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LOCALIZED FREE AND FORCED VIBRATIONS OF NEARLY PERIODIC DISORDERED STRUCTURES

*Christophe Pierre**
The University of Michigan
Ann Arbor, Michigan

ABSTRACT

An investigation of the effects of disorder on the dynamics of nearly periodic structures is presented. Emphasis is placed on the study of mode localization and vibration confinement phenomena for mistuned assemblies of coupled, multi-degree-of-freedom component systems. Perturbation methods are developed to predict the occurrence of localized modes, and strong localization is shown to occur for weak coupling between component systems. Furthermore, a "modal" coupling parameter is defined that determines the possibility for localization in a given mode. Generally speaking, higher modes are shown to be more susceptible to localization than lower ones, and localization is unavoidable if the mode number is large enough. The occurrence of localization is also shown to be dependent upon the location of the coupling constraint between the component systems.

1. INTRODUCTION

Structural systems are commonly investigated by assuming that their parameters are known precisely, even though small manufacturing and material tolerances result in parameter uncertainties and structural irregularities that can affect their dynamics significantly. The assumption of structural regularity is perhaps most critical when studying nominally periodic structures, as it has been observed that the presence of irregularities in such structures may inhibit the propagation of vibration. Depending on the magnitudes of both the disorder and the internal coupling, the irregularities may localize the vibration modes and confine the energy to a region close to the source. This phenomenon, referred to as normal mode localization, has excited considerable interest in solid state physics[1-4] over the years, and more recently was rediscovered in the field of structural dynamics[5-12]. A survey of mode localization phenomena in structures can be found in the review paper by Ibrahim[13].

Nearly periodic structures made of weakly coupled component systems have closely spaced eigenvalues that make them highly sensitive to small irregularities. The modes of such tuned, or periodic, structures extend throughout the structure, while the modes of the corresponding mistuned, or disordered, structures undergo drastic changes to become localized about a few component systems. Research by the present author and others has shown that strong mode localization occurs for assemblies of weakly coupled component systems such as chains of coupled pendula[5,10,11], bladed-disk assemblies[7,8], and some large space structures[9]. Disordered multi-span structures with irregularly spaced constraints have been shown to be susceptible to mode localization as well, through both theoretical and experimental studies[6,12]. Most of the previous work on localized vibrations of assemblies of component systems concerns *single*-degree-of-freedom (DOF) substructures, although the research by Valero and Bendiksen[8] on the occurrence of localization in mistuned shrouded blade assemblies should be noted.

Localized modes in disordered structural systems may result in either beneficial or damaging consequences. For instance, localized vibrations in mistuned blade assemblies may be damaging because the stresses remain localized and thus lead to blade fatigue. Serious negative consequences can also be expected for the robustness and stability of active control systems for large space structures if their design is based upon erroneous extended modes of the ordered structure. However, for the same large space structures, localization could also be utilized as a means of confining vibrations to a region close to the source of disturbance, leading to a passive control of vibrations by irregularities. In all cases, because small irregularities lead to such drastic phenomena, it is important to investigate the underlying physical mechanisms of mode localization in order to be able to predict its occurrence and to analyze its characteristics.

This paper investigates the localization of both the free and forced responses of disordered structures made of coupled *multi-DOF* component systems. In the first part of the paper, mode localization and vibration confinement phenomena are illustrated with a simple structure, namely a chain of single-DOF oscillators whose vibration modes have been previously studied in the literature[5,11]. The second part of the paper develops a general theory of mode localization for disordered assemblies of coupled component systems, where each subsystem is described by its modal representation. Perturbation methods are developed to predict the occurrence of localized modes. These methods, in addition to their low cost, provide an important physical insight into localization. A "modal" coupling is defined that determines the possibility for localization in a given mode. In the third part of the paper the general theory is applied to a system made of coupled component beams. The motivation for this study is that a closed assembly can be regarded as a simple model of a bladed-disk. Numerical results are presented and the localization of the modes is discussed in terms of the strength of the interbeam coupling, the mode number, and the location of the coupling constraint.

2. LOCALIZATION — A SIMPLE EXAMPLE

When localization occurs, *small* irregularities result in *drastic* changes in the dynamics of the nearly periodic structure. To illustrate this change, consider the system of coupled single-DOF oscillators shown in Fig. 1, whose equations of motion are given in Appendix A. The two key parameters of the system are the dimensionless coupling between pendula, R^2, and the dimensionless length deviation from the nominal value for the i-th pendulum, Δl_i. The tuned, or ordered, system consists of identical oscillators. Its modes of vibration are given in Appendix A, and three of the mode shapes are represented in Fig. 1.a. Note that the modes are *collective*, or *extended*, and that the motion amplitude varies sinusoidally in space. Since the tuned mode shapes are independent of the interpendulum coupling, this extended pattern is valid for arbitrarily small coupling. Also note that as R^2 becomes small, the tuned eigenvalues become clustered in a small passband. On the other hand, the mistuned system, shown in Fig. 1.b, consists of oscillators with slightly different natural frequencies. For small interpendulum coupling and small disorder, the natural modes of the mistuned system are drastically different from the tuned modes, with each of the mistuned system's modes *localized* about one pendulum.

The modes of the pendulum system have been extensively studied in Refs. 5 and 11, and localization has been shown to occur when both the irregularities and the interpendulum coupling are small. In other words, localization is susceptible to occur if the tuned eigenvalues are clustered in a small passband. It was also proven that the degree of localization only depends upon the ratio of disorder to coupling and that localization becomes more pronounced as this ratio increases. In Ref. 11 perturba-

* Assistant Professor, Mechanical Engineering and Applied Mechanics. Member AIAA

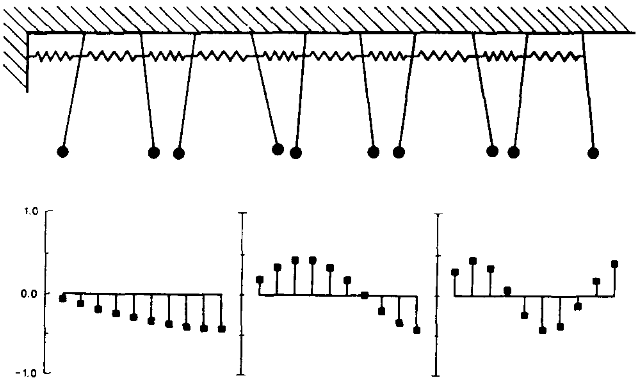


Fig. 1.a Tuned, or ordered system.

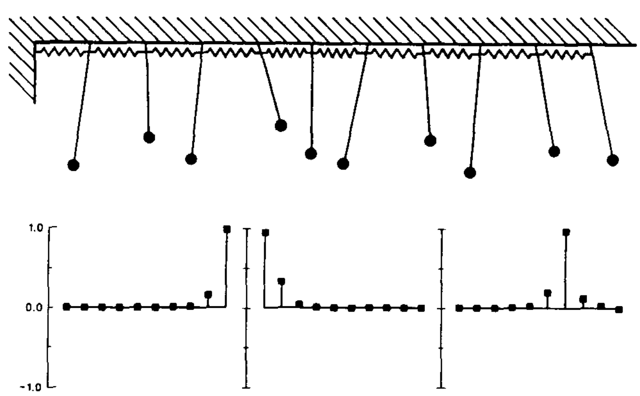


Fig. 1.b Mistuned, or disordered system. The length deviations have been exaggerated for the purpose of clarity. The standard deviation of the irregularities is 3.7%.

Fig. 1. Three of the ten modes of vibration of a ten-DOF pendulum system. The length of a vertical bar represents the angular amplitude of the corresponding pendulum. The coupling between pendula is such that $R = \sqrt{k/m} \,/\, \sqrt{g/l} = .1$.

tion methods were used systematically to conduct the investigation. However, since small disorder results in drastic changes, a classical perturbation analysis that treats the irregularities as perturbations could not be used. Therefore, a so-called Modified Perturbation Method (MPM) suitable to the analysis of strongly localized modes was developed[11]. The main characteristic of this method is that since localization occurs for small interpendulum coupling, this coupling ought to be treated as a perturbation as well. Furthermore, in order to avoid multiple eigenvalues for the unperturbed system, the key idea introduced in Ref. 11 was to ***introduce disorder in the unperturbed system.*** Then, the (modified) unperturbed system is a mistuned chain of uncoupled pendula, and the perturbation consists of the coupling. Since the perturbed modes are perturbations of decoupled oscillations, they are localized, as can be seen in Fig. 1.b. Therefore, perturbation methods provide physical insight into the occurrence of mode localization.

Next, the effects of disorder on the behavior of the system under forced excitation are considered. It has been shown by Hodges[5,6] that irregularities may result in the confinement of the vibration near the source of excitation. To illustrate this confinement effect, single harmonic excitation is applied at the left end of the pendulum chain. In order to obtain finite resonant amplitudes, viscous dampers are added between the oscillators. Only the steady-state response is considered, and the relevant equations can be found in Appendix A. Typical frequency responses of both tuned and mistuned systems are shown in Figs. 2 and 3 for strong and weak interpendulum coupling, respectively. It is observed in Fig. 2 that mistuning has very little effect on the response of the system and that the vibration propagates along the chain for frequencies close to the resonant frequencies. Note that the modes of this system do not become localized. On the other hand, Fig. 3 is for a weakly coupled system whose modes do become localized as in Fig. 1.b. It is observed that for the tuned system the vibration propagates along the chain if the excitation frequency lies within the small passband. However, for the mistuned system, the vibration amplitude decreases rapidly along the chain for *all* excitation frequencies; thus, there is no propagation of vibration. In other words, small disorder *confines* the vibration near the source of excitation. The decay of the vibration amplitude can be shown to be exponential on the average[5].

