LONGITUDINAL DYNAMIC STABILITY OF A SHUTTLE VEHICLE

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LONGITUDINAL DYNAMIC STABILITY OF A SHUTTLE VEHICLE

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Abstract

This paper presents an analytical study of the longitudinal dynamic stability of a non-rolling, lifting vehicle that is gliding at hypersonic speeds. The analysis applies to shuttle vehicles that are designed for operating up to the rim of a planetary atmosphere. A general non-dimensional time transformation is introduced to derive a unified second order linear differential equation for the angle-of-attack, valid for all types of reentry of a general type of vehicle. The stability of motion is discussed for two fundamental regimes of flight that are based on various different assumptions. For nearly ballistic entry along a straight line trajectory, the equation reduces to a confluent hypergeometric equation, the solution of which can be expressed in terms of the Whittaker's function. Using a theorem in the theory of stability of differential equation, criteria for damped oscillations are derived. The critical case of small static stability derivatives is discussed briefly, in which case instability is pitch, is discussed in detail, and the critical altitude below which the vehicle is unstable is given in explicit form. For entry at small flight path angle, the unified equation reduced to a damped Mathieu's equation with periodic forcing term. Using the method of Krylov-Bogoliubov, an approximate solution is constructed. It is shown that the aerodynamic criteria for stability are the same as for the case of ballistic entry. In addition, for each vehicle configuration, and specified planetary atmosphere, there exists an altitude range where the angle-of-attack frequency is nearly equal to the orbital frequency, causing instability in pitch. This resonance instability is due to the ellipticity of the orbit. Criteria for eccentric stability are derived.

Note

The purpose of this paper is to discuss the longitudinal dynamic stability of a non-rolling, lifting vehicle that is gliding at hypersonic speeds. The analysis applies to shuttle vehicles that are designed for operating up to the rim of a planetary atmosphere.

In earlier studies, it is customary to formulate assumptions for a specific flight regime before deriving the dynamic stability equation. Brill and Dobbins 

Theoretical and experimental work by Dobbins and others have developed the aerodynamic force equations. This means that the analysis can be applied to a portion of ballistic entry along which the deceleration rate is being developed. Laitone then discussed the range of validity of the assumptions by deriving the classical second-order linear ordinary differential equation. This equation is integrated by the method of Krylov-Bogoliubov. It is shown that the aerodynamic criteria for stability are the same as for the case of straight line entry. In addition, there is always present a strong instability due to aerodynamic drag. Furthermore, for each vehicle configuration and specified planetary atmosphere, there exists an altitude range where the angle-of-attack frequency is nearly equal to the orbital frequency, causing instability in pitch. It is shown that the range of instability increases with the orbit eccentricity. Criteria for eccentric stability are then derived.
II. Unified Dynamic Equation

The motion of a non-rolling, lifting vehicle in a resisting medium and subject to the gravity force of a spherical planet is governed by the system of equations, for an axis system that rotates about the vehicle center of mass (as indicated in Fig. 1) so that the z-axis is always tangent to the instantaneous flight path.

\[ \frac{dv}{dt} = -\frac{1}{2} C_D v^2 + g \sin \gamma \]  
\[ \frac{d\gamma}{dt} = \frac{1}{2} C_L v^2 - g \sin \gamma \cos \gamma \]  
\[ \frac{d\psi}{dt} = \frac{1}{2} C_L v^2 - \frac{1}{2} g \sin \gamma \cos \gamma \]  
\[ \frac{d\theta}{dt} = v \sin \gamma \]  
\[ \theta = \psi + \theta \]  
\[ \phi = \text{constant} \]

The first two equations are, respectively, the drag and lift equations along the tangent and normal to the flight path. The first term on the right-hand side of the pitching moment equation, Eq. 3, expresses the restoring aerodynamic torque, while the second term corresponds to the gravity torque. The last three equations are kinematic relations. The mass density, \( \rho \), of the atmosphere, and the acceleration of the gravity, \( g \), are altitude dependent.

