_{13}\hat{\mathbf{r}}_1 + \mathbf{J}^2_{13}\hat{\mathbf{r}}_2 + \mathbf{J}^2_{33}\hat{\mathbf{r}}_3 = \mathbf{J}^3_{131}\hat{\mathbf{r}}_1^2 + 2\mathbf{J}^3_{231}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_2 + 2\mathbf{J}^3_{331}\hat{\mathbf{r}}_1\hat{\mathbf{r}}_3 \\ +\mathbf{J}^3_{232}\hat{\mathbf{r}}_2^2 + 2\mathbf{J}^3_{332}\hat{\mathbf{r}}_2\hat{\mathbf{r}}_3 + \mathbf{J}^3_{333}\hat{\mathbf{r}}_3^2 = (\mathbf{J}^1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}_3\end{aligned} \tag{91}$$

we can say that whenever these requirements are met, the partial derivatives of the force with respect to ρ and s are linearly dependent since the vectors $\mathbf{J}^2\cdot\hat{\mathbf{r}}$, $(\mathbf{J}^3\cdot\hat{\mathbf{r}})\cdot\hat{\mathbf{r}}$, and $\hat{\mathbf{r}}$ will point in the same direction.

The force partial derivatives with respect to the rest of the optical parameters B_f, B_b, ϵ_f, and ϵ_b are:

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial B_f} &= P(r)\Big[\mathbf{J}^2\cdot\hat{\mathbf{r}}\frac{\partial a_2}{\partial B_f}\Big] \\ &= P(r)\Big[\mathbf{J}^2\cdot\hat{\mathbf{r}}\Big((1-s)\rho + (1-\rho)\frac{\epsilon_f}{\epsilon_f+\epsilon_b}\Big)\Big]\end{aligned} \tag{92}$$

The force partial derivative with respect to B_b, ϵ_f, and ϵ_b are given by:

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial B_b} &= P(r)\Big[\mathbf{J}^2 \cdot \hat{\mathbf{r}}\frac{\partial a_2}{\partial B_b}\Big] \\ &= -P(r)\Big[\mathbf{J}^2 \cdot \hat{\mathbf{r}}(1-\rho)\frac{\epsilon_b}{\epsilon_f+\epsilon_b}\Big]\end{aligned} \tag{93}$$

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial \epsilon_f} &= P(r)\Big[\mathbf{J}^2 \cdot \hat{\mathbf{r}}\frac{\partial a_2}{\partial \epsilon_f}\Big] \\ &= P(r)\Big[\mathbf{J}^2 \cdot \hat{\mathbf{r}}(1-\rho)\frac{B_f}{\epsilon_f+\epsilon_b} - \mathbf{J}^2 \cdot \hat{\mathbf{r}}(1-\rho)\frac{\epsilon_f B_f - \epsilon_b B_b}{(\epsilon_f+\epsilon_b)^2}\Big]\end{aligned} \tag{94}$$

$$\begin{aligned}\frac{\partial \mathbf{F}}{\partial \epsilon_f} &= P(r)\Big[\mathbf{J}^2 \cdot \hat{\mathbf{r}}\frac{\partial a_2}{\partial \epsilon_b}\Big] \\ &= P(r)\Big[-\mathbf{J}^2 \cdot \hat{\mathbf{r}}(1-\rho)\frac{B_b}{\epsilon_f+\epsilon_b} - \mathbf{J}^2 \cdot \hat{\mathbf{r}}(1-\rho)\frac{\epsilon_f B_f - \epsilon_b B_b}{(\epsilon_f+\epsilon_b)^2}\Big]\end{aligned} \tag{95}$$

