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ANALYSIS OF THREE DIMENSIONAL FLOW FROM A JET UP TO A CONICAL FLAME FRONT WHEN THE NORMAL VELOCITY AT THE FLAME FRONT IS A CONSTANT.

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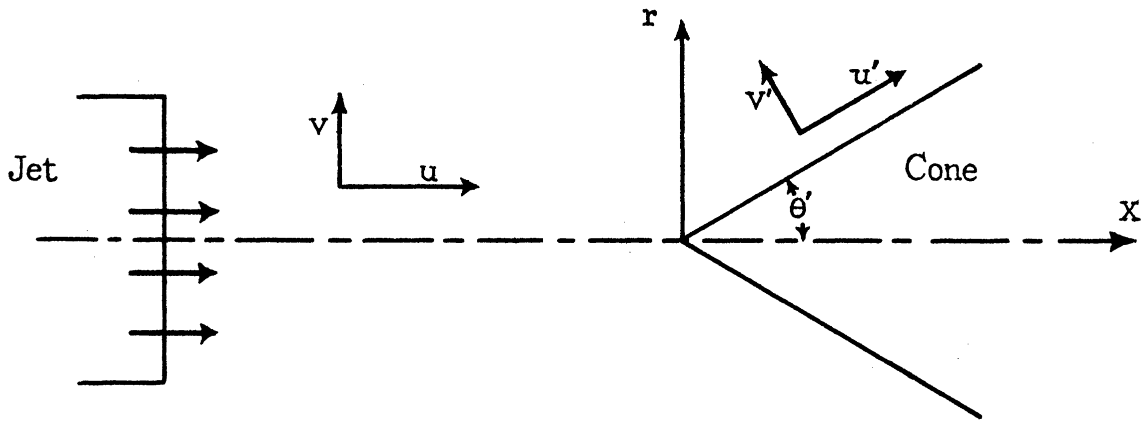
Analysis of three dimensional flow from a jet up to a conical flame front when the normal velocity at the flame front is a constant.

It will be shown that the problem cannot be solved by present hydrodynamic and aerodynamic methods.



$$u = u' \cos \theta' - v' \sin \theta' \qquad (1)$$

$$v = u' \sin \theta' + v' \cos \theta' \qquad (2)$$

where u and v are velocities parallel to the x-and r-axes respectively and u' and v' are velocities tangent to and normal to the flame front respectively.

At the flame front

$$r = x \tan \theta' \qquad (3)$$

From the new boundary condition

$$v' = f(\theta') \qquad (4)$$

A question may be raised as to why the axially symmetric cone problem cannot be handled as we did the wedge (see EMV3). Both problems are essentially two dimensional. In the wedge there is cross sectional symmetry. That is, if you can handle the flow for one cross section, you are handling the flow for all cross sections and thus the total solution in the plane of the cross-section is the solution for all streamlines. In the axially symmetric cone case, if you can handle the solution in the plane containing the axis of the cone, because of rotational symmetry one would expect that you would be handling the total solution for all such planes of similar cross-section. The answer to this question is not obvious, because the standard answer, that the differential equations are different, is not necessarily the correct one. This implies that the classical methods of complex variable cannot be applied to the spatial axially symmetrical problem. There are cases in the literature where the use of the complex variable has been successfully applied to the axially symmetric supersonic differential equations. (Ref. 1)

Since the technique employed in the wedge-like flame problem was to add the solution due to the solid wedge to the solution for flow if the wedge were not there, we will try to duplicate that technique. Let us then start out by trying to find the solution for flow about a finite solid cone. We will not use complex variable methods. However, if we do find the solution it would be wise to then convert to a complex potential function.

In the axially symmetric case,

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial r} \qquad (5)$$

$$v = \frac{1}{r}\frac{\partial\psi}{\partial X} \qquad (6)$$

$$\frac{\partial\psi}{\partial r} = -ur \qquad (7)$$

$$\frac{\partial\psi}{\partial X} = vr \qquad (8)$$

$$\psi = -\int ur\,dr \qquad (7)'$$

Since our plan is to find the solution for the solid cone first, we know that v' is equal to 0 at the boundary of the cone, which is the zero streamline.