The elimination of \( \phi \) and \( \theta \) results in the following exact equation for the angle-of-attack \( \alpha \):

\[ \frac{d^2 \alpha}{dt^2} = 2C_D v^2 + 2C_L v^2 - \frac{1}{2} g \cos \gamma \]

\[ + \frac{1}{2} \rho g \frac{dv}{dt} \frac{d^2 v}{dt^2} \cdot \frac{d^2 v}{dt^2} - \frac{1}{2} g \cos \gamma \cos 2\gamma \]

(8)

Let

\( \zeta = \frac{v}{C_L} \), \( \psi = \frac{1}{C_L} \), \( \phi = \frac{1}{C_L} \frac{dv}{dt} \)

and use the time transformation

\[ \xi = \frac{1}{2} \sqrt{\int (v \cos \psi, \int (dt)^2 + (dt)^2) \int (dt)^2 \int (dt)^2} \]

(9)

The new independent variable, \( \xi \), can be identified as the reference length travelled along the trajectory of the center of mass. For small variations of the angle-of-attack, we may assume

\[ C_D \approx C_D + C_D \alpha \]

\[ C_L \approx C_L + C_L \alpha \]

In hypersonic flight, the aerodynamic derivatives in the linearization of \( C_D \) and \( C_L \) are approximately independent of flight speed and Mach number. This same assumption also applies to \( C_D \) and aerodynamic forces and moments have been shown to manifest this effect as

\[ C_{m_1} = C_{m_1} + C_{m_1} \alpha \]

\[ C_{m_2} = C_{m_2} + C_{m_2} \alpha \]

Upon substitution into Eq. (7), and within the validity of the linearized theory, we have the linear differential equation of the second order which governs the angle of attack oscillations:

\[ \alpha + 4\alpha + 2\alpha = f(t) \]  

where

\[ f(t) = f_1(t) + f_2(t) \]

(10)

(11)

The elimination of \( \alpha \) and \( \beta \) results in the following equation for the angle-of-attack \( \alpha \):

\[ \frac{d^2 \alpha}{dt^2} = 2C_D v^2 + 2C_L v^2 - \frac{1}{2} g \cos \gamma \]

\[ + \frac{1}{2} \rho g \frac{dv}{dt} \frac{d^2 v}{dt^2} \cdot \frac{d^2 v}{dt^2} - \frac{1}{2} g \cos \gamma \cos 2\gamma \]

(12)

The trajectory of the center of mass is known, the elements of the flight path, \( x, y, z \), and the atmospheric mass density, \( \rho \), can be evaluated as functions of the independent variable \( \xi \) and the unified equation, Eq. 12, uniquely determines the time history of the angle-of-attack oscillations for each prescribed set of initial conditions on \( \alpha \) and \( \alpha' \).

This assumes the so-called limited problem, that is the angle-of-attack oscillations have negligible effect on the trajectory. A successful analytical integration of Eq. (12) depends on the forms of the functions \( f_1(t) \), and in general the equation cannot be integrated in closed form. Fortunately, the use of the time transformation defined by Eq. (9) allows the reduction of the unified equation, Eq. (12), to well-known equations in mathematical physics, at least for the two fundamental types of entry trajectory discussed in this paper. Furthermore, the new time variable, \( \xi \), which is monotonically increasing for any flight trajectory, renders the coefficients \( f_1(t) \) well behaved and most often, with the aid of the vast literature in the theory of stability of ordinary differential equations, qualitative criteria can be derived without having to solve the equation. An illustrative example will be given in the next section.

III. Stability of Nearly Ballistic Entry

For nearly ballistic entry along a flight path with small curvature, at high deceleration rate, the most fundamental assumption is that in Eq. (1) we neglect the contribution of the gravity force. Then

\[ \frac{v^2}{2} = -\frac{1}{2} C_D v^2 \]

(13)

For this type of entry, the aerodynamic force is mainly drag force, and we can write

\[ C_D = C_D + C_D \alpha \]

\[ C_L = C_L \alpha \]

Thus, from Eq. (14), with the definition (9)

\[ v' = \frac{v}{2} \]

(15)

Also, for nearly straight line flight path, \( z \approx 1 \), and \( v \ll v' \). Equation (12) reduces to the classical form

\[ \alpha + 4\alpha + 2\alpha = 0 \]

