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A First Order Analytical Solution for Spacecraft Motion about (433) Eros

J.F. San-Juan,* A. Abad,† D.J. Scheeres,‡ and M. Lara§

The orbital motion of spacecraft about asteroids is highly perturbed, and classical theories for motion close to spheroidal bodies cannot be applied. In particular, this is the case for motion about (433) Eros: its large ellipticity coefficient (having the same order as the oblateness coefficient) and its fast rotation rate dominate the dynamics. In this paper we obtain a first order theory for the motion of a satellite around Eros by means of two Lie transformations. The first one is a simplification of the Hamiltonian expressed in polar-nodal variables by using a new technique, the algorithm of relegation. The second one is the classical Delaunay normalization. After both transformations we replace the actual nonintegrable Hamiltonian by an integrable approximation to it.

Introduction

THE nature of orbital motion about asteroids is highly nonlinear. Large ellipticity coefficients (having the same order as the oblateness coefficients) and fast rotation rates can produce chaotic motion in the vicinity of these celestial bodies that is difficult to understand.

Numerical approaches have been used to understand the dynamics of the problem. Thus, a global criterion for the stability of motion is given in,[20] where numerically determined periodic orbits are used to explore the stability of three dimensional trajectories around asteroids. It is found there that families of three dimensional periodic orbits change their stability type at certain critical inclinations enabling the drawing of a line —relating orbital inclination and mean radius of the orbiter— that separates regions of stable motion from unstable ones. The transitions to instability are associated with a 5% fluctuation in energy over each orbit.

Contrary to numerical solutions, the determination of action-angle variables reflecting the actual dynamics of a chaotic system enables the replacement of the (actual) nonintegrable Hamiltonian by an integrable approximation that is designed to give good agreement with the real dynamics. Classical theories for motion close to spheroidal bodies assume that the Keplerian attraction clearly prevails over other forces. That is not necessarily the case for highly perturbed dynamical systems, and therefore classical theories are not valid for these systems.

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In this paper we study the dynamics about the asteroid (433) Eros, and obtain a first order analytical theory of an spacecraft about the asteroid. We restrict our study to the second degree and order gravity field, and assume that Eros is in uniform rotation around its axis of greatest inertia. This simplified model includes all the main perturbations that act on an orbiter in this system,[24,26] namely the Keplerian attraction, the Coriolis force, the oblateness, and the ellipticity perturbations. By computing the relative influence of a wide range of initial conditions we conclude that the addition of the Keplerian plus the Coriolis term dominates the dynamics, while the oblateness and the ellipticity perturbations, both of the same order, remain at a higher order.

Our analytical theory is found by averaging. First, we determine action-angle variables suitable to average the Hamiltonian over one of the fast angle variables. Then, the averaging is done by the expedient finding of generating functions by means of Lie transformations. After the contact transformation is computed the new Hamiltonian (in new canonical variables) appears as an explicit series depending on a small parameter. In the process of computing the canonical transformations, the influence of the angle variables is put off to higher orders of the small parameter. To this end we select the oblateness (J_2) coefficient as a small parameter, and use Deprit's method[10] for constructing the Lie transformations.

The usual technique when implementing closed form analytical theories, the Delaunay normalization[13] —that converts the principal part of the Hamiltonian into an integral of the transformed system— cannot be directly applied to our problem. Since the argument of the node is present in the Hamiltonian through the ellipticity perturbation, the Coriolis term adds a term to the Lie derivative that prevents the computation in the usual way of the generator of the Lie transformation. To overcome this inconvenience we first perform a simplification of the Hamiltonian making use of the

Deprit's relegation algorithm[15] (see also[22,27]). This procedure uses repeated iterations of a transformation after which the desired perturbation (the ellipticity perturbation in our case) appears in the new Hamiltonian with a lower influence. When the perturbation is small enough we may neglect it.

After the elimination of the ellipticity term the Hamiltonian becomes equal to the Hamiltonian of the main problem of the artificial satellite, in which the longitude of the node is cyclic and, hence, the Coriolis term becomes constant and may be deleted. Then a Delaunay normalization may be performed transforming the Hamiltonian into an integrable one. The Delaunay normalization is made in closed form, without using series expansion in the eccentricity.

All these operations have been made symbolically by using the Poisson series processor PSPC[1] included in the software ATESAT.[2,3,23]

Dynamical model

In order to formulate the motion of a satellite around the asteroid Eros, let us consider the asteroid as a solid rotating around the z-axis with constant velocity ω, and let us take up to the second order in the potential expansion. The satellite motion will be referred to a rotating frame with origin at the center of mass of Eros, and whose axes coincide with its principal axes of inertia defined by the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.

