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Upwind and High-Resolution Methods for Compressible Flow:

From Donor Cell to Residual-Distribution Schemes

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In this paper I review three key topics in CFD that have kept researchers busy for half a century. First, the concept of upwind differencing, evident for 1-D linear advection. Second, its implementation for nonlinear systems in the form of high-resolution schemes, now regarded as classical. Third, its genuinely multidimensional implementation in the form of residual-distribution schemes, the most recent addition. This lecture focuses on historical developments; it is not intended as a technical review of methods, hence the lack of formulas and absence of figures.

Why upwind differencing?

Upwind differencing is a way of differencing the spatial-derivative terms in the advection equation, and is almost as old as CFD, starting with the work of Courant, Isaacson and Rees (1952[1]). In their paper, the choice of an upwind-biased stencil follows rather naturally from the "backward" variant of the Method of Characteristics. In the course of the decades further evidence has been gathered in support of upwind discretizations.

- *Godunov, 1959.* The Russian mathematician S. K. Godunov[2] favored the first-order-accurate upwind scheme among a family of simple descretizations, because it is the most accurate one that preserves the monotonicity of an initially monotone discrete solution.

- *Fromm, 1968.* IBM researcher Jacob Fromm[3] constructed higher-order advection schemes with low dispersive error, by combining schemes with predominantly negative and predominantly positive phase errors: "Zero Average Phase Error Method." The resulting schemes turn out to be upwind biased.

- *Wesseling, 1973.* Dutch aerospace engineer (turned numerical analyst) Pieter Wesseling[4] used Parseval's theorem to relate the numerical error committed by advection schemes to the Fourier transform of the intial-value distribution. For two different families of advection schemes he found it is an upwind scheme that minimizes the L_2-error made in one time step *if the initial values contain a discontinuity.* This would indicate upwind schemes may be the preferred choice for compressible flow, where shock discontinuities are common and arise even from the smoothest initial data.

- *Van Leer, 1986.* Reversing Fromm's procedure, Dutch astrophysicist (turned aerospace engineer) Bram van Leer[5] developed an operational definition of upwind schemes. For example, a linear update scheme of the form

$$u^j = \sum_k C_k(\nu) u_{j+k}, \tag{1}$$

is called *upwind-biased with respect to the CFL-number range (0,1)* if and only if its coefficents satisfy the symmetry relation

$$C_k(1-\nu) = C_{-k-1}(\nu); \tag{2}$$

here ν is the CFL number. For any such scheme, the result of one step with CFL number ν followed by a step with CFL-number $1-\nu$, is free of dispersion, a quantitative expression of Fromm's idea. It further follows that such a scheme is free of dispersion for $\nu = 1/2$.

- *Jeltsch, 1987.* Mathematicians including Swiss Rolf Jeltsch,[6] searching for advection stencils with the greatest potential accuracy for a given

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number of grid-points, have proved that these stencils are upwind biased.

The price one has to pay for all this goodness is the computational effort in determining the advection direction. That is trivial for a linear 1-D advection equation, but a major effort for nonlinear advection operators hidden in nonlinear systems of multidimensional conservation laws.

Implementation for nonlinear conservation laws

Riemann solvers

Having resigned to upwind differencing for advection, I will now discuss the step of implementing this concept in finite-volume fashion for a nonlinear system of conservation laws such as the Euler equations.

The finite-volume interpretation of the simplest upwind advection scheme is the "donor-cell" scheme, where the advected quantity streams from the upwind "donor" cell into the cell to be updated. This terminology stems from the Los Alamos and Livermore National Laboratories. In his landmark 1959 paper, Godunov developed an ingenious interpretation of the donor-cell scheme, which could immediately be generalized to the 1-D Euler equations, or any other hyperbolic system of nonlinear conservation laws. The key to this generalization is the solution to Riemann's initial-value problem, that is, the problem of the inviscid interaction of two uniform gases at a plane interface. Differences between these states will be resolved by a system of plane waves moving away from the interface. If the equation of state of the gas is simple, the exact solution in the disturbed region can be obtained to any precision with medium computational effort, and in an approximate way with little effort. Often an approximation is sufficient for use in a finite-volume scheme, since only an interface flux is needed, and the details of the sub-grid solution are averaged out anyway after each time step.

