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Predicting localization phenomena via Lyapunov exponent statistics

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PREDICTING LOCALIZATION PHENOMENA VIA LYAPUNOV EXPONENT STATISTICS

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Abstract

Lyapunov exponents can be used to analyze localization in a disordered system for which the system's transfer matrix is known. Lyapunov exponents provide a measure of the exponential spatial amplitude decay of the various wave types. However, this decay may be due to several mechanisms. In this work, the standard deviations of the Lyapunov exponents are shown to predict frequency regions where the decay is due to localization, as opposed to off-resonance or dissipation. In this way, critical frequency regions where increased vibration amplitudes may occur can be systematically identified. The covariance of the Lyapunov exponents is also considered, and proposed as an indicator of wave conversion.

1. Introduction

A periodic structure consists of an assembly of identical substructures, or *bays*, which are dynamically coupled in some identical manner. As an example of a periodic structure, consider a simply supported multi-span beam with uniform physical properties. If the spans are of equal length, then this is a periodic structure. Each span is an identical substructure, and the spans are coupled through the rotation at each support.

A periodic system that has only one type of coupling between adjacent bays is called *mono-coupled*, while one that has more than one type of coupling is called *multi-coupled*. The multi-span beam on simple supports is a mono-coupled system. But if the supports were flexible — and, for instance, modeled by linear springs — then this would be a multi-coupled system, since the spans would be coupled through both a rotation and a vertical displacement at each support.

For each coupling type in a periodic structure undergoing harmonic motion, there is a wave pair consisting of a left- and right-traveling harmonic wave. For certain frequency ranges, called *passbands*, the waves propagate without attenuation through a periodic (undamped) system, with only a phase change from one bay to the next. There exists one passband for each degree of freedom of a subsystem, and the natural frequencies of a finite periodic system are found inside of the passbands. Outside the passbands, the wave amplitudes decay exponentially along the system. In general, these frequency regions are *stopbands*, which are characterized by the spatial decay of the wave amplitudes and a phase change per bay of 0 or π radians[1]. For multi-coupled systems, there may also exist *complexbands*, in which two different wave types traveling in the same direction have identical decay rates and opposite phase change per bay.

In an actual engineering structure, however, the substructures are not identical. This break in the periodicity, called *disorder* or *mistuning*, may have a dramatic effect on the dynamics of the system. In particular, disorder causes partial reflections of the waves at each bay, even at frequencies within the passband of the associated periodic system. The effect of these multiple reflections over many bays is to confine waves to one part of the structure, leading to larger wave (vibration) amplitudes for some bays than would be found in the ordered structure. This phenomenon is known as *localization*.

The partial reflections at each bay also lead to an (asymptotically) exponential spatial decay of the wave amplitudes away from the localized region. For a mono-coupled system, there is a single wave type, so the spatial decay rate of this wave type corresponds exactly to the spatial decay rate of the vibration amplitudes. The associated exponential decay constant is known as the *localization factor*.

Localization in multi-coupled structures is much more difficult to analyze than in mono-coupled structures. This is because in multi-coupled structures the decay rates of all the different wave types must be considered. Furthermore, at any given bay a wave type may be partially reflected and transmitted into waves of the same type or into other wave types as well. We refer to the partial reflection/transmission of one wave type into another as *wave interaction* or *mixing*. If one wave dies out but leaks energy into another wave which may propagate further, we call this *wave conversion*.

One way to quantify localization in a multi-coupled system is to find the Lyapunov exponents of the (stochastic) global wave transfer matrix[2], which is the matrix that relates the wave amplitude vector at one end of the structure to that at the other end. For the mono-coupled case, it has been shown[3,4] that the largest Lyapunov exponent is equivalent to the localization factor.

In Ref. 5, Lyapunov exponents were shown to provide a valuable tool for analyzing wave decay in multi-coupled systems. In this paper, the work of Ref. 5 is extended. Here, we consider the various statistics of the Lyapunov exponents. In particular, the standard deviations of the Lyapunov exponents are calculated, and they are shown to determine frequency regions where localization effects, rather than off-resonance or damping effects, cause the wave amplitude decay. We also consider the covariance of the Lyapunov exponents as a possible predictor of wave conversion.