Under the previous assumptions, the Hamiltonian defining the motion is

$$\mathcal{H} = \tfrac{1}{2}(\boldsymbol{X}\cdot\boldsymbol{X}) - \boldsymbol{\omega}\cdot(\boldsymbol{x}\times\boldsymbol{X}) + \mathcal{V}(\boldsymbol{x}), \qquad (1)$$

where $\boldsymbol{X} = (X, Y, Z)$ are the conjugate momenta of the Cartesian variables in the rotating frame $\boldsymbol{x} = (x, y, z)$, and $\mathcal{V}$ is the potential

$$\mathcal{V} = -\frac{\mu}{r} + \frac{\mu\alpha^2}{r^3}\left[C_{2,0}\left(\frac{1}{2} - \frac{3}{2}\frac{z^2}{r^2}\right) - 3C_{2,2}\frac{x^2-y^2}{r^2}\right], \qquad (2)$$

where μ is the gravitational constant, α the equatorial radius, $r = \sqrt{x^2+y^2+z^2}$ is the radial distance of the satellite and the harmonic coefficients are $C_{2,0} < 0 < C_{2,2}$ since Eros spins around its axis of greatest inertia.

The equations of motion corresponding to the Hamiltonian (1) are

$$\begin{aligned} \dot{\boldsymbol{x}} + \boldsymbol{\omega}\times\boldsymbol{x} &= \boldsymbol{X}, \\ \dot{\boldsymbol{X}} - \boldsymbol{\omega}\times\boldsymbol{X} &= -\nabla_{\boldsymbol{x}}\mathcal{V}, \end{aligned} \qquad (3)$$

and $\mathcal{H}(\boldsymbol{X}, \boldsymbol{x}) = h$ is an integral of the motion.

The simplified model of Eqs. (3) with $\mathcal{V}$ given by Eq. (2) —second degree and order gravity field and uniform rotation around the axis of greatest inertia— includes all the main perturbations that act on an orbiter in this system,[24,26] namely the Keplerian attraction, the Coriolis force, and the oblateness and ellipticity perturbations. To the well known effects of the oblateness perturbation, the main effect of the $C_{2,2}$ term is a change in orbit energy and angular momentum that produces noticeable variations in the orbital elements.[25] When the asteroid rotates slowly as compared to the spacecraft orbit period, the averaged problem of orbital motion can be integrated in the formal sense.[17] But the situation is very different for fast rotation rates, where the motion is far from being integrable. First order analytical theories based on averaging of the orbital elements may not be accurate at all, and instead of trying to give approximate solutions, the analytical efforts have taken the direction of giving estimates of the stability of motion based on energy and angular momentum variations over one orbit.[26] On the other hand, extensive numerical computations have been performed in order to understand the chaotic dynamics.[18,20]

The numerical values of Eros we use in this paper are

$$\begin{aligned} \alpha &= 16.5\,\mathrm{km} \\ \mu &= 4.463\times10^{-4}\,\mathrm{km}^3/\mathrm{s}^2 \\ \omega &= 3.31182\,10^{-4}\mathrm{s}^{-1} \\ C_{2,0} &= -0.110231 \\ C_{2,2} &= 0.052826 \end{aligned}$$

and are taken from.[21]

Ordering the Hamiltonian

Inertially referenced orbital elements have been traditionally used for studying the long term evolution of dynamical systems. By formulating the perturbing function in orbital elements —the semimajor axis a, the eccentricity e, the inclination I, the argument of the pericenter g, the argument of the node Ω, and the mean anomaly ℓ—, the (averaged) Lagrange planetary equations can be integrated providing approximate solutions for the secular motion of the satellite.

The usual averaging procedure is done developing the perturbation function as a Fourier series in the mean anomaly ℓ with coefficients as series in powers of the eccentricity e and the inclination function $\sin I$, the validity of such solutions are constrained to small values of the eccentricity. Contrary, in order to avoid expansions in powers of the eccentricity we formulate the orbital problem in Whittaker variables and carry all developements in closed form.