Note that the last four equations are linearly dependent and can be expressed in terms of a known partial derivative. For instance, taking the partial of the force with respect to B_f as our basis, the other partial derivatives can be expressed as:

$$\frac{\partial \mathbf{F}}{\partial B_f} = c_1 \frac{\partial \mathbf{F}}{\partial B_b} \tag{96}$$

$$\frac{\partial \mathbf{F}}{\partial B_f} = c_2 \frac{\partial \mathbf{F}}{\partial B_b} \tag{97}$$

$$\frac{\partial \mathbf{F}}{\partial \epsilon_f} = c_3 \frac{\partial \mathbf{F}}{\partial B_b} \tag{98}$$

where

$$c_1 = -\frac{1-\rho}{(1-s)\rho + \frac{\epsilon_f}{\epsilon_b}(1-s\rho)} \tag{99}$$

$$c_2 = -\frac{B_b+B_f}{\epsilon_f+\epsilon_b}\frac{1-\rho}{(1-s)\rho + \frac{\epsilon_f}{\epsilon_b}(1-s\rho)} \tag{100}$$

$$c_3 = \frac{B_b+B_f}{\epsilon_f+\epsilon_b}\frac{1-\rho}{\frac{\epsilon_f}{\epsilon_b}(1-s)\rho + (1-s\rho)} \tag{101}$$

The partial derivatives of the moment with respect to these previous parameter can be obtained by replacing $\frac{\partial \mathbf{F}}{\partial \hat{\mathbf{r}}}$ by $\frac{\partial \mathbf{M}}{\partial \hat{\mathbf{r}}}$ where they appear.

IV Locally Optimal Control Laws

One advantage to having closed-form equations for the force acting on a solar sail is that we are able to easily define explicit control and guidance laws. In the following examples we take the advantage of this to implement guidance and orientation laws for a non-ideal sail.

A Maximum Energy Increase

In [6], a guidance law, using the sun-sail angle as the controller, was developed to find optimum escape trajectories from the sun using flat, ideal sails. In this section we extend the locally optimal control law developed in [6] to a four-quadrant non-ideal billowed solar sail. To accomplish this, the Gauss variational equation relating the semi-major axis change with respect to the true anomaly can be written as:

$$\frac{\partial \mathbf{a}}{\partial f} = \frac{2pr^2}{\mu(1-e^2)^2} \begin{bmatrix} e\sin f \\ 0 \\ 1+e\cos f \end{bmatrix} \cdot \mathbf{F}_p \tag{102}$$

where μ is the sun's gravitational parameter, e is the orbit eccentricity, f is the true anomaly, and $\mathbf{F}_p$ is the force expressed in local polar coordinates. A coordinate transformation T is needed to obtain $\mathbf{F}_p$, as $\mathbf{F}_p = T \cdot \mathbf{F}$. We only consider changes in the sail position and attitude in the orbit plane (i.e., set $\delta_f \equiv 0$). Then T would be given by:

$$T = \begin{bmatrix} -\sin\alpha & 0 & -\cos\alpha \\ 0 & -1 & 0 \\ \cos\alpha & 0 & -\sin\alpha \end{bmatrix} \tag{103}$$

where $\mathbf{F}_p$ has radial F_r, out of plane $F_{r\times\theta}$, and transverse F_θ force components. Taking the partial derivative of Eq. (104) yields:

$$\frac{\partial}{\partial \alpha}\Big(\frac{\partial \mathbf{a}}{\partial f}\Big) = \frac{2pr^2}{\mu(1-e^2)^2} \begin{bmatrix} e\sin f \\ 0 \\ 1+e\cos f \end{bmatrix} \cdot \Big(\frac{\partial T}{\partial \alpha} \cdot \mathbf{F} + T \cdot \frac{\partial \mathbf{F}}{\partial \alpha}\Big) \tag{104}$$

The partial of T with respect to α is readily obtained from Eq. (103) and the partial of $\mathbf{F}$ with respect to α is given by Eq. (78). Setting the above equation equal to zero for the square-billowed sail model coefficients found in Appendix B, the optimal angle satisfies the relation:

$$\begin{aligned} 0 \;=\;& 1.9201(1+e\cos f) - 5.7604e\sin f\tan\alpha - 3.7804(1+e\cos f)\tan^2\alpha \\ & -0.0599e\sin f\tan^3\alpha \end{aligned} \tag{105}$$

The solution of Eq. (105) is chosen so that Eq. (102) is maximized. The control law equation is close to the solution for an ideal flat solar sail. The force acting on an ideal flat sail in the local polar frame is given by:

$$\mathbf{F}_p = 2P(r)A \begin{bmatrix} \cos^3\alpha \\ 0 \\ \sin\alpha\cos^2\alpha \end{bmatrix} \tag{106}$$

