

## A SYMPLECTIC KEPLERIAN MAP FOR PERTURBED TWO-BODY DYNAMICS

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### Abstract

A perturbation theory for solving initial value problems is presented. Based on solutions to perturbed two-point boundary value problems this theory analytically solves for the state of a perturbed system. Applications of the theory are found primarily in the fields of orbital mechanics and optimal control. Examples showing the accuracy of the theory in the two-body and restricted three body problems are presented.

## INTRODUCTION

Recently a perturbation technique that allows one to analytically solve perturbed two-point boundary value problems (2BVP) was presented.<sup>1</sup> This technique was derived from Hamilton's principle and uses Hamilton's principal function (HPF), which embodies the dynamics of the system. This perturbation technique was used to solve the perturbed two-body problem.

This paper contains further development of this technique, obtaining a general analytical expression for the initial value problem (IVP). This allows one to obtain an analytical expression for the dynamics of the perturbed state. This idea has been explored by Ross and Scheeres (2007) to predict the change in orbit elements in the restricted three body problem.<sup>2</sup>

This method requires to first obtain the time history of the nominal system. This solution is used to solve a simple quadrature to obtain the perturbed trajectory in phase space. This method eliminates the need to fully integrate the equations of motion, and it only requires to solve Kepler's equation and a quadrature to obtain the perturbed system's solution

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## HAMILTON'S PRINCIPAL FUNCTION

Consider a system with a Hamiltonian function  $H^{(0)}(\vec{x}, t)$ ,  $\vec{x}$  being the state. Let this system be perturbed by a force potential  $R(\vec{x}, t)$  that preserves the Hamiltonian structure. The Hamiltonian of the perturbed system is therefore

$$H(\vec{x}, t) = H^{(0)}(\vec{x}, t) + R(\vec{x}, t) \quad (1)$$

where  $\vec{x} = \vec{x}^{(0)} + \epsilon\vec{x}^{(1)}$  being a small parameter. Consider Hamilton's principal function defined as the action integral<sup>3</sup>

$$W = \int_{t_1}^{t_2} L(\vec{q}, \dot{\vec{q}}) dt \quad (2)$$

where  $\vec{q}$  is the generalized coordinate vector, and  $\vec{p}$  the generalized momenta, which are related to the principal function through by the following relationships:

$$\begin{aligned} \vec{p}_1 &= -\frac{\partial W}{\partial \vec{q}_1} \\ \vec{p}_2 &= \frac{\partial W}{\partial \vec{q}_2} \end{aligned} \quad (3)$$

and

$$\begin{aligned} -\frac{\partial W}{\partial t_1} + H(\vec{q}_1, \vec{p}_1, t_1) &= 0 \\ \frac{\partial W}{\partial t_2} + H(\vec{q}_2, \vec{p}_2, t_2) &= 0 \end{aligned} \quad (4)$$

The expressions in Eq. 4 are the two boundary conditions at times  $t_1$  and  $t_2$  that the principal function must satisfy, while Eq. 5 shows the pair of partial differential equations that must hold true for  $W$ . The principal function for the perturbed system is:

$$W = W^{(0)} + \epsilon W^{(1)} + \epsilon^2 W^{(2)} + \dots + \epsilon^\infty W^{(\infty)} \quad (5)$$

The first order perturbed Hamiltonian is

$$H(x, t) = H^{(0)}(\vec{x}^{(0)}, t) + \epsilon \left[ \frac{\partial H^{(0)}}{\partial \vec{x}} \Big|_{\vec{x}^{(0)}} \vec{x}^{(1)} + R(\vec{x}^{(0)}, t) \right] \quad (6)$$

From the unperturbed system we know that the nominal solution  $W_0$  satisfies

$$\frac{\partial W_0}{\partial t_1} - H_0(\vec{x}_1^{(0)}, t) = 0 \quad \frac{\partial W_0}{\partial t_2} + H_0(\vec{x}_2^{(0)}, t) = 0 \quad (7)$$

Hence, ignoring higher order terms, Eq. 5 will become

$$\begin{aligned} \frac{\partial W_1}{\partial t_1} - \frac{\partial H_0}{\partial \vec{p}_1} \Big|_{\vec{x}_1^{(0)}} \vec{p}_1^{(1)} - R(\vec{x}_1^{(0)}, t) &= 0 \\ \frac{\partial W_1}{\partial t_2} + \frac{\partial H_0}{\partial \vec{p}_2} \Big|_{\vec{x}_2^{(0)}} \vec{p}_2^{(1)} + R(\vec{x}_2^{(0)}, t) &= 0 \end{aligned} \quad (8)$$

Following Hamilton, we note that the total derivative of  $W_\alpha$  with respect to  $t_1$  and  $t_2$ , can be expressed as the following:<sup>4</sup>

$$\begin{aligned} \frac{dW_\alpha}{dt_1} &= \frac{\partial W_\alpha}{\partial t_1} + \frac{\partial W_\alpha}{\partial q_1} \frac{dq_1}{dt_1} \\ \frac{dW_\alpha}{dt_2} &= \frac{\partial W_\alpha}{\partial t_2} + \frac{\partial W_\alpha}{\partial q_2} \frac{dq_2}{dt_2} \end{aligned} \quad (9)$$

Where  $\vec{q}_1$  and  $\vec{q}_2$  lie along the nominal trajectory. By definition  $\frac{d\vec{q}_1}{dt_1} = \frac{\partial H}{\partial \vec{p}_1} \Big|_{\vec{x}_1^{(0)}}$  and  $\frac{d\vec{q}_2}{dt_2} = \frac{\partial H}{\partial \vec{p}_2} \Big|_{\vec{x}_2^{(0)}}$  since the expansion is about this nominal solution. Also from Eq. 5  $\frac{\partial W_1}{\partial \vec{q}_1} = -\vec{p}_1^{(1)}$  and  $\frac{\partial W_1}{\partial \vec{q}_2} = \vec{p}_2^{(1)}$ , therefore Eq. 10 becomes