Calling $\boldsymbol{G} = \boldsymbol{x}\times\boldsymbol{X}$ the angular momentum, we materialize the ascending node by the vector $\boldsymbol{n} = \boldsymbol{k}\times\boldsymbol{G}$. The Whittaker canonical variables $(r, \theta, \nu, R, \Theta, N)$ are: the distance r of the satellite, the argument of latitude θ —the angle between $\boldsymbol{n}$ and $\boldsymbol{x}$—, the argument of the node ν —the angle between $\boldsymbol{i}$ and $\boldsymbol{n}$—, the modulus Θ of $\boldsymbol{G}$, $N = \boldsymbol{G}\cdot\boldsymbol{k}$, and the radial velocity $R = \boldsymbol{X}\cdot\boldsymbol{x}/r$ in the inertial frame.

Using these variables, we distinguish four terms in the Hamiltonian: the Keplerian term $\mathcal{H}_{\mathrm{K}}$, Coriolis one $\mathcal{H}_{\mathrm{C}}$, the oblateness or *main problem* term $\mathcal{H}_{\mathrm{o}}$ and the

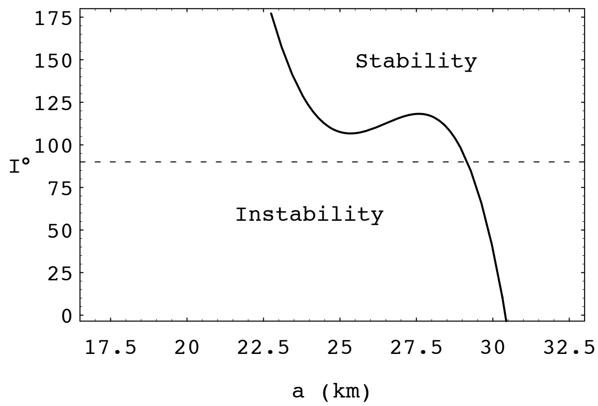

Fig. 1 Stability regions for three-dimensional motion around the asteroid Eros (after[19,20]).

ellipticity one $\mathcal{H}_{\rm e}$

$$\mathcal{H}_{\rm K} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} \tag{4}$$

$$\mathcal{H}_{\rm C} = -\omega N \tag{5}$$

$$\mathcal{H}_{\rm o} = \frac{\mu\alpha^2 J_2}{4r^3}\left(2 - 3s_i^2 + 3s_i^2\cos 2\theta\right) \tag{6}$$

$$\begin{aligned}\mathcal{H}_{\rm e} = \frac{\mu\alpha^2 J_2 C'_{2,2}}{4r^3}\Big[&12 s_i^2\cos 2\nu - \\ &- 3\left(2 - 2c_i - s_i^2\right)\cos(2\theta - 2\nu) \\ &- 3\left(2 + 2c_i - s_i^2\right)\cos(2\theta + 2\nu)\Big]\end{aligned} \tag{7}$$

where $c_i = N/\Theta, s_i = \sqrt{1-c_i^2}$ are functions of the momenta Θ, N, we take the usual convention $J_2 = -C_{2,0}$, while the harmonic coefficient $C_{2,2}$ is replaced by $C'_{2,2} = C_{2,2}/J_2$.

In order to check the relative influence of these four terms in the Hamiltonian, we resort to the global criterium for stability given in,[20] where the stability character of low-eccentricity orbits is shown to depend on their inclination. Thus, in Fig. 1 the line

$$I^\circ = 1447.87a - 163.846a^2 + 6.19364a^3 - 0.0779807a^4,$$

where a must be in km, separates stable almost circular motion from unstable one.

We compute the respective values of Eqs. (4)–(7) for a wide range of initial conditions sweeping part of the stability area provided in Fig. 1, more precisely, we inspect the nonchaotic region corresponding to (stable) almost circular periodic motion further than 26 km away from the center of mass of Eros. We conclude that the addition of the Keplerian term plus the Coriolis one dominates the dynamics, while the influence of the oblateness and the ellipticity perturbations, both of the same order, is lower and remains at a higher order. Therefore, at least in that region, the Hamiltonian admits the following asymptotic expansion:

$$\mathcal{H} = \mathcal{H}_0 + \epsilon\,\mathcal{H}_1, \quad \left\{\begin{array}{ll}\mathcal{H}_0 = & \mathcal{H}_{\rm K} + \mathcal{H}_{\rm C}, \\ \mathcal{H}_1 = & (\mathcal{H}_{\rm o} + \mathcal{H}_{\rm e})/J_2,\end{array}\right.$$

in which we selected $\epsilon = J_2$ as the small parameter.

Now we can proceed in computing the infinitesimal contact transformations that will provide our analytical theory. This will be done using Lie transformations as detailed below.