For two decades Godunov's finite-volume scheme was of little influence in Western numerical circles, although it was mentioned in the book by R. D. Richtmyer and K. W. Morton.[7] This changed with the advent of Godunov-type high-resolution schemes,[8] when some of the world's most talented numerical analysts spent their efforts on, among other things, developing "approximate Riemann solvers."

Let me dwell upon the most popular approximate Riemann solvers developed in those days, and their individual pros and cons. They can be ordered according to how closely they adhere to the wave structure of the exact Riemann solution. The most detailed approximations of the wave system are found in the solvers of British aeronautical engineer Phil Roe (1981[9]) and American mathematician Stan Osher (1981[10]). The former solver is based on a local lineaization of the flow equations; the latter replaces shock waves by simple compression waves.[11] With these models the flux imbalance between two cells is split into contributions from forward-moving and backward-moving waves; hence we speak, rather clumsily, of "flux-difference splitting," or, more elegantly, of "fluctuation splitting."

A family of solvers in which a smaller or larger number of waves are "lumped" was presented by the Israeli mathematician Ami Harten († 1994), CFD's American nestor Peter Lax, and Bram van Leer (1983[12]); this approach is now indicated by the initials HLL. It is particularly useful when the detailed Riemann solution is complicated, as for extended hydrodynamics[13] and magnetohydrodynamics,[14] or when a steady flow solution is sought in which certain kinds of waves never appear.[15] The latest and most efficient descendant in this family is HLLL, a flux due to aerospace engineer (turned space scientist) Timur Linde (2000[16]).

In "flux-vector splitting" or simply "flux splitting," the fluxes themselves are are split into forward and backward contributions. This may interpreted as the result of transport by particles rather than by waves, and is related to methodology for integrating the Boltzmann equation. Representative of such methods are the "Beam Scheme" of astronomer Kevin Prendergast (1974[17]), a true "Boltzmann solver," the splitting of NASA Ames' Joe Steger († 1992) and Bob Warming (1981[18]), which turned out to be a special case of the Beam Scheme, and the differentiable splitting of Van Leer (1982[19]). Currently, the most advanced CFD methods founded on the Boltzmann equation are due to Hong-Kong-based Kun Xu,[20] a former student of Prendergast.

Computational aerodynamicists became particularly fond of upwind flux formulas because these produced the narrowest possible shock structures in transonic flow simulations.[21, 22] It can be shown that the upwind flux formula based on Roe's approximate Riemann solver yields a steady normal-shock structure (if aligned with the grid) that contains at most one internal cell, whereas the differential flux formulas of Osher[10] and Van Leer[19] include one or two internal cells. This property is lost for shocks oblique to the grid, which serves as a motivation for the search of truly multi-D upwind methods (see the last section).

While scoring high in shock representation, flux splitting can be shown to lead to diffusion across contact discontinuities and shear layers, even if these are steady and aligned with the grid.[23] This type of numerical error is absent for detailed Riemann solvers like Roe's or Osher's, which explicitely recognize such waves. A hybrid splitting, combining

the best of fluctuation splitting (accuracy) and flux splitting (simplicity) is AUSM (Advection Upwind Split Method) developed by NASA Lewis' Meng-Sing Liou and Chris Steffen (1993) under the name AUSM (Advection Upwind Split Method). It has become a very popular numerical flux function; for the latest update on the AUSM family see Liou (2001[24]).

Helpful Hint Nr. 1. Riemann-solver-based fluxes have nothing to do with limiters, used in higher-order schemes to avoid numerical oscillations. There is no such thing as a "TVD Riemann solver;" using such language reveals ignorance.

This, though, leads us to the next subject.

High-resolution schemes

Interlaced with the evolution of upwind methods is the history of high-resolution schemes: schemes that are at least second-order accurate in regions where the solution is smooth, while capturing discontinuities as narrow, monotone structures.