Only a few papers have considered the statistics of Lyapunov exponents. Cha and Morganti[6] investigated the mean, variance, and probability density of the localization factor for the mono-coupled system considered here. However, they used this information to make correct inferences about the rate of exponential decay of a typical system, while we use the standard deviation for a different purpose. Cusumano and Lin[7] calculated the covariance matrices for the Lyapunov *vectors* of a nonlinear system in order to determine modal interaction. We adopt a similar approach here by using the covariance of the Lyapunov *exponents* to explore wave conversion.

This paper is organized as follows. In Section 2, we review the concept of Lyapunov exponents as well as the algorithm used to calculate them. In Section 3, an example mono-coupled structure and multi-coupled structure are introduced. In Section 4, we examine the various statistics of the Lyapunov exponents for these example systems. In Section 5, conclusions are drawn from this study.

2. Lyapunov Exponents

We now provide a basic introduction to Lyapunov exponents for nearly periodic systems. A more detailed review was presented in Ref. 5.

Let us consider a periodic or nearly periodic system undergoing harmonic motion. The vector $\mathbf{u}_i$ contains amplitudes of physical coordinates taken at the junction to the right of an arbitrary bay i. This vector is of size $2m$, where m is the number of coupling coordinates between adjacent bays[8]. The harmonic dynamics of bay i may be represented by a $2m \times 2m$ transfer matrix, $\mathbf{T}_i$, which depends on the frequency of motion. If the system is perfectly periodic, then each transfer matrix is identical, $\mathbf{T}_i \equiv \mathbf{T}$, so that adjacent states are related by:

$$\mathbf{u}_i = \mathbf{T}\mathbf{u}_{i-1} \qquad (1)$$

for all i. In a disordered system, however, $\mathbf{T}_i$ will be a random matrix:

$$\mathbf{u}_i = \mathbf{T}_i\mathbf{u}_{i-1} \qquad (2)$$

We can use $\mathbf{X}$, the matrix of eigenvectors of $\mathbf{T}$, to define the transformation from physical to wave coordinates:

$$\mathbf{u} = \mathbf{X}\mathbf{v} \qquad (3)$$

where $\mathbf{v}$ is a vector of wave amplitudes. Substituting Eq. (3) into Eq. (2), we retrieve the following relation for wave vectors at adjacent bays:

$$\mathbf{v}_i = \left[\mathbf{X}^{-1}\mathbf{T}_i\mathbf{X}\right]\mathbf{v}_{i-1} = \mathbf{W}_i\mathbf{v}_{i-1} \qquad (4)$$

where $\mathbf{W}_i$, the wave transfer matrix for bay i, is simply the representation of $\mathbf{T}_i$ in the wave coordinate basis. Note that in an ordered system, $\mathbf{W}_i = \mathbf{W}$ is diagonal, which implies that the $2m$ characteristic waves are independent for this case. For a disordered system, however, there will be off-diagonal terms indicating both wave interactions and reflections.

Our goal here is to describe the asymptotic behavior of the wave vector, $\mathbf{v}$, as it is taken at subsequent bays along the system. If we have an arbitrary initial wave vector, $\mathbf{v}_0$, at the left end of a nearly periodic structure, the wave vector after N bays, $\mathbf{v}_N$, is

$$\mathbf{v}_N = [\mathbf{W}_N\mathbf{W}_{N-1}\cdots\mathbf{W}_1]\,\mathbf{v}_0 = \mathcal{W}_N\mathbf{v}_0 \qquad (5)$$

where $\mathcal{W}_N$ is the global wave transfer matrix for the N-bay segment. We take the norm of a vector $\mathbf{v}$ to be:

$$||\mathbf{v}|| \equiv [\langle \mathbf{v}, \mathbf{v} \rangle]^{\frac{1}{2}} = [\mathbf{v}^* \mathbf{v}]^{\frac{1}{2}} \tag{6}$$

where $||\bullet||$ is the norm of the argument, $\langle \bullet, \bullet \rangle$ denotes an inner product, and $*$ denotes the complex conjugate of the transpose. From Eqs. (5) and (6), the norm of $\mathbf{v}_N$ is:

$$||\mathbf{v}_N|| = \left[\mathbf{v}_0{}^* \mathcal{W}_N^* \mathcal{W}_N \mathbf{v}_0\right]^{\frac{1}{2}} \tag{7}$$

Clearly, the magnitude of this norm will depend on the eigenvalues of $\mathcal{W}_N^* \mathcal{W}_N$, the square root of which are the singular values of $\mathcal{W}_N$.