Lie transformations

A Lie transformation[10] is a one-parameter family of mappings

$$\varphi\,:\,(\boldsymbol{x}', \boldsymbol{X}'; \varepsilon) \longmapsto (\boldsymbol{x}, \boldsymbol{X}),$$

defined by the solution $\boldsymbol{x}(\boldsymbol{x}', \boldsymbol{X}'; \varepsilon)$ and $\boldsymbol{X}(\boldsymbol{x}', \boldsymbol{X}'; \varepsilon)$ to the Hamiltonian system

$$\frac{d\boldsymbol{x}}{d\varepsilon} = \frac{\partial\mathcal{W}}{\partial\boldsymbol{X}}, \qquad \frac{d\boldsymbol{X}}{d\varepsilon} = -\frac{\partial\mathcal{W}}{\partial\boldsymbol{x}},$$

satisfying the initial conditions $\boldsymbol{x}(\boldsymbol{x}', \boldsymbol{X}'; 0) = \boldsymbol{x}'$ and $\boldsymbol{X}(\boldsymbol{x}', \boldsymbol{X}'; 0) = \boldsymbol{X}'$. The function

$$\mathcal{W}(\boldsymbol{x}, \boldsymbol{X}\,;\,\varepsilon) = \sum_{n\geq 0}\frac{\varepsilon^n}{n!}\mathcal{W}_{n+1}(\boldsymbol{x}, \boldsymbol{X}), \tag{8}$$

is the generator of the transformation.

Due to the properties of the Hamiltonian systems, the Lie transformation φ is a completely canonical transformation that maps the Hamiltonian

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{X}\,;\,\varepsilon) = \sum_{n\geq 0}\frac{\varepsilon^n}{n!}\mathcal{H}_n(\boldsymbol{x}, \boldsymbol{X}); \tag{9}$$

onto a new one

$$\mathcal{K}(\boldsymbol{x}', \boldsymbol{X}'\,;\,\varepsilon) = \sum_{n\geq 0}\frac{\varepsilon^n}{n!}\mathcal{K}_n(\boldsymbol{x}', \boldsymbol{X}'). \tag{10}$$

For the sake of simplicity we drop the "primes" and use again the notation $(\boldsymbol{x}, \boldsymbol{X})$ to represent the variables $(\boldsymbol{x}', \boldsymbol{X}')$ after the transformation. Thus, there is no ambiguity in writing $\mathcal{K} = \mathcal{K}(\boldsymbol{x}, \boldsymbol{X}; \varepsilon)$.

The Lie-Deprit method[10] gives a way to find both the Lie transformation and the transformed Hamiltonian (10) by solving, term by term, the homological equation

$$\mathcal{L}_{\mathcal{H}_0}(\mathcal{W}_n) = \mathcal{K}_n - \tilde{\mathcal{H}}_n, \tag{11}$$

in which $\mathcal{L}_{\mathcal{H}_0}(\mathcal{W}_n)$ is the Poisson bracket $(\mathcal{W}_n; \mathcal{H}_0)$, and $\tilde{\mathcal{H}}_n$ is a linear combination of Poisson brackets involving the coefficients

$$\begin{array}{ll}\mathcal{W}_j, & (j = 1, \ldots, n-1), \\ \mathcal{H}_k, & (k = 1, \ldots, n).\end{array}$$

Briefly, the procedure is as follows. First, we compute $\widetilde{\mathcal{H}}_1$, and choose $\mathcal{K}_1$ in such a way that, while being free from the short period effects that we want to average, it makes possible to integrate $\mathcal{W}_1$ in the partial differential equation (11). Then, and iterative procedure based on the "Lie triangle" is performed where, after choosing the order n of the transformed Hamiltonian, the coefficients $\widetilde{\mathcal{H}}_j$, $\mathcal{K}_j$ $(j = 2, \ldots, n)$ are obtained, and $\widetilde{\mathcal{W}}_j$ is solved for in the homological equation. More details about the method can be found in.[10, 16]

Relegation of the longitude of the node

The relegation of the node tries to get rid of the angle ν by using a transformation after which the desired perturbation appears in the new Hamiltonian with a lower influence. After repeated iterations of the transformation the perturbation should be small enough so that we may neglect it.

In our case, the longitude of the node appears only in the ellipticity perturbation Eq. (7). Thus, one can look for Lie transformations that increase the power p of the factor $(1/r)^p$. Depending on the initial conditions, after successive iterations we eventually find a value of the exponent of $(1/r)$ for which the effect of the term that contains the angle ν is sufficiently small and can be neglected.

