of various functions are depicted in Fig. 1, and the heat-transfer ratio is found to be

$$\dot{q}_w/(\dot{q}_w)_{\omega=0} = 0.08025/0.08027 \approx 1$$

which indicates negligible spinning effect on convective heat transfer for flows in the stagnation-point region under hypersonic flight conditions. This result was predicted by Scala and Workman,[3] and experimentally verified by Whitesel.[5]

Another interesting case is that concerning small but finite crossflow for which the higher-order terms in $\hat{g}$, α, K_1, and their lateral derivative $\partial/\partial\zeta$ are neglected. Then, Eqs. (16–18) become (for $P_r = 1$)

$$f''' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) ff'' + \beta(\theta - f'^2) = 0 \quad (22)$$

$$\hat{g}''' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) f\hat{g}'' + K_1\hat{m}(f'^2 - \theta) - \left(K_2 + \frac{\beta}{\hat{m}}\right) f'g' = 0 \quad (23)$$

$$\theta'' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) f\theta' = 0 \quad (24)$$

If we multiply Eq. (23) by e_2 and substitute $(e_2\hat{g})' = V(\eta)G(\xi)$ therein, we obtain

$$V'' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) fV' - \left(\frac{\beta}{\hat{m}} + \frac{2\xi}{G}\frac{dG}{d\xi}\right) f'V = \frac{\hat{m}K_1e_2}{G}(\theta - f'^2) \quad (25)$$

It is noted that Eqs. (22, 24, and 25) are essentially those obtained by Beckwith[1] except for the coefficient constants.

Finally, for the case of similar flows in the plane of symmetry of an inclined axisymmetric body with zero streamwise pressure gradient and insulated walls, the following conditions prevail: $e_1 = 1$, $e_2 = r(x)$, $K_1 = 0$, $\beta = 0$, and $\partial g/\partial\eta = 0$. In order to transform the resulting equations into a familiar form, we first differentiate Eq. (14) with respect to ζ^*, which is defined as $r\zeta^* = \zeta$. Then, with the aid of the following definitions:

$$\hat{F} = (1 + K_2)f \qquad \hat{G} = \frac{\partial g}{\partial \zeta^*} \qquad K^* = \frac{\partial}{\partial\zeta^*}\left(\frac{\beta^*}{rZ}\right)$$

$$C_1 = \frac{2\xi}{r} \qquad C_2 = \frac{2\xi}{r}(1 + K_2)^{-1}$$

we obtain

$$\hat{F}''' + (\hat{F} + C_1\hat{G})\hat{F}'' = 0 \quad (26)$$

$$\hat{G}''' + (\hat{F} + C_1\hat{G})\hat{G}'' + K^*(\rho_e/\rho) - C_1\hat{G}'^2 - C_2\hat{F}\hat{G}' = 0 \quad (27)$$

which are the governing equations for supersonic flows in the plane of symmetry of a yawed cone with insulated surface.[6]

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Large Amplitude Vibration of Buckled Beams and Rectangular Plates

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Nomenclature

a, b, h	=	plate width, length, and thickness (x, y, z directions), respectively
r	=	a/b, plate aspect ratio
t	=	time
u, v, w	=	displacements in the x, y, z directions, respectively
D	=	plate flexural rigidity, $Eh^3/12(1 - \nu^2)$
EI	=	beam flexural rigidity
F	=	stress function
ρ	=	mass density
ν	=	Poisson's ratio

Introduction

In recent years, a number of investigations of the large amplitude vibration of beams[1–4] and flat rectangular plates[5–8] have been reported in which the ends of the beams and the edges of the plates have been assumed to remain a fixed distance apart during vibration. In particular, Burgreen[2] has considered the free vibration of a simply supported beam that has been given an initial end displacement, and the author[8] has considered free and forced vibration of simply supported and clamped beams and rectangular plates for which initial end and edge displacements have been prescribed. In both reports, a one-degree-of-freedom representation of the equations of motion is used. Results are obtained for edge displacements in the postbuckling as well as the prebuckling region. In the case of forced motion, however, the results were restricted to symmetrical motion about the flat position of the beam or plate. For the buckled beam or plate, it is also possible to have vibration about the static buckled position. This has been discussed for free vibration in the forementioned reports, and it is the purpose of the following remarks to extend the discussion to a case of forced motion.