Since $\mathcal{W}_N$ is a product of independent and identically distributed random matrices, asymptotically the Nth root of the singular values are nonrandom with probability one[9,10]. The Nth root of an arbitrary kth singular value, σ_k, is of the form[11]

$$[\sigma_k(\mathcal{W}_N)]^{1/N} \to e^{\gamma_k} \quad \text{as} \quad N \to \infty \tag{8}$$

where γ_k is the kth *Lyapunov exponent*. Equation (8) leads to the following definition of the kth Lyapunov exponent:

$$\gamma_k \equiv \lim_{N \to \infty} \frac{1}{N} \ln |\sigma_k(\mathcal{W}_N)| \tag{9}$$

where $|\bullet|$ is the modulus of the argument. For an *ordered* system, Eq. (9) simplifies to

$$\gamma_k \equiv \ln |\mathbf{W}_{(k,k)}| \equiv \ln |\lambda_k(\mathbf{T})| \tag{10}$$

where $\mathbf{W}_{(k,k)}$ is the (k,k) element of $\mathbf{W}$.

Our notation assumes that the $2m$ Lyapunov exponents are ordered in the following fashion:

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{2m} \tag{11}$$

We refer to γ_1 as the first Lyapunov exponent, γ_2 as the second Lyapunov exponent, and so forth.

For periodic or nearly periodic systems, the Lyapunov exponents have the following properties:

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m \geq 0 \tag{12}$$

$$\gamma_{2m+1-i} = -\gamma_i \qquad i = 1, 2, \ldots, m \tag{13}$$

In Eq. (12), a strict inequality holds if the system is disordered[11]. In the disordered case, we refer to $\gamma_1, \ldots, \gamma_m$ as the *positive Lyapunov exponents*, and to $\gamma_{m+1}, \ldots, \gamma_{2m}$ as the *negative Lyapunov exponents*. The fact that these Lyapunov exponents appear in opposite pairs is a serendipitous result, since therefore only m Lyapunov exponents need to be calculated.

The relevance of Lyapunov exponents might best be demonstrated by the special case of a disordered mono-coupled system. Here, we have only one pair of waves (one wave type). The first Lyapunov exponent, γ_1, will therefore correspond exactly to the asymptotic growth rate of a left-traveling wave in the system as we go from left to right, which is the opposite of the decay rate of the wave in its direction of travel. Recalling that the localization factor, γ_{loc}, of a disordered mono-coupled system is the absolute value of the asymptotic exponential decay constant of a wave, we have:

$$\gamma_1 \equiv \gamma_{loc} \tag{14}$$

which holds with probability one[3,4].

We now present an algorithm due to Wolf *et al.*[12] which can compute the full spectrum of Lyapunov exponents. Given an initial wave vector, $\mathbf{v}_0$, the vector will, in general, align itself with the fastest growing one-dimensional subspace after many bays. Therefore, the first Lyapunov exponent dictates the growth of the vector after many bays. This yields the the following relation

$$\gamma_1 \equiv \lim_{N \to \infty} \frac{1}{N} \ln \left[\frac{||\mathbf{v}_N||}{||\mathbf{v}_0||}\right] \tag{15}$$

Equation (15) leads to the Wolf algorithm for finding the first Lyapunov exponent[5,12]:

$$\gamma_1 \approx \frac{1}{N} \sum_{i=1}^{N} \ln \left|\left|\mathbf{W}_i \frac{\mathbf{v}_{i-1}}{||\mathbf{v}_{i-1}||}\right|\right| \tag{16}$$

Note that the final Lyapunov exponent is simply the average of the iterate values. The result of this algorithm converges to the exact first Lyapunov exponent as N becomes large.