For an appreciation of the problem of designing a high-resolution scheme for the Euler equations it is useful to first consider modeling the linear advection of a step function. Here we immediately run into a famous theorem included by Godunov in his 1959 paper: *if an advection scheme preserves the monotonicity of the solution it is at most first-order accurate.* This result could discourage anyone attempting to improve advection schems; fortunately, there is a way to circumvent it. In the proof of this theorem it is tacitly assumed that the linear advection equation is approximated by a linear discretization; once *nonlinear* discretizations are admitted the theorem no longer stands and high-resolution schemes become possible.

The realization that Godunov's theorem could be circumvented came at the start of the 1970s, when, within the span of one year, three independent approaches were launched for the construction of oscillation-free higher-order advection schemes. Astrophysicist Jay Boris (Naval Research Laboratory) presented the first non-oscillatory second-order scheme SHASTA - Sharp and Smooth Transport Algorithm - at a seminar course in Trieste in August 1971,[25] followed by another astrophysicist, Bram van Leer (then at Leiden Observatory, Netherlands), who presented a non-oscillatory modification of the Lax-Wendroff scheme at the 3rd International Conference on Numerical Methods in Fluid Dynamics (ICNMFD) in Paris in July 1972.[26] These approaches, while quite distinct, are similar in their use of nonlinearity to prevent numerical oscillations in regions of strongly varying solution gradients, a theme that has persisted in hyperbolic method development ever since. It is useful to discuss the two approaches separately in some detail. As to the third, Russian approach, see the subsection "A historical injustice."

The algorithms developed by Boris, D. L. Book *et al.* are known as Flux-Corrected Transport (FCT) methods and have a predictor-corrector structure. A first-order, non-oscillatory scheme is used to estimate the solution at the advanced time level; a correction step then removes the large dissipative error made in the first step, uncovering a solution with second- or third-order accuracy. During the second step the corrective fluxes are compared to the provisional solution values and limited where necessary, in order to ensure that no new extrema will arise, nor existing extrema grow. The comparison step makes the overall method nonlinear: the coefficients in the scheme depend on the solution itself, even when applied to a linear equation.

The development of FCT methods is described in four key papers published in the period 1973-79 in the Journal of Computational Physics (JCP), *viz.* a sequence of three by Boris *et al.*[27–29] and a fourth by S. Zalesak,[30] also at NRL. FCT methods are widely used for simulating violent time-dependent flows, but are less suited for steady-state calculations and therefore have had little influence on computational aerodynamics.

The methodology of Van Leer was published in a series of five papers, starting with the above conference paper; the other four appeared in JCP in the period 1974-79.[8, 31–33] In this work, oscillations are regarded as the result of oscillatory interpolation of the discrete intitial values; the remedy therefore consists of introducing non-oscillatory or monotonicity-preserving interpolation. The simplest higher-order schemes reconstruct a linear or quadratic distribution in a cell using three contiguous cell averages. Monotonicity is preserved by limiting the values of the first and second derivatives of the distribution.

Following Godunov, Van Leer's schemes include fluxes derived from the solution of Riemann problems. When combined with higher-order reconstruction this leads to upwind-biased differencing, hence the name MUSCL - Monotone Upstream Scheme for Conservation Laws - given to the first computer code based on the above principles. This code was written at Leiden Observatory by astrophysicist Paul Woodward. It uses piecewise linear interpolation and achieves second-order accuracy for smooth flow. MUSCL and its sequel PPM (Piecewise Parabolic Method) were advanced and popularized in the early 1980s by Woodward and numerical analyst Phil Colella at Lawrence Livermore Laboratory.[34, 35] A landmark paper is their 1984 JCP review,[36] an elaborate comparative study in which Godunov-type schemes, FCT and more traditional

methods are pitted against one another.

The emergence of high-resolution Godunov-type methods created yet another research trend: designing effective limiters for use in one-dimensional higher-order reconstruction.[37–39] Particularly interesting is the history of the involvement of Ami Harten.

