SWITCHING CONDITIONS AND A SYNTHESIS TECHNIQUE FOR THE SINGULAR SATURN GUIDANCE PROBLEM

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NOTES

Abstract

A singular optimal guidance problem which was motivated by difficulties encountered in the Saturn V SA-501 flight has been studied. It is shown that if the guidance equations are based upon a singular version of the flat-earth problem, then the control must be discontinuous at a junction of singular and nonsingular subcases for almost all cases. A good suboptimal guidance scheme based upon a nonsingular approximation of the singular problem is presented. The resultant suboptimal control is continuous, which is more desirable than a discontinuous control, and causes only a noise-level difference in payload.

I. Introduction

In the second flight of the Saturn V vehicle (SA-501), two engines shut down early in the S-II stage. The measurements received by the on-board guidance scheme, the Iterative Guidance Mode (IGM), indicated that only one engine was out. This resulted in a steep plane steering program in the S-IIV stage which caused the time rate of change of the steering angle to reach its limiting value for a large portion of the S-IIV flight. Since the IGM is based on uncontrolled variational theory, the resultant trajectory did not reach the desired terminal orbit.

In the flight mentioned above a large disturbance caused the guidance law to determine a steering angle rate of change which was too large. Thus, it would be desirable to design a guidance law in such a way that the time rate of change of the steering angle is a bounded control variable, say, with \( |\dot{\phi}| \leq \text{K} \), such that the steering angle is a state variable since it cannot change rapidly because of physical and reliability constraints. However, the resultant optimal control problem is a singular problem, and the variational and computational theory for such problems is far from satisfactory.

As is well-known, the variational and numerical theory for totally nonsingular optimal control problems is well developed, and recently McDaniell and Powers, 3, Speyer and Jacobson,4, and Gol(0) developed new necessary conditions and sufficient conditions for totally singular problems. Thus, the main problems to be resolved are: (1) The determination of necessary and sufficient conditions for optimal trajectories which possess both singular and nonsingular subcases which is the case in the Saturn guidance problem; and (2) The development of a computational scheme for the generation of optimal trajectories which possess both singular and nonsingular subcases. Partial results in this direction have been obtained by Jacobson, Gershowitz, and Lele(6) and Pagurek and Woodside(5).

In the following analysis, recently developed necessary conditions for composite optimal trajectories (i.e., trajectories which contain both singular and nonsingular subcases) are used to characterize the local switching behavior of the singular Saturn guidance problem. Since the resultant behavior is not physically desirable, the problem is transformed into a good suboptimal nonsingular representation of the problem which could be incorporated easily into a recently proposed guidance scheme for Saturn class vehicles(6).

II. Singular Optimal Control Theory

In this section, properties from singular optimal control theory which we shall apply later will be summarized.

Consider the problem of minimizing

\[
J = \int_{t_0}^{t_f} \left[ g(y, u) + \frac{1}{2} \dot{y}^T Q \dot{y} + 2 \dot{y}^T R u + u^T (0, R) u \right] \, dt
\]

subject to the following conditions

\[
\dot{x}(t) = f(x, u)
\]

\[
x(0) = x_0
\]

\[
\phi^T(x, u) = 0
\]

\[
|u| \leq \text{K}. \tag{2.5}
\]

The state \( x \) is a \( n \)-dimensional, \( u \) is a scalar control variable, and \( \phi \) is a \( p \)-dimensional vector function which defines the terminal surface, \( p \geq n + 1 \).

Along an optimal trajectory the following necessary conditions hold:

\[
\dot{x}(t) = -H^T \beta(x, u, \dot{x}, \phi) \tag{2.6}
\]

\[
H^T \beta(x, u, \dot{x}, \phi) = -C_1 \phi(x, u, \dot{x}, \phi) + C_2 \dot{x}(x, u, \dot{x}, \phi) \tag{2.7}
\]
where \(t\) and \(\nu\) are Lagrange multipliers and
\[
\mathbf{H}(\xi,\tau,\omega) = \min_{\xi} \mathbf{H}(\xi,\tau,\omega)
\]
(2.9)

This, of course, is the familiar Pontryagin maximum principle in a minimum form (i.e., the Hamiltonian is minimal with respect to the control if the performance index is to be minimized).