From (7)' and (1) $\int u' \cos\theta' \, r\,dr = 0$ (9)

since $\theta' \neq \frac{\pi}{2}$ (as cone would no longer exist)

then $\int u' \, r\,dr = 0$ (10)

Since this equation holds for all limits of the integral finite or infinite. (See appendix I)

$$u' \equiv 0 \qquad (11)$$

Thus we have shown that in the axially symmetric case if we assume the normal velocity to the flame front is equal to zero we find that we are forcing the tangential velocity also to be zero.

Thus the axially symmetric solution for subsonic flow about cones with $\theta' > 0$ cannot be found by this method. It might be well to add that for cones of very small solid angle, this method will yield a result.

For if $\theta' \cong 0$

Equations (1) and (2) become

$$u = u' \qquad (1)'$$

$$v = u'\theta' \qquad (2)'$$

and thus equation (10) becomes

$$\int u' r \, dr = 0 \qquad (10)'$$

This equation holds for all u, thus (10)' must have the solution

$$r \equiv 0$$

But this is the X axis which is the streamline of a thin cone ($\theta' \cong 0$). Thus this method does yield approximate solutions for thin cones.

Returning to the case in which $\theta' > 0$, we must try to find a solution which satisfies the boundary conditions and also LaPlace's equation.

LaPlace's equations for axial symmetry are,

$$\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = 0 \qquad (12)$$

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = 0 \qquad (13)$$

Let us pick the most general solution satisfying (13) which doesn't contain the Bessel solution* of EMV2.

$$\psi = C_1 X + C_2 r^2 + C_3 X r^2 + C_4 \qquad (14)$$

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial r} = -2C_2 - 2C_3 X \qquad (15)$$

$$v = \frac{1}{r}\frac{\partial \psi}{\partial X} = \frac{C_1}{r} + C_3 r \qquad (16)$$

$$u = -\frac{\partial \phi}{\partial X} \qquad (17)$$

$$\phi = -\int u \, dx = +2C_2 X + C_3 X^2 + f(r) \qquad (18)$$

$$v = -\frac{\partial \phi}{\partial r} \qquad (19)$$

* Ruled out in EMV-5

$$\phi = -\int v\, dr = -C_1 \ln r - \frac{C_3 r^2}{2} + g(X) \qquad (20)$$

from (18) and (20)

$$\phi = -C_1 \ln r - \frac{C_3 r^2}{2} + 2\, C_2\, X + C_3\, X^2 + C_5 \qquad (21)$$

but ϕ (0,0) =0 (Boundary condition 1) (22)

thus $C_1 = 0 \quad C_5 = 0$ (23)

but $\psi(0,0) = 0$ (boundary condition 2) (24)

$C_4 = 0$ (25)

if $\theta' = 0 \quad u = U$
(boundary condition 3) (26)

This means that if the cone does not exist the flow will remain uniform rectilinear flow.

thus at $\theta' = 0$, u is a constant

and thus independent of X

but $u = -2C_2 \ -2C_3\, X$ (15)

this would only be true if at $\theta' = 0$ that $C_3 = 0$

$C_3 = h\ (\theta')$ where $h(0) = 0$ (27)

thus $C_3 = h\ (\theta\ ')$ (28)

thus $-2\, C_2 = U$

$$C_2 = -\frac{U}{2} \qquad (29)$$

now to regroup our knowledge

$$\psi = -\frac{U}{2} r^2 + h\ (\theta')\ X\ r^2 \qquad (30)$$

$$\phi = -h\ (\theta')\ \frac{r^2}{2} - U\ X + h\ (\theta')\ X^2 \qquad (31)$$

$$u = U - 2\ h(\theta')\ X \qquad (32)$$

$$v = h\ (\theta')\ r \qquad (33)$$

now rewriting (1) and (2) under condition (4) we have

$$u = u' \cos\theta' - f(\theta') \sin\theta' \tag{34}$$

$$v = u' \sin\theta' + f(\theta') \cos\theta' \tag{35}$$

now substituting (32) in (34) we obtain

$$u' = \frac{U - 2h(\theta')X + f(\theta')\sin\theta'}{\cos\theta'} \tag{36}$$

now substituting (33) in (35) we obtain

$$u' = \frac{h(\theta')r - f(\theta')\cos\theta'}{\sin\theta'} \tag{37}$$

now combining (36) and (37) and applying (3)

$$X = \frac{+f(\theta')\cot\theta' + \frac{U}{\cos\theta'} + f(\theta')\tan\theta'}{\frac{3h(\theta')}{\cos\theta'}} \tag{38}$$