Initially Harten did not embrace the concept of non-oscillatory interpolation, because it seemed restricted to one space dimension. Instead he was inspired by the work of Lax's student James Glimm,[40] who had shown that the total variation of the solution of a scalar 1-D conservation law can not increase, and actually decreases in a shock. By analogy, Harten introduced the total variation of a discrete function as a measure of its oscillatory nature,[41,42] a quantity with the promise of multidimensional applicability. This led to the formulation of Total-Variation-Diminishing (TVD) schemes for scalar nonlinear conservation laws; the acronym TVD quickly became synonymous with high-resolution, usually upwind, schemes (see *Helpful Hint Nr. 1* above). Harten derived local conditions sufficient for ensuring the TVD property; for a linear conservation law these reduce to the constraint of monotone interpolation. It later was shown by numerical analysts Jonathan Goodman and Randy LeVeque[43] that the total variation is too blunt a tool to be of use in constraining multi-dimensional discrete functions: a multi-dimensional TVD advection scheme can be no better than first-order accurate.

Harten then returned to non-oscillatory interpolation theory, and, from 1985 onward, developed the theory of Essentially Non-Oscillatory (ENO) interpolation, joined by Osher, numerical analyst Björn Engquist, mechanical engineer Sukumar Chakravarthy, Osher's student Chi-Wang Shu[44–47] and others. ENO schemes are only Total-Variation-Bounded (TVB) and do generalize to multi-dimensional equations.

In these schemes a systematic procedure selects the discrete stencil whose data will give the smoothest interpolant, i. e. the function with the lowest values of its derivatives. This procedure is nonlinear everywhere, that is, even in regions where there is not the slightest danger of numerical oscillations arising. In later years ENO has been replaced by the milder version WENO (Weighted ENO[48]). A particularly attractive formulation of limiters comparable to WENO is due to Hung Huynh at NASA Glenn. His limiters are based on three arguments, but the notation is terse owing to the introduction of the concept of the "median" of three numbers.

The ENO procedure is the only known non-oscillatory interpolation that allows a truly multidimensional extension, albeit very costly. The single paper about this subject is due to the French numerical analyst Rémi Abgrall.[49]

Helpful Hint Nr. 2. A limiter is a nonlinear algorithm that reduces the high-derivative content of an subgrid interpolant in order to make it non-oscillatory; it is *not* the interpolant itself. Calling standard polynomial interpolation a "limiter" is a bad practice, again revealing the ignorance of the speaker. This habit was probably caused by the appearance of interpolants with built-in limiter, such as "harmonic gradient averaging" due to Van Leer (1977[8]), and similar techniques due to Dutch astronomer Dick van Albada (1982[50]) and Dutch aerospace engineer Barry Koren (1989[39]). Such interpolants can still be written as a target polynomial interpolant modified by a limiter; for instance, the harmonic average of successive finite-differences,

$$(\Delta W)_j = \left\{ \frac{1}{2} \left(\frac{1}{\Delta_- W_j} + \frac{1}{\Delta_+ W_j} \right) \right\}^{-1}, \quad (3)$$

may be written as

$$(\Delta W)_j = \frac{\Delta_+ W_j + \Delta_- W_j}{2} \times \quad (4)$$

$$\left\{ 1 - \left(\frac{\Delta_+ W_j - \Delta_- W_j}{\Delta_+ W_j + \Delta_- W_j} \right)^2 \right\}. \quad (5)$$

The first factor is the standard central difference, the second factor is the limiter. Note that the limiter contains a discretization of the logarithmic second derivative of the solution:

$$\left(\frac{\Delta_+ W_j - \Delta_- W_j}{\Delta_+ W_j + \Delta_- W_j} \right)^2 \approx \frac{(\Delta x)^2}{4} \left(\frac{\partial \ln \left| \frac{\partial W}{\partial x} \right|}{\partial x} \right)^2. \quad (6)$$

It is this quantity that senses the high rate of change of the solution gradient at the foot and head of a captured discontinuty.