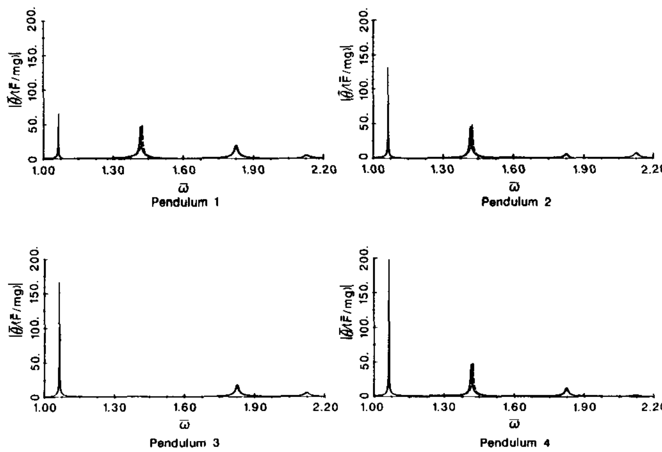


Fig. 2. Frequency responses of tuned (——) and mistuned (- - -) chains of four pendula under single-harmonic excitation at one end, for strong coupling $R = 1$. The first pendulum is excited. The mistuning is such that $\Delta l_1 = .01$, $\Delta l_2 = -.055$, $\Delta l_3 = .05$, $\Delta l_4 = -.02$. The dimensionless damping is $c/(m\omega_o) = .005$.

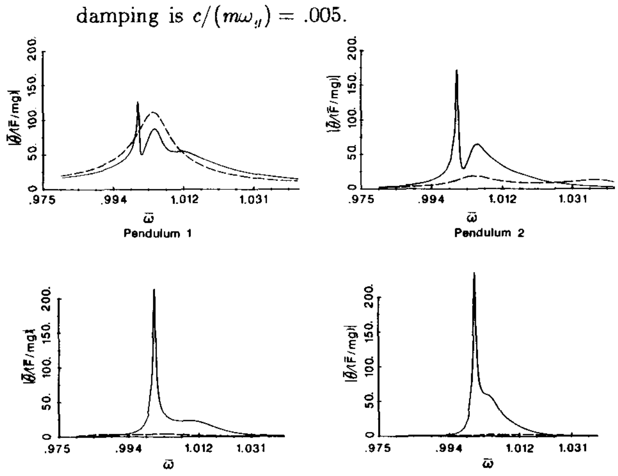


Fig. 3. Frequency responses of tuned (——) and mistuned (- - -) chains of pendula for weak coupling $R = .1$. The mistuning and damping values are as in Fig. 2.

Therefore, for weakly coupled systems, the introduction of disorder results in the strong localization of the free modes and the confinement of the forced vibrations. This confinement effect can be explained by noting that for an end excitation of the disordered system, the force vector is almost orthogonal to all but one of the strongly localized mode shapes. In fact, the excitation vector is approximately proportional to the mode shape localized about the first pendulum. Therefore the system vibrates primarily in this mode, leading to a response confined near the first pendulum (*i.e.*, near the source of excitation). On the other hand, for the tuned system all extended modes are excited by the force vector, leading to the propagation of the vibration within the frequency band. Finally, note that the forced response of the weakly coupled mistuned system is a perturbation of that of the decoupled system (for which there is no propagation at all), and not of that of the tuned system. Therefore, modified perturbation method concepts could also be applied to the forced response.

3. DISORDERED ASSEMBLIES OF COMPONENT SYSTEMS

3.1 Equations of Motion

The pendulum system of Fig. 1 can be generalized by replacing each pendulum with a multi-DOF subsystem. In this part, the modes of vibration of a disordered assembly of N coupled, continuous (or possibly discrete), component systems are studied. Each structural component system is described by its modal representation, hence the approach is very general. The equations of free motion are derived by a Component Mode Analysis[14], as follows.

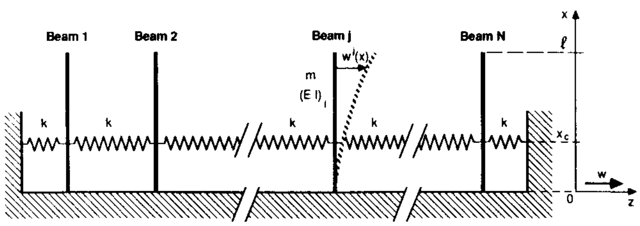


Fig. 4. Assembly of coupled, component systems (beams).

Assemblies such as the one shown in Fig. 4 are studied. For the sake of simplicity, one-dimensional component systems are considered, although this can be generalized (for example see Ref. 10). The ordered, or tuned, assembly consists of identical subsystems. The free modes of the nominal (unperturbed) component system are $(\lambda_i, \phi_i(x))_{i=1,\cdots,M}$, where M is the number of component modes included in the analysis, λ_i the square of the natural frequency, and ϕ_i the associated mode shape. When small disorder is introduced, the modes of the j-th (decoupled) component system become:

$$\lambda_i^j = \lambda_i\left(1 + d\lambda_i^j\right) \quad i = 1,\cdots,M \quad j = 1,\cdots,N \tag{1}$$

$$\phi_i^j(x) = \phi_i(x) + d\phi_i^j(x) \quad i = 1,\cdots,M \quad j = 1,\cdots,N \tag{2}$$

where the subscript i refers to the component mode number, while the superscript j designates the component system number; x is the position along the component systems; $d\lambda_i^j$ and $d\phi_i^j$ are the i-th mode perturbations for the j-th component system, resulting from structural and material irregularities.* Note that $d\lambda_i^j$ is a dimensionless mistuning. These component mode perturbations can be easily determined from the structural parameter perturbations by applying the classical perturbation method for the eigenvalue problem[15,16], and clearly they include first but also higher order terms in the parameter perturbations.

The coupling between the component systems is represented by identical springs of stiffness k. Small variations of this stiffness throughout the assembly are believed to have small effects, hence this is not considered. For the j-th component system the coupling constraint is located at x_j, and the corresponding deflection, $w^j(x_j)$, is denoted by w^j.

The deflection of the j-th component system is expanded as

$$w^j(x,t) = \sum_{i=1}^{M} \phi_i^j(x)\,\eta_i^j(t) \tag{3}$$

where η_i^j is the i-th normal coordinate for the j-th component system. The kinetic energy of the system is

$$T = \frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{M} M_i\left(\dot{\eta}_i^j\right)^2 \tag{4}$$

* For simplicity, the domain of the component systems is assumed to remain unperturbed.

Note that the component modes have not been normalized. However, for a given component mode number, the generalized masses have been chosen to be identical for all component systems. This is natural if perturbation methods for the eigenvalue problem are used to obtain the modes of the mistuned component systems from the ones of the nominal component system, as these methods conserve the generalized masses. The strain energy of the system is

$$U = \frac{1}{2}\sum_{j=0}^{N} k\left(w^j - w^{j+1}\right)^2 + \frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{M} M_i\lambda_i^j\left(\eta_i^j\right)^2 \tag{5}$$

Recall that w^j is the deflection at the j-th constraint location. By definition, $w^0 = w^{N+1} = 0$. The N constraints are expressed by

$$f^j = w^j - \sum_{i=1}^{M}\eta_i^j(t)\,\phi_i^j(x_j) = 0 \qquad j = 1,\cdots,N \tag{6}$$

and the Lagrangian of the system is given by

$$L = T - U + \sum_{j=1}^{N}\alpha^j f^j \tag{7}$$

where the α^j's are the Lagrange multipliers. Applying Hamilton's principle, the equations of free motion are found to be, in addition to the constraints (6):

$$M_i\ddot{\eta}_i^j + M_i\lambda_i^j\eta_i^j + \alpha^j\phi_i^j(x_j) = 0 \quad i = 1,\cdots,M \quad j = 1,\cdots,N \tag{8}$$

$$k\left(-w^{j-1} + 2w^j - w^{j+1}\right) - \alpha^j = 0 \qquad j = 1,\cdots,N \tag{9}$$

Eliminating the Lagrange multipliers and assuming single harmonic motion of frequency, ω, yields

$$M_i\left(-\omega^2 + \lambda_i^j\right)\bar{\eta}_i^j + k\phi_i^j(x_j)\sum_{l=1}^{M}\left[-\bar{\eta}_l^{j-1}\phi_l^{j-1}(x_{j-1}) + 2\bar{\eta}_l^j\phi_l^j(x_j) - \bar{\eta}_l^{j+1}\phi_l^{j+1}(x_{j+1})\right] = 0 \qquad i = 1,\cdots,M \quad j = 1,\cdots,N \tag{10}$$

Defining the vector of normal coordinate amplitudes, $\underline{\bar{\eta}}$, of dimension NM, by

$$\underline{\bar{\eta}} = \left[\bar{\eta}_1^1,\cdots,\bar{\eta}_i^1,\cdots,\bar{\eta}_M^1 \,|\cdots\cdots|\, \bar{\eta}_1^N,\cdots,\bar{\eta}_i^N,\cdots,\bar{\eta}_M^N\right]^T \tag{11}$$

Eq. (10) can be written in matrix form as

$$\left\{[A] - \omega^2[I]\right\}\underline{\bar{\eta}} = \underline{0} \tag{12}$$

where $[I]$ is the identity matrix and $[A]$ is an NM by NM block tridiagonal matrix given by

$$[A] = \begin{bmatrix} \ddots & \ddots & & 0 \\ \ddots & [a^{j,j-1}] & [a^{j,j}] & [a^{j,j+1}] \\ & & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \tag{13a}$$

The submatrices $[a^{k,l}]$ are M by M and defined by

$$\begin{cases} \left[a^{j,j-1}\right] = -\left[\mathrm{diag}(k/M_i)\right]\,\underline{\phi}_j\,\underline{\phi}_{j-1}^T & j = 2,\cdots,N \\ \left[a^{j,j}\right] = \left[\mathrm{diag}(\lambda_i^j)\right] + 2\left[\mathrm{diag}(k/M_i)\right]\,\underline{\phi}_j\,\underline{\phi}_j^T & j = 1,\cdots,N \\ \left[a^{j,j+1}\right] = -\left[\mathrm{diag}(k/M_i)\right]\,\underline{\phi}_j\,\underline{\phi}_{j+1}^T & j = 1,\cdots,N-1 \end{cases} \tag{13b}$$

where $\underline{\phi}_j$ is the M vector of modal deflections at x_j:

$$\underline{\phi}_j = \left[\phi_1^j(x_j),\cdots,\phi_i^j(x_j),\cdots,\phi_M^j(x_j)\right]^T \tag{14}$$

Note that $\underline{\phi}_k\,\underline{\phi}_l^T$ is an M by M matrix that can be regarded as a matrix of "modal" influence coefficients between the constraint locations of two component systems. (k/M_i) is the coupling natural frequency for the i-th component mode. The submatrices composing $[A]$ correspond to component systems, while the M lines and columns of these submatrices refer to component modes. For an assembly of single-DOF systems, $M = 1$, and the submatrices reduce to scalar quantities.