To find, say, n non-negative Lyapunov exponents, we begin with an arbitrary set of independent wave vectors: $\mathbf{v}_{01}, \mathbf{v}_{02}, \ldots, \mathbf{v}_{0n}$. The first index refers to the bay (or iteration) number, while the second index is used to distinguish between the n vectors. We then use Gram-Schmidt reorthonormalization (GSR) to construct a set of orthonormal vectors, $\mathbf{v}'_{01}, \mathbf{v}'_{02}, \ldots, \mathbf{v}'_{0n}$, that span the same n-dimensional subspace. Next, we multiply each of these vectors by the same random wave transfer matrix, which is equivalent to sending each wave vector through one random bay of a disordered system. Then GSR is used to construct the new orthonormal set. This process is repeated for many iterations.

For each k-dimensional subspace, the effect of the first $k-1$ Lyapunov exponents is removed by projecting $\mathbf{v}_{\mathrm{ik}}$ onto $\mathbf{v}'_{\mathrm{ik}}$. We therefore write the following *general formulation of the Wolf algorithm*[12]:

$$\gamma_k \approx \frac{1}{N}\sum_{i=1}^{N} \ln\langle \mathbf{v}_{\mathrm{ik}}, \mathbf{v}'_{\mathrm{ik}}\rangle \qquad k = 1,2,\ldots,n \quad (17)$$

This is equivalent to Eq. (16) for $k = 1$. As for Eq. (16), the results converge to the exact Lyapunov exponents as N becomes large.

3. Example Systems

3.1 A Mono-Coupled System

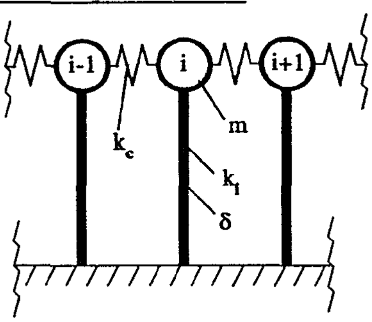

Figure 1 An example mono-coupled system: a chain of oscillators coupled by linear springs. Each oscillator is considered to have a one degree-of-freedom tip deflection.

As an example of a mono-coupled system, we consider a chain of one degree-of-freedom oscillators coupled by linear springs, as shown in Fig. 1. Each oscillator has mass m and structural damping factor δ, and each coupling spring has stiffness k_c. An arbitrary ith oscillator is considered to have a random stiffness, $k_i = k(1 + f_i)$, where k is the nominal stiffness, and we assume that f_i is a value taken from a uniformly distributed random variable of disorder with mean zero and variance S^2. For the ordered case, of course, $f_i \equiv 0$. The equation of steady state harmonic motion for an arbitrary ith oscillator is:

$$\begin{aligned} -\omega^2 m x_i + [k(1 + f_i + j\delta) + 2k_c]x_i \\ -k_c x_{i-1} - k_c x_{i+1} = 0 \end{aligned} \quad (18)$$

where x_i is the displacement of oscillator i, and $j = \sqrt{-1}$. If we define $\mathbf{u}_i \equiv [x_{i+1}, x_i]^T$ then the transfer matrix of Eq. (2) is:

$$\mathbf{T}_i = \begin{bmatrix} \frac{1+2R-\bar{\omega}^2+f_i+j\delta}{R} & -1 \\ 1 & 0 \end{bmatrix} \quad (19)$$

where $\bar{\omega} = \omega/\sqrt{k/m}$ is a dimensionless frequency and $R = k_c/k$ is a dimensionless coupling strength. We may now use the Wolf algorithm to find the Lyapunov exponents. Results will be shown in Section 4.

3.2 A Bi-Coupled System

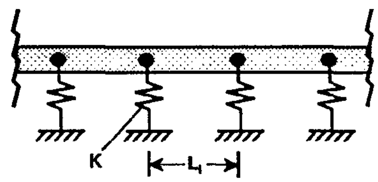

Figure 2 An example bi-coupled system: a multi-span beam pinned to elastic supports.