Helpful Hint Nr. 3. Limiters of the type introduced by Van Leer reduce the higher-order content of the sub-grid interpolant for the sake of monotonicity, regardless of what conservation law is to be integrated. Using the expression "flux limiter" is therefore inappropriate and reveals ignorance.

It is true that limiting the interpolant has an effect on the fluxes; Osher has actually published shemes in which the limiting is applied to interpolants of split fluxes. This approach, motivated by doubtful computational savings, is undesirable, as split fluxes are less smooth than the solution itself. In particular, they are nondifferentiable in a sonic point, unlike the solution.

In the corrector step of an FCT scheme, though, the fluxes indeed are limited; in this case the term "flux limiting" is appropriate.

A historical injustice

It has been pointed out to me by Dr. Vladimir Sabelnikov, formerly of TsAGI, the Central Aerodynamical National Laboratory near Moscow, that a scheme closely resembling MUSCL (including limiting) was developed in this laboratory by V. P. Kolgan (1972[51]). Kolgan died young; his work apparently received little notice outside TsAGI. When I visited Novosibirsk in 1978 in order to present MUSCL to Godunov and to find out if anything similar had been developed in the Soviet Union, the approach appeared to be new to Godunov. He was not aware of Kolgan's work.

Hancock's scheme

Soon after its publication (1979[8]), MUSCL was greatly simplified by UC Berkeley graduate (fluid mechanics) Steve Hancock 1980), while developing the PISCES industrial simulation code for Physics International (later absorbed by MacNeal-Schwendler). Hancock's predictor-corrector version would have remained buried in a bulky code manual if not for its description and use in a paper* by Van Albada, Van Leer and Bill Roberts (astronomer, UVA) on numerical methods for cosmic gas dynamics. (NB: the same paper includes the first description of Van Albada's limiter.) I will use the present occasion to bring it once more to the attention of the CFD community, as it has been a regular workhorse in my department, both in the Keck CFD Laboratory and in the graduate CFD curriculum. The method is remarkably robust; for example, it handles the double-blast-wave problem of Woodward and Colella[36] without any special attention.

Below are coding instructions for the 1-D description given to all advanced CFD students for use in their first computing project; the trivial 2-D extension is regularly used in the second project (see Van Leer[52]).

Hancock's predictor-corrector scheme.

Hancock's scheme is a MUSCL scheme implemented in predictor-corrector fashion, similar to the Richtmyer version of Lax-Wendroff. It includes the following steps (conversions among state quantities omitted).

1. Reconstruct linear subcell distributions of a complete set of state variables; these may be the conserved variables $\rho, \rho u, \rho E$. In practice, though, it is not such a good idea to use ρE, which must remain greater than $\rho u^2/2$; independent interpolation of $\rho, \rho u$ and ρE does not guarantee this. Better use $\rho, \rho u, p$, or even primitive variables ρ, u, p; call the latter set $\mathbf{W}$.

   You'll work only with uniform grids, so you don't have to worry about variations in Δx. In this case it suffices to define

   $$\delta W_j = \text{ave}\,(W_j - W_{j-1}, W_{j+1} - W_j)\,, \qquad (7)$$

   where the average may include your favorite limiter. Keep the following averages available:

   $$\text{ave}(a,b) = \frac{a+b}{2}; \quad \text{(algebraic)} \qquad (8)$$

   $$\text{ave}(a,b) = \begin{cases} \text{minmod}\left(\frac{a+b}{2}, 2a, 2b\right), & ab > 0, \\ 0, & ab \le 0; \\ \text{(double minmod)} \end{cases} \qquad (9)$$

   $$\text{ave}(a,b) = \begin{cases} \text{minmod}\{\text{maxmod}(a,b), & \\ \quad \text{minmod}(2a, 2b)\}, & ab > 0, \\ 0, & ab \le 0. \\ \text{(Superbee)} \end{cases} \qquad (10)$$

   Observe that these three form a hierarchy. Double minmod limits the algebraic average of a and b to twice the smaller of the two: the weakest limiting still avoiding under/overshoots. Superbee, moreover, discards any unlimited value of the algebraic average in favor of the larger of the two, yielding artificial steepening wherever there is no danger of under/overshooting.