With this information the equation for the optimum angle is obtained from:

$$0 = 2(1+e\cos f) - 6e\sin f\tan\alpha - 4(1+e\cos f)\tan^2\alpha \tag{107}$$

The solution of the above equation is obtained by solving the quadratic equation for $\tan\alpha$ and is presented in [6]. Both the ideal and the optimum control laws can be compared now on the squared-billowed sail model. Fig. 2 is polar plot of the orbit change using both guidance laws starting at 1 au during a time of one year. The optimum control law has a faster energy increase, as expected. The orbital energy increase for both guidance laws is shown in Fig. 3, which clearly shows that the non-ideal guidance law is optimum. The values for the optical parameters used in the simulation were $\rho = 0.9$, $s = 1$, $B_f = 0.8$, $B_b = 0.5$, $\epsilon_f = 0.05$, $\epsilon_b = 0.3$ and the mass chosen was $80kg$.

B Maximum Propulsive Force

As another example, we can compute the planar orientation that gives the maximum propulsive force on the sail. This can be found by differentiating the square of the force magnitude with respect to the sun-sail angle and setting the resulting expression equal to zero:

$$\frac{\partial(\mathbf{F}\cdot\mathbf{F})}{\partial\alpha} = 2\mathbf{F}\cdot\frac{\partial\mathbf{F}}{\partial\alpha} = 0 \tag{108}$$

The corresponding equation for the four-quadrant sail example is:

$$\Big(-3.3735 - 3.3040\cos 2\alpha\Big)\cos\alpha\sin\alpha = 0 \tag{109}$$

which has its extrema when α equals 0, $\pi/2$, π, and $3\pi/2$. The term in the parenthesis is zero for α complex. These solutions are the same as that of an ideal solar sail. When $\alpha = 0$, the maximum force is achieved, and with α equal to $\pi/2$ or $3\pi/2$, the force is zero. Finally, the solution $\alpha = \pi$ implies that the sail is facing away from the sun, an orientation we do not consider. This simple result occurs due to the overall symmetry of the four quadrant sail.

Now we consider a more general situation. Let us find the sun-sail angle that provides the maximum force for only one quadrant of the square-billowed sail whose force tensors coefficients are given in Appendix B. Following the same procedure as for the complete sail, the equation that needs to be satisfied is now:

$$-83.806 - 759.262\tan\alpha + 248.428\tan^2\alpha - 27.134\tan^3\alpha + \tan^4\alpha = 0 \tag{110}$$

This equation has only two real solutions, -6.096^0 and 83.913^0. The first solution maximizes the force on the sail; the sign is negative due to the quadrant position with respect to the overall sail (the opposite quadrant would have the reverse sign on the solutions). The second solution minimizes the force on the sail. These examples showcase the ease with which we can work with complex sail shapes using the generalized sail model.

V Conclusions

In this paper the Generalized Sail Model was studied in more detail. Conditions are found to determine when the force tensor coefficients are guaranteed to be zero or non-zero. The concept of generalized centers of pressure is defined, which allows us reduce the coefficients needed to characterized the moment from 36 to 9. These 9 coefficients are distributed into three three-dimensional vectors, which, together with the force tensors, characterize the moment acting on the sail.

The equations for finding the generalized centers of pressure are non-standard linear equations and ideas on how to solve for them are developed. Using these ideas, the generalized centers of pressure for a flat sail, billowed circular sail, and four-quadrant billowed sail are computed.

The partial derivatives of the force and moment equations are computed with respect to each parameter affecting the force and moment. Some of the force partial derivatives such as the partial with respect to B_f, B_b, ϵ_f, and ϵ_b were found to be linearly dependent. As an application of the analytic partial derivatives, a guidance law was developed to maximally increase the orbit energy for a four-quadrant non-ideal sail. This guidance law was contrasted to the guidance law established for a flat sail by applying both laws to the four-quadrant non-ideal sail; the non-ideal guidance law had the better performance. It was also shown that the Generalized Sail Model is capable of handling complex sail models by finding the angle α that generated the most thrust for the four-quadrant sail and for a single quadrant of the same model. The four-quadrant sail had the same solutions as a flat sail with $\alpha = 0$ being the angle that generated the most thrust. The single-quadrant model had an optimum sun-sail line angle equal to -6.096^0.