3.2 Perturbation Methods

The occurrence (*resp.* nonoccurrence) of localized modes can be predicted by generalizing the Modified (*resp.* Classical) Perturbation Method developed in Ref. 11 for a chain of coupled pendula. In order to study the relative orders of magnitude of its elements, it is convenient to nondimensionalize $[A]$ by dividing the i-th rows of the submatrices $[a^{k,l}]$ by the corresponding eigenvalue of the nominal component system, λ_i. This yields the generalized eigenvalue problem:

$$\left\{\left[\tilde{A}\right] - \omega^2 \left[\text{DIAG}\left[\text{diag}(1/\lambda_i)\right]\right]\right\} \underline{\eta} = \underline{0} \tag{15}$$

where $[\text{DIAG}\,[\text{diag}(1/\lambda_i)]]$ is an NM by NM diagonal matrix made of N identical M by M diagonal submatrices of generic element $1/\lambda_i$; $[\tilde{A}]$ is given by

$$\left[\tilde{A}\right] = \begin{bmatrix} \ddots & \ddots & & 0 \\ \left[\tilde{a}^{j,j-1}\right] & \left[\tilde{a}^{j,j}\right] & \left[\tilde{a}^{j,j+1}\right] & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \tag{16a}$$

where

$$\begin{cases} \left[\tilde{a}^{j,j-1}\right] = -\left[\text{diag}(R_i^2)\right] \underline{\phi}_j \underline{\phi}_{j-1}^T \\ \left[\tilde{a}^{j,j}\right] = \left[\text{diag}(1 + d\lambda_i^j)\right] + 2\left[\text{diag}(R_i^2)\right] \underline{\phi}_j \underline{\phi}_j^T \\ \left[\tilde{a}^{j,j+1}\right] = -\left[\text{diag}(R_i^2)\right] \underline{\phi}_j \underline{\phi}_{j+1}^T \end{cases} \tag{16b}$$

where $R_i^2 = k/(M_i\lambda_i)$ is the dimensionless ratio of the spring stiffness to the modal stiffness for the i-th component mode, or equivalently, the ratio of the square of the modal coupling natural frequency, k/M_i, to the square of the i-th component frequency, λ_i; hence R_i^2 can be regarded as a modal stiffness ratio. It is important to note that R_i^2 decreases as the component mode number increases, because λ_i increases with i. Thus one may expect localization to occur more easily in the higher modes. Also note that Eqs. (15-16) are a generalization of the equations of a chain of coupled pendula (see Appendix A).

For simplicity, identical constraint locations are considered in the subsequent developments, *i.e.*, $x_c \equiv x_j$; $j = 1, \cdots, N$. Several cases are distinguished.

Strong Coupling in All Component Modes

Here finite or large values of the stiffness k are considered. It is also assumed that all the modal stiffness ratios, $(R_i^2)_{i=1,\cdots,M}$, are finite or large, that is, on the order of one or larger. Furthermore, it is assumed that the modal deflections at the constraint location are not small. In brief, it is assumed that the elements of the submatrices $[\text{diag}(R_i^2)]\, \underline{\phi}_k \underline{\phi}_l^T$ are not small, but on the order of one or larger. Then all the elements of $[\tilde{A}]$ have approximately the same order of magnitude and the classical perturbation approach (CPM) must be used to study mistuning. The unperturbed system is the tuned one, and the perturbation consists of the (modal) mistuning among the component systems. The unperturbed matrix is given by

$$\left[\tilde{A}_0^{(c)}\right] = \begin{bmatrix} \ddots & \ddots & & 0 \\ -\left[\tilde{a}_0\right] & [I] + 2\left[\tilde{a}_0\right] & -\left[\tilde{a}_0\right] & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \tag{17a}$$

where

$$\left[\tilde{a}_0\right] = \left[\text{diag}(R_i^2)\right] \underline{\phi}\, \underline{\phi}^T \tag{17b}$$

and $\underline{\phi} = [\phi_1(x_c), \cdots, \phi_M(x_c)]^T$ is the vector of modal deflections at the constraint for the nominal component system.
The first order perturbation matrix is:

$$\left[\widetilde{\delta A}^{(c)}\right] = \begin{bmatrix} \ddots & \ddots & & 0 \\ \left[\widetilde{\delta a}^{j,j-1}\right] & \left[\widetilde{\delta a}^{j,j}\right] & \left[\widetilde{\delta a}^{j,j+1}\right] & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \tag{18a}$$

where

$$\begin{cases} \left[\widetilde{\delta a}^{j,j-1}\right] = -\left[\text{diag}(R_i^2)\right] \left[\delta\phi^{j,j-1}\right] \\ \left[\widetilde{\delta a}^{j,j}\right] = \left[\text{diag}(d\lambda_i^j)\right] + 2\left[\text{diag}(R_i^2)\right] \left[\delta\phi^{j,j}\right] \\ \left[\widetilde{\delta a}^{j,j+1}\right] = -\left[\text{diag}(R_i^2)\right] \left[\delta\phi^{j,j+1}\right] \end{cases} \tag{18b}$$

and $[\delta\phi^{k,l}]$ is given by, to the first order

$$\left[\delta\phi^{k,l}\right] = \underline{\phi}(\underline{d\phi}^l)^T + \underline{d\phi}^k \underline{\phi}^T \quad \text{where } \underline{d\phi}^k = \left[\cdots, d\phi_i^k(x_c), \cdots\right]^T \tag{18c}$$

The second order perturbation matrix could also be easily obtained. The modes of the mistuned assembly are perturbations of the ones of the tuned system and are certainly not localized. The key advantage of the perturbation approach is that it only requires the solution of the tuned system, which can be obtained by taking advantage of its periodicity properties.

Weak Coupling in All Component Modes

Here the stiffness k and the modal stiffness ratios, $(R_i^2)_{i=1,\cdots,M}$, are assumed to be small, on the order of or smaller than the dimensionless mistunings $d\lambda_i^j$:

$$R_i^2 = \frac{k}{M_i\lambda_i} \ll 1 \qquad R_i^2 \leq \mathcal{O}(d\lambda_i^j) \qquad i = 1, \cdots, M \tag{19}$$

Note that if the condition (19) is met for $i = 1$ it is likely to be verified for $i > 1$ as well, since R_i^2 decreases as i increases.

For k small the natural frequencies of the tuned assembly are clustered in M groups, or bands, with N frequencies in each narrow band. The width of these clusters decreases with k, and the introduction of mistuning makes the frequencies constituting the passbands move apart. The classical perturbation approach fails to analyze this small coupling case, because due to the closeness of the eigenvalues, the mode shapes perturbations can be shown to become large when disorder is introduced, which violates the underlying assumptions of asymptotic expansions[9,10].

A modified perturbation method (MPM) can be developed as follows for the small coupling case. Since in Eq. (16) all the elements of $[\tilde{A}]$ but the diagonal ones are small, the coupling between component systems, as well as the mistuning, ought to be treated as a perturbation. If both mistuning and coupling are considered perturbations, though, the unperturbed matrix becomes the identity matrix, thereby leading to M N-fold multiple eigenvalues for the unperturbed system, λ_i, corresponding to the modes of identical decoupled component systems. Even though the perturbation theory could still be applied to these multiple eigenvalues, physical insight into mode localization would be lost. Therefore, similarly to the approach developed in Ref. 11, the key idea is to *introduce mistuning in the unperturbed system* in order to split the unperturbed eigenvalues. This results into the (modified) unperturbed matrix

$$\left[\tilde{A}_0^{(m)}\right] = \left[\text{DIAG}\left[\text{diag}(1 + d\lambda_i^j)\right]\right] \tag{20}$$

and into the (modified) perturbation matrix

$$\left[\widetilde{dA}^{(m)}\right] = \begin{bmatrix} \ddots & \ddots & & 0 \\ -\left[\widetilde{da}^{j,j-1}\right] & 2\left[\widetilde{da}^{j,j}\right] & -\left[\widetilde{da}^{j,j+1}\right] & \\ & 0 & \ddots & \ddots \end{bmatrix} \tag{21a}$$

where

$$\left[\widetilde{da}^{k,l}\right] = \left[\mathrm{diag}(R_i^2)\right] \underline{\phi}_k \underline{\phi}_l^T \tag{21b}$$

The elements of $[\widetilde{A}_0^{(m)}]$ and $[\widetilde{dA}^{(m)}]$ are on the order of one and of the first order, respectively. Since mistuning among subsystems is usually random, the unperturbed eigenvalues, $\lambda_i(1 + d\lambda_i^j)$, are likely to be simple. Also note that Eqs. (20,21) are a straightforward generalization of the Modified Perturbation Method developed in Ref. 11 for a chain of coupled pendula.