As an example of a multi-coupled system, we present the structure shown in Fig. 2. This is an undamped multi-span beam undergoing transverse bending motion, pinned at flexible supports which are modeled by linear springs of stiffness K. Each span (bay) has length L for the ordered case. Disorder is introduced by allowing the length of bay i to be $L_i = L(1 + f_i)$, where f_i is a value taken from a uniformly distributed random variable with mean 0 and variance S^2. Since the beam has a vertical displacement and a rotation at each support, the bays are coupled through two coupling coordinates, and this is a bi-coupled structure.

The beam has Young's modulus E, cross-sectional inertia I, and mass per unit length μ. We define the following dimensionless parameters:

$$\bar{\omega} \equiv \frac{\omega}{\sqrt{EI/\mu L^4}} \qquad P \equiv \sqrt{\bar{\omega}} \qquad \bar{K} \equiv \frac{KL^3}{2EI} \quad (20)$$

The equations of motion and the system transfer matrix are not derived here. We refer the interested reader to Ref. 13, where this system is considered in detail.

Let us now consider the Lyapunov exponents of an ordered multi-span beam with $\bar{K} = 50$, which are shown versus the frequency parameter P in Fig. 3. The Lyapunov exponents indicate four distinct frequency regions. For $P < 3.027$, the waves are in a *complexband*. For $3.027 < P < 3.056$, both wave types are in a passband (*double passband*). For $3.056 < P < \pi$, one wave type lies in a stopband, while the other belongs still to a passband (*stopband-passband*). We refer to the wave type which has $\gamma_1 > 0$ here as wave type I, and to the other which has $\gamma_2 = 0$ as wave type II. For $P > \pi$, both wave types are in stopbands, (*double stopband*).

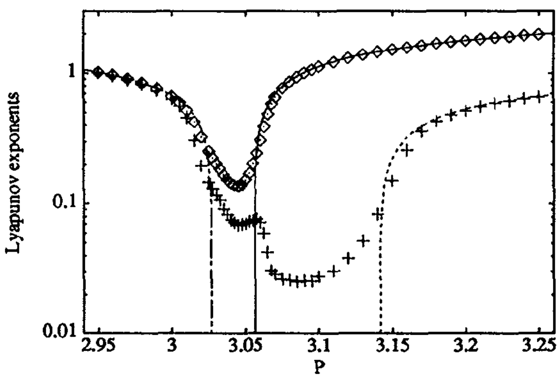

Figure 3 The Lyapunov exponents for an ordered multi-span beam , $\bar{K} = 50$: γ_1 (—), γ_2 (- - -), and both γ_1 and γ_2 (— - — -). Also shown are the Lyapunov exponents γ_1 (◇) and γ_2 (+) calculated for the disordered case, $\bar{K} = 50$, $S = 1\%$.

In the case of an ordered system, γ_1 (γ_2) is the *exact* spatial amplitude decay rate of wave type I (II). For a disordered system, where the wave transfer matrix for each bay is random, the Lyapunov exponents indicate wave decay which may be due to attenuation and/or localization.

In Fig. 3, the Lyapunov exponents calculated using the Wolf algorithm (200,000 iterations) for a disordered case ($S = 1\%$) are also shown. In Ref. 5, the authors examined simulations of right-traveling wave amplitudes, and concluded the following. In frequency regions where the Lyapunov exponents of the disordered system are close to those of the ordered system (complexband and double stopband), the decay is due primarily to off-resonance effects. In the double passband, the wave decay is due to localization, and the two waves have similar decay rates. In the stopband-passband region, the two waves have distinct decay rates, with that of type I being largely due to attenuation. In this region (results were shown at $P = 3.08$), the *wave conversion* phenomenon was observed: wave type I vanished quickly, but leaked some of it's energy to wave type II, which propagated more readily through the system.

4. Statistics of Lyapunov Exponents

While wave decay can be caused by several mechanisms — localization, damping, or off-resonance — we are most interested in those frequency regions in which the Lyapunov exponents indicate localization, since the decay is thus due to a confinement of wave energy which may lead to large vibration amplitudes. We now extend the work of Ref. 5 to include a more systematic identification of frequency regions where localization occurs. In addition, we suggest a statistical measure for predicting wave conversion.