Appendix A

Tensor and Vector Notation

The force surface normal distribution integrals are defined as the integral of the outer product of the normal vector:

$$\mathbf{J}^{m}_{i_1 i_2 \ldots i_m} = \int_A \hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_m}\,dA \quad (111)$$

$$i_j = 1,2,3 \quad (112)$$

where the entries $\hat{n}_i$ are the elements of the normal vector evaluated at the surface element dA. The moment surface normal distribution integrals contain an additional multiplier, which contains the information of the area element moment arm with respect to the origin, and expressed are as:

$$\mathbf{K}^{m}_{i_1 i_2 \ldots i_m} = \int_A \tilde{\varrho}\cdot\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_m}\,dA \quad (113)$$

$$\mathbf{L}_{i_1 i_2} = \int_A \hat{n}_{i_1}\vec{\varrho}_{i_2}\,dA \quad (114)$$

$$i_j = 1,2,3 \quad (115)$$

where $\vec{\varrho}$ is the vector from the sail coordinate origin to the the center of pressure of the differential element dA, and tilde over a vector implies a transformation from a given three-dimensional in vector $\mathbf{V}$ into a square skew-symmetric matrix $\tilde{\mathbf{V}}$ specified as:

$$\tilde{\mathbf{V}} = \begin{bmatrix} 0 & -V_3 & V_2 \\ V_3 & 0 & -V_1 \\ -V_2 & V_1 & 0 \end{bmatrix} \quad (116)$$

where the index inside the matrix denotes the element of the vector $\mathbf{V}$.

The products of the force and moment tensors and the sun's position unit vector can be defined using the summation convention:

$$\hat{\mathbf{r}}\cdot\mathbf{T}^3\cdot\hat{\mathbf{r}} = \mathbf{T}^3_{ijk}\hat{\mathbf{r}}_j\hat{\mathbf{r}}_k$$
$$\hat{\mathbf{r}}\cdot\mathbf{T}^2 = \mathbf{T}^2_{ij}\hat{\mathbf{r}}_i$$
$$\mathbf{T}^2\cdot\hat{\mathbf{r}} = \mathbf{T}^2_{ij}\hat{\mathbf{r}}_j$$
$$\hat{\mathbf{r}}\cdot\mathbf{T}^1 = \mathbf{T}^1_{i}\hat{\mathbf{r}}_i$$

where the superscript determines the tensor rank and equal indices imply summation, i.e., $a_i b_i = \sum_{i=1}^{3} a_i b_i$. The force and moment tensors follow these conditions. In the same manner, the product of a rank-2 and rank-3 tensor can be stated as:

$$\mathbf{T}^2\cdot\mathbf{T}^3 = \mathbf{T}^2_{il}\mathbf{T}^3_{ljk}$$

Appendix B

Force and Moment Tensors for Common Sail Geometries

In this section force and tensor coefficients are presented for a number of sail geometries. These results have been derived in [4], and are given here for completeness.

Flat Sail. The most basic shape is the flat sail. The force coefficients of a flat sail about the geometric center are:

$$\mathbf{J}^1 = \begin{bmatrix} 0 \\ 0 \\ A \end{bmatrix} \tag{117}$$

$$\mathbf{J}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{bmatrix} \tag{118}$$

$$\mathbf{J}^3_{ij2} = \mathbf{J}^3_{ij1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{119}$$

$$\mathbf{J}^3_{ij3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{bmatrix} \tag{120}$$

where A is the sail area. The moment tensors, taken at the geometric center, are identically equal to zero. Now, if we assume a square sail with the reference point taken at one of the sail corners, let's say the lower-left corner so that the sail is in the first quadrant of the $x - y$ set of axis, then the force tensors will have the same values as above, however, the moment tensors will be:

$$\mathbf{K}^2 = \frac{l^3}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{121}$$

$$\mathbf{K}^3_{ij1} = \mathbf{K}^3_{ij2} = \mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{122}$$

$$\mathbf{K}^3_{ij3} = \frac{l^3}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{123}$$

where l is the length the sail's side.