The unperturbed solution consists of the modes of the decoupled mistuned component systems, which have slightly different natural frequencies. The perturbation matrix introduces (small) coupling between the component systems. Since the perturbed modes are perturbations of decoupled oscillations, they are *localized* about individual component systems. The modes corresponding to the j-th component system consist primarily of oscillations of the j-th system, and of oscillations of lesser magnitude of its nearest neighbors, the propagation range depending on the "modal" coupling and mistuning. It is remarkable that the Modified Perturbation Method predicts localized modes without requiring one to solve the eigenvalue problem. The first and second order perturbation theory for the eigenvalue problem[15,16] can be readily applied to Eqs. (20-21), and formulae are given in Appendix B for the first order eigensolution perturbation. From these perturbation expressions, it could be easily shown that the degree of localization in the i-th group of modes depends primarily on the ratios $R_i^2\,\phi_i(x_c)/d\lambda_i^j$ of modal coupling to modal disorder — a result similar to the one derived in Ref. 11 for a chain of coupled pendula.

General Case

The coupling is unlikely to be either strong or weak in all component modes. In general, k can be considered to be finite (not small), leading to finite values of R_i^2 for the lower groups of modes and to small values for the higher groups. Then the lower modes remain extended, while for i greater than some threshold value localization occurs. Moreover, assuming constant modal mistuning $d\lambda_i^j$ when i varies, localization is seen to become more and more pronounced as the group number i increases, since R_i^2 decreases as λ_i increases. In the limit $i \to \infty$, the mistuned modes become decoupled, while the tuned ones remain extended. Furthermore, for any k (even arbitrarily large) and for any mistuning (no matter how small), there exists a group number i^* such that for $i > i^*$, *all modes become strongly localized.* The conclusion is that the higher groups of modes are always localized, the lower bound for the occurrence of localization depending on k and on the mistuning strength.

Another interesting comment concerns the values of the modal deflections at the constraint location, $\phi_i^j(x_c)$. As much as the modal stiffness ratios, these modal deflections determine the magnitude of the coupling terms in $[\widetilde{A}]$, and therefore the suitable perturbation schemes. Finite values of $\phi_i^j(x_c)$ have little effect on the coupling between component systems and thus on mode localization. On the other hand, if the constraint is close to a node of the i-th mode, then $\phi_i^j(x_c)$ is very small (or zero), leading to small coupling between the component systems for the i-th group of modes. In this case, even though R_i^2 may be finite or large, the i-th lines of the submatrices $[\mathrm{diag}(R_i^2)]\,\underline{\phi}_k\,\underline{\phi}_l^T$ are small, leading to small effective coupling in the i-th group of modes, and therefore to a localized i-th group of modes. Also, the i-th group of modes may be localized while the neighboring $(i-1)$-th and $(i+1)$-th groups remain extended. This is achieved if R_{i+1}^2 is finite and the constraint is located close to one of the nodes of the i-th component mode.

Therefore, the occurrence of localization in a group of modes is determined by the relative magnitudes of (1) the modal mistuning and (2) the modal coupling, defined as the product of the modal stiffness ratio and the modal deflection at the constraint, $R_i^2\phi_i^j(x_c)$. Strong localization occurs if the latter is of the order of, or smaller than the former. In general a variety of cases can be encountered: *all modes strongly localized; extended lower modes and localized higher modes; extended lower modes except for one group of localized modes due to the location of the constraint* It is important to emphasize that if M is chosen large enough, the higher groups of modes will *always* be localized.

The analysis of both extended and localized modes for a given mistuning configuration can be easily performed by combining the CPM and the MPM. If the interbeam coupling for the i-th group, $R_i^2\phi_i^j(x_c)$, is finite, only $d\lambda_i^j$ must be considered as a perturbation in the i-th lines of the submatrices (Eq. (16)). Conversely, if $R_i^2\phi_i^j(x_c)$ is on the order of $d\lambda_i^j$ or smaller, the coupling terms in the i-th lines of the submatrices are considered as perturbations and $d\lambda_i^j$ belongs to the unperturbed matrix. This provides a consistent analysis of both extended and localized modes. An illustration of this general theory is presented in the next section.

4. LOCALIZATION FOR COUPLED BEAM ASSEMBLIES

4.1 Equations of Motion

The theory presented in Section 3 is applied to the disordered assembly of coupled component beams shown in Fig. 4. Each component system is a cantilevered-free beam. The natural frequencies of the nominal component beam (assumed uniform) are given by

$$\lambda_i = \overline{\omega}_i^2 \left(\frac{EI}{ml^4}\right) \qquad i = 1, \cdots, \infty \tag{22}$$

where $\overline{\omega}_i$ are the dimensionless natural frequencies of a uniform clamped-free beam, whose first six values are listed in Appendix C; EI, m, and l are the beam's stiffness, mass per unit length, and length, respectively. It is assumed that mistuning originates from discrepancies in the terms (EI/ml^4). The eigenvalues of the individual component systems can then be written as

$$\lambda_i^j = \overline{\omega}_i^2 \left(\frac{EI}{ml^4}\right)_j = \overline{\omega}_i^2 \left(\frac{EI}{ml^4}\right)(1 + d\lambda^j) \tag{23}$$

where $d\lambda^j$ is the dimensionless mistuning for the j-th component system. Note that mistuning is the same in all component modes. Nondimensionalizing the eigenvalues by the nominal value, (EI/ml^4), yields

$$\overline{\lambda}_i^j = \overline{\lambda}_i\,(1 + d\lambda^j) \tag{24}$$

Since no domain perturbation is considered, all beams have the same length. For simplicity, it is further assumed that mistuning originates from discrepancies in the stiffness EI, not in the mass m. This results in identical generalized masses for all component systems without introducing a mode shape perturbation $d\phi_i^j$. Moreover, the generalized masses $M_i = ml \equiv M$ are also independent of the mode number. In general, of course, mistuning could originate from local inhomogeneities or boundary condition variations. This would result in different component mode perturbations than the ones given in Eq. (23), but these could be easily calculated by applying the perturbation theory for the continuous eigenvalue problem[15-17]. Therefore, the mode shapes of the clamped-free component beams are given by

$$\phi_i^j(x) = \cos\beta_i x - \cosh\beta_i x + \frac{\sin\beta_i l - \sinh\beta_i l}{\cos\beta_i l + \cosh\beta_i l}\left(\sin\beta_i x - \sinh\beta_i x\right) \tag{25}$$

where $(\beta_i l)^4 = \overline{\omega}_i^2$, and $\phi_i^j \equiv \phi_i$ independent of j, *i.e.*, there is no component mode shape perturbation due to mistuning.

In the next step, Eqs. (12–14) are readily applied to the beam system. An important simplification is that the modal deflection vector, $\underline{\phi}$, is now the same for all component systems. Furthermore, the equations are made dimensionless by dividing them by the nominal value EI/ml^4, resulting in the introduction of dimensionless eigenvalues, $\overline{\omega}^2$, and of the dimensionless coupling stiffness, $\overline{k} = k/\left(\frac{EI}{l^3}\right)$. Note that for $\overline{k} = 3.$ the spring stiffness equals the static stiffness of the nominal (clamped–free) component beam. The equations of free motion are

$$\left\{[\overline{A}] - \overline{\omega}^2[I]\right\}\underline{\overline{\eta}} = \underline{0} \tag{26}$$

where

$$[\overline{A}] = \begin{bmatrix} \ddots & \ddots & \ddots \\ -\overline{k}\underline{\phi}\,\underline{\phi}^T & \left[\mathrm{diag}\left(\overline{\lambda}_i\left(1 + d\lambda^j\right)\right)\right] + 2\overline{k}\underline{\phi}\,\underline{\phi}^T & -\overline{k}\underline{\phi}\,\underline{\phi}^T \\ \ddots & \ddots & \ddots \end{bmatrix} \tag{27}$$

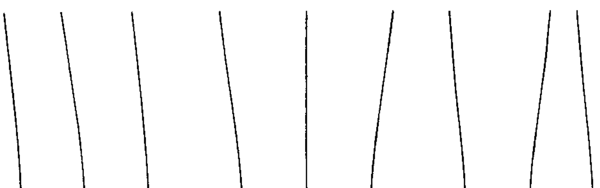
Eq. (26) could also be written in the form of Eqs. (15–16) by dividing the i-th lines of the submatrices by $\overline{\lambda}_i$. This results in the modal stiffness ratios, $R_i^2 = \overline{k}/\overline{\lambda}_i$, which decrease rapidly as the component mode number increases.
Once the eigensolution of $[\overline{A}]$ has been obtained, the continuous mode shapes are given by, for the j-th component system

$$\overline{w}^j(x) = \sum_{i=1}^{M} \overline{\eta}_i^j \phi_i(x) \qquad j = 1, \cdots, N \tag{28}$$

4.2 Results

Unless otherwise stated, results were obtained by direct solution procedure, not by perturbation methods. Tuned and mistuned assemblies of three and five component beams were studied. The random–like mistuning distributions had small standard deviations 2.5% and 4.5% for three and five beams, respectively, and are listed in the figure captions. Only the value of the coupling stiffness, $\overline{k}$, was varied for a given small mistuning, because the degree of localization only depends upon the ratio of coupling to mistuning; therefore, when localization occurs, an increase in coupling is equivalent to a decrease in mistuning. Moreover, several constraint locations were considered.

Fig. 5 shows the lower 12 modes of tuned and mistuned assemblies of three beams connected at their free ends, for $\overline{k} = 3.$, that is, the spring stiffness equals the static stiffness of the nominal beam. The corresponding natural frequencies are also listed. Observe that the high frequencies are clustered in groups of three. Even though the lower three frequencies do not have a small passband character, the 4th to 6th, 7th to 9th, and 10th to 12th frequencies constitute three narrow passbands. The passband width is seen to decrease rapidly as the group number increases, which is a consequence of the corresponding decrease in the modal stiffness ratio, R_i^2. The lower three frequencies do not exhibit a small passband character because the coupling in the first group of modes is not small.