(16)

where

\( b(\xi) = [C_D + C_L - 2C_D \alpha] \)

(17)

\[ c(\xi) = \frac{1}{2} [C_D + C_L] \]

(18)

The coefficients \( b \) and \( c \) are solely functions of the atmospheric mass ratio, which can be easily expressed in terms of the independent variable \( \xi \). In this case of straight line ballistic entry, Eq. (17) is equivalent to Allen's equation (6), and the variable \( \xi \) is equivalent to the altitude variable. To show this equivalence, let us consider the case of an isothermal atmosphere.

\[ \alpha = \rho \frac{v^2}{2} \]

(19)

and use the time transformation

\[ \frac{\partial x}{\partial t} = \frac{v}{C_D} \]

(20)

from the initial time

\[ x = 2 \frac{\rho}{C_D} \frac{v^2}{2} \]

and we have the linear relation between \( y \) and \( \xi \)

\[ y = \frac{\rho v^2}{2} \]

(21)

Consequently we have for the function \( b(\xi) \)

\[ b(\xi) = \rho \frac{v^2}{2} \]
The appropriate solution of Eq. (29) that will lead to a physically valid angle-of-attack variation for Eq. (24) is given by the confluent hypergeometric function

$$w(a) = I_1(a) + a I_0(a)$$

and

$$(a) = \frac{a}{a^2 + 1} \begin{pmatrix} a^2 + 1 & 1 \\ 1 & a^2 + 1 \end{pmatrix}$$

where

$$a = \frac{1}{k}, \; k = \frac{1}{k} + \frac{1}{k} + \frac{1}{k}.$$ 

This asymptotic representation of Eq. (30) is given by

$$w(a) = \frac{1}{2} I_0(2 \sqrt{a})$$

Upon introducing the above asymptotic solution of Eq. (27) we obtain an asymptotic solution of Eq. (24) identical to Eq. (25) which was obtained by Allen (1973) under more restrictive assumptions. As shown by Eqs. (31) and (32) the asymptotic representation, given by Eq. (35), is valid for large values of $a$ which occur whenever either $k_1$ or $k_2$ are much smaller than $k_2$. For the solid nose cone considered by Allen (1973), these values are given in Fig. 2, and it is seen that $k > 10^5$ for $10^6 < x < 10^6$, primarily because $k_2 < 10^3$ and $k_1 < 1$. Consequently, Allen's approximate solution as given in Eq. (46) is very satisfactory for this type of body at hypersonic speeds. However, for other types of bodies for which $k$, as defined by Eqs. (25) and (31) is not large, we must return to the exact solution of Eq. (24) which has been given by Eqs. (27) and (30).

$$w(a) = \frac{a^2}{4} I_0(2 \sqrt{a})$$

This exact solution provides an oscillatory variation for the angle-of-attack only if

$$2k \sqrt{a} > 2k \sqrt{a} + 2k \sqrt{a}$$

as shown by Slater (1960, p. 216). However, this relation, and the question of stability of the oscillation, are better analyzed by putting the confluent hypergeometric equation, Eq. (29), into the form of Whittaker's equation. Whittaker's transformation is

$$w(a) = \frac{1}{2} \left( 1 + \frac{k}{k} + \frac{k}{k} \right) - \frac{1}{2} \left( 1 + \frac{k}{k} + \frac{k}{k} \right)$$

The final equation is the Whittaker's equation

$$\nabla^2 W + \frac{\nu}{\nu^2 + 1} W = 0$$

In this form, we can see the importance of the constant coefficients $k$. By a proper selection of the signs of $k$, as defined by Eq. (28), we can always make both the constant $k$ and the argument of the confluent hypergeometric function $w(a)$ positive in the differential equation. It now appears that the damping constant $k_1$ and the parameter $k$ are the two important parameters for the stability of the entry vehicle. Allen's assumption fails when the moment stability derivative $CM_\phi$ is approaching zero. In this case the constant $k_1$ can be small. Explicitly, we have, by taking $CM_\phi = 0$