In general, the optimal trajectory for this problem consists of some combination of singular arcs and nonsingular (bang-bang) arcs. A singular arc is one along which
\[
\mathbf{H}(\xi,\tau,\omega) = 0
\]
(2.11)
on a nonzero time interval. A nonsingular arc is one along which \(\mathbf{H}(\xi,\tau,\omega) \neq 0\) except possibly at a countable number of points \(t_1, t_2, \ldots \in [0, t]\). On a nonsingular arc (2.9) implies \(d\xi/du = \mathbf{H}(\xi,\tau,\omega)\) which indicates the bang-bang character of nonsingular arcs.

The defining feature of a singular arc is that Eq. (2.9) of the minimum principle is satisfied trivially, and, thus, it cannot be used to distinguish between maxima and minima. In [46] Kelley [110] developed a new necessary condition for singular arcs which allows one to distinguish between maxima and minima.

Kelley Condition

(Généralized Legendre-Clebsch Condition)

Let \(\mathbf{H}(\xi,\tau,\omega) = 0\) be a weak, relative minimum for Eq. (2.1). Then
\[
-\mathbf{H}(\xi,\tau,\omega) \leq 0 \quad \forall u \neq 0
\]
(2.12)
where \(\mathbf{H}(\xi,\tau,\omega) = 0\) is the lower order time derivative of \(u\) in which \(u\) appears explicitly. (Note: if \(q = 0\), then the arc is nonsingular; if \(q \neq 0\), then \(q\) is called the order of the singular arc.)

The Kelley condition is a pointwise or local necessary condition. Recently, a new necessary condition (a generalization of the classical Jacobi condition) was developed for totally singular problems [102]. A strengthened form of this condition (along with the strengthened Kelley condition and Eqs. (2.0) - (2.1)) leads to a sufficient condition for a totally singular arc. However, a useful sufficient condition for composite singular problems is still lacking.

In Reference 21, Kelley, Kopp, and Meyer used Taylor series expansions in the neighborhood of a singular-nonsingular junction along with the maximum principle to obtain necessary conditions at the junction. Recently these results have been generalized [111] as follows:

**Theorem 1.** Let \(\mathbf{x}(\tau)\) be an optimal trajectory which contains both singular and nonsingular subarcs, and let the singular subarcs be of \(q\)th order, i.e.,
\[
\mathbf{H}(\xi,\tau,\omega) = \partial_1 \mathbf{H}(\xi,\tau,\omega) = 0
\]
(2.13)

Suppose the optimal control is piecewise analytic in a neighborhood of a junction (this is not always the case as is shown in Reference 11), and \(\mathbf{H}(\xi,\tau,\omega) \neq 0\) at the junction. If \(\mathbf{H}(\xi,\tau,\omega) = 0\), where \(\mathbf{H}(\xi,\tau,\omega) = 0\) is the lowest order derivative of \(\mathbf{x}\) which is discontinuous at the junction, then \(q = 0\) must be an integer.

The main conclusion of this theorem is that if a Taylor series expansion is valid in the neighborhood of a junction and the control is discontinuous at the junction, then the singular subarc must be of odd order. The singular Saturn guidance problem contains odd order (\(q = 1\)) singular arcs. Note that the theorem does not imply that the optimal control must jump if \(q = 0\). Also, there exist well known cases of \(q\)-even problems with discontinuous controls but the controls are not piecewise analytic (e.g., an infinite number of switches between \(u = 0\) and \(u = 1\) on the nondiscontinuous side of the junction in finite time interval).

With the analyticity assumption removed, the following result can be obtained:

**Theorem 2.** Let \(\mathbf{x}(\tau)\) be an optimal trajectory which contains both singular and nonsingular subarcs, where the singular subarcs are of \(q\)th order. Then,

(a) \(\mathbf{H}(\xi,\tau,\omega) = 0\) on the nonsingular side of the junction implies the control must be discontinuous;

(b) \(\mathbf{H}(\xi,\tau,\omega) = 0\) on the singular side of the junction and \(\mathbf{H}(\xi,\tau,\omega) = 0\) at the junction imply the control must be continuous.

Theorem 2 has the desirable quality of determining if the control "jumps" or is continuous at a junction without an analyticity assumption. However, the conditions are more difficult to verify than those of Theorem 1 if analyticity is a valid assumption. In some cases the theorems can be used together to indicate what one cannot assume, e.g., \(\mathbf{H}(\xi,\tau,\omega) = 0\) on the nonsingular side of the junction and \(q = 0\) even imply that \(u = 0\) is discontinuous by Theorem 2, but by Theorem 1, \(u = 0\) must be continuous.