$$\begin{aligned} \frac{dW_1}{dt_1} &= \frac{\partial W_1}{\partial t_1} - \vec{p}_1^{(1)} \frac{\partial H}{\partial \vec{p}_1} \Big|_{\vec{x}_1^{(0)}} \\ \frac{dW_1}{dt_2} &= \frac{\partial W_1}{\partial t_2} + \vec{p}_2^{(1)} \frac{\partial H}{\partial \vec{p}_2} \Big|_{\vec{x}_2^{(0)}} \end{aligned} \quad (10)$$

where  $\frac{\partial H}{\partial \vec{p}_1} \Big|_{\vec{x}_1^{(0)}}$  is a function of  $\vec{q}_1, \vec{p}_1^{(0)}$  and  $\frac{\partial H}{\partial \vec{p}_2} \Big|_{\vec{x}_2^{(0)}}$  is a function of  $\vec{q}_2, \vec{p}_2^{(0)}$ . Therefore, by substitution the right hand side of Eq. 11 into Eq. 9 we obtain total time derivative

expressions for the  $W_1$ , therefore

$$\begin{aligned}\frac{dW_1}{dt_1} - R(\vec{x}_1^{(0)}, t) &= 0 \\ \frac{dW_1}{dt_2} + R(\vec{x}_2^{(0)}, t) &= 0\end{aligned}\tag{11}$$

which leads to

$$W^{(1)} = - \int_{t_0}^{t_1} R(\vec{x}^{(0)}(t), t) dt\tag{12}$$

In order to calculate the required change in the initial and final velocities to solve the two-point boundary value problem, we simply need to take the partial derivative of the principal function with respect to the generalized coordinates:

$$\vec{p}_0^{(1)} = - \frac{\partial W^{(1)}}{\partial \vec{q}_0} = \int_{t_0}^{t_1} \frac{\partial R}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \vec{q}_0}\tag{13}$$

$$\vec{p}_1^{(1)} = \frac{\partial W^{(1)}}{\partial \vec{q}_1} = - \int_{t_0}^{t_1} \frac{\partial R}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \vec{q}_1}\tag{14}$$

Care must be taken in performing the partial derivatives as the endpoints remain fixed.<sup>1</sup>

## HAMILTON'S CHARACTERISTIC FUNCTION

Recall that Hamilton's principal function is defined as:<sup>4</sup>

$$W(\vec{q}_0, \vec{q}_1, t_0, t_1) = \int_{t_0}^{t_1} L(\vec{q}, \dot{\vec{q}}, \tau) d\tau = \int_{t_0}^{t_1} \left[ \vec{p} \dot{\vec{q}} - H(\vec{q}, \vec{p}, \tau) \right] d\tau\tag{15}$$

Hamilton's principal function and characteristic function are related by

$$Q(\vec{p}_0, \vec{p}_1, t_0, t_1) = W(\vec{q}_0, \vec{q}_1, t_0, t_1) + \vec{p}_0 \vec{q}_0 - \vec{p}_1 \vec{q}_1\tag{16}$$

The variation of the characteristic function is therefore

$$\delta Q = \delta W + \vec{p}_0 \delta \vec{q}_0 - \vec{p}_1 \delta \vec{q}_1 + \vec{q}_0 \delta \vec{p}_0 - \vec{q}_1 \delta \vec{p}_1 \quad (17)$$

where

$$\delta W = \vec{p}_1 \delta \vec{q}_1 - \vec{p}_0 \delta \vec{q}_0 - H(\vec{q}_1, \vec{p}_1) \delta t_1 + H(\vec{q}_0, \vec{p}_0) \delta t_0 \quad (18)$$

Therefore

$$\delta Q = \vec{q}_0 \delta \vec{p}_0 - \vec{q}_1 \delta \vec{p}_1 - H(\vec{q}_1, \vec{p}_1) \delta t_1 + H(\vec{q}_0, \vec{p}_0) \delta t_0 \quad (19)$$

On the other hand if we simply take the variation of  $Q(\vec{p}_0, \vec{p}_1, t_0, t_1)$

$$\delta Q = \frac{\partial Q}{\partial \vec{p}_0} \delta \vec{p}_0 + \frac{\partial Q}{\partial \vec{p}_1} \delta \vec{p}_1 + \frac{\partial Q}{\partial t_0} \delta t_0 + \frac{\partial Q}{\partial t_1} \delta t_1 \quad (20)$$

Combining the two equations together:

$$\vec{q}_0 = \frac{\partial Q}{\partial \vec{p}_0} \quad \vec{q}_1 = -\frac{\partial Q}{\partial \vec{p}_1} \quad (21)$$

and

$$\frac{\partial Q}{\partial t_0} - H(\vec{q}_0, \vec{p}_0, t_0) = 0 \quad \frac{\partial Q}{\partial t_1} + H(\vec{q}_1, \vec{p}_1, t_1) = 0 \quad (22)$$

These equations represent the boundary conditions and partial differential equations that Hamilton's characteristic function must satisfy.

## PERTURBATION THEORY FOR HAMILTON'S CHARACTERISTIC FUNCTION

Due to basic existence theorems we know that the system  $H(\vec{x}, t) = H^{(0)}(\vec{x}, t) + \epsilon R(\vec{x}, t)$  has a solution defined by both Hamilton's principal and characteristic functions. Consider a Taylor series expansion of the solution to the first order, allowing the characteristic function to take the form

$$Q = Q^{(0)} + \epsilon Q^{(1)} \quad (23)$$

Hamilton's characteristic function satisfies the following boundary conditions

$$\vec{q}_0 = \frac{\partial Q}{\partial \vec{p}_0} \quad \vec{q}_1 = -\frac{\partial Q}{\partial \vec{p}_1} \quad (24)$$

Define

$$\vec{q} = \vec{q}^{(0)} + \epsilon \vec{q}^{(1)} \quad (25)$$

Hence,

$$\vec{q}_0^{(1)} = \frac{\partial Q^{(1)}}{\partial \vec{p}_0} \quad \vec{q}_1^{(1)} = -\frac{\partial Q^{(1)}}{\partial \vec{p}_1} \quad (26)$$

The first order perturbed Hamiltonian is

$$H(\vec{x}, t) = H^{(0)}(\vec{x}^{(0)}, t) + \epsilon \left[ \frac{\partial H^{(0)}}{\partial x} \Big|_{\vec{x}^{(0)}} \vec{x}^{(1)} + R(\vec{x}^{(0)}, t) \right] \quad (27)$$