This of course only holds at the flame front because of the use of θ' instead of θ and because we used conditions (3) and (4). However, equation (38) tells us that X is a constant at the flame front and this of course is not correct as the flame front has length and is not just a point. This must mean that equation (38) is really an indeterminate form.

$$\therefore\ \theta' \neq 0, \frac{\pi}{2} \qquad h(\theta') = 0 \tag{39}$$

$$f(\theta')\cot\theta' + \frac{U}{\cos\theta'} + f(\theta')\tan\theta' = 0 \tag{40}$$

$$f(\theta')\left(\frac{1}{\sin\theta'\cos\theta'}\right) + \frac{U}{\cos\theta'} = 0$$

$$f(\theta') + U\sin\theta' = 0$$

$$f(\theta') = -U\sin\theta' \tag{41}$$

now apply (39) and (41) to (37)

$$u' = U\cos\theta' \tag{42}$$

from (34)

$$u = U \cos^2 \theta' + U \sin^2 \theta' = U \qquad (43)$$

$$v = U \cos \theta' \sin \theta' - U \sin \theta' \cos \theta' = 0 \qquad (44)$$

Thus at the front we just have uniform rectilinear flow. This obviously means that the cone does not exist as it has had no effect upon the flow.

This just means that the conical flame problem cannot be solved by these hydrodynamic and aerodynamic methods. We expected this result as soon as we showed that the solid cone problem was unsolvable by these methods.

REFERENCES

1. An Investigation of the Exact Solution of the Linearized Equations for the Flow Past Conical Bodies.
By O. Laporte and R. C. F. Bartels
Bumblebee Series Report No. 75 February 1948

APPENDIX I

The potential equation is

$$\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0 \qquad (45)$$

At the flame front

$$r = X \tan \theta' \qquad (3)$$

thus $$\frac{\partial^2\phi}{\partial X^2} = \frac{\frac{\partial^2\phi}{\partial r^2}}{\tan^2\theta'} = \tan^2\theta' \frac{\partial^2\phi}{\partial r^2} \qquad (46)$$

Equation (45) becomes

$$(\tan^2\theta' + 1)\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0 \qquad (47)$$

$$\sec^2\theta' \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0$$

$$-\sec^2\theta' \frac{\partial\frac{\partial\phi}{\partial r}}{\partial r} - \frac{1}{r}\frac{\partial\phi}{\partial r} = 0$$

$$\sec^2\theta' \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \qquad (48)$$

but applying (2)

$$v = u' \sin\theta' + v' \cos\theta' \qquad (2)$$

and for the solid cone $v' = 0$

$$v = u' \sin\theta' \qquad (49)$$

$$\frac{\partial v}{\partial r} = \sin\theta' \frac{\partial u'}{\partial r} \qquad (50)$$

thus by substituting (50) and (49)

into (48)

$$\sec^2\theta' \sin\theta' \frac{\partial u'}{\partial r} + \frac{u' \sin\theta'}{r} = 0 \qquad (51)$$

$$\sec^2\theta' \frac{\partial u'}{\partial r} + \frac{u'}{r} = 0$$

but at the front u' is only a function of X and r and from (3) $r = X \tan\theta'$ thus u' is a function of r only and the partial derivative is really the ordinary derivative.

$$\sec^2\theta' \frac{du'}{dr} + \frac{u'}{r} = 0 \qquad (52)$$

$$\frac{du'}{u'} = -\frac{dr}{r} \cos^2\theta'$$

$$u' = Cr^{-\cos^2\theta'} \qquad (53)$$

but $\int u' \, r \, dr = 0$ (10)

now substitute (53) into (10)

$$\int C \, r^{1-\cos^2\theta'} \, dr = \int C \, r^{\sin^2\theta'} \, dr = 0$$

now integrating from 0 to L .

$$C \left. \frac{r^{\sin^2\theta' + 1}}{\sin^2\theta' + 1} \right|_0^L = 0$$

$$C_L^{\sin^2\theta' + 1} = 0 \qquad (55)$$

where $\theta' > 0$ and $L > 0$

thus equation (55) becomes a contradiction

unless $C = 0$ (56)

but $u' = C \, r^{-\cos^2\theta'}$ (53)

thus $u' = 0$ (54)

Q.E.D.