Finally, another useful property [111] is:

**Property 1.** If \(A(x,\tau,\omega) \geq 0\) at a junction and
\[
[u \leq 0 \leq K, u + \mathbf{H}(\xi,\tau,\omega) \leq 0 \leq K, u + \mathbf{H}(\xi,\tau,\omega) \leq 0 \leq K]
\]
(2.9)

The system is then a singular control.

**Property 2.** If \(A(x,\tau,\omega) > 0\) at a junction and
\[
[u \leq 0 \leq K, u + \mathbf{H}(\xi,\tau,\omega) \leq 0 \leq K, u + \mathbf{H}(\xi,\tau,\omega) \leq 0 \leq K]
\]
(2.10)

The system is then a nonsingular control.

**Property 3.** If \(A(x,\tau,\omega) = 0\) at a junction and
\[
[u \leq 0 \leq K, u + \mathbf{H}(\xi,\tau,\omega) \leq 0 \leq K, u + \mathbf{H}(\xi,\tau,\omega) \leq 0 \leq K]
\]
(2.11)

Then the system is a switching control.

**property 4.** If \(A(x,\tau,\omega) < 0\) at a junction and
\[
[u \leq 0 \leq K, u + \mathbf{H}(\xi,\tau,\omega) \leq 0 \leq K, u + \mathbf{H}(\xi,\tau,\omega) \leq 0 \leq K]
\]
(2.12)

Then the system is a switching control.

Because of the simplicity of this problem, the optimal control may be determined by inspection. At \(t_1\), the steering angle is \(\psi_1 = \psi_0\), so if \(\psi_0\) could change to any other value instantaneously, then the optimal trajectory would be a straight line connecting \(\psi_0, \psi_1, \psi_2, \ldots\) and the initial steering angle could be \(\psi_0 = \tan^{-1} \mathbf{y}(0, \mathbf{z}(0))\), i.e., the velocity vector would swing to the dashed arrow at the origin in Figure 1. Since the best steering angle at any state is the angle between the x-axis and the line connecting the state with \(\psi_0, \psi_1, \psi_2, \ldots\), the optimal control \(u \leq 0 \leq K\) is the control which will cause \(\mathbf{y} = 0\) to arrive at the "best" point of a fast as possible. Thus, on the subarc \((t_1, \mathbf{y}(t_1)\), \(u = \mathbf{K}\) is the optimal control, which is nonsingular. At \(t_1, \mathbf{y}(t_1)\), \(u = \mathbf{K}\) and \(\mathbf{y}(t_1)\) is the true steering angle \(\psi_0\) and \(\psi_1\), which is the steering angle that the vehicle may now be steered by \(\psi = \tan^{-1} \mathbf{y}(t_1)\) with \(\mathbf{K}\) as the steering angle, which is a singular control. Since the steering angle is mainly dependent upon \(\mathbf{K}\), \(K\), and the terminal conditions. For example, if \(\psi_0 < 0\), then the optimal control is totally singular, whereas the other control is a nonsingular control for \(\mathbf{K}\), which is a singular control. The main conclusion of these statements is that the joining of a singular subarc to a nonsingular subarc is a function of "nonsingular" information. This complicates considerably the procedure of synthesizing optimal singularity guidance laws on a board vehicle.

**IV. Saturn Guidance. A Singular Flat-Earth Problem**

In this section, an analysis of the switching procedure for a flat-earth representation of circular trajectory will be given. The results obtained will be directly applicable to the Saturn V vehicle since the IGMs are based upon the flat-earth approximation. However, the condition of an exceptional case, except for the flat-earth approximation. The case occurs when the control does not "ride" onto or off of the control boundary.

The planar equations of motion and boundary conditions for the singular flat-earth problem are (see Figure 2):

\[
sin \theta = \frac{x}{r}, \quad \cos \theta = \frac{y}{r}, \quad \frac{dx}{dt} = v \cos \theta, \quad \frac{dy}{dt} = v \sin \theta
\]
(4.1)
\begin{align*}
\dot{q} &= \frac{p}{m} \sin \gamma - g q, \quad q(0) = q_0, \quad q(\infty) = 0 \\
\dot{\psi} &= u, \quad u(0) = u_0, \quad |u| \leq 2K \\
\text{min} &= u^2 + m \dot{\psi}^2 - w^2 - \omega^2 
\end{align*}

where \( e \) is the radius of the circular orbit, \( v_e \) is circular velocity at \( e \). It is desired to transfer the vehicle from the given initial conditions into the circular orbit in minimum time.