Taking into account the zero-order of our Hamiltonian we may express $\mathcal{L}_{\mathcal{H}_0}$ as

$$\mathcal{L}_{\mathcal{H}_0} = \mathcal{L}_{\mathrm{K}} + \mathcal{L}_{\mathrm{C}}$$

where $\mathcal{L}_{\mathrm{K}}$ represents the Lie derivative in the Keplerian flow

$$\mathcal{L}_{\mathrm{K}} = R\frac{\partial}{\partial r} - \left(\frac{\mu}{r^2} - \frac{\Theta^2}{r^3}\right)\frac{\partial}{\partial R} + \frac{\Theta}{r^2}\frac{\partial}{\partial \theta}$$

and $\mathcal{L}_{\mathrm{C}} = -\omega\, \partial\_/\partial\nu$. Then, the homological equation (11) has the form

$$\mathcal{L}_{\mathrm{K}}(\mathcal{W}_n) - \omega\frac{\partial \mathcal{W}_n}{\partial \nu} = \mathcal{K}_n - \tilde{\mathcal{H}}_n. \tag{12}$$

At first order we have

$$\widetilde{\mathcal{H}}_1 = \mathcal{H}_1 = \mathcal{H}_{\mathrm{o}} + \mathcal{H}_{\mathrm{e}}.$$

In order to compute $\mathcal{W}_1$ and $\mathcal{K}_1$ we split them into the form

$$\begin{aligned} \mathcal{W}_1 &= \mathcal{W}_{1,0} + \mathcal{W}^*_{1,0}, \\ \mathcal{K}_1 &= \mathcal{K}_{1,0} + \mathcal{K}^*_{1,0} \end{aligned}$$

so that $\mathcal{K}_{1,0}$ is the averaging of $\widetilde{\mathcal{H}}_1$ over the angle ν, and $\mathcal{W}_{1,0}$ verifies

$$\mathcal{L}_{\mathrm{C}}(\mathcal{W}_{1,0}) = \mathcal{K}_{1,0} - \widetilde{\mathcal{H}}_1,$$

that is to say:

$$\begin{aligned} \mathcal{K}_{1,0} &= \frac{1}{2\pi}\int_0^{2\pi} (\mathcal{H}_{\mathrm{o}} + \mathcal{H}_{\mathrm{e}})\, d\nu, \\ \mathcal{W}_{1,0} &= -\frac{1}{\omega}\int (\mathcal{K}_{1,0} - \mathcal{H}_{\mathrm{o}} - \mathcal{H}_{\mathrm{e}})\, d\nu. \end{aligned}$$

Let us note that $\mathcal{H}_{\mathrm{o}}$ does not depend on ν and $\mathcal{H}_{\mathrm{e}}$ is periodic in ν, then we have $\mathcal{K}_{1,0} = \mathcal{H}_{\mathrm{o}}$ and $\mathcal{W}_{1,0}$ becomes a periodic function in ν:

$$\begin{aligned} \mathcal{W}_{1,0} = {} & \frac{3\mu\alpha^2 C'_{2,2}}{8\omega r^3} \times \\ & \times \Big[2s_i^2 \sin 2\nu - \left(2 - 2c_i - s_i^2\right)\sin(2\theta - 2\nu) \\ & + \left(2 + 2c_i - s_i^2\right)\sin(2\theta + 2\nu)\Big]. \end{aligned} \tag{13}$$

With this choice, the fundamental equation (12) becomes into

$$\mathcal{L}_{\mathrm{K}}(\mathcal{W}^*_{1,0}) - \omega\frac{\partial \mathcal{W}^*_{1,0}}{\partial \nu} = \mathcal{K}^*_{1,0} - \mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,0}),$$

which is the same expression as before (12) with the unknowns $\mathcal{W}^*_{1,0}$, $\mathcal{K}^*_{1,0}$, and the computable function $\mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,0})$. The process is iterative and we may decompose $\mathcal{W}^*_{1,0}, \mathcal{K}^*_{1,0}$ in the form

$$\begin{aligned} \mathcal{W}^*_{1,0} &= \mathcal{W}_{1,1} + \mathcal{W}^*_{1,1}, \\ \mathcal{K}^*_{1,0} &= \mathcal{K}_{1,1} + \mathcal{K}^*_{1,1}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{K}_{1,1} &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,0})\, d\nu, \\ \mathcal{W}_{1,1} &= -\frac{1}{\omega}\int (\mathcal{K}_{1,1} - \mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,0}))\, d\nu. \end{aligned}$$