The Lyapunov exponents of the disordered systems which we have considered are actually *average* values of the Lyapunov exponents calculated at each iteration of the algorithm. If the decay is due mostly to off-resonance or damping, then these values should be nearly the same for each iteration. If the decay is due to disorder, however, then the Lyapunov exponents for each iteration will vary. Recall that the variance of a random variable measures the spread of the random variable about its mean. Therefore, the variance (or the standard deviation) of a Lyapunov exponent provides a measure of the influence of disorder on the decay rate.

In Fig. 4, we consider the multi-span beam with 1% disorder. The averages and the standard deviations of the Lyapunov exponents found at each iteration of the Wolf algorithm are shown.

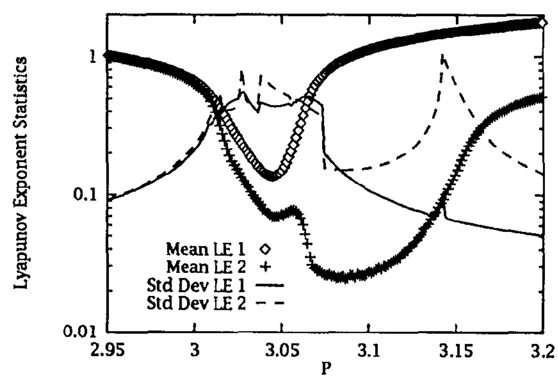

Figure 4 The averages and standard deviations of the Lyapunov exponents (found at each iteration of the Wolf algorithm) for a disordered multi-span beam, $\bar{K} = 50$, $S = 1\%$.

Each standard deviation has a local maximum around $P = \pi$, which is the passband edge for wave type II of the ordered system. These peaks seem to be due to a numerical singularity at this frequency. If we ignore this apparent anomaly, we see that the standard deviations are greatest in the frequency region around the double passband. This indicates that localization plays a prominent role in the decay here, as we expected. Note that as P increases, the standard deviations suddenly drop. Thus, we might consider the frequency where this drop occurs to be an upper bound for the frequency region of greatest interest for localization.

We now return to the mono-coupled example in order to consider the use of the standard deviation to separate damping and disorder effects. The average of the first Lyapunov exponent (the localization factor) as well as its standard deviation are shown in Fig. 5 for the mistuned-undamped system with strong coupling.

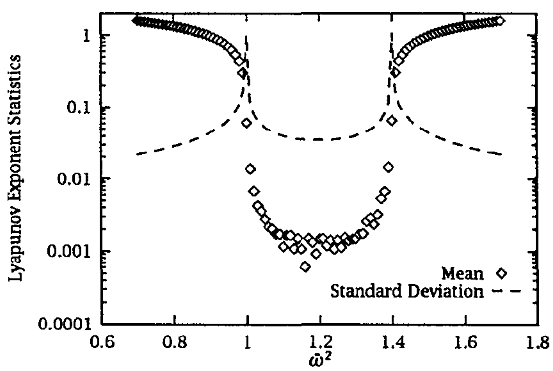

Figure 5 The statistics of the first Lyapunov exponent for a mistuned-*undamped* mono-coupled system, $R = 0.1$, $\frac{S}{R} = 0.1$.

Again, we see peaks in the standard deviation at the passband edges due to numerical singularities. Otherwise, the standard deviation is greatest inside the passband region, and decreases outside (note the log scale). This decrease does not seem especially dramatic, unless we compare the standard deviation to the mean value. Inside the passband region, the standard deviation is more than an order of magnitude larger than the mean, while in the stopband regions the mean is more than order of magnitude larger than the standard deviation. We thus see that localization effects are only significant in the passband region.

In Fig. 6, we consider damping, $\frac{\delta}{R} = 0.1$, in the same mono-coupled system. Now we see that the standard deviation *decreases* inside the passband. This is due to the influence of damping decreasing the variation of the Lyapunov exponent. Outside the passband, where off-resonance effects are most significant, the standard deviation is similar to that of the undamped case. For all frequencies (ignoring the peaks at the passband edges), the mean is an order of magnitude greater than the standard deviation. This supports the conclusion of Ref. 14 that damping effects (versus disorder effects) dominate the decay rate for the strong coupling case.