The Hamiltonian is

\begin{equation}
H = \lambda_1 p + \lambda_2 q + \frac{p^2}{2m} \sin \gamma + \lambda_4 q
\end{equation}

which implies, by Eqs. (4) and (5),

\begin{equation}
\lambda_1 = 0 \\
\lambda_2 = c_2 \\
\lambda_4 = c_4 \\
\lambda_5 = -\lambda_3 \cos \alpha - \lambda_4 \sin \alpha \quad (\alpha = 0, \text{at a continuous junction})
\end{equation}

If an optimal singular subarch exists, then by the Kelley condition:

\begin{equation}
H = \frac{p}{m} \sin \gamma - \lambda_4 q \\
\frac{p}{m} \lambda_4 \cos \gamma - \lambda_3 \sin \alpha = 0
\end{equation}

implies

\begin{equation}
H, \lambda_1 \geq 0. \quad (\text{at a singular arc})
\end{equation}

By Eq. (4), if \( u \neq 0 \) on a singular arc,

\begin{equation}
\tan \alpha = \frac{c_4 + c_2}{c_3} \quad (\sin \gamma = 0, \quad \lambda_3 \neq 0)
\end{equation}

and by Eq. (4),

\begin{equation}
H, \lambda_1 \geq 0 \quad (\text{at a singular arc})
\end{equation}

which implies that the minima should be chosen in Eq. (4) for the minimum time problem. Upon substitution of Eqs. (4) and (5) into Eq. (4),

\begin{equation}
H = \lambda_1 p + \lambda_2 q + \frac{p^2}{2m} \sin \gamma \geq 0 \quad (\text{at a singular arc})
\end{equation}

We shall now consider under what conditions a saturation junction is possible, i.e., the control rides on or off the boundary (or, is continuous at the junction).

If the control is continuous and well-behaved at the junction, then Theorem I of Section 2 is applicable, i.e., the control is piecewise analytic in a neighborhood of the junction. Since \( q = 1 \) for this problem, if the control is assumed to be continuous at the junction, then by Theorem I, \( r \leq 1 \) (i.e., if \( r \neq 0, \text{then } \psi \leq 1 \text{ since } \psi \text{ must be odd} \)). Thus, if \( u \text{ is continuous, then } u + i \text{ is continuous also},

By Eq. (4), the expression for \( u \) on the singular arc may be determined.

\begin{equation}
\dot{u} = \frac{c_4 + c_2}{c_3} \lambda_3 (\text{on a singular arc}).
\end{equation}

Since \( \dot{u} \geq 0 \) on the singular arc, it follows that \( u \geq u_0 \) (on a singular arc).

\begin{equation}
0 = (u - u_0) \geq 0 \quad (\text{at a continuous junction})
\end{equation}

Therefore, the continuity of \( u \) at the junction requires \( u = u_0 \) at the junction, which implies

\begin{equation}
c_3 = 0 \quad (\text{at a continuous junction})
\end{equation}

If \( c_2 = 0 \), then

\begin{equation}
\lambda_2 = c_2 \quad (\text{at a continuous junction})
\end{equation}

or by Eq. (4),

\begin{equation}
\sin \gamma = 0 \quad (\text{at a continuous junction})
\end{equation}

\begin{equation}
\sin \gamma = 0 \quad (\text{at a continuous junction})
\end{equation}

\end{align*}

\begin{align*}
\text{Eq. (5.6) can be transformed into an equality constraint by introducing a slack variable, } z, \text{ for } \gamma \leq 0. \\
\text{is an equality constraint which enforces the desired inequality constraint. By defining the augmented function}
\end{align*}

\begin{equation}
h(u, z) = \lambda_4 u + l^T \lambda_3 (\lambda_3 + z),
\end{equation}

and forming

\begin{equation}
\begin{bmatrix}
\dot{u} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
0 \\
\frac{\gamma}{\gamma}
\end{bmatrix}
\end{equation}

and then checking the second-order sufficient condition for ordinary minimization problems, the following optimal control is determined:

\begin{equation}
u = \begin{cases}
\lambda_4 \leq 0 \text{ or } & 2K \leq 2K \\
\lambda_4 \geq 0 \text{ or } & 2K \leq 2K
\end{cases}
\end{equation}

Note that the control is continuous at the junction points \( \lambda_4 = \pm 2K \). Since \( \lambda_4 \) must be continuous by the Weierstrass-Erdmann corner conditions.