Taking into account the value of $\mathcal{W}_{1,0}$ given by Eq. (13), and the relations

$$\begin{aligned} \mathcal{L}_{\mathrm{K}}\left[\left(\frac{1}{r}\right)^n\right] &= -nR\left(\frac{1}{r}\right)^{n+1}, \\ \mathcal{L}_{\mathrm{K}}\left[\begin{array}{c} \sin(l\,\theta + k\,\nu) \\ \cos(l\,\theta + k\,\nu) \end{array}\right] &= \left(\begin{array}{c} l\dfrac{\Theta}{r^2}\cos(l\,\theta + k\,\nu) \\ -l\dfrac{\Theta}{r^2}\sin(l\,\theta + k\,\nu) \end{array}\right) \\ \mathcal{L}_{\mathrm{K}}\big[R^m\big] &= -m\left(\frac{\mu}{r^2} - \frac{\Theta}{r^2}\right)R^{m-1}, \end{aligned} \tag{14}$$

where k, l, m, n are integer numbers, one can see that $\mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,0})$ has terms in $(1/r)^4, (1/r)^5$ instead of the terms in $(1/r)^3$ appearing in $\mathcal{H}_{\mathrm{e}}$. Moreover, it is purely periodic in ν, that means its average $\mathcal{K}_{1,1}$ over ν is zero, and $\mathcal{W}_{1,1}$ is similar to $\mathcal{W}_{1,0}$ with terms of powers 4 and 5 in $(1/r)$.

After $(m+1)$ iterations, we have

$$\mathcal{L}_{\mathrm{K}}(\mathcal{W}^*_{1,m}) - \omega\frac{\partial \mathcal{W}^*_{1,m}}{\partial \nu} = \mathcal{K}^*_{1,m} - \mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,m}),$$

and we may stop the process by choosing $\mathcal{W}^*_{1,m} = 0$ and $\mathcal{K}^*_{1,m} = \mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,m})$. Eventually

$$\begin{aligned} \mathcal{K}_1 &= \sum_{i=0}^{m} \mathcal{K}_{1,i} + \mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,m}), \\ \mathcal{W}_1 &= \sum_{i=0}^{m} \mathcal{W}_{1,i}. \end{aligned}$$

Following the same argumentation as in the first iteration we may conclude that every term $\mathcal{K}_{1,i} = 0$ with i between 1 and m, and $\mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,m})$ contains terms with powers in $(1/r)$ with exponent $(m+3)$ and greater. In our case, after three iterations, the new Hamiltonian is

$$\mathcal{K}_1 = \mathcal{H}_{\mathrm{o}} + \mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,2}),$$

where

$$\begin{aligned}\mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,2}) = {} & \frac{3\mu\alpha^2 C'_{2,2}}{16\omega^3 r^6}\times \\ \times \Bigg[& \Theta(c_i-1)^2\left(\frac{15\Theta^2}{r^3}-\frac{11\mu}{r^2}-\frac{60R^2}{r}\right)\cos(2\theta-2\nu) \\ & +\Theta(c_i+1)^2\left(\frac{15\Theta^2}{r^3}+\frac{11\mu}{r^2}+\frac{60R^2}{r}\right)\cos(2\theta+2\nu) \\ & -R(c_i^2-1)\left(\frac{45\Theta^2}{r^2}-\frac{45\mu}{r}-60R^2\right)\sin 2\nu \\ & -R(c_i-1)^2\left(\frac{105\Theta^2}{2r^2}-\frac{21\mu}{r}-30R^2\right)\sin(2\theta-2\nu) \\ & +R(c_i+1)^2\left(\frac{105\Theta^2}{2r^2}-\frac{21\mu}{r}-30R^2\right)\sin(2\theta+2\nu)\Bigg]\end{aligned}$$

that contains only powers greater than six in $(1/r)$. Therefore, for the given values of Eros the contribution of $\mathcal{L}_{\mathrm{K}}(\mathcal{W}_{1,2})$ will remain at second order of J_2 for any distance r, and it can be neglected from our first order approach. Note that the term $\mathcal{H}_{\mathrm{o}}$ is not affected by the transformation since it does not depend on the argument of the node.