2. Advance the solution inside cell j by half the full time-step, using the nonconservative equations in terms of the primitive variables,

   $$\mathbf{W}_t + (\mathbf{A}_W)_j \mathbf{W}_x = \mathbf{0}, \qquad (11)$$

   where the coefficient matrix $\mathbf{A}_W$ is similar to $\mathbf{A}_U$ and reads

   $$\mathbf{A}_W = \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{pmatrix}. \qquad (12)$$

   The predictor step thus becomes

   $$\tilde{\mathbf{W}}_j = \mathbf{W}_j - \frac{\Delta t}{2\Delta x} \mathbf{A}_W \delta \mathbf{W}_j. \qquad (13)$$

3. Now compute time-centered interface values using the *old* gradients (this is OK in a second-order scheme):

   $$\tilde{\mathbf{W}}_{j-\frac{1}{2}R} = \tilde{\mathbf{W}}_j - \frac{1}{2}\delta \mathbf{W}_j, \qquad (14)$$

   $$\tilde{\mathbf{W}}_{j+\frac{1}{2}L} = \tilde{\mathbf{W}}_j + \frac{1}{2}\delta \mathbf{W}_j. \qquad (15)$$

4. Compute time-centered interface fluxes by solving a Riemann problem, exactly or approximately, at each interface:

   $$\tilde{\mathbf{F}}_{j+\frac{1}{2}} = \mathbf{F}\left(\tilde{\mathbf{W}}_{j+\frac{1}{2}L}, \tilde{\mathbf{W}}_{j+\frac{1}{2}R}\right). \qquad (16)$$

*The same paper includes the first description of Van Albada's limiter.

5. Finally, advance the solution over the full time-step, using the time-centered interface fluxes:

$$\mathbf{U}^j = \mathbf{U}_j - \frac{\Delta t}{\Delta x}\left(\tilde{\mathbf{F}}_{j+\frac{1}{2}} - \tilde{\mathbf{F}}_{j-\frac{1}{2}}\right). \qquad (17)$$

Variation 1: if you set $\delta\mathbf{W} = 0$ everywhere you'll get back the first-order upwind scheme; this is useful for debugging, or for studying the quality of your numerical flux near a sonic point. Of course there are cheaper ways to embed the first-order scheme in your program. In this project, though, we are not interested in comparing second- to first-order accuracy; that's too elementary.

Variation 2: instead of the primitive variables you may use the characteristic variables V_k in the predictor step, which actually are the most appropriate choice when gradient limiting is expected. They also give the "cleanest" results. Consult me.

Helpful Hint Nr. 4. The coefficient matrix $\mathbf{A}_W$ in the primitive equations is often referred to as the "primitive Jacobian," This is incorrect use of the term Jacobian. A Jacobian matrix is the derivative of one vector to another one, and $\mathbf{A}_W$ is not the derivative of a vector. The confusion stems from the fact that the coefficient matrix $\mathbf{A}_U$ in the *conservative* equations is a true Jacobian, since it results from differentiating the flux vector with respect to the conservative state vector ("flux Jacobian"). Granted, $\mathbf{A}_W$ does follow from $\mathbf{A}_U$ by a similarity transformation. I will therefore not go so far as to say that using the term "primitive Jacobian" reveals the ignorance of the speaker, since the mistake is sufficiently subtle. But I encourage avoiding it.

Multidimensional extensions

The traditional way of extending upwind differencing to multidimensional equations is by doing it dimension by dimension. This means that numerically all transport is done by waves moving normal to the cell faces. The finite-volume technique will still be consistent and may even be highly accurate for smooth flow, but discontinuities oblique to grid faces will be worse resolved than those aligned with the grid. More diffusive schemes may not show such directionality in their resolution, but that does not make them any better. On the contrary, the sensitivity of the upwind scheme's numerical error to the direction of a captured discontinutiy is revealing and contains the key to its remedy.