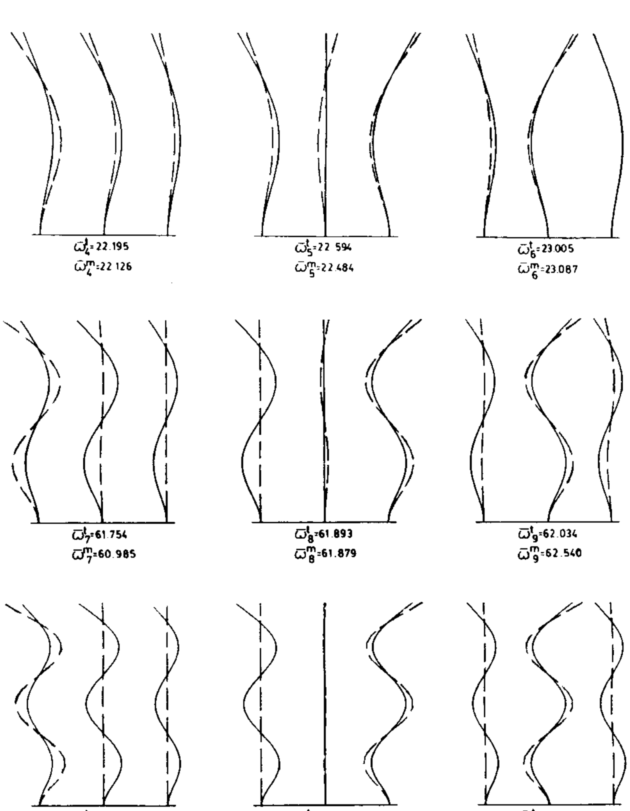


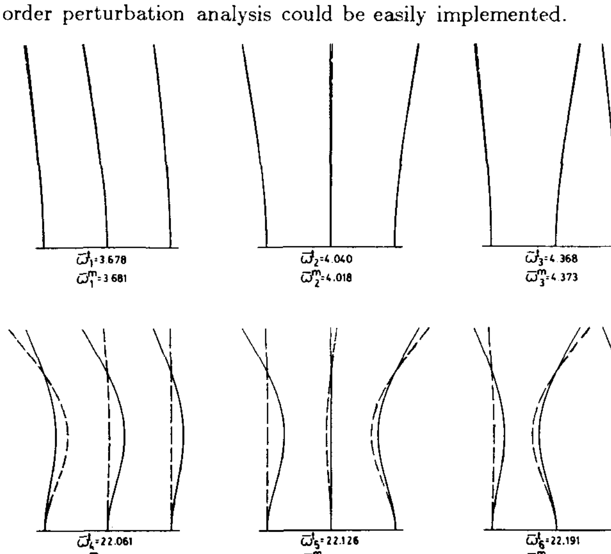
Fig. 5. Modes of tuned (——, $\overline{\omega}^t$) and mistuned (- - -, $\overline{\omega}^m$) assemblies of three beams, for $\overline{k} = 3.$ and $x_c = 1$. Six component modes are included in the analysis ($M = 6$). The mistuning is $d\lambda^1 = .03$, $d\lambda^2 = -.02$, $d\lambda^3 = 0$.

One observes in Fig. 5 that all the modes of the tuned system are extended, that is, all beams vibrate with comparable amplitudes (except when symmetry causes the middle beam to have a zero deflection). Also, in a given group of modes, a single component mode contributes largely to the motion. For instance, in the lower three modes the beams deflect primarily in the first component mode. In general, in the i-th group of modes the beams vibrate primarily in the i-th component mode. This can be explained by noting that the interbeam coupling is not strong enough to distort significantly the deflection shapes of the coupled assembly from the decoupled component modes. As will be seen later, for large values of $\overline{k}$ several component modes may contribute significantly to the modes of the coupled assembly.

Considering the mistuned modes in Fig. 5, it is seen that the lower three modes (first group) are very similar to the corresponding tuned ones, and do not become localized. In the second group, the modes undergo significant changes when small mistuning is introduced and become partially localized. In the third and fourth passbands the mistuned modes are drastically different from the tuned ones, each mistuned mode becoming strongly localized about one component beam. In the fourth passband localization is so pronounced that the modes are almost decoupled, that is, they consist of oscillations of decoupled mistuned beams. The conclusion is that localization becomes more severe as the group number increases. This can be explained by noting that the modal coupling decreases rapidly as the component mode number increases, which causes the off-diagonal coupling terms in Eq. (27) to become small relatively to the diagonal terms $\overline{\lambda}_i$. For the system of Fig. 5 the modal stiffness ratios are $R_1^2 = .24$, $R_2^2 = .006$, $R_3^2 = 8.10^{-4}$, and $R_4^2 = 2.10^{-4}$, explaining that very strong localization occurs for $i = 3$ and $i = 4$. As i keeps increasing, stronger and stronger localization occurs. Therefore, for the mistuned system, all the higher modes are strongly localized, and an analysis not accounting for small disorder would yield erroneous results.

Since the mistuned modes of the first group remain extended, they are merely perturbations of the tuned modes. Conversely, the mistuned modes of the third and fourth groups are perturbations of the modes of the decoupled mistuned systems. This justifies the use of the classical and modified perturbation procedures developed in Section 3.2 for the analysis of extended and localized modes, respectively.

To illustrate this point, a combined classical and modified perturbation method has been applied to a three beam assembly whose first group of modes remains extended while the second group becomes strongly localized when mistuning is present. For simplicity, two component modes have been considered in the analysis, although this could be easily generalized. Good convergence of the component mode analysis was already achieved in this case for $M = 2$. The CPM defined by Eqs. (17–18) has been applied to the first group of modes, while the MPM given by Eqs. (20–21) has been applied to the second group. Therefore, considering Eq. (27), only the mistuning $\overline{\lambda}_1\, d\lambda^j$ was included in the first lines of the submatrices constituting the perturbation matrix, while all the coupling terms were included in their second lines. The unperturbed modes consist of the lower three modes of the tuned system and of the three second modes of the decoupled mistuned beams. The formulae given in Appendix B for the first order eigensolution perturbation were applied. The "exact" modes of the tuned and mistuned systems are displayed in Fig. 6 for $\overline{k} = .5$ and $x_c = 1$. Observe that the first group remain extended while the higher groups are strongly localized. The mistuned modes obtained by both the exact method and the first order perturbation method are displayed in Fig. 7 for the first two groups. Excellent agreement is observed. For instance, the maximum error is .2% for the natural frequencies. As predicted, the first group is not localized by mistuning, while the second group undergoes strong localization. Of course, higher passbands would even be more strongly localized. This example demonstrates that perturbation methods have the ability to predict and analyze the occurrence of localization without solving the mistuned eigenvalue problem. Moreover, important physical insight is gained through these methods. Also note that a second order perturbation analysis could be easily implemented.



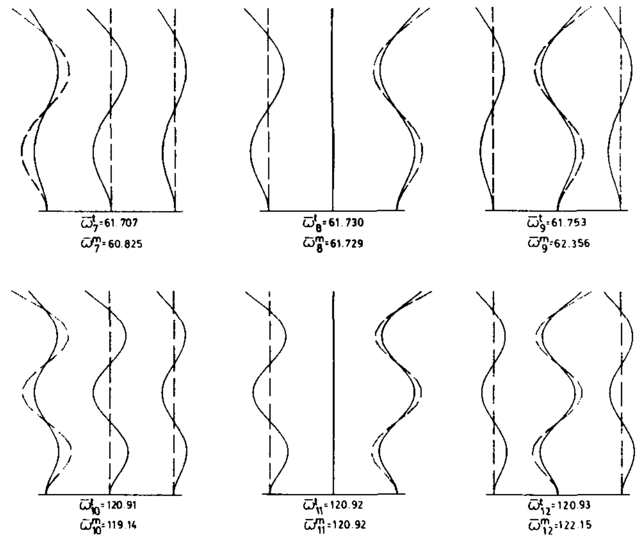


Fig. 6. Modes of tuned (—–, $\overline{\omega}^t$) and mistuned (- - -, $\overline{\omega}^m$) assemblies of three beams, for $\overline{k} = .5$, $x_c = 1.$, and $M = 6$. Mistuning is as in Fig. 5.

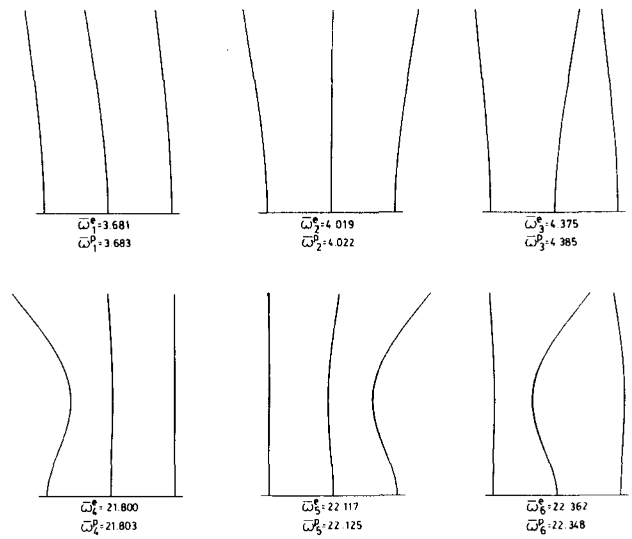


Fig. 7. Comparison of the modes of a mistuned assembly of three beams obtained by the exact method (—–, $\overline{\omega}^e$) and the combined classical and modified perturbation method (- - -, $\overline{\omega}^p$). $\overline{k} = .5$, $x_c = 1.$, and $M = 2$. Mistuning is as in Fig. 5.

Fig. 8 displays the tuned and mistuned modes of a three beam assembly for $\overline{k} = 10.$ and $x_c = .1$, that is, the coupling stiffness is large and the constraint is located near the clamped base of the beams. One observes that all groups of modes exhibit a small passband character and that all the mistuned modes, even the lower three, become strongly localized. Again, localization is stronger in the higher modes. The occurrence of localization in the first frequency cluster can be explained by noting that, even though $\overline{k}$ is large, the coupling between component systems in the first group of modes is determined by $\overline{k}\phi_1(x_c)/\overline{\lambda}_1$, which is small because of the small value of $\phi_1(.1)$. Thus, even though the coupling stiffness is more than three times the one of Fig. 5, all modes are strongly localized because the coupling is rendered small by the small modal deflections at the constraint location.