Thus, at first order of ϵ, after the relegation of the node we obtain the Hamiltonian

$$\mathcal{K} = \mathcal{K}_0 + \epsilon\,\mathcal{K}_1 = \mathcal{H}_{\mathrm{K}} + \mathcal{H}_{\mathrm{C}} + \mathcal{H}_{\mathrm{o}}$$

where the $\mathcal{H}$'s are the ones given in Eqs. (4)–(6), but now expressed in new variables. The argument of the node ν became a cyclic variable and, therefore, $\mathcal{H}_{\mathrm{C}}$ represents a constant of the motion that can be dropped from the new Hamiltonian $\mathcal{K}$. Then, the transformed Hamiltonian is

$$\mathcal{H}_{\mathrm{K}} + \mathcal{H}_{\mathrm{o}},$$

that is formally equal to the Hamiltonian of the main problem of an earth satellite but, of course, with different values for the constants. Therefore, taking into account the scale invariance of the problem[6] the dynamics should be the well known dynamics of the main problem of the artificial satellite except for a different length scale corresponding to the quantitative value of the Eros's J_2.

At this point we must note that the main problem of the artificial satellite enjoys cylindrical symmetry, that is not at all the case of Eros. Consequently, the validity of a first order theory will be limited to certain regions of phase space as we will see later. More general analytical theories must include the ellipticity effect, that appears free from ν in high order theories, and will be the topic of a future paper.

Delaunay normalization

To obtain a first order theory we will perform now a Delaunay normalization.[13] Therefore, instead of the Whittaker variables we will use the Delaunay ones (ℓ, g, h, L, G, H), where ℓ is the mean anomaly, g is the argument of the pericenter, h is the argument of the node, and the conjugated momenta are

$$L = \sqrt{\mu a}, \quad G = L\sqrt{1-e^2}, \quad H = G\cos I,$$

The asymptotic Hamiltonian

$$\mathcal{K} = \mathcal{K}_0 + \epsilon\,\mathcal{K}_1,$$

obtained after the relegation of the node may be written in Delaunay variables by means of the expressions

$$\begin{aligned}\mathcal{K}_0 &= -\frac{\mu^2}{2L^2}, \\ \mathcal{K}_1 &= \mathcal{M}_0\frac{a^3}{r^3} + \mathcal{M}_1\frac{a^3}{r^3}\cos(2g+2f),\end{aligned}$$

where R, f are implicit functions of the mean anomaly ℓ, and

$$\begin{aligned}\mathcal{M}_0 = \mathcal{M}_0(L,G,H) = {} & \frac{\alpha^2\mu^4(3H^2-G^2)}{4G^2L^6}, \\ \mathcal{M}_1 = \mathcal{M}_1(L,G,H) = {} & \frac{3\alpha^2\mu^4(G^2-H^2)}{4G^2L^6},\end{aligned}$$

are functions of the momenta.

The Delaunay normalization is a Lie transformation that maps

$$\mathcal{K} = \sum_{n\geq 0}\frac{\varepsilon^n}{n!}\,\mathcal{K}_n$$

into a new one

$$\mathcal{R} = \sum_{n\geq 0}\frac{\varepsilon^n}{n!}\,\mathcal{R}_n$$

in which $\mathcal{R}_n$ belongs to the kernel of the Lie derivative, i.e. $\mathcal{R}_n$ is the average of $\tilde{\mathcal{K}}_n$ over the mean anomaly ℓ.

Instead of using the expansions of r and f in powers of the eccentricity we will compute the integrals with respect to ℓ. But, taking into account the relation

$$a\sqrt{1-e^2}\,d\ell = r df,$$

we change the independent variable to be the true anomaly. Then, at first order we find

$$\mathcal{R}_1 = \mathcal{M}_0(1-e^2)^{-3/2} = \mathcal{M}_0\frac{L^3}{G^3}.$$

where ℓ and g simultaneously disappear.

With this election the homological equation has the form

$$n\frac{\partial W_1}{\partial \ell} = \mathcal{M}_0\left(\frac{a^3}{r^3}-\frac{L^3}{G^3}\right) + \mathcal{M}_1\frac{a^3}{r^3}\cos(2g+2f).$$

A simple quadrature gives

$$W_1 = \frac{\mu}{G^3}\Big[\mathcal{M}_0(f-\ell)+ \\ +\mathcal{M}_0 e\sin f + \frac{\mathcal{M}_1}{2} e\sin(f+2g) \\ +\frac{\mathcal{M}_1}{2} e\sin(2f+2g) + \frac{\mathcal{M}_1}{6} e\sin(3f+2g)\Big].$$

The Hamiltonian after Delaunay normalization is

$$\mathcal{R} = -\frac{\mu^2}{2L^2} + \epsilon\frac{L^3}{G^3}\mathcal{M}_0(L,G,H),$$

that depends only of the momenta and is trivially integrable.

Summary and Conclusions

Classical theories for motion close to spheroidal bodies assume the prevalence of the Keplerian attraction over other forces. That is not true for the case of many asteroids, where the Coriolis force can be of the same order of the Keplerian attraction.