$$k_1 \frac{C_{D_a} + C_{L_B} + C_{M_\phi}}{4(B - h)}$$

or

$$k_1 \left( \frac{C_{D_a} + C_{L_B} + C_{M_\phi}}{4(B - h)} \right)$$

with the constant $k$ defined as

$$k = \frac{C_{D_a} + C_{L_B} + C_{M_\phi}}{4(B - h)}$$

In sign convention, $\gamma > 0$, and for the validity of the high deceleration assumption $\gamma = \gamma_0 = 45^\circ$. For a reference length of the order of $10^4$, and $CM_\phi > 10^3$, the coefficient $2/FL \sin \gamma_0 / 10^4$, would be the order of $10^3$. For practical cases, the static stability derivative $CM_\phi$ is negative, the constant $k_1$ is large, and the solution for $a$ is valid. Since it is unlikely that $CM_\phi$ be allowed to have positive value, the minimum value of $CM_\phi$ for a certain position of the center of mass would be zero and the limiting value of $k$ is $1/10^4$. For small values of $k$, Whittaker's solution can be unstable at a certain altitude and it is interesting to investigate the case of instability of flight due to the position of the center of mass. This is the case where Allen's assumption is no longer valid. This can be done by using Whittaker's equation, Eq. (30), and a theorem in the foundations of solutions of ordinary linear differential equations due to Lotz (1960). Following Lotz, we consider the linear differential equation

$$w(a) + \frac{a^2}{2} \frac{\nu}{\nu^2 + 1} W = 0$$

Then it is shown that

$$\phi(a) = \frac{\nu}{\nu^2 + 1} \phi(1) + \frac{\nu}{1} \phi(2)$$

and as long as $\phi(1)$ varies monotonically, that is it is either never decreasing or never increasing, even though varying, the magnitude of $W$ is bounded by

$$W(a) = M \exp \left( -\frac{1}{\nu^2 + 1} \right)$$

where

$$M = \sqrt{2 \pi^2 W(a)}$$

Now, applying criterion (40) to Eq. (30), we have

$$\phi(a) = \frac{a^2}{2} \frac{\nu}{\nu^2 + 1} \phi(2)$$

Since both $k$ and $\nu$ are positive, the monotonicity of the solution is assured. For positive values of $\phi(a)$, we must have

$$k^2 - 4k - 1 < 0$$

That is

$$k^2 - 2k - 1 < 0$$

This inequality gives the limit for the altitude, under which the boundedness of the Whittaker's solution is not assured. As has been mentioned, the criterion applies to the case where $CM_\phi$ is negligibly small. In general, $k_1 + k_2 < k$ and condition (41) holds at all altitudes, and Lotz's criteria lead to the bound

$$\gamma_0 \leq M \phi(1) \psi(2)$$

with $\gamma_0$ is decreasing, when $k_1 < 0$, the angle-of-attack oscillation is certainly stable. Then, we can say that, when $CM_\phi$ is not nearly zero, as long as

$$k_1 \leq k_2 \frac{CM_\phi}{4(B - h)}$$

the angle-of-attack is bounded, and a damped oscillation will occur during rapid descent along a straight line through an isothermal earth atmosphere.

When $k_1$ is positive, $\phi(a)$ can still be bounded by a decreasing function. However, in the exponential (47) is decreasing. This function passes through a minimum when

$$k_1 = k_2 \frac{CM_\phi}{4(B - h)}$$

This critical altitude for stability

$$k_1 = \frac{k_2}{k_2} \frac{CM_\phi}{4(B - h)}$$

The oscillation is stable above this altitude, and unstable below. The critical altitude obtained by Allen (1960) is

$$k_1 = \frac{k_2}{k_2} \frac{CM_\phi}{4(B - h)}$$

Thus, our criteria gives a larger margin, even for the general case, for the critical altitude than the one obtained with Allen's simplified analysis.

In summary, when $CM_\phi$ is nearly zero, the inequality (40) gives the stability criteria. However, the stability criterion is still valid for $CM_\phi$ near zero, condition (41) has to be replaced by the more restricted condition (46).