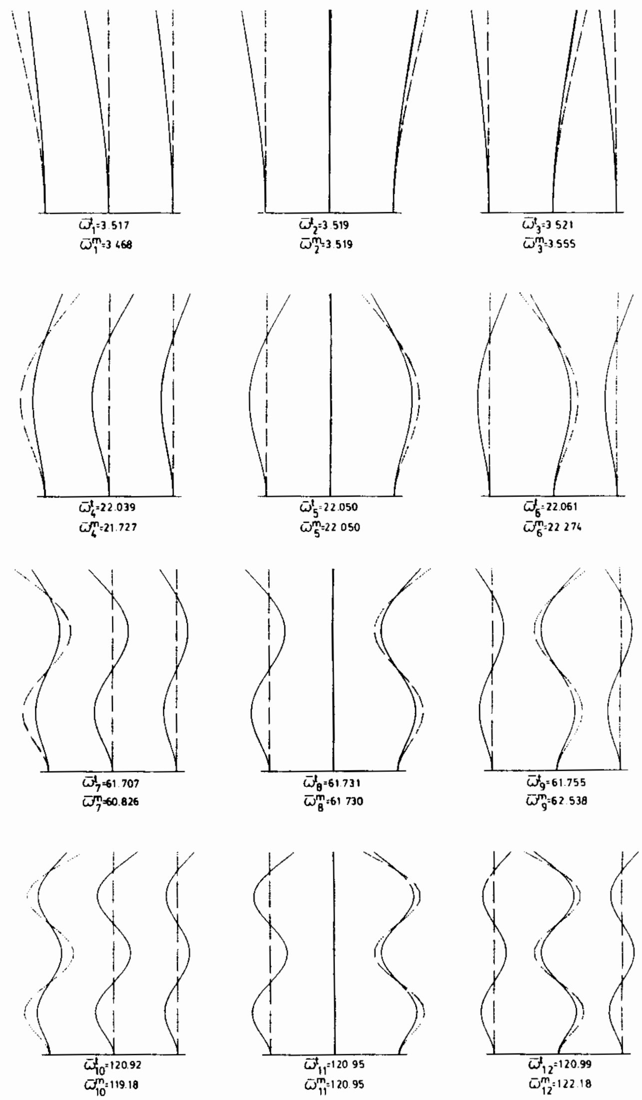
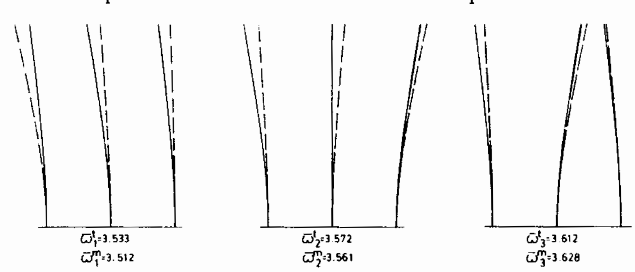


Fig. 8. Modes of tuned (—–, $\bar{\omega}^t$) and mistuned (- - -, $\bar{\omega}^m$) assemblies of three beams, for $\bar{k}$ = 10., x_c = .1, and M = 6. Mistuning is as in Fig. 5.

Fig. 9 displays the lower 12 modes of tuned and mistuned assemblies of three very weakly coupled beams connected at their tip ($\bar{k}$ = .05). Again, all frequencies are clustered in groups of three in narrow bands, and strong localization occurs in all mistuned modes. This can be easily explained by noting that all the coupling terms $R_i^2 \underline{\phi}\, \underline{\phi}^T$ are small in Eq. (27), thus the mistuned modes are perturbations of the mistuned decoupled modes.



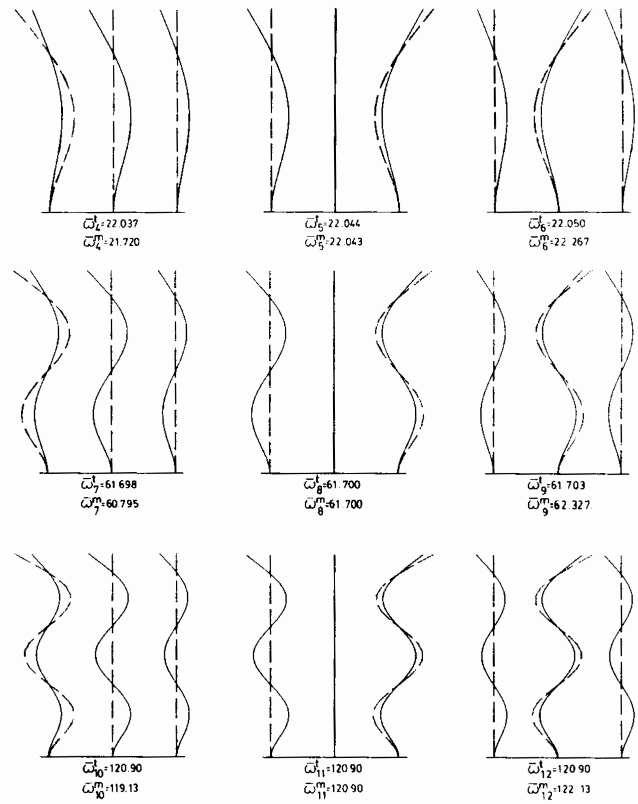
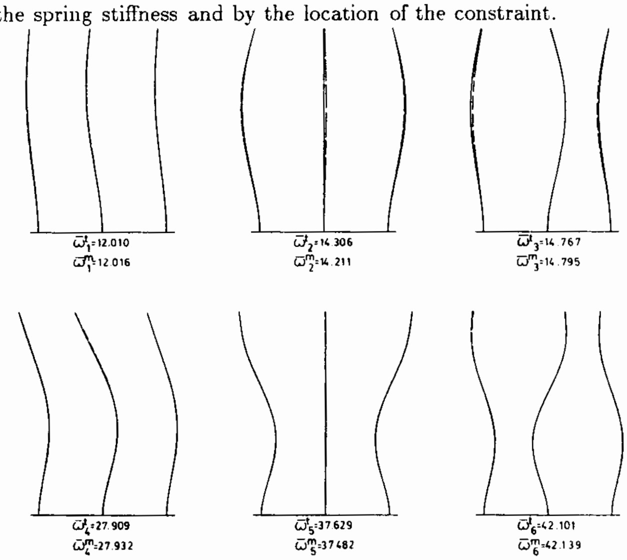


Fig. 9. Modes of tuned (—–, $\bar{\omega}^t$) and mistuned (- - -, $\bar{\omega}^m$) assemblies of three beams, for $\bar{k}$ = .05, x_c = 1., and M = 6. Mistuning is as in Fig. 5.

Fig. 10 is for beams very strongly coupled at their tips ($\bar{k}$ = 100.). In order to insure the convergence of the component mode analysis, seven component modes were considered. There is no small passband character for the first nine frequencies, and accordingly the lower nine modes do not become localized. However, the 10th to 12th and 13th to 15th frequencies are clustered in two groups of small widths. Weak localization occurs in the fourth passband, while the modes of the fifth passband are strongly localized. This can be easily explained by considering Eq. (27), as the modal stiffness ratios $\bar{k}/\bar{\lambda}_i$ are very large for the lower modes, but become small in the fourth and fifth passbands, therefore allowing localization to occur. The conclusion is that even for (arbitrarily) large coupling, localized modes always occur at high frequencies, the threshold value being determined by the value of the spring stiffness and by the location of the constraint.



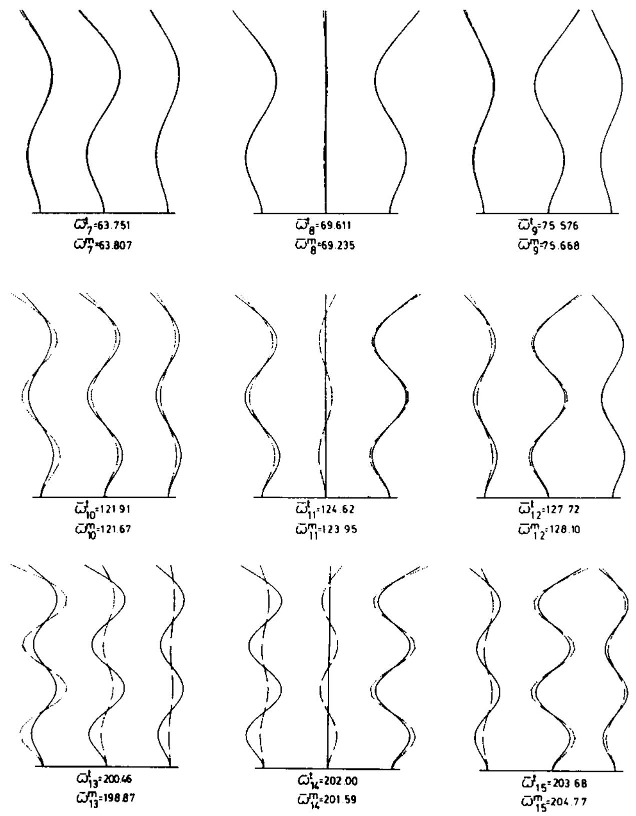


Fig. 10. Modes of tuned (—, $\bar{\omega}^t$) and mistuned (- - -, $\bar{\omega}^m$) assemblies of three beams, for $\bar{k}$ = 100., x_c = 1., and $M = 7$. Mistuning is as in Fig. 5.

It is also worth noting that in Fig. 10 the lower six modes are quite different from the ones shown in Fig. 5 for $\bar{k} = 3.$, both in terms of mode shapes and natural frequencies. This is because the high value of the spring stiffness causes a distortion in the lower modes, thereby making several component modes participate significantly in the global modes. However, this distortion becomes smaller in the fourth passband and negligible in the fifth one. This is explained by noting that the modal coupling becomes small in these passbands, in which case the global modes vibrate primarily in the single corresponding component mode. Another interpretation is that the higher modes are stiffer than the lower ones, hence the coupling stiffness has much less effect on the former than on the latter.