For these kind of highly nonlinear dynamical systems analytical theories can be constructed with a more sophisticated averaging using suitable action-angle variables. After relegating the node, we obtain the Hamiltonian of the main problem of the artificial satellite —a well known problem. Then, we perform a Delaunay normalization and arrive at an integrable Hamiltonian that solves the problem and permits the generation of approximate ephemerides for any desired solution.

Thus, from our first order theory we see that the qualitative behavior should be similar to the main problem, where low-eccentricity, frozen orbits exist for most inclinations, and which suffer from biffurcations of eccentric frozen orbits at the critical (retrograde and direct) inclinations.[6,9] From[20] we know that this is true for orbits further than ≈ 30.2 km away from Eros. As presented in Fig. 2, one stability index of a family of three-dimensional periodic (frozen) orbits approaches the critical value $k = 2$ twice (at inclinations $I \approx 53°$ and $I \approx 117°$ for the example presented), corresponding to the bifurcation of eccentric orbits. But, also from,[20] we see that behavior suffers radical changes for orbits close to the asteroid. Numerical computations give strong indications that, contrary to the main problem, frozen orbits no longer exist for direct inclinations in the close neighborhood of Eros. The typical behavior of almost circular periodic orbits close to Eros is presented in Fig. 3. While they show stability for retrograde inclinations, there is a limiting inclination where the periodic orbit stability index grows very large. Periodic orbits at direct inclinations, if they exist, are so chaotic that they are of no practical interest.

Therefore, although our results are not conclusive, the analytical solution could be useful for mission purposes. In this paper we only proceed to first order.

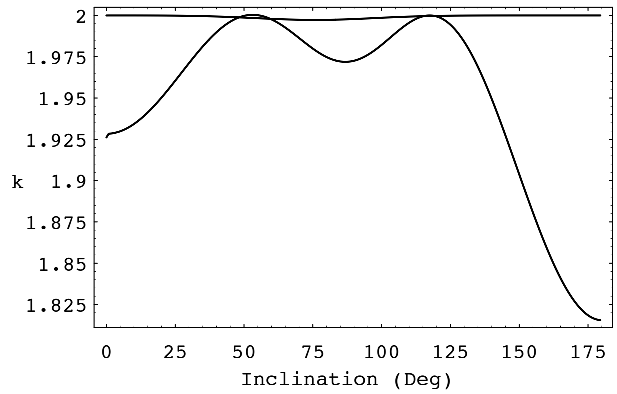

Fig. 2 Evolution of the stability indices of a typical family of three-dimensional periodic orbits far away from Eros (after[20]).

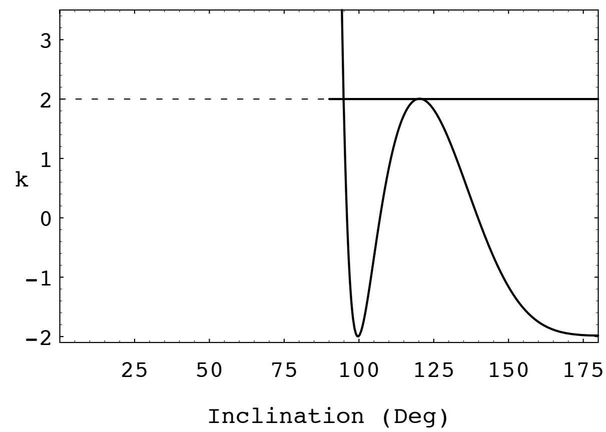

Fig. 3 Evolution of the stability indices of a typical family of three-dimensional periodic orbits close to Eros (after[20]).

More sophisticated analytical theories including the ellipticity effect, which should appear free from the argument of the node in higher order theories, may be much more accurate.

When pursuing a higher order theory, the number of iterations that are necessary to relegate the node grows considerably, with the consequent increase in size of the generating function. Moreover, the transformed Hamiltonian will no longer resemble the main problem Hamiltonian, despite that its form will be similar to the zonal problem.[8] When trying to obtain a closed form theory in that case, the Delaunay normalization could be problematic.[4] Then, the need arises of performing (nontrivial) simplifications[14] of the Hamiltonian —elimination of the parallax,[7,11,12] elimination of the perigee[5]—, but these kinds of problems can be solved very efficiently with the methods provided in the powerful tool ATESAT.[2,3]

The computation of high order analytical theories for spacecraft motion in the vicinity of Eros is in progress, we have obtained stimulating partial results and, hopefully, a high order theory will be provided very soon.

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