Billowed Circular Sail. Next we define a billowed circular sail with a slope varying linearly from its edge and zero at its center as shown in Fig. 4. Its surface is given by the equation:

$$z_b = -\frac{\alpha_{max}}{2R_0} r_s^2 + \frac{\alpha_{max} R_0}{2} \tag{124}$$

where α_{max} is the slope at the outer rim and is negative for a concave shape, R_0 is the sail radius, and r_s is the radial polar coordinate. Then computing the force coefficients we obtain:

$$\mathbf{J}^1 = \begin{bmatrix} 0 \\ 0 \\ \pi R_0^2 \end{bmatrix} \tag{125}$$

$$\mathbf{J}^2 = \frac{\pi R_0^2}{\alpha_{max}^2}\begin{bmatrix} \frac{2+(-2+\alpha_{max})\sqrt{1+\alpha_{max}^2}}{3} & 0 & 0 \\ 0 & \frac{2+(-2+\alpha_{max})\sqrt{1+\alpha_{max}^2}}{3} & 0 \\ 0 & 0 & 2\Big(-1+\sqrt{1+\alpha_{max}^2}\Big) \end{bmatrix} \tag{126}$$

$$\mathbf{J}_{ij1}^3 = \frac{\pi R_0^2}{2\alpha_{max}^2}\Big(\alpha_{max}^2 - \log(1+\alpha_{max}^2)\Big)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{127}$$

$$\mathbf{J}_{ij2}^3 = \frac{\pi R_0^2}{2\alpha_{max}^2}\Big(\alpha_{max}^2 - \log(1+\alpha_{max}^2)\Big)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{128}$$

$$\mathbf{J}_{ij3}^3 = \frac{\pi R_0^2}{\alpha_{max}^2}\begin{bmatrix} \frac{\alpha_{max}^2-\log(1+\alpha_{max}^2)}{2} & 0 & 0 \\ 0 & \frac{\alpha_{max}^2-\log(1+\alpha_{max}^2)}{2} & 0 \\ 0 & 0 & \log(1+\alpha_{max}^2) \end{bmatrix} \tag{129}$$

and the moment coefficients with respect to the origin are:

$$\mathbf{L} = \frac{1}{4}\pi R_0^3\alpha_{max}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{130}$$

$$\mathbf{K}^2 = \frac{\pi R_0^3\Big(6-5\alpha_{max}^2-\sqrt{1+\alpha_{max}^2}(6-8\alpha_{max}^2+\alpha_{max}^4)\Big)}{15\alpha_{max}^3}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{131}$$

$$\mathbf{K}_{ij1}^3 = -\frac{\pi R_0^3}{8\alpha_{max}}\Big(\alpha_{max}^2(-2+\alpha_{max}^2)-2(-1+\alpha_{max}^2)\log(1+\alpha_{max}^2)\Big)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \tag{132}$$

$$\mathbf{K}_{ij2}^3 = -\frac{\pi R_0^3}{8\alpha_{max}}\Big(\alpha_{max}^2(-2+\alpha_{max}^2)-2(-1+\alpha_{max}^2)\log(1+\alpha_{max}^2)\Big)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{133}$$

$$\mathbf{K}_{ij3}^3 = -\frac{\pi R_0^3}{8\alpha_{max}}\Big(\alpha_{max}^2(-2+\alpha_{max}^2)-2(-1+\alpha_{max}^2)\log(1+\alpha_{max}^2)\Big)\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{134}$$

Four-Quadrant Sail. For an increase in the complexity of a sail, we can define a square solar sail with billow depicted in Fig. 5 . The sail will be composed of four quadrants each being a section of an oblique cone. The force and moment tensors need to be computed for a single quadrant and the complete geometry will be obtained by rotating the results of the single quadrant through a sequence of four angles and adding the results. The results for a single quadrant were generated numerically for a square sail with sides l of $100m$ and a maximum billow of 4% of l; the results are as follows:

$$\mathbf{J}^{1*} \quad = \quad \begin{bmatrix} -2.6701e+002 \\ 1.5632e-013 \\ 2.5000e+003 \end{bmatrix} \tag{135}$$

$$\mathbf{J}^{2*} = \begin{bmatrix} 2.9579e+001 & -6.3949e-014 & -2.6394e+002 \\ -6.3949e-014 & 2.1050e+001 & 2.4158e-013 \\ -2.6394e+002 & 2.4158e-013 & 2.4749e+003 \end{bmatrix} \tag{136}$$