Already in the early 1980s a start was made with the development of directional upwinding. Davis (1984[53]) rotated the frame of the Riemann solver to the shock-normal direction, in order to get better shock resolution. Various methods of this type followed, e.g., those by Levy *et al.* (1993[54]) and Dadone and Grossman (1991[55]). A more radical approach is that of Rumsey *et al.* (1993[56]), in which the numerical flux is based on a rotated Riemann solver plus a shear wave normal to the other waves. Unfortunately, flux formulas based on rotated frames, where the rotation angle follows from flow gradients, are not robust when used in higher-order schems. They do achieve the goal of uniform resolution of shocks and shears independent of direction.

A more fundamental and robust approach, which also has its roots in the 1980s and is due to Roe (1986[57]), is that of the "genuinely multidimensional" upwind schemes. These may be regarded as the true multi-D generalization of 1-D fluctuation splitting. The multi-D residual is decomposed into contributions of different physical origin; these are then sent downwind in order to represent advection, or distributed omnidirectionally when representing subsonic acoustic propagation. These methods are best formulated on simplex-type (finite-element) grids and include newly developed, compact limiters for avoiding oscillations.

The schemes have almost exclusively been designed for marching to a steady solution; only recently there has been some activity in formulating time-accurate extensions, e.g. by Hubbard and Roe (2000[58]).

Genuinely multi-D schemes have matured during the 1990s owing to the efforts of Roe *et al.* at the University of Michigan in Ann Arbor and Herman Deconinck *et al.* at the Von Kármán Institute near Brussels. Milestones were the doctoral theses of Lisa Mesaros (1995, UMich[59]) and Henri Paillère (1995, VKI[60]). The approach has become popular only in Europe, where it is actually used for solving industrial flow problems. A comprehensive report describing the European effort up to 1996 is the BRITE/EURAM project book edited by Deconinck and Koren (1997[61]).

The state of the art is probably still defined by the theses of Dutchman Erwin van der Weide (1998, TU Delft, NL[62]) and Roumanian Doru Caraeni (2000, TI Lund, S[63]). Van der Weide solves complex steady viscous rocket-base-flow problems. His findings are that the method does fulfil its promise of uniform resolution regardless of direction, but that convergence to a steady solution suffers, probably because of the compact, highly nonlinear limiters. Caraeni develops a third-order-accurate scheme for Large-Eddy Simulation (LES). Temporal accuracy is achieved by using the scheme only in an inner iterative (pseudo-time) loop, which solves an outer, implicit update scheme.

The newer name for the multi-D upwind methods is "residual-distribution schemes." Roe, Abgrall and Z.-J. Wang (Michigan State University) are currently working on achieving high-order accuracy

without going outside one vertex-defined cell. These techniques are starting to look more and more like Discontinuous-Galerkin methods. Their justification lies in large-eddy and turbulence simulation, as well as in climate modeling.

Roe and students have also been studying the coupling of residual-distribution schemes to grid adaptation.[64, 65]

While the research on residual-distribution schemes has greatly increased our insight in the structure of the flow equations and our ability to numerically represent the flow physics with minimal grid bias, the reality is that the American aeronautical community has not bought into this methodology. Presumably, the advent of high-performance computing and promise of massively parallel computing has quelled any drive toward a systematic modernization of CFD algorithms. It is easy to be cynical or at least sceptical regarding the impact of current cutting-edge algorithm research on the CFD-users community. I personally believe, though, that the next round of gains in CFD will not come from hardware improvement but, once again, from method development.

In this paper I have made a point of indicating the disciplines and nationalities of important contributors to the development of CFD in the past half century. In doing this I wanted to emphasize that CFD was developed by people of all walks of life, not just by aeronautical engineers. It is a tribute to our aerospace community that it has had the openness of mind to absorb the very best other disciplines had to offer. Where excellence is the prime goal, xenophobia has no place. May other disciplines follow in our footsteps.

Those with further interest in upwind and high-resolution schemes are encouraged to consult the book *"Upwind and High-Resolution Schemes"*,[66] an anthology of key papers on the subject, preceded by a technical introduction by Phil Roe and a historical review by myself, from which I have heavily borrowed for this paper.

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