The same type of analysis applied to the case where $C_{D_a}$ is zero. In this case a small forcing term is present due to the perturbed lifting force. Also, it should be mentioned that, as has been pointed out by Allen, for high drag shapes, the velocity quickly approaches the so-called terminal velocity for which the drag and the weight

are equal, the recovery angle is getting steeper as the altitude increases, and below a certain altitude the assumptions of high deceleration rate, straight line trajectory are no longer valid. The transonic-hypersonic flight is in itself a different flight regime and deserves a separate study.

IV Stability of Shallow Gliding Entry

For gliding entry with a flight path that is nearly parallel to the earth's surface, it has been shown by Laitone and Choe (1955), and Vignes and Dubosz (1955) that the trajectory is a descending spiral which, for a few revolutions required for a stability analysis, can be approximated by a nearly circular orbit with equation

$$r = a(1 - \frac{1}{2})$$

where $r_0$ is the radius of the reference circular orbit, $\psi$ a small quantity which denotes here the eccentricity of the orbit and $r$ the true anomaly which defines the position of the vehicle along the orbit. For nearly circular orbit, $r$ is equal to the mean anomaly and we have

$$r = a \psi, \quad \psi = (1 - a^2)(1 - a^2) = a^2$$

where subscript zero denotes the condition along the reference circular orbit. The quantity $\psi$ is the one-dimensional orbit frequency, and the ratio $\psi$ of the velocity along the reference circular orbit to the circular velocity with drag, at the distance $r_0$, can be evaluated from

$$\psi = \frac{1}{\sqrt{a}} \frac{\sqrt{1 + a^2}}{1 - a^2}$$

The coefficient $\psi_0$ denotes the first important mass density gradient in general

$$\psi_0 = 1 + \frac{1}{2} \frac{\sqrt{1 + a^2}}{1 - a^2}$$

It is important, for an order of magnitude analysis, to mention that the coefficients $\psi_0$ are large, so that $\psi_0$ is of the order of $10^3$ as shown in Fig. 2. The coefficients are calculated from an inverse polynomial representation of the earth's atmosphere as given in the U.S. Standard Atmosphere Supplement, 1966. The values of $\psi_0$ are in excellent agreement with the values calculated from tabulated data in the altitude range 100-600 thousand feet.

To the order $r$, along the gliding trajectory, we have

$$r = a(1 - \frac{1}{2})$$
After each revolution, the contribution of the periodic terms in the preceding revolution averages to zero, and to the accuracy of this analysis we can take

\[ \epsilon \approx 0 \]  

Thus, for small entry, our variable is proportional to the mean anomaly along the orbit. Using the elements of the orbit as given by Eq. (55), and the new variable \( \epsilon \), as related to Eq. (57), we can rewrite the unified equation, Eq. (52), to apply to the case of small entry. It is important to mention the following facts:

The elements of the orbit, as given in Eq. (55), are good approximations above 100 thousand feet. They will be used mainly to evaluate the forcing terms, namely the function \( f(\epsilon) \) in the unified equation, Eq. (21). For the damping and frequency, namely the functions \( J_1(\epsilon) \) and \( J_2(\epsilon) \), it is important to know the effect of the drag by using the drag equation, Eq. 1, with the small gravity component neglected. Then we have

\[ \frac{d^2 \epsilon}{d\tau^2} + 2 \omega \frac{d \epsilon}{d\tau} + \epsilon = c_3 \]  

Neglecting the second order of the damping \( \epsilon \), the general solution for the angle of attack is

\[ \epsilon = c_1 + c_2 e^{i \omega_0 \tau} \]  

The condition for stability is

\[ |C_1| = \frac{c_1}{c_3} > 0 \]  

which is the same as condition (48) for ballistic entry. The angle-of-attack has a damped oscillation with frequency \( \omega_0 \) and its value tends asymptotically to \( \omega_0 \) as \( \tau \to \infty \). The constant forcing term \( c_3 \) can be seen as the drag force which induces the spiral decay of the orbit. Another instability with dynamic nature will arise when we consider the ellipticity of the orbit. For \( \epsilon \) non vanishing, we first integrate the homogeneous equation in Eq. (59)

\[ \frac{d^2 \epsilon}{d\tau^2} + 2 \omega \frac{d \epsilon}{d\tau} + \epsilon = 0 \]  