$$\mathbf{J}^{3*}_{ij1} = \begin{bmatrix} -3.4320e+000 & 9.5479e-015 & 2.9196e+001 \\ 9.5479e-015 & -2.6571e+000 & -7.1054e-014 \\ 2.9196e+001 & -7.1054e-014 & -2.6092e+002 \end{bmatrix} \tag{137}$$

$$\mathbf{J}^{3*}_{ij2} = \begin{bmatrix} 9.5479e-015 & -2.6571e+000 & -7.1054e-014 \\ -2.6571e+000 & 7.3275e-015 & 2.0719e+001 \\ -7.1054e-014 & 2.0719e+001 & 5.9686e-013 \end{bmatrix} \tag{138}$$

$$\mathbf{J}^{3*}_{ij3} = \begin{bmatrix} 2.9196e+001 & -7.1054e-014 & -2.6092e+002 \\ -7.1054e-014 & 2.0719e+001 & 5.9686e-013 \\ -2.6092e+002 & 5.9686e-013 & 2.4501e+003 \end{bmatrix} \tag{139}$$

$$\mathbf{K}^{2*} = \begin{bmatrix} 1.1156e-011 & -4.3574e+003 & -7.2198e-011 \\ -8.8400e+003 & 1.1433e-011 & 8.2920e+004 \\ 2.9261e-012 & -1.2636e+003 & -1.6097e-011 \end{bmatrix} \tag{140}$$

$$\mathbf{K}^{3*}_{ij1} = \begin{bmatrix} -1.7564e-012 & 5.5001e+002 & 1.0612e-011 \\ 9.7746e+002 & -1.9098e-012 & -8.7389e+003 \\ -5.4279e-013 & 1.6134e+002 & 2.5815e-012 \end{bmatrix} \tag{141}$$

$$\mathbf{K}^{3*}_{ij2} = \begin{bmatrix} 5.5001e+002 & -1.4786e-012 & -4.2888e+003 \\ -1.8259e-012 & 6.9317e+002 & 8.8951e-012 \\ 1.6134e+002 & -5.5911e-013 & -1.2432e+003 \end{bmatrix} \tag{142}$$

$$\mathbf{K}^{3*}_{ij3} = \begin{bmatrix} 1.0713e-011 & -4.2888e+003 & -4.9958e-011 \\ -8.7389e+003 & 8.6153e-012 & 8.2090e+004 \\ 2.4487e-012 & -1.2432e+003 & -1.6893e-011 \end{bmatrix} \tag{143}$$

$$\mathbf{L}^{*} = \begin{bmatrix} 8.9003e+003 & 9.5577e-012 & 4.2745e+002 \\ -7.6721e-012 & -4.4501e+003 & 2.3528e-014 \\ -8.3333e+004 & -7.0465e-011 & -4.4501e+003 \end{bmatrix} \tag{144}$$

The tensors for the complete square sail are:

$$\mathbf{J}^{1} \quad = \quad \begin{bmatrix} 5.6843e-014 \\ 0 \\ 1.0000e+004 \end{bmatrix} \tag{145}$$

$$\mathbf{J}^2 = \begin{bmatrix} 1.0126e+002 & -1.7764e-015 & 5.6843e-014 \\ -1.7764e-015 & 1.0126e+002 & -5.6843e-014 \\ 5.6843e-014 & -5.6843e-014 & 9.8995e+003 \end{bmatrix} \tag{146}$$

$$\mathbf{J}^3_{ij1} = \begin{bmatrix} 0 & 0 & 9.9829e+001 \\ 0 & 1.1102e-015 & -1.7764e-015 \\ 9.9829e+001 & -1.7764e-015 & 5.6843e-014 \end{bmatrix} \tag{147}$$

$$\mathbf{J}^3_{ij2} = \begin{bmatrix} -2.2204e-016 & 1.2212e-015 & -1.7764e-015 \\ 9.9920e-016 & -8.8818e-016 & 9.9829e+001 \\ -1.7764e-015 & 9.9829e+001 & 0 \end{bmatrix} \tag{148}$$

$$\mathbf{J}^3_{ij3} = \begin{bmatrix} 9.9829e+001 & -1.7764e-015 & 5.6843e-014 \\ -1.7764e-015 & 9.9829e+001 & 0 \\ 5.6843e-014 & 0 & 9.8003e+003 \end{bmatrix} \tag{149}$$