Equations of Motion

The differential equation of motion for a beam of unit width is

$$\rho h w_{,tt} + (EIw_{,yy})_{,yy} - \frac{Eh}{b}\left[v_0 + \frac{1}{2}\int_0^b (w_{,y})^2 dy\right] w_{,yy} = P(y, t) \quad (1)$$

where v_0 represents an initial axial displacement measured from the unstressed state. For a plate, the dynamic von Kármán equations are

$$\begin{aligned} \nabla^4 F &= E(w_{,xy}{}^2 - w_{,xx}w_{,yy}) \\ D\nabla^4 w - h(F_{,yy}w_{,xx} + F_{,xx}w_{,yy} - 2F_{,xy}w_{,xy}) + \rho h w_{,tt} &= P(x, y, t) \end{aligned} \quad (2)$$

where

$$\sigma_x = F_{,yy} \qquad \sigma_y = F_{,xx} \qquad \tau_{xy} = -F_{,xy}$$

are the membrane stresses. When a single mode is assumed and Galerkin's method is applied, the problem reduces to the solution of a single ordinary differential equation in time.

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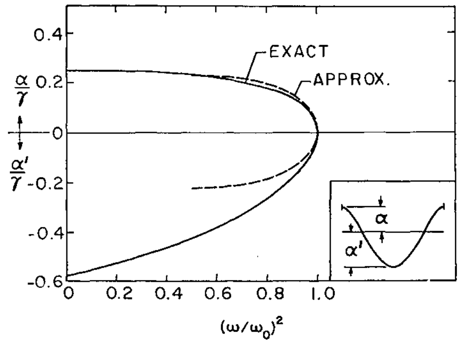

Fig. 1 Free vibration.

In the case of a simply supported beam, for example, we assume

$$w(y, t) = b\xi(t)\sin(\pi y/b) \tag{3}$$

and obtain the following equation in nondimensional form:

$$\xi_{,\tau\tau} + p\xi + q\xi^3 = f(\tau) \tag{4}$$

where

$$p = \frac{\pi^4}{12}\gamma^2(1-\lambda) \qquad q = \frac{\pi^4}{4}$$

$$\tau = \left(\frac{E}{\rho}\right)^{1/2}\frac{t}{b} \qquad \gamma = \frac{h}{b}$$

The parameter λ is a measure of the initial axial displacement and is defined as

$$\lambda = v_0/v_{0_{cr}} \tag{5}$$

where $v_{0_{cr}}$ is the axial displacement that produces the buckling load. Thus $\lambda > 1$ refers to the postbuckling region. An equation of the same form is obtained for other beam boundary conditions and for plates as well. The coefficients p and q for simply supported and clamped beams and rectangular plates are given in Ref. 8. The remarks that follow apply to these cases as well as others that may be defined.

To study the motion about the static buckled position, it is convenient to change to the variable

$$\delta = \xi - \xi_1 \tag{6}$$

where ξ_1 is the static buckle amplitude and δ is the variation from that position. If Eq. (6) is substituted into Eq. (4), it follows that for harmonic forcing

$$\delta_{,\tau\tau} + \omega_0^2\delta + c_2\delta^2 + c_3\delta^3 = \bar{f}\cos\omega\tau \tag{7}$$

where

$$\omega_0^2 = p + 3q\xi_1^2 \qquad c_2 = 3q\xi_1 \qquad c_3 = q$$

Note that ω_0 is the linear vibration frequency about the buckled position. The problem of small amplitude vibration of a buckled plate has been more fully discussed elsewhere.[9, 10]

This equation is of similar form to an equation derived for the vibration of initially curved plates and shells.[11–13] The Linstedt-Duffing perturbation technique[11,13] used in two of the preceding reports may be applied here. Let

$$\begin{aligned} \delta &= \delta_0 + \alpha\delta_1 + \alpha^2\delta_2 + \alpha^3\delta_3 + \dots \\ \omega^2 &= \omega_0^2 + \alpha\omega_1^2 + \alpha^2\omega_2^2 + \alpha^3\omega_3^2 + \dots \\ \bar{f} &= \bar{f}_0 + \alpha\bar{f}_1 + \alpha^2\bar{f}_2 + \alpha^3\bar{f}_3 + \dots \end{aligned} \tag{8}$$

where the initial conditions on Eq. (7) are taken to be

$$\delta(0) = \alpha \qquad \delta_{,\tau}(0) = 0 \tag{9}$$

from which it follows that

$$\delta_1(0) = 1 \qquad \delta_0(0) = \delta_2(0) = \delta_3(0) = \dots = 0 \tag{10}$$