Therefore,

$$\frac{\partial Q^{(1)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial \vec{x}_0} \Big|_{\vec{x}_0^{(0)}} \vec{x}_0^{(1)} - R(\vec{x}_0^{(0)}, t) = 0 \quad (28)$$

$$\left. \frac{\partial Q^{(1)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial \vec{x}_1} \right|_{\vec{x}^{(0)}} \vec{x}_0^{(1)} + R(\vec{x}_1^{(0)}, t_1) = 0 \quad (29)$$

Performing the same manipulation as with Hamilton's principal function we obtain the following result:

$$\frac{dQ^{(1)}}{dt_0} - R(\vec{x}_0^{(0)}, t_0) = 0 \quad (30)$$

$$\frac{dQ^{(1)}}{dt_1} + R(\vec{x}_1^{(0)}, t_1) = 0 \quad (31)$$

which leads to

$$Q^{(1)} = - \int_{t_0}^{t_1} R(\vec{x}^{(0)}, \tau) d\tau \quad (32)$$

We can now solve for the necessary change in generalized coordinates to solve the two-point boundary value problem:

$$\vec{q}_0^{(1)} = \frac{\partial Q^{(1)}}{\partial \vec{p}_0} \quad \vec{q}_1^{(1)} = - \frac{\partial Q^{(1)}}{\partial \vec{p}_1} \quad (33)$$

therefore

$$\vec{q}_0^{(1)} = - \int_{t_0}^{t_1} \frac{\partial R}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \vec{p}_0} d\tau \quad \vec{q}_1^{(1)} = \int_{t_0}^{t_1} \frac{\partial R}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \vec{p}_1} d\tau \quad (34)$$

## RELATIONSHIP BETWEEN IVP AND 2BVP

Hamilton's principal function,  $W(\vec{q}_0, \vec{q}_1, t_0, t_1)$  allows us to solve the 2BVP. Similarly, we can solve a different kind of 2BVP (given the momenta we can solve for the coordinates) using Hamilton's characteristic function  $Q(\vec{p}_0, \vec{p}_1, t_0, t_1)$ . From previously derived theory, we

know that

$$W^{(1)} = - \int_{t_0}^{t_1} R(\vec{x}^{(0)}, \tau) d\tau \quad Q^{(1)} = - \int_{t_0}^{t_1} R(\vec{x}^{(0)}, \tau) d\tau \quad (35)$$

Therefore

$$\vec{q}_1^{(1)} = - \frac{\partial Q^{(1)}}{\partial \vec{p}_1} = \int_{t_0}^{t_1} \frac{\partial R}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \vec{p}_1} d\tau \quad \vec{p}_1^{(1)} = \frac{\partial W^{(1)}}{\partial \vec{q}_1} = - \int_{t_0}^{t_1} \frac{\partial R}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \vec{q}_1} d\tau \quad (36)$$

Combining these two equations

$$\begin{bmatrix} -\vec{p}_1^{(1)} \\ \vec{q}_1^{(1)} \end{bmatrix} J = \int_{t_0}^{t_1} R_x \frac{\partial \vec{x}}{\partial \vec{x}_1} J d\tau \quad (37)$$

When solving the 2BVP we fixed the endpoints creating a constraint for the system. Then, we proceeded to solve that constraint by using the fact that we knew what the fixed coordinate at time  $t_1$  had to be. If we remove the constraint of the state at the final time, the term  $\frac{\partial x}{\partial x_1}$  should be the backwards state transition matrix  $\Phi(\tau, t_1)$ . Therefore:

$$\begin{bmatrix} \vec{q}_1^{(1)} \\ \vec{p}_1^{(1)} \end{bmatrix} = \int_{t_0}^{t_1} R_x \Phi(\tau, t_1) J d\tau \quad (38)$$

where

$$\Phi(\tau, t_1) = \begin{bmatrix} \Phi_{qq} & \Phi_{qp} \\ \Phi_{pq} & \Phi_{pp} \end{bmatrix} \quad (39)$$

Now use the fact

$$\Phi(\tau, t_1) = \Phi^{-1}(t_1, \tau) = -J\Phi^T J \quad (40)$$



Hence:

$$\begin{bmatrix} \vec{q}_1^{(1)} \\ \vec{p}_1^{(1)} \end{bmatrix} = \int_{t_0}^{t_1} J \begin{bmatrix} R_q & R_p \end{bmatrix} \begin{bmatrix} \Phi_{pp}^T & -\Phi_{qp}^T \\ -\Phi_{pq}^T & \Phi_{qq}^T \end{bmatrix} \Big|_{(t_1, \tau)} d\tau \quad (41)$$

Therefore the first order perturbed state at time  $t_1$  can be expressed as:

$$x_1^{(1)} = \int_{t_0}^{t_1} \begin{bmatrix} -R_q & R_p \end{bmatrix} \begin{bmatrix} \Phi_{qp}^T & \Phi_{pp}^T \\ \Phi_{pq}^T & \Phi_{qq}^T \end{bmatrix} d\tau \quad (42)$$

which can be rewritten as

$$\vec{x}_1^{(1)} = \int_{t_0}^{t_1} [\Phi(t_1, \tau) J R_x]^T d\tau \quad (43)$$

The first order solution to the perturbed state is then:

$$\vec{x}_1 = \Phi(t_1, t_0) \vec{x}_0^{(0)} + \epsilon \left[ \int_{t_0}^{t_1} N(t_1, \tau) d\tau \right] \quad (44)$$

where  $N(t_1, \tau) = [\Phi(t_1, \tau) J R_x]^T$

The integration is carried out over the time defined by the nominal system, and along the nominal path in phase space. Hence, in order to calculate the perturbed state of the system the knowledge of the nominal system and the perturbed Hamiltonian is needed.