Using the Liouville transformation

\[ \epsilon = e^{i \omega_0 \tau} \]  

the equation is transformed into a Mathieu's equation

\[ \frac{d^2 u}{d\tau^2} + \left( \omega_0^2 + 2 \omega \right) u = 0 \]  

where a small constant of order \( \epsilon^2 \) has been omitted in the coefficient of \( \omega \). The Mathieu's equation possesses periodic solutions only when the constants satisfy a certain relation. In general, the solution is not periodic and if the damping condition (64) is satisfied, the solution for \( \epsilon \) in (55) is a damped Mathieu's solution with negligible damping at high altitude. In Eqs. (67) for \( \epsilon \), it can been seen that, when \( \epsilon = 0 \), the solution is a pure harmonic function. For small \( \epsilon \), the solution is oscillatory although not periodic in general. For large \( \epsilon \) the solution may become unstable. It is possible to determine the zone of instability by the method of Krylov-Bogoliubov (40). Following Bogoliubov, for the second approximation, we have

\[ \omega = \omega_0 + \alpha \omega_0 \epsilon \]  

where the amplitude \( \alpha \), and the phase angle \( \alpha \), considered as functions of \( \epsilon \), must be determined by the equations of second approximation

\[ \frac{d^2 u}{d\tau^2} + \left( \omega_0^2 + 2 \omega \right) u = 0 \]  

The governing equation in a linear, second order differential equation with periodic coefficients, and periodic forcing terms. Its form is to be expected by the almost periodic nature of the trajectory. For an understanding of the different terms, let us consider the case of circular orbit, \( \epsilon = 0 \). Of course, this can happen only when the aerodynamic forces are vanishingly small. Then, we have

\[ \frac{d^2 \epsilon}{d\tau^2} + 2 \omega \frac{d \epsilon}{d\tau} + \epsilon = c_3 \]  

To solve the system of equations (69) we introduce new variables \( x_1 \) and \( x_2 \) according to the relations

\[ x_1 = \epsilon \]  

\[ x_2 = \epsilon \]  

Then, it can be shown that the system reduces to the linear system with constant coefficients

\[ \frac{d^2 x_1}{d\tau^2} + \left( \omega_0^2 + 2 \omega \right) x_1 = 0 \]  

The characteristic equation of the system is

\[ x^2 + 2 \omega \frac{d x}{d\tau} + \omega_0^2 = \frac{1}{2} \]  

Hence, the general solution of the system is

\[ x_1 = c_1 e^{i \omega_0 \tau} + c_2 e^{-i \omega_0 \tau} \]  

\[ x_2 = \omega_0 \left( c_1 e^{i \omega_0 \tau} - c_2 e^{-i \omega_0 \tau} \right) \]  

where \( c_1 \) and \( c_2 \) are constants of integration. The amplitude \( \epsilon \) and the phase angle \( \alpha \) of the solution of (68) for \( \epsilon = 0 \) are then given by

\[ \epsilon = \sqrt{x_1^2 + x_2^2} \]  

\[ \alpha = \tan^{-1} \frac{x_2}{x_1} \]  

Following Poincaré (40), we seek the following series solutions for the different forcing functions

\[ c_1 = c_1 + \sum g(p) \epsilon \cos \omega_0 \tau \]  

\[ c_2 = \sum P \epsilon \sin \omega_0 \tau \]  

where \( g(p) \) and \( P \) are functions of \( \epsilon \). It is clear from (71) that, if the roots of the characteristic equation (62) are imaginary, the amplitude \( \epsilon \) will be a bounded function of the time. The condition for dynamic instability, that is the condition (64) is satisfied, the solution for \( \epsilon \) in (55) is a damped Mathieu's solution with negligible damping at high frequency \( \omega \). Therefore, since \( A < B \) near resonance, condition (75) can be shown to be satisfied and

\[ 1 + \frac{1}{A^2} < B^2 \]  