$$\mathbf{K}^2 = \begin{bmatrix} 4.1837e-011 & 8.9651e+003 & 0 \\ -8.9651e+003 & 4.5475e-011 & -1.4552e-011 \\ 0 & 3.4106e-013 & -6.4389e-011 \end{bmatrix} \tag{150}$$

$$\mathbf{K}^3_{ij1} = \begin{bmatrix} -4.5475e-013 & -1.9895e-013 & 3.9108e-011 \\ -2.8422e-013 & -2.8422e-014 & -8.9003e+003 \\ -2.1600e-012 & -2.8422e-014 & 1.1369e-013 \end{bmatrix} \tag{151}$$

$$\mathbf{K}^3_{ij2} = \begin{bmatrix} -1.9895e-013 & -1.1369e-013 & 8.9003e+003 \\ -2.8422e-014 & -4.5475e-013 & 4.0927e-011 \\ -2.8422e-014 & -2.2169e-012 & 2.2737e-013 \end{bmatrix} \tag{152}$$

$$\mathbf{K}^3_{ij3} = \begin{bmatrix} 3.9108e-011 & 8.9003e+003 & 0 \\ -8.9003e+003 & 4.0927e-011 & -2.1828e-011 \\ 0 & 2.2737e-013 & -6.7573e-011 \end{bmatrix} \tag{153}$$

$$\mathbf{L} = \begin{bmatrix} 8.9003e+003 & 3.0923e-011 & -1.1369e-013 \\ -3.4561e-011 & 8.9003e+003 & 1.1369e-013 \\ 2.1828e-011 & -2.1828e-011 & -1.7801e+004 \end{bmatrix} \tag{154}$$

Acknowledgments

Leonel Rios-Reyes and Daniel J. Scheeres acknowledge the support for this research from the Jet Propulsion Laboratory/California Institute of Technology, which is under contract with the National Aeronautics and Space Administration.

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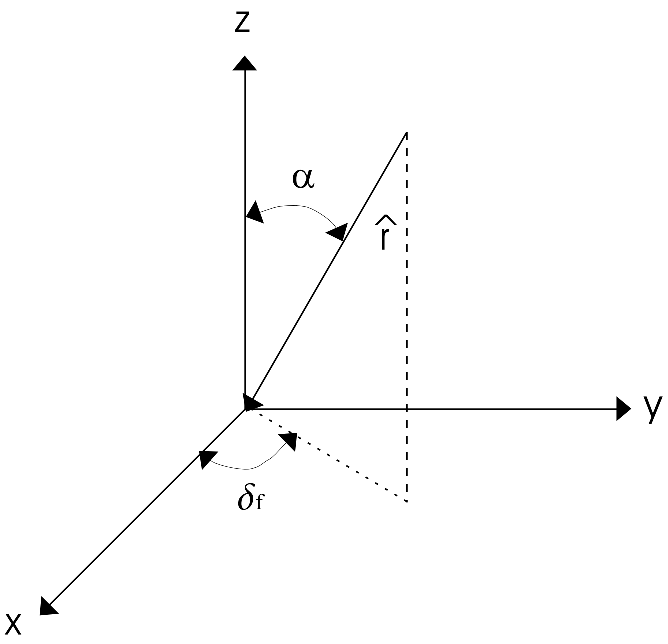

Figure 1: Attitude angles and $\hat{\mathbf{r}}$ in sail body-fixed frame.

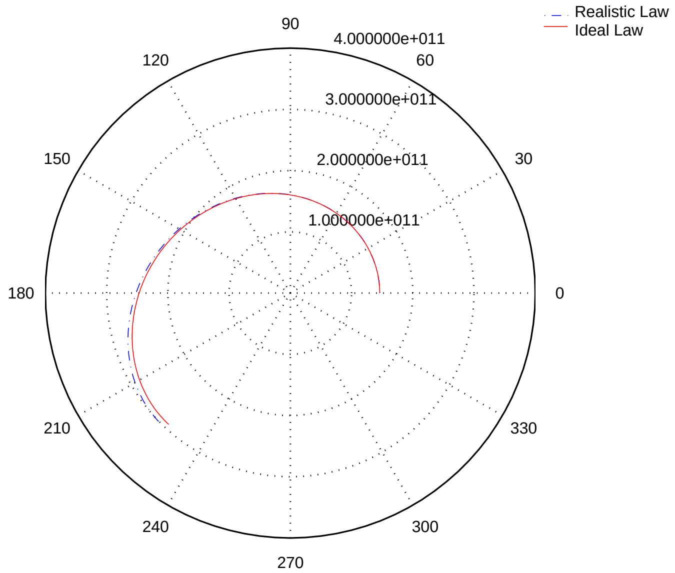

Figure 2: Trajectories for realistic and ideal guidance laws.