We have assumed that the perturbation theory preserves the Hamiltonian structure of the nominal system. In order to verify that we must check that the system is still symplectic. Consider the equations of motion in a Hamiltonian form:

$$\dot{x} = J \frac{\partial H}{\partial x} \quad (45)$$

where  $x = x^{(0)} + \epsilon x^{(1)}$ ,  $H(x, t) = H^{(0)}(x^{(0)}, t) + \epsilon H^{(1)}(x, t)$ , and  $J$  is

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \quad (46)$$

For the nominal system, the dynamics must satisfy

$$\dot{x}^{(0)} = J \frac{\partial H^{(0)}}{\partial x} \quad (47)$$

which is symplectic and from which we can get the solution  $x^{(0)}(t; x_0)$ . Consider now the equations of motion of the full (perturbed) system:

$$\dot{x}(t; x_0, \epsilon) = \dot{x}^{(0)} + \epsilon \dot{x}^{(1)} = J \frac{\partial H^{(0)}}{\partial x} \Big|_{x^{(0)}} + \epsilon \left[ J \frac{\partial R}{\partial x} \Big|_{x^{(0)}} \right] \quad (48)$$

We know that the solution to the initial value problem will be of the form:

$$x_1 = \Phi(t_1, t_0)x_0 + \epsilon \int_{t_0}^{t_1} \Phi(t_1, \tau) J R_x d\tau \quad (49)$$

where  $R_x = \frac{\partial R}{\partial x} \Big|_{x^{(0)}}$ . The system will be symplectic if the Jacobian of the system  $M = \frac{\partial x_1}{\partial x_0}$  satisfies

$$M^T J M = J \quad (50)$$

Since we are solving the perturbed initial value problem, the state at the initial time will be the same for the nominal solution and the perturbed solution,  $x_0 = x_0^{(0)}$ . Therefore we can rewrite the solution as:

$$x_1 = \Phi(t_1, t_0) \left[ x_0^{(0)} + \epsilon \int_{t_0}^{t_1} \Phi(t_0, \tau) J R_x d\tau \right] \quad (51)$$

The Jacobian then becomes

$$M = \frac{\partial x_1}{\partial x_0} = \Phi(t_1, t_0) \left[ I + \epsilon \int_{t_0}^{t_1} \Phi^{-1}(\tau, t_0) J R_{xx} \Phi(\tau, t_0) d\tau \right] \quad (52)$$

Therefore

$$M^T J M = [I + \epsilon \int \Phi^T(\tau, t_0) R_{xx} \Phi(\tau, t_0)] \Phi^T(t_1, t_0) J \Phi(t_1, t_0) \left[ I + \epsilon \int_{t_0}^{t_1} \Phi^T(\tau, t_0) R_{xx} J^T \Phi^{-T}(\tau, t_0) \right] \quad (53)$$

Since we assume that the unperturbed system is symplectic,  $\Phi^T(t_1, t_0) J \Phi(t_1, t_0) = J$  and the above equation becomes

$$M^T J M = \left[ I + \epsilon \int \Phi^T(\tau, t_0) R_{xx} \Phi(\tau, t_0) \right] J \left[ I + \epsilon \int_{t_0}^{t_1} \Phi^T(\tau, t_0) R_{xx} J^T \Phi^{-T}(\tau, t_0) \right] \quad (54)$$

We are studying the effect of a perturbation to the first order, therefore for the system to be symplectic to the first order the following condition must be satisfied:

$$\int_{t_0}^{t_1} \Phi^T(\tau, t_0) R_{xx} J^T \Phi^{-T}(\tau, t_0) J d\tau + \int_{t_0}^{t_1} J \Phi^{-1}(\tau, t_0) J R_{xx} \Phi(\tau, t_0) d\tau = 0 \quad (55)$$

The following relationships can be obtained from  $M^T J M = J$ :

$$\begin{aligned} J^T \Phi^{(-T)}(\tau, t_0) J &= \Phi(\tau, t_0) \\ -\Phi &= J \Phi^{-1} J \end{aligned} \quad (56)$$

Therefore the above equation becomes

$$\int_{t_0}^{t_1} \Phi^T(\tau, t_0) R_{xx} \Phi(\tau, t_0) d\tau - \int_{t_0}^{t_1} \Phi^T(\tau, t_0) R_{xx} \Phi(\tau, t_0) d\tau = 0 \quad (57)$$

The system is then symplectic to the first order.

## SOLVING THE IVP: DELAUNAY VARIABLES IN THE 2BP

Consider the set of Delaunay variables for the two-body problem:

$$l = \sqrt{\frac{\mu}{a^3}} t \quad L = \sqrt{\mu a} \quad (58)$$

$$g = \omega \quad G = L \sqrt{1 - e^2} \quad (59)$$

$$h = \Omega \quad H = G \cos i \quad (60)$$

where  $\{a, e, i, \Omega, \omega, l\}$  are the classical orbit elements. The dynamics of the nominal solution are:

$$\dot{l} = \frac{\mu^2}{L^3} \quad \dot{L} = 0 \quad (61)$$

$$\dot{g} = 0 \quad \dot{G} = 0 \quad (62)$$

$$\dot{h} = 0 \quad \dot{H} = 0 \quad (63)$$

The state transition matrix for this system is

$$\Phi(t, t_0) = \begin{bmatrix} 1 & 0 & 0 & \Phi_{14} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (64)$$

where  $\Phi_{14}(t_1, \tau) = -\frac{3}{2}(t_1 - \tau)$ . The Hamiltonian of the nominal system can be expressed as the Keplerian energy

$$H^{(0)} = -\frac{1}{2} \frac{\mu^2}{L^2} \quad (65)$$

Let the system be altered by a perturbing potential while preserving the Hamiltonian structure:

$$H(\vec{x}, t) = H^{(0)}(\vec{x}^{(0)}) + \epsilon R(\vec{x}, t) = -\frac{1}{2} \frac{\mu^2}{L^2} + \epsilon R(\vec{x}, t) \quad (66)$$

where  $\vec{x} = \vec{x}^{(0)} + \epsilon \vec{x}^{(1)}$ . From the theory, the solution to the initial value problem can be written as:

$$\vec{x}_1 = \Phi(t_1, t_0) \left[ \vec{x}_0^{(0)} + \epsilon \int_{t_0}^{t_1} \Phi(t_0, \tau) J R_x d\tau \right] \quad (67)$$