It is convenient to introduce the forcing function as follows: let

$$\bar{f}_0 = \bar{f}_1 = \bar{f}_2 = 0 \tag{11}$$

so that

$$\bar{f} = \alpha^3\bar{f}_3 \tag{12}$$

Then it follows that when Eqs. (8) are substituted into Eq. (7) and terms are collected according to the power of α, a series of equations are obtained. The first is

$$\alpha^0: \quad \delta_{0,\tau\tau} + \omega^2\delta_0 + c_2\delta_0^2 + c_3\delta_0^3 = 0 \tag{13}$$

which has as its solution, in view of Eqs. (10),

$$\delta_0(\tau) = 0 \tag{14}$$

and next

$$\alpha^1: \quad \delta_{1,\tau\tau} + \omega^2\delta_1 = 0 \tag{15}$$

which has as its solution

$$\delta_1(\tau) = \cos\omega\tau \tag{16}$$

Continuing, we obtain

$$\alpha^2: \quad \delta_{2,\tau\tau} + \omega^2\delta_2 = -(c_2/2) + \omega_1^2\cos\omega\tau - (c_2/2)\cos2\omega\tau \tag{17}$$

To insure a periodic solution, it is necessary that

$$\omega_1^2 = 0 \tag{18}$$

thus the solution to Eq. (17) becomes

$$\delta_2(\tau) = (c_2/6\omega^2)(-3 + 2\cos\omega\tau + \cos2\omega\tau) \tag{19}$$

Finally,

$$\alpha^3: \quad \delta_{3,\tau\tau} + \omega^2\delta_3 = -\frac{c_2^2}{3\omega^2} + \left(\omega_2^2 + \frac{5}{6}\frac{c_2^2}{\omega^2} - \frac{3}{4}c_3 + \bar{f}_3\right)\cos\omega\tau - \frac{c_2^2}{3\omega^2}\cos2\omega\tau - \left(\frac{c_2^2}{3\omega^2} + \frac{c_3}{4}\right)\cos3\omega\tau \tag{20}$$

Once again, to insure a periodic solution, it is necessary that

$$\omega_2^2 = -\frac{5}{6}\frac{c_2^2}{\omega^2} + \frac{3c_3}{4} - \bar{f}_3 \tag{21}$$

which, from Eqs. (8) and (12), may be written

$$\omega^2 = \omega_0^2 + \alpha^2\left(\frac{3c_3}{4} - \frac{5}{6}\frac{c_2^2}{\omega^2} - \frac{\bar{f}}{\alpha}\right) \tag{22}$$

It is worth noting that the Ritz-Galerkin and related methods as they are commonly applied are inadequate for obtaining an approximate solution to Eq. (7). It is common practice to use the solution of the corresponding linear equation as an assumed solution of the nonlinear equation. The frequency-amplitude relation is obtained by means of a certain time integration over a cycle of the motion which minimizes the error introduced by this assumption. Unfortunately the restriction imposed by the assumed solution is such that all contributions of the term $c_2\delta^2$ are lost in the integration, regardless of its actual influence. If, however, more care is used in the selection of an assumed function, this difficulty may be overcome.

In our case, let

$$\delta(\tau) = A + B\cos\omega\tau + C\cos2\omega\tau \tag{23}$$

and proceed with the Ritz-Galerkin method as described, for example, in Ref. 14. We then obtain four algebraic equations in the four unknowns ω^2, A, B, and C, but because of the complexity of the equations, no general algebraic solution is possible. If, in addition, we let

$$\begin{aligned} \omega^2 &= \omega_0^2 + \alpha\omega_1^2 + \alpha^2\omega_2^2 + \dots \\ A &= \alpha A_1 + \alpha^2 A_2 + \alpha^3 A_3 + \dots \\ B &= \alpha B_1 + \alpha^2 B_2 + \alpha^3 B_3 + \dots \\ C &= \alpha C_1 + \alpha^2 C_2 + \alpha^3 C_3 + \dots \end{aligned} \tag{24}$$