Fig. 11 is an interesting illustration of the importance of the constraint location. An assembly of three strongly coupled beams ($\bar{k}$ = 100.) interconnected at x_c = .7829 is considered. The constraint is located at the node of the second component mode, *i.e.*, $\phi_2(x_c) = 0$. The second passband of the tuned assembly has a three-fold multiple eigenvalue, as can be seen from Fig. 11 and Eq. (27). This multiple eigenvalue is split by mistuning, yielding a narrow passband. Also note that the last mode of the first passband changes position to actually become the sixth mode. This is due to the fact that the frequencies of the first group are much further apart for x_c = .78 than for x_c = 1., and also because the frequencies in the second group are lower for x_c = .78 (due to zero coupling in the second group) than for x_c = 1. As expected because of the large coupling stiffness, the modes of the first group (1st, 2nd, and 6th modes) remain extended. On the other hand, the modes of the second passband (3rd,4th, and 5th modes) become very strongly localized when mistuning is introduced, because the coupling terms in the second lines of the submatrices in Eq. (27) are zero, since $\phi_2(x_c) = 0$. The modes of the third and fourth passbands are only partially localized. Hence, if the beams are interconnected at or near a component mode's node, the corresponding group of global modes become strongly localized when small mistuning is introduced. Note that this is the case for arbitrarily large coupling. Moreover, if the coupling stiffness is large, some higher groups of modes do not become localized, therefore leading to the concept of "transient" localization (in the modal domain).

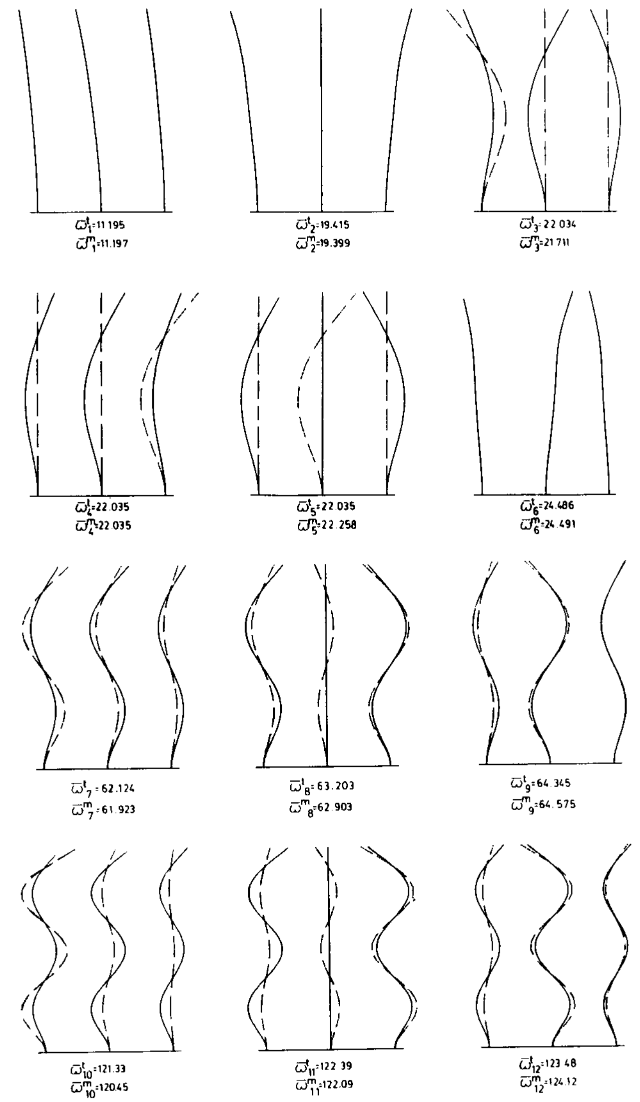


Fig. 11. Modes of tuned (—, $\bar{\omega}^t$) and mistuned (- - -, $\bar{\omega}^m$) assemblies of three beams, for $\bar{k}$ = 100., x_c = .7829, and $M = 7$. Mistuning is as in Fig. 5.

Finally, Fig. 12 is for an assembly of five beams coupled at their tips, for $\bar{k} = 3$. The mistuning listed in the figure caption is larger than for the three beam assembly, hence more pronounced localization can be expected. Three passbands are plotted. It is shown that the first group of mistuned modes are extended, while strong localization already occurs in the second group. The third group becomes totally localized. Here the confinement effect is, indeed, spectacular, as the mistuning standard deviation is only 4.5% and the coupling stiffness is not small but equal to the beam's static stiffness.

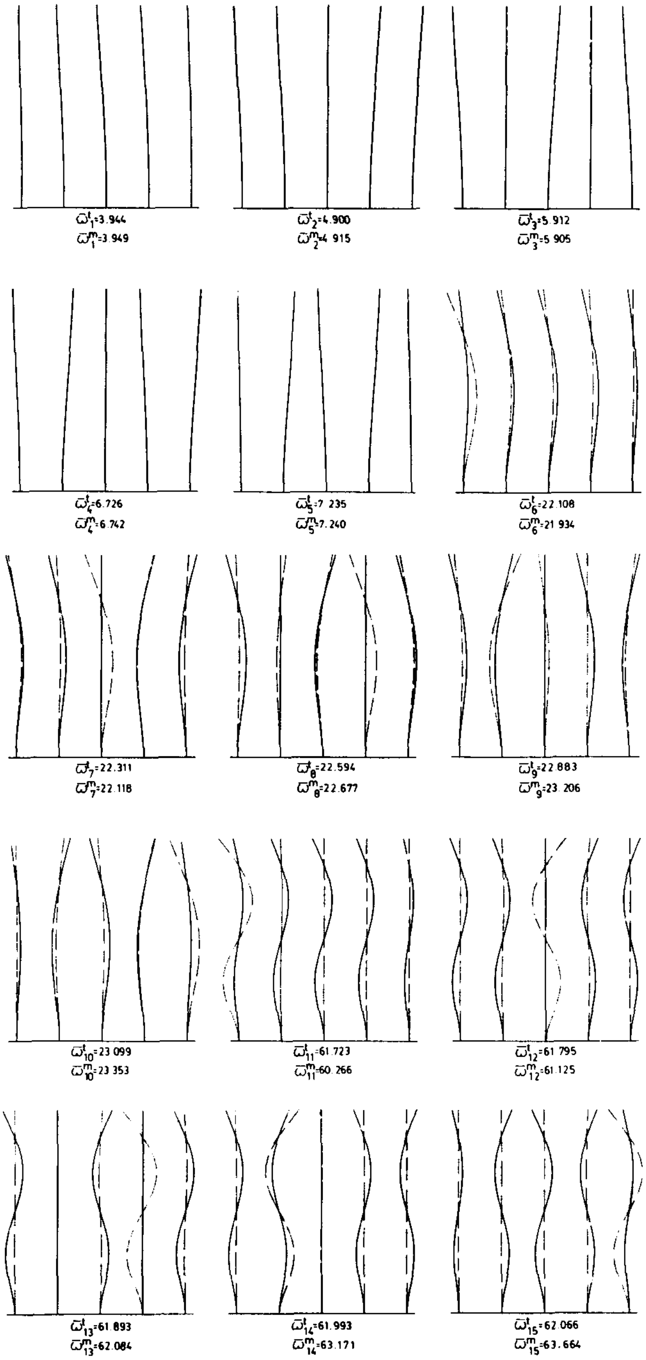


Fig. 12. Modes of tuned (—–, $\bar{\omega}^t$) and mistuned (- - -, $\bar{\omega}^m$) assemblies of five beams, for $\bar{k} = 3.$, $x_c = 1.$, and $M = 6$. The mistuning is $d\lambda^1 = .055$, $d\lambda^2 = -.04$, $d\lambda^3 = .025$, $d\lambda^4 = -.006$, $d\lambda^5 = -.055$.

4.3 Discussion

The above results show that small disorder may have drastic effects on the dynamics of the system. The degree of localization depends upon the ratio of modal mistuning (λ_i^j) to modal coupling ($R_i^2\phi_i(x_c)$), localization occurring if both quantities are small and if this ratio is on the order of one or larger. For the beam system the modal mistuning was considered to be constant, which may or may not be the case for physical systems. The modal coupling is determined by the spring stiffness constant, the component mode number, and the modal deflection at the constraint location. The modal coupling decreases rapidly as the component mode number increases, hence for arbitrarily large coupling stiffness and arbitrarily small mistuning strong localization occurs in all modes higher than some threshold number. Also, strong localization occurs in the groups of modes whose primary component mode has a node at the constraint location. Localization may or may not occur in the neighboring groups of modes depending on the coupling strength. Hence localization may appear in a group of modes, then disappear, to eventually reappear in the higher modes.

Finally, the criterion formulated in Refs. 5 and 11 for a chain of coupled pendula can be seen to apply readily to assemblies of component systems, as follows. The modal mistuning determines directly the spread in the natural frequencies of the individual mistuned component systems. Furthermore, the value of the modal coupling, $R_i^2\phi_i(x_c)$, determines the width of the frequency passbands* of the tuned system, as can be seen readily from Eq. (27) and from the natural frequencies listed in Figs. 5–12. For instance for small R_i^2 the i-th passband of the tuned system becomes small, and for $\phi_i(x_c) = 0$. multiple eigenvalues occur for the tuned system. Therefore it can be stated that strong localization occurs in a given group of modes if the corresponding passband width of the tuned system if on the order of, or smaller than the spread in individual frequencies (due to mistuning) of the component systems, and if both quantities are small. This criterion, initially stated in Refs. 5 and 11, applies readily to assemblies of component systems.