For this problem  $R_x$  will be a 6 dimensional vector evaluated along the nominal solution:

$$R_x = \begin{bmatrix} \frac{\partial R}{\partial l} \\ \frac{\partial R}{\partial R} \\ \frac{\partial R}{\partial q} \\ \frac{\partial R}{\partial h} \\ \frac{\partial R}{\partial L} \\ \frac{\partial R}{\partial G} \\ \frac{\partial R}{\partial H} \end{bmatrix}_{\vec{x}^{(0)}} \quad (68)$$

Therefore, the solution to the perturbed system can be expressed as:

$$\vec{x}_1 = \Phi(t_1, t_0) \vec{x}_0^{(0)} + \epsilon \left[ \int_{t_0}^{t_1} N(t_1, \tau) d\tau \right] \quad (69)$$

where

$$N(t_1, \tau) = \begin{bmatrix} \frac{\partial R}{\partial L} - \Phi_{14}(t_1, \tau) \frac{\partial R}{\partial l} \\ \frac{\partial R}{\partial G} \\ \frac{\partial H}{\partial R} \\ -\frac{\partial l}{\partial R} \\ -\frac{\partial q}{\partial R} \\ -\frac{\partial h}{\partial R} \end{bmatrix}_{\vec{x}^{(0)}} \quad (70)$$

In order to solve the perturbed two-body problem one needs the nominal solution and the analytical form of the perturbing Hamiltonian  $R$ . The orbit is evaluated at the nominal order, The relationship between the orbit and time is given by Kepler's equation, which has no analytical solution and it must be numerically solved.

Therefore, the nominal two-body solution must be obtained through numerical methods, and feed the solution into the perturbation expression. This will result in a solution of the perturbed orbit to the first order. In order to obtain a higher order solution, time can be divided into smaller fragments and feed the solution into the integral, using the result of the perturbed orbit as the initial condition for the nominal orbit in the next time segment. Making the time segments small will result in the exact solution of the the perturbed system.

## PERIAPSIS MAPPING

For problems that span over long time periods is of great interest to observe the change in orbit elements over that time. For that purpose we can build a periapsis map showing the evolution of the orbit element. Ross and Scheeres constructed a periapse (Poincare) map that shows the relationship between the semimajor axis and argument of periapse of the orbit over time.

However, for every nominal periapsis passage, the actual periapse will be perturbed, and therefore shifted from its nominal position. In order to build a Poincare map for the actual orbit, this change in periapse has to be taken into account in order to build a Keplerian map.

Let  $X_0$  be the nominal state at periapse passage  $t_p$ , and  $X_1$  be the next periapsis passage. The nominal periapse occurs at time  $T$ , however the actual periapse is slightly perturbed and occurs at time  $T + \delta T$ . Therefore, in order to calculate the actual periapse passage:

$$X_1 = y(t; X_0, T) = X_0 + \int_T^{T+\delta T} N(T + \delta T, \tau) d\tau \quad (71)$$

$$X_1 = X_0 + \Delta X + \int_T^{T+\delta T} N(T + \delta T, \tau) d\tau \cong X_0 + \Delta X + N(\delta T, 0)\delta T \quad (72)$$

for small  $\delta T$ . In order to solve for the small deviation  $\delta T$ , use the fact that at the new periapse,  $\Delta l = 2\pi + \delta l$ . Therefore

$$\Delta l + N_l(\delta T, 0)\delta T = 0 \quad (73)$$

$$\delta T = -\frac{\Delta l}{N_l(\delta T, 0)} \quad (74)$$

And finally,

$$X_1 = X_0 + \int_{t_p}^{T - \frac{\Delta l}{N_l}} N\left(T - \frac{\Delta l}{N_l}, \tau\right) d\tau \quad (75)$$

where  $t_p$  is the previous periapsis passage. Then  $X_1$  is the state at the subsequent periapsis passage,

## RESTRICTED THREE-BODY PROBLEM

In this section we introduce the restricted three body problem, where a massless body orbits two large point-masses that are orbiting around their barycenter. We rewrite the problem in Delaunay variables for the general 3-D case, and then specialize it to the planar case, which simplifies the perturbation expressions.

In the restricted three-body problem (R3BP) the Hamiltonian function be written as

$$H = \frac{1}{2}\vec{v} \cdot \vec{v} - U(\vec{r}_1, \vec{D}, \epsilon) \quad (76)$$

where  $\vec{v}$  is the velocity,  $\vec{r}_1$  is the radial position vector from the center of the primary to the particle, and  $\vec{D}$  is the position vector of the secondary body from the primary. The

force potential  $U$  is defined as

$$U = \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \quad (77)$$

where  $r_1, r_2$  are the absolute distances from the primary and secondary bodies to the particle,  $\mu_1, \mu_2$  are the gravitational parameters of the primary and secondary, and  $\mu_1 > \mu_2$ . Define the mass ratio  $\epsilon$  as:

$$\epsilon = \frac{\mu_2}{\mu_1 + \mu_2} \quad (78)$$

Using the law of sines,  $r_2$  can be expressed in terms of  $r_1$ ,  $D$ , and  $\theta$  the angle between the inertial and rotating frames

$$r_2(r_1, \theta) = \sqrt{r_1^2 + D^2 - 2r_1D \cos(u - \theta)} \quad (79)$$

where  $u$  is the angle between the particle and the ascending node. Therefore the Hamiltonian can be written as

$$H = \frac{1}{2} \vec{v} \cdot \vec{v} - (1 - \epsilon) \frac{\mu}{r_1} - \epsilon \frac{\mu}{\sqrt{r_1^2 + D^2 - 2r_1D \cos(u - \theta)}} = K - \epsilon \frac{\mu}{\sqrt{r_1^2 + D^2 - 2r_1D \cos(u - \theta)}} \quad (80)$$

where  $\mu = \mu_1 + \mu_2$  is the total gravitational parameter of the system, and  $K$  is the Keplerian energy of the two-body system (primary-particle system). In the rotating frame, the Hamiltonian can be expressed as:

$$H_{rot} = K - \dot{\theta}G - \epsilon \frac{\mu}{\sqrt{r_1^2 + D^2 - 2r_1D \cos(u - \theta)}} \quad (81)$$

We can view the PRC3BP as a rotating two-body problem with third-body perturbation effects.

$$\begin{aligned} H_{in} &= K + \epsilon R \\ H_{rot} &= K - \dot{\theta}G + \epsilon R \end{aligned} \quad (82)$$



where

$$R = -\frac{\mu}{\sqrt{r_1^2 + D^2 - 2r_1 D \cos(u - \theta)}} \quad (83)$$

where  $G$  is the angular momentum of the particle about the primary.