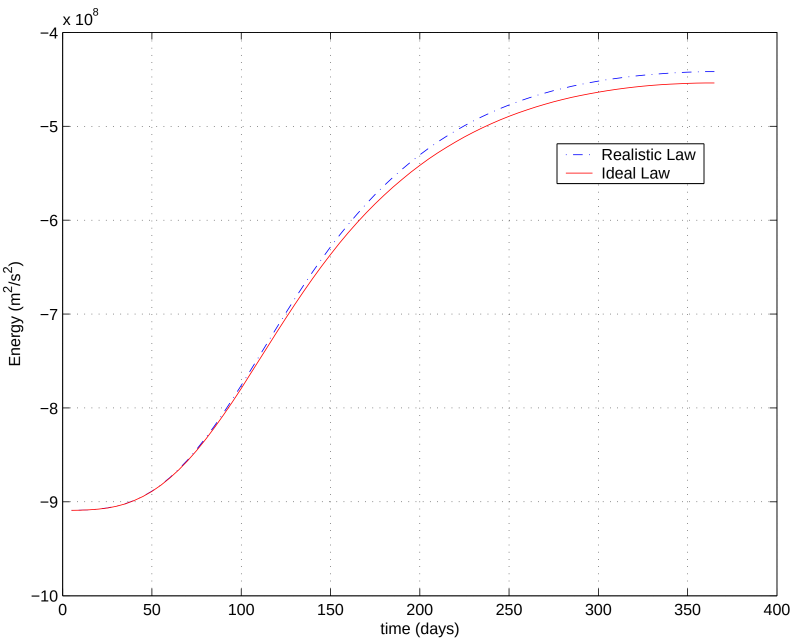

Figure 3: Energy increase for realistic and ideal guidance laws.

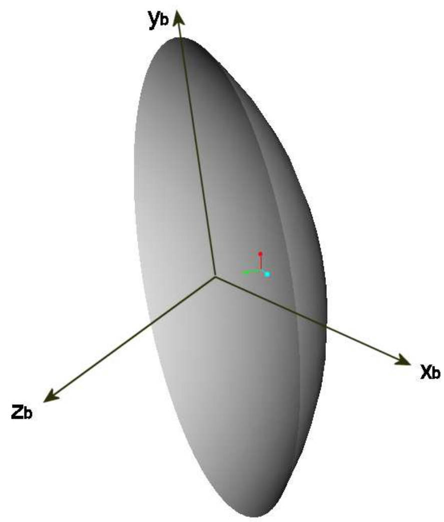

Figure 4: Circular Sail Geometry.

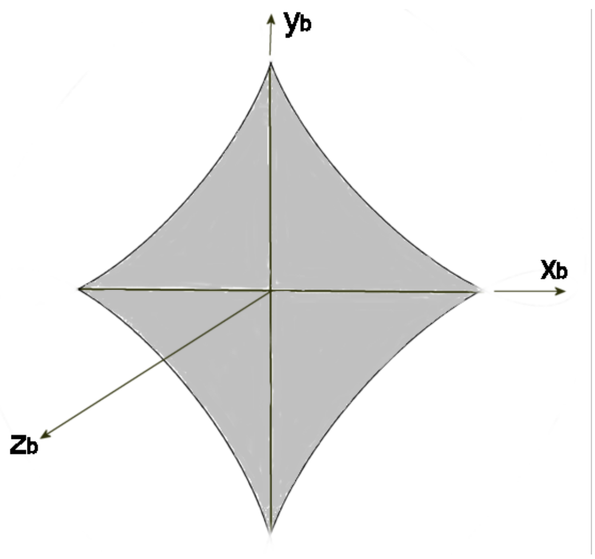

Figure 5: Four-quadrant Sail.