Recall that for multi-span beams[12] the above criterion was shown to be valid only for the first group of modes, and was unable to predict localization for the higher groups. This suggests a fundamental difference between the mechanisms of localization for assemblies of component systems and multi-span structures. This fascinating paradox requires further insight into mode localization, and is left for future research.

4.4 Convergence

The convergence of the component mode analysis has been checked carefully for each of the above calculations. In general the convergence is excellent, and for the values of the coupling stiffness considered in this paper the maximum number of component modes necessary to insure converged frequency values is $M = 7$. Note that the component mode analysis converges very quickly when the dimensionless modal couplings, $R_i^2\phi_i(x_c)$, are small, since then a single component mode primarily contributes to the global modes. The least rapid convergence occurs in the lower modes for large values of $\bar{k}$. Then six or seven modes are necessary to insure a very good convergence. For example, for $\bar{k} = 100.$, using seven component modes leads to natural frequencies of the first group of modes converged up to the second decimal place.

5. CONCLUSIONS

The following conclusions can be drawn from this study:

- Nearly periodic structures such as assemblies of component systems are highly sensitive to small disorder among the component systems. Under certain conditions the extended modes of the tuned system become strongly localized when small mistuning is introduced. Confinement of forced vibrations is also a consequence of disorder.
- Combined classical and modified perturbation methods have been developed, that predict the occurrence of strong mode localization and analyze the characteristics of localized modes.
- Localization occurs in a given group of modes if the corresponding modal coupling is on the order of, or smaller than the modal mistuning.
- Since the modal coupling decreases as the component mode number increases, localization occurs more easily in the higher

* note that the passband character is lost for large values of $R_i^2\phi(x_c)$.

groups of modes than in the lower ones. For arbitrarily large coupling stiffness and small mistuning, mode localization is unavoidable if the mode number is large enough. If the coupling constraint is located at the node of a component mode, the corresponding group of modes becomes very strongly localized.

ACKNOWLEDGEMENT

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REFERENCES

1. Anderson, P. W., "Absence of Diffusion in Certain Random Lattices," *Physical Review*, Vol. 109, 1958, pp. 1492–1505.
2. Rosenstock, H. B., and McGill, R. E., "Vibrational Modes of Disordered Linear Chains," *Journal of Mathematical Physics*, Vol. 3, 1962, pp. 200–202.
3. Rosenstock, H. B., and McGill, R. E., "Vibrations of Disordered Solids," *Physical Review*, Vol. 176, 1968, pp. 1004–1014.
4. Dean, P., "Vibrational Spectra of Diatomic Chains," *Proceedings of the Royal Society of London*, Vol. A254, 1960, pp. 507–521.
5. Hodges, C. H., "Confinement of Vibration by Structural Irregularity," *Journal of Sound and Vibration*, Vol. 82, No. 3, 1982, pp. 441–424.
6. Hodges, C. H., and Woodhouse, J., "Vibration Isolation from Irregularity in a Nearly Periodic Structure: Theory and Measurements," *Journal of the Acoustical Society of America*, Vol. 74, No. 3, 1983, pp. 894–905.
7. Bendiksen, O. O., "Aeroelastic Stabilization by Disorder and Imperfections," *presented at the XVIth International Congress of Theoretical and Applied Mechanics*, Lyngby, Denmark, August 19–25, 1984.
8. Valero, N. A., and Bendiksen, O. O., "Vibration Characteristics of Mistuned Shrouded Blade Assemblies," *ASME Journal of Engineering for Gas Turbines and Power*, Vol. 108, No. 2, 1986, pp. 293–299.
9. Bendiksen, O. O., "Mode Localization Phenomena in Large Space Structures," *Proceedings of the 27th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference*, May 19–21, 1986, San Antonio, Texas.
10. Pierre, C., "Analysis of Structural Systems with Parameter Uncertainties," *Ph.D. Dissertation*, Duke University, August 1985.
11. Pierre, C., and Dowell, E. H., "Localization of Vibrations by Structural Irregularity," *Journal of Sound and Vibration*, Vol. 114, No. 3, May 1987, to appear.
12. Pierre, C., Tang, D. M., and Dowell, E. H., "Localized Vibrations of Disordered Multi-Span Beams: Theory and Experiment," Accepted for publication in the *AIAA Journal* (also, *Proceedings of the 27th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference*, May 19–21, 1986, San Antonio, Texas).
13. Ibrahim, R. A., "Structural Dynamics with Parameter Uncertainties," *Applied Mechanics Reviews*, in print.
14. Dowell, E. H., "Free Vibrations of an Arbitrary Structure in Terms of Component Modes," *ASME Journal of Applied Mechanics*, Vol. 39, No. 3, Sep. 1972, pp. 727–732.
15. Meirovitch, L., *Computational Methods in Structural Dynamics*, Sijthoff & Noordhoff, The Netherlands, 1980.
16. Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, Volume I, Interscience Publishers, Inc., New York, 1953.
17. Pierre, C., "Eigensolution Perturbation for Systems with Perturbed Boundary Conditions," *Journal of Sound and Vibration*, in print.

APPENDIX A

The disordered chain of n coupled pendula shown in Fig. 1 is considered. The Δl_i's are the dimensionless length deviations from the nominal length, l. The free vibration eigenvalue problem is

$$\left([A] - \bar{\omega}^2[I]\right)\underline{\bar{\theta}} = \underline{0} \tag{A1}$$

$$[A] = [\text{tridiag}\,(\alpha_i;\beta_i;\gamma_i)] \tag{A2}$$

where

$$\begin{aligned}\alpha_i &= -R^2\,\frac{1+\Delta l_{i-1}}{1+\Delta l_i}\\ \beta_i &= \frac{1}{1+\Delta l_i} + (2-\delta_i^n)R^2\\ \gamma_i &= -R^2\frac{1+\Delta l_{i+1}}{1+\Delta l_i}\end{aligned} \tag{A3}$$

where $\delta_i^n = 1$ if $i = n$, $= 0$ otherwise. $\bar{\omega}^2 = \omega^2/(g/l)$ is a dimensionless eigenvalue, $\underline{\bar{\theta}}$ the corresponding eigenvector of pendulum angular amplitudes. $[A]$ is an n by n tridiagonal matrix, where α_i is the element of the lower diagonal (i-th line, (i-1)-th column), β_i is the element on the main diagonal (i-th line, i-th column), and γ_i is the element on the upper diagonal (i-th line, (i+1)-th column). By definition, $\alpha_1 = \gamma_n = 0$. $R^2 = \omega_k^2/\omega_g^2 = (k/m)/(g/l)$ is the dimensionless coupling, where g is the gravitational acceleration, m the mass of a pendulum, and k the stiffness constant of the springs connecting the pendula at their tips.
The tuned eigensolution is

$$\bar{\omega}_{0i}^2 = 1 + 2R^2\left[1 - \cos\frac{(2i-1)\pi}{2n+1}\right] \qquad i = 1,\cdots,n \tag{A4}$$

$$\underline{\bar{\theta}}_{0i} = \left[\sin\frac{(2i-1)\pi}{2n+1},\cdots\cdots,\sin\frac{n(2i-1)\pi}{2n+1}\right]^T \tag{A5}$$

The complex amplitude of the steady-state response under single harmonic excitation is

$$\underline{\bar{\theta}} = \left\{[A] - \bar{\omega}^2[I] + i\bar{\omega}[C]\right\}^{-1}\underline{\bar{F}}' \tag{A6}$$

$[C]$ is a damping matrix due to viscous dampers of coefficients c placed between the pendula, given by

$$[C] = \left(\frac{c}{m\omega_g}\right)[\text{tridiag}\,(a_i;b_i;c_i)] \tag{A7}$$

$$\begin{aligned}a_i &= -\frac{1+\Delta l_{i-1}}{1+\Delta l_i}\\ b_i &= 2-\delta_i^n\\ c_i &= -\frac{1+\Delta l_{i+1}}{1+\Delta l_i}\end{aligned} \tag{A8}$$

$\bar{F}_i' = \bar{F}_i/[mg(1+\Delta l_i)]$ are the dimensionless complex amplitudes of the force components. The frequency responses are obtained by taking the modulus of the complex amplitude $\underline{\bar{\theta}}$.

APPENDIX B
FIRST-ORDER EIGENSOLUTION PERTURBATION

The n by n unperturbed real matrix $[A_0]$ has n eigenvalues and right and left eigenvectors denoted by λ_{0i}, $\underline{x}_{0i}$ and $\underline{y}_{0i}$, respectively, assumed to be real. Distinct eigenvalues are assumed. The perturbed matrix $[A]$ is

$$[A] = [A_0] + [\delta A] + \cdots \tag{B1}$$

where $[\delta A]$ is the first order perturbation matrix in the structural parameters perturbations. Considering Taylor expansions of the perturbed eigensolution

$$\lambda_i = \lambda_{0i} + \delta\lambda_i \tag{B2}$$

$$\underline{x}_i = \underline{x}_{0i} + \underline{\delta x}_i \tag{B3}$$

the first order perturbations $\delta\lambda_i$ and $\underline{\delta x}_i$ can be shown to be:

$$\delta\lambda_i = \frac{\underline{y}_{0i}^T\,[\delta A]\,\underline{x}_{0i}}{\underline{y}_{0i}^T\,\underline{x}_{0i}} \qquad i = 1, \cdots, n \tag{B4}$$

$$\underline{\delta x}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\underline{y}_{0j}^T\,\underline{x}_{0j}}\,\frac{\underline{y}_{0j}^T\,[\delta A]\,\underline{x}_{0i}}{\lambda_{0i} - \lambda_{0j}}\,\underline{x}_{0j} \qquad i = 1, \cdots, n \tag{B5}$$

Detailed derivations can be found in Refs. 15 and 16.

APPENDIX C

$$\sqrt{\omega_1} = 1.875 \quad \sqrt{\omega_2} = 4.694 \quad \sqrt{\omega_3} = 7.855$$

$$\sqrt{\omega_4} = 10.996 \quad \sqrt{\omega_5} = 14.137 \quad \sqrt{\omega_6} = 17.279$$