## THE JACOBI INTEGRAL

In the restricted three body problem the Jacobi integral can be expressed as

$$C_J = K - \dot{\theta}H + \epsilon R(l, g, h, L, G, H) \quad (84)$$

The time rate of change of the Jacobi integral is therefore

$$\dot{C}_J = \dot{K} - \dot{\theta}\dot{H} + \epsilon\dot{R}(l, g, h, L, G, H) \quad (85)$$

$$\dot{C}_J = \frac{\mu^2}{L^3}\dot{L} - \dot{\theta}\dot{H} + \epsilon \left[ \frac{\partial R}{\partial l}\dot{l} + \frac{\partial R}{\partial g}\dot{g} + \frac{\partial R}{\partial h}\dot{h} + \frac{\partial R}{\partial L}\dot{L} + \frac{\partial R}{\partial G}\dot{G} + \frac{\partial R}{\partial H}\dot{H} \right] \quad (86)$$

From the relationship  $\dot{x} = J \frac{\partial C_J}{\partial x}$  we know that

$$\begin{aligned} \dot{l} &= \frac{\partial C_J}{\partial L} = \frac{\mu^2}{L^3} + \epsilon \frac{\partial R}{\partial L} & \dot{L} &= -\frac{\partial C_J}{\partial l} = -\epsilon \frac{\partial R}{\partial l} \\ \dot{g} &= \frac{\partial C_J}{\partial G} = \epsilon \frac{\partial R}{\partial G} & \dot{G} &= -\frac{\partial C_J}{\partial g} = -\epsilon \frac{\partial R}{\partial g} \\ \dot{h} &= \frac{\partial C_J}{\partial H} = -\dot{\theta} + \epsilon \frac{\partial R}{\partial H} & \dot{H} &= -\frac{\partial C_J}{\partial h} = -\epsilon \frac{\partial R}{\partial h} \end{aligned} \quad (87)$$

Therefore

$$\begin{aligned} \dot{C}_J &= \epsilon \left[ -\frac{\mu^2}{L^3} \frac{\partial R}{\partial l} + \dot{\theta} \frac{\partial R}{\partial h} + \frac{\partial R}{\partial l} \frac{\mu^2}{L^3} - \frac{\partial R}{\partial h} \dot{\theta} \right] + \\ &+ \epsilon^2 \left[ \frac{\partial R}{\partial l} \frac{\partial R}{\partial L} + \frac{\partial R}{\partial g} \frac{\partial R}{\partial G} + \frac{\partial R}{\partial h} \frac{\partial R}{\partial H} - \frac{\partial R}{\partial L} \frac{\partial R}{\partial l} - \frac{\partial R}{\partial G} \frac{\partial R}{\partial g} - \frac{\partial R}{\partial H} \frac{\partial R}{\partial h} \right] = 0 \end{aligned} \quad (88)$$

Therefore the Jacobian for the perturbed two-body problem is constant.

## PLANAR RESTRICTED CIRCULAR THREE-BODY PROBLEM

In the PRC3BP,  $H = G$  and  $g' = h + g$ , therefore the perturbing potential becomes

$$R = -\frac{\mu}{\sqrt{r^2 + D^2 - 2rD \cos(g' + f - \theta)}} \quad (89)$$

and

$$r = \frac{1}{\mu} \frac{G^2 L}{L + \sqrt{L^2 - G^2} \cos f} \quad (90)$$

The Jacobi integral for the PRC3BP can be expressed as

$$C_J = K - \dot{\theta}G + \epsilon R = -\frac{\mu}{2L^2} - \dot{\theta}G + \epsilon R \quad (91)$$

where  $g = g' - \dot{\theta}t$ . The Jacobian is constant to the first order.

## PERTURBED ROTATING TWO-BODY PROBLEM

The planar restricted circular three-body problem can be viewed as a perturbed rotating two-body problem. The equations of motion for the PRC3BP expressed in the synodic frame centered at the barycenter are:

$$\begin{aligned} \ddot{x} &= 2\dot{y} + x - \mu_1 \frac{(x+\epsilon)}{\sqrt{(x+\epsilon)^2 + y^2}} - \mu_2 \frac{(x-1+\epsilon)}{\sqrt{(x-1+\epsilon)^2 + y^2}} \\ \ddot{y} &= -2\dot{x} + y - \mu_1 \frac{y}{\sqrt{(x+\epsilon)^2 + y^2}} - \mu_2 \frac{y}{\sqrt{(x-1+\epsilon)^2 + y^2}} \end{aligned} \quad (92)$$

where  $\epsilon = m_2/(m_1 + m_2)$ . The trajectories of the nominal and perturbed systems can be obtained by numerically integrating the above equations. In order to transform the synodic trajectory  $(Q_s, P_s)$  back to an inertial trajectory  $(Q_i, P_i)$  centered at the main primary, allowing to express the trajectory using Delaunay elements:

$$\begin{aligned} Q_p &= Q_s + \epsilon \hat{x}_p \\ P_p &= P_s \end{aligned} \quad (93)$$

$$Q'_p = Q_p \tag{94}$$

$$P'_p = P_p + \dot{\Theta} Q_p$$

$$Q_i = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} Q'_p \tag{95}$$

$$P_i = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} P'_p$$

The perturbation of the orbit elements with respect to the nominal system can be obtained by the following integral:

$$\begin{bmatrix} \Delta l \\ \Delta g \\ \Delta L \\ \Delta G \end{bmatrix} = \int_{t_0}^{t_1} \begin{bmatrix} \frac{\partial R}{\partial L} + \frac{3}{2}(t_1 - \tau) \frac{\partial R}{\partial t} \\ \frac{\partial R}{\partial G} \\ -\frac{\partial R}{\partial L} \\ -\frac{\partial R}{\partial g} \end{bmatrix}_{x^{(0)}} d\tau \tag{96}$$

where

$$R = -\frac{\mu_2}{r_1^2 + D^2 - 2rD \cos u} \tag{97}$$

and  $u = g + f - \dot{\Theta}t$ .