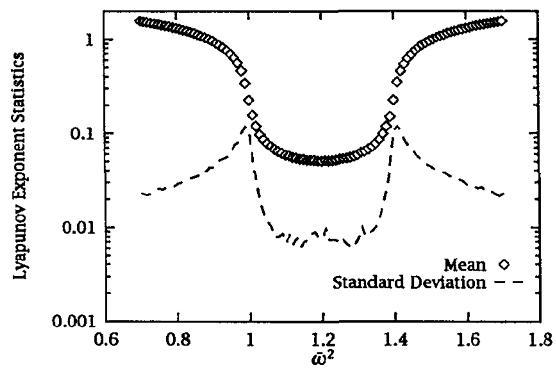

Figure 6 The statistics of the first Lyapunov exponent for a mistuned-*damped* mono-coupled system, $R = 0.1$, $\frac{S}{R} = 0.1$, $\frac{\delta}{R} = 0.1$.

We now turn our attention to the multi-span beam and the wave conversion phenomenon. We have seen that in the region of greatest localization, there was a mixing of the wave types[5]. Outside this region, we observed wave conversion, where one wave died quickly, but leaked energy to the other wave. We propose here that the covariance of the Lyapunov exponents may provide a statistical measure which is capable of predicting wave conversion.

The covariance provides a measure of the statistical dependence of two random variables. If the covariance is zero, then these random variables are independent. Therefore, it seems that the covariance of two Lyapunov exponents would be greatest in regions of wave conversion, since here the decay rates are directly related, and that relationship is governed by the disorder effects.

In Fig. 7, we show the covariance of the first and second Lyapunov exponents for the disordered multi-span beam. The frequencies where we might predict wave conversion based on the covariance include $P = 3.08$, where we observed the wave conversion phenomenon[5]. Note that we might also expect wave conversion in the complexband region.

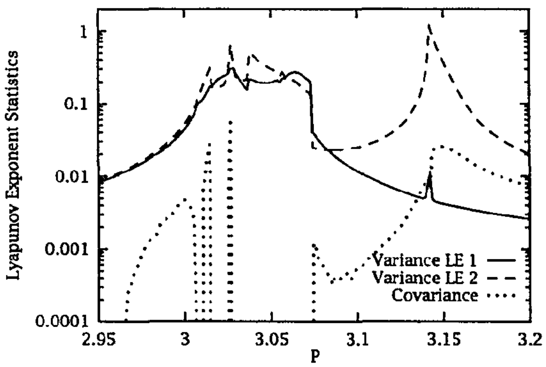

Figure 7 The variances and covariance of the first and second Lyapunov exponents for the disordered beam, $\bar{K} = 50$, $S = 1\%$.

In general, we see that the non-trivial values of the covariance occur outside the frequency region where the variances are greatest. This would suggest that in different frequency regions, disorder has very different effects on the waves. In the frequency band where the variances are highest, disorder causes a mixing of waves, but confines the energy of each. Outside this band, disorder allows a leakage of energy from the wave which decays most quickly due to off-resonance — the wave conversion phenomenon. In this case, by leaking energy to the wave which decays least, disorder allows a *better* transmission of energy.

5. Conclusions

Lyapunov exponents can be used to analyze wave decay, localization, and even wave conversion in nearly periodic structures for which a transfer matrix is known. In multi-coupled systems, Lyapunov exponents provide a measure of the decay of the multiple wave types.

The frequency regions of greatest interest are those in which the Lyapunov exponents indicate decay due to localization, because of the attendant increase in response amplitudes. To this end, it was found that the various statistics of the Lyapunov exponents calculated for each bay provide useful information when considered together. In particular, the standard deviations of the Lyapunov exponents determine frequency bands where localization occurs. The averages can then be used to compare the expected strength of localization in the various frequency ranges of interest. From a practical engineering standpoint, this systematic identification of critical frequency regions may be the most useful aspect of a Lyapunov exponent analysis.

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