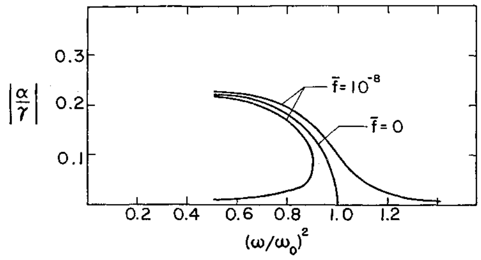

Fig. 2 Forced vibration.

and solve the resulting equations, we arrive at the identical frequency-amplitude relation given by Eq. (22).†

Free Vibration

The relation for free vibration may be obtained by setting = 0 in Eq. (22). An exact solution for free vibration is also possible in this case in terms of elliptic functions. It is

$$\delta(\tau) = (\alpha + \mu)dn(\bar{\omega}\tau, k) - \mu \tag{25}$$

where

$$\mu = \frac{c_2}{3c_3} \qquad \bar{\omega}^2 = \frac{c_3(\alpha + \mu)^2}{2} \qquad k^2 = \frac{(\omega_0{}^2 - c_2\mu)}{\bar{\omega}^2} + 2$$

The initial conditions are

$$\delta(0) = \alpha \qquad \delta_{,\tau}(0) = 0 \tag{26}$$

where α is positive and subject to the condition

$$-2 < \frac{(\omega_0{}^2 - c_2\mu)}{c_3(\alpha + \mu)^2} < -1 \tag{27}$$

Physically this restricts consideration to motion about the buckled position on one side of the flat position. The period of the motion is given by the elliptic integral

$$T = \frac{2K}{\bar{\omega}} = \frac{2}{\bar{\omega}}\int_0^{\pi/2} \frac{d\phi}{1 - k^2 \sin^2\phi} \tag{28}$$

It should be noted that Eq. (25) is not a general solution to Eq. (7) for arbitrary values of the coefficients $\omega_0{}^2$, c_2, and c_3 but only for free motion when the relation

$$9\omega_0{}^2c_3 - 2c_2{}^2 = 0 \tag{29}$$

holds. This condition is satisfied in this case because Eq. (7) was obtained from Eq. (4). An exact solution in terms of elliptic functions is still possible, however, as described in Ref. 12, for example.

Numerical Results

Numerical results have been obtained from Eqs. (22) and (28) for the special case of a buckled beam with $\lambda = 2$ and $\gamma = 0.005$ and are presented in Fig. 1 for free motion. In this figure, the amplitude is given in terms of the number of beam thicknesses and the frequency in terms of the square of the ratio of the nonlinear to the linear frequency. Since for the exact solution the motion is not symmetrical about the undeflected position, the curve for a negative initial condition differs from that for a positive initial condition. A typical deflection curve for a cycle of the motion is shown in the lower right of the figure. The negative amplitude α' is related to α by

$$\alpha' = (\alpha + \mu)(1 - k^2)^{1/2} - \mu \tag{30}$$

In Fig. 2, the dynamic response of the same beam to harmonic forcing is shown as obtained from Eq. (22).

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† This was pointed out to the author by E. F. Masur.

Approximate Absorption Coefficients for Vibrational Electronic Band Systems

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Introduction

It is frequently necessary to calculate both spectral and total gas radiation arising from the shock layers, boundary layers, wakes, or exhaust plumes associated with hypersonic vehicles. Such calculations may be required in order to assess radiant heat transfer to vehicle surfaces, interference with onboard optical devices, or detection by remotely located instruments. In addition, gas radiation emission may be used as a diagnostic device in hypervelocity testing.

There are a number of sources of detailed spectral absorption or emission data for the major constituents of equilibrium air.[1–3] Similar information about other gas compositions such as air-ablation product mixtures, rocket exhausts, and extraterrestrial atmospheres is not so readily available, however. It is the purpose of this note to describe an approximate method by which spectral radiation can be estimated rapidly for an important class of radiators, namely, vibrational-electronic band systems of diatomic molecules.

Detailed Spectral Absorption Estimates

Meyerott et al.[2] present the radiative contributions from individual band systems in terms of average spectral absorption coefficients from which spectral emission may be

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