## EXAMPLE SIMULATIONS

Take as an example a spacecraft orbiting the central body with a semimajor axis  $a = 0.67$  and eccentricity  $e = 0.3$ . The distance between the primary and secondary masses is  $D = 1$ , and the mass ratio  $\epsilon = 0.00005667$ . Let the spacecraft be initially at periaipse. Figures ?? though 10 show the Delaunay elements after 100 orbits. The integrations of the elements are carried out in 1, 8, and 1000 steps per orbit.

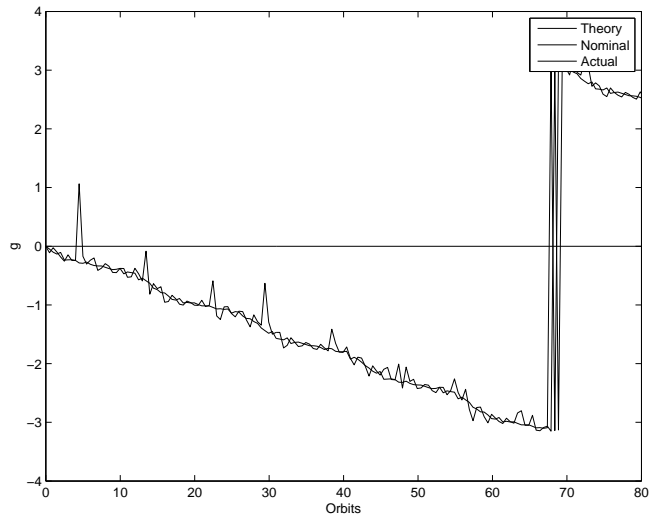


Figure 1: Change in  $g$  in 2 steps per orbit

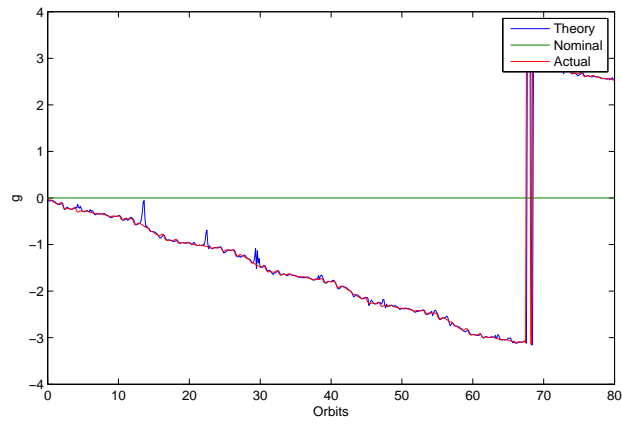


Figure 2: Change in  $g$  in 8 steps per orbit

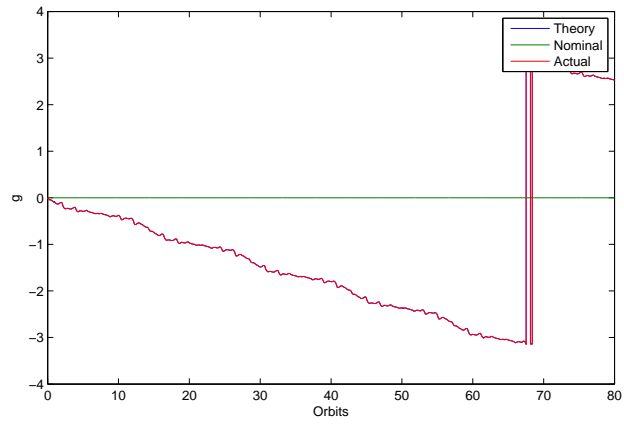


Figure 3: Change in  $g$  in 1000 steps per orbit

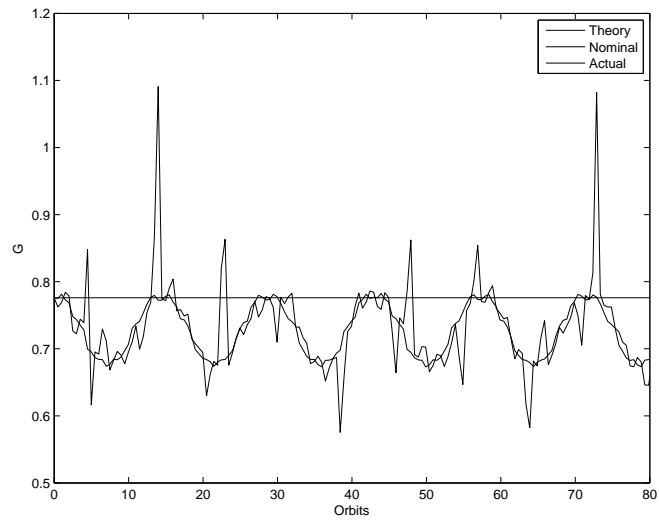


Figure 4: Change in  $G$  in 2 steps per orbit

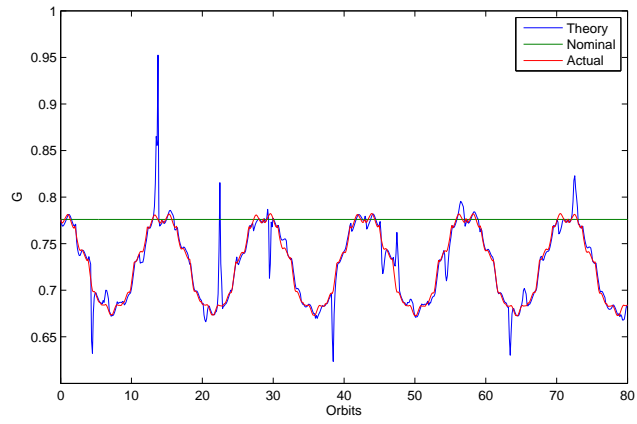


Figure 5: Change in G in 8 steps per orbit

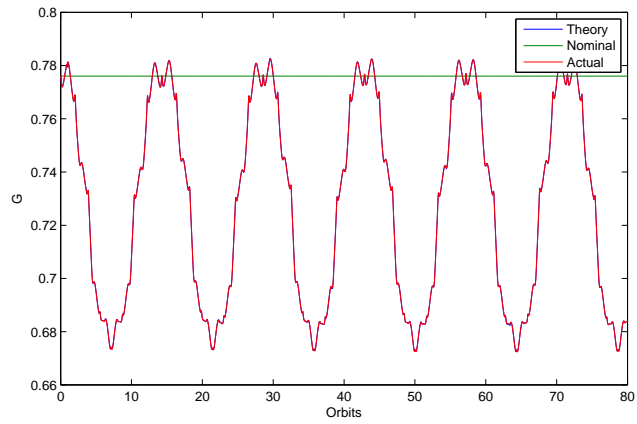


Figure 6: Change in G in 1000 steps per orbit

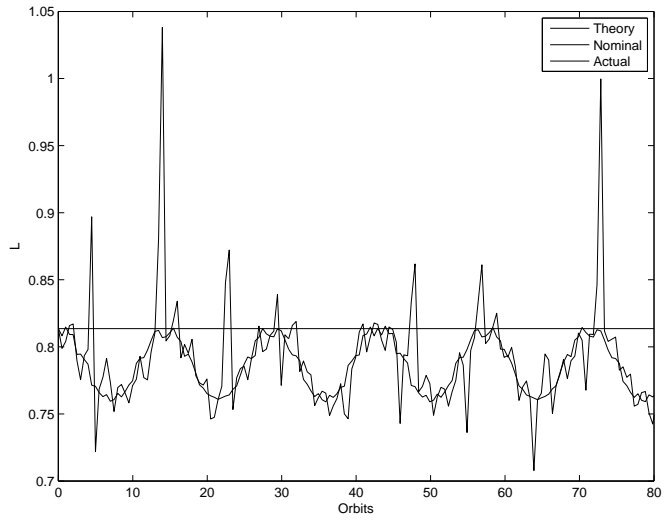


Figure 7: Change in L in 2 steps per orbit

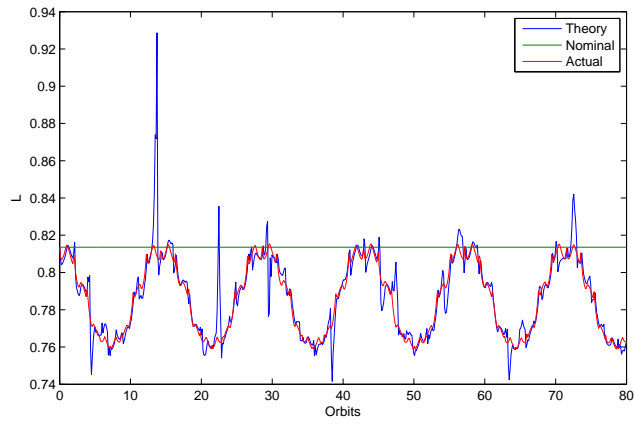


Figure 8: Change in L in 8 steps per orbit

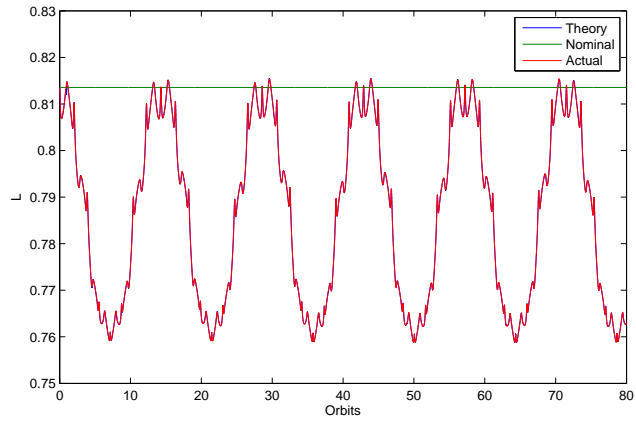


Figure 9: Change in L in 1000 steps per orbit

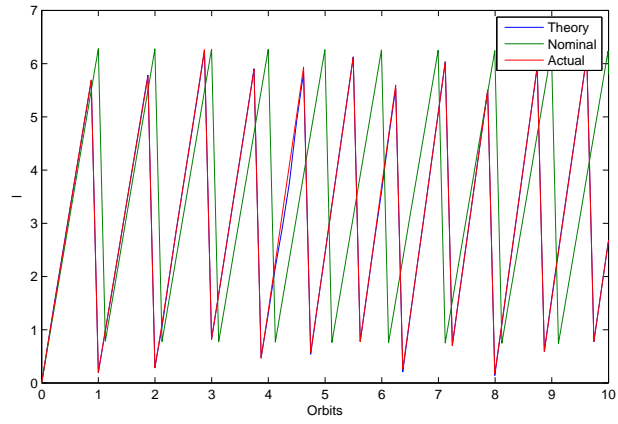


Figure 10: Change in l in 8 steps per orbit



## CONCLUSIONS

We have presented a perturbation technique for Hamiltonian dynamical systems. This technique is a further development from a perturbation theory to solve two-point boundary value problems, and can be used to obtain an analytical expression for the perturbed initial value problem.

We have shown that this theory allows one to obtain the full trajectory of a space-craft in phase space by quadratures. With small enough step sizes the theory predicts the actual trajectory of the particle. We have also developed a periapse map that shows the evolution of orbit elements as they pass through the actual periapsis of the perturbed orbit.

Future research will include a robust method of finding trajectories of interest for inter-planetary missions. This should result in an algorithm for mission planning tools that will decrease the cost of a space mission significantly.

This can also be applied to optimal control problems. We envision to solve continuous low-thrust optimal control problems with the perturbation theory as the underlying basis.

## ACKNOWLEDGEMENT

We would like to acknowledge *Eusko Jaurlaritza / Gobierno Vasco* (Basque Government) for the full financial support of the research project.

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