Fig. 4 plots the zone of instability as function of the eccentricity. It is clear that, for circular orbit, resonance is not observed, and the zone of instability gets larger when the orbit eccentricity increases. It should be mentioned that this analysis applies to nearly circular orbit. When \( e \) is large, the trajectory extends over a large range of the altitude, and higher order gradients of the mass density of the atmosphere should be included. The altitude where \( \epsilon = 0 \), called the resonance altitude, is obtained by solving

\[ \omega^2 = \frac{3}{2} \frac{\Delta \rho}{\rho} - \frac{1}{2} \frac{\Delta \rho}{\rho} \]  

Resonance occurs at high altitude, and we can approximately take \( \omega^2 \approx 1 \). This leads to the relation, by neglecting the contribution of \( \Delta \rho \), and using subscript \( s \) to denote constant sea level

\[ \frac{\Delta \rho}{\rho} = \frac{2 \tan \left( \frac{\omega_0}{2} \right)}{\omega_0} \]  

The left-hand side of the formula (68) above is solely dependent on the planetary atmosphere. This very simple function, as varying with the altitude, for the earth's atmosphere, is plotted in Fig. 5. The plot can be used to compute graphically the resonance altitude for any type of vehicle, as characterized by the right-hand side of Eq. (70). The criterion (70) shows that, near this resonance altitude the angle-of-attack oscillation is dynamically unstable. For any given vehicle, and orbit eccentricity, the exact altitude range for instability can be computed numerically by using equality signs in (70).

The complete solution for the angle-of-attack is the sum of the general solution of the damped Mathieu's equation, Eq. (65), and a particular solution of the non-homogeneous equation, Eq. (59). To construct the particular solution we can neglect the small damping and consider the equation

\[ \frac{d^2 \epsilon}{d\tau^2} + \left( \omega_0^2 - 2 \omega \right) \epsilon = 0 \]  

Following Poincaré (40), we seek the following series solutions for the different forcing functions
3. Lnifoe. E. V.,”Dynamic Longitudinal Stability 
   and resonance at high altitude. Respectively equations 
   for the Reentry Ballistic Missile.” J. Aerospace Si. .Vol.26.pp.94-98, 
   Feb. 1959. 

   Again, the form of the solutions (81) we can see that resonance occurs when we have $\lambda \neq 0$. The amplitude of the force oscillations becomes increasing large, and linearized theory no longer holds. Fortunately for practical aerodynamic configuration, resonance occurs at high altitude, and during entry, the vehicle is quickly spiraling through this critical altitude in less time than for the resonance to build up.

V. Conclusion

In this paper, we have presented an analytical study of the longitudinal dynamic stability of a hypervelocity vehicle during its descent through a planetary atmosphere. A general non-dimensional variable has been introduced to replace the real time. This new variable, defined as the number of reference length traveled in time allows the derivation of a unified equation of motion for the angle-of-attack, valid for all types of reentry of a general type of reentry vehicle. Two flight regimes of fundamental importance have been discussed.

In the first case, the steep reentry along a nearly straight line flight path is analyzed. It is shown that the general solution can be obtained in terms of the confluent hypergeometric function. Using a theorem in the theory of stability of differential equations, simple criteria for boundedness of the oscillations have been obtained. In general, for normal, nearly ballistic vehicle configuration, the motion of the angle-of-attack is stable. When the position of the center of mass of the vehicle is such that the static stability derivative $C_{m}$ is small, the vehicle motion becomes unstable below a certain altitude. Explicit expression for this critical altitude is derived.

In the second case, the descent is achieved along a spiral flight path with small angle of inclination. The general equation is reduced to a damped Mathieu's equation with periodic forcing term. Using the method of Krylov-Bogoliubov, approximate solutions are constructed, and criteria for stability derived. It is shown that, first the vehicle should be designed such that a certain aerodynamic criterion be satisfied. It is the same as the stability criterion for ballistic entry. There is always present a small spiral instability due to the effect of drag. Also, there exists an altitude range in which the motion is unstable if the vehicle is uncontrolled for a certain length of time. This altitude range is a function of both the characteris-
Fig. 4. Zone of Dynamic Instability as Function of the Eccentricity

Fig. 5. Resonance Altitude

\[
K = \frac{2 k_y^2 (3 k_y^{-1})(W/S)}{L c m_\alpha}
\]