The usual Euler-Lagrange equations hold for the multipliers. The only other new condition of interest is the transversality condition for \( \lambda_4 \). Since \( \lambda_4 \) is unspecified, then

\begin{equation}
\lambda_4 l = 0,
\end{equation}

which implies that \( -2K \leq \lambda_4 \leq 2K \). or

\begin{equation}
\lambda_4 (2K) \geq 0 \text{ or } \lambda_4 (2K) \leq 0
\end{equation}

This states that the control must have an interior segment in a neighborhood of \( u \). However, in some numerical studies for small terminal interior arc was very short, e.g., 0.1 seconds of a 160 second trajectory.

Since the guidance scheme of Reference 7 and 8 involves an iteration scheme which uses initial Lagrange multiplier estimates, a similar scheme was used to converge the optimal trajectories of this study. Since a sufficient condition for composite singular arcs does not exist, we can only use physical reasoning to argue that the resultant singular extremals are indeed optimal.

In Figure 4, the "best" steering angle history from the initial position and velocity of the vehicle (see Appendix A for the numerical values used in this study) is shown. The initial position and velocity represent a point on the 512 B state trajectory of the Saturn SA-502 flight. Note that the desired steering angle at the given initial position and velocity is \( \gamma = 17.1 \). Since the steering range on the Saturn is constrained to approximately one degree per second, one cannot assume that the steering angle can change instantaneously to the desired value.
In Figure 5, the suspected optimal time rate of change of the steering angle is presented. The optimal control is nonsingular on the interval $t_i$, $t_f$ and singular on the interval $(t_i, t_f)$. Note that the control is discontinuous at the junction, which is expected since the flat-earth problem is an excellent approximation of the problem of control. Note also the difference in the initial value of $y$ causes the constrained trajectory to be approximately 6.5 seconds longer than the trajectory of Figure 4, which results in a 3500 pound fuel burn.

The optimal control for a nonsingular approximation of the given singular problem is presented in Figure 5, also. The value $t_2 = 100.000$ was found to give good results with respect to optimality and ease of convergence. Note that the suboptimal control is continuous, and in some sense approximates the optimal singular control. The final time of the suboptimal trajectory is $t_i = 157.352$, which represents a fuel penalty of only 46.5 pounds.

A puzzling trend was encountered when the $c$-method was used for converging the suboptimal trajectories of this study. It was found that lower values of the original performance index, i.e., Eq. (5.2), were obtained as $c$ increased instead of the usual decreased, which at first glance seems contrary to intuition. Of course this trend may be due to the fact that an initial multiplier guessing scheme was used to convert the trajectories instead of a function space method. However, the Pontryagin minimum principle was satisfied numerically in each case.

A possible explanation of the above trend is simply that the augmented performance index of Eq. (5.3) does not converge to the minimum value of the original performance index $J_0$ for this particular problem as $c \to 0$. The proof of convergence for the $c$-algorithm in Reference 5 for fixed $t_f$ and $u$ is in control of the problem, convergence cannot be assumed. Indeed, further analysis revealed that the augmented performance index decreased to a minimum value, and appeared to be converging to a value considerably larger than the minimum value of the original performance index. In other words, as $c$ decreased, $J_2$ decreased, but $t_f$ increased, indicating that $lim J_2(t_f) \neq 0$.

To lend support to our contention that the $c$-algorithm may not converge to the optimal singular solution for a minimum time problem, consider the following argument. From Eq. (5.3) the minimum value of $J_2$ for a particular value of $c$ can be written

$$J_2 = J_2(t_i) + c \omega(t_i)$$

Therefore $c(t) > 0$ is the average value of the optimal $c$-solution over the interval $[t_i, t_f]$ for the given value of $c$, and for simplicity we have taken $t_i = 0$. Differentiating Eq. (5.13) by the chain rule, we get

$$\frac{dJ_2}{dt} = \frac{dJ_2}{dt_f} + \omega(t_f)$$

**References**


