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Optimal Trajectories for Maximum Endurance Gliding in a Horizontal Plane

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Abstract

This paper presents the optimal solutions to the maximum endurance subsonic gliding trajectories in a horizontal plane by the application of Pontryagin's maximum principle. The gliding vehicle is assumed to be carried by an aeroplane which has a velocity vector heading along the line connecting aeroplane and target. Then at a certain point when the aeroplane is approaching, directly above, or leaving the target, the vehicle is released. The vehicle then glides to the target in the horizontal plane with a maximum flight time. The formulation is reduced to a minimum number of aerodynamic characteristics by the use of a set of dimensionless variables so that the results are applicable to a whole class of vehicles. Bank angle is the control variable. There are two constraints on the turning maneuverability, one is the load factor and the other is the maximum lift coefficient. These two constraints are discussed in detail. Several optimal trajectories with different points of release are computed. For a specified altitude, it is found that there are a maximum radius of penetration and an absolute maximum endurance. Also, the influence of the altitude on the gliding flight in a horizontal plane is discussed.

Nomenclature

C_0, C_1, C_2, C_3	constants of integration
C_D	drag coefficient
C_{D0}	zero lift drag coefficient
C_L	lift coefficient
C_L^*	lift coefficient for maximum lift-to-drag ratio
D	drag force
E^*	maximum lift-to-drag ratio
h	altitude
H	Hamiltonian function
$\bar{H}$	part of the Hamiltonian containing μ
k_1, k_2, k_3	constants, $=C_1/C_0, C_2/C_0, C_3/C_0$, respectively
K	induced drag factor
L	lift force
m	mass of the vehicle
M_1, M_2	$=\tan\mu$, $1/\cos^2\mu$, respectively
$p_x, p_y, p_u, p_\psi, p_\theta$	adjoint variables associated to state variables
P_1, P_2	$=p_\psi/u$, $-(\omega p_u)/2E^*u^2$, respectively
r	radius of penetration
S	reference area
t	time
u	dimensionless speed
V	speed of vehicle
W	weight of vehicle
x, y	dimensionless coordinates
X, Y	position coordinates of vehicle
Δ	$=\tan\mu$
ε	small value
θ	dimensionless time
λ	normalized lift coefficient
μ	bank angle
ρ	density of atmosphere
τ	normalized time
ψ	velocity yaw angle
ω	dimensionless wing loading

Subscripts

0	initial conditions
f	final conditions

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Introduction

The problem of determining minimum and maximum endurance trajectories for subsonic gliding flight in a horizontal plane has been considered previously.[1,2] In Ref.1, both minimum and maximum endurance gliding flights were formulated and solved numerically. In the formulation, the control variable is parametrized as a polynomial of the normalized time. Hence the problem becomes a parameter optimization problem. In Ref. 2, the case of maximum endurance trajectories return to the original starting point is further considered by using a piecewise continuous polynomial control approximation. In both references, a specified small lifting vehicle which can be used as a low-level weapons delivery system is used for the numerical integration.

This paper presents the optimal solutions to the maximum endurance trajectories by the application of Pontryagin's maximum principle.[3,4] Furthermore, by the use of a set of dimensionless variables, the solutions obtained are applicable to a whole class of glide vehicles.

Variational Formulation

The geometry of gliding flight in a horizontal plane is depicted in Fig. 1 and the motion is governed by the state equations:[5]

$$\dot{X} = V \cos\psi \tag{1a}$$

$$\dot{Y} = V \sin\psi \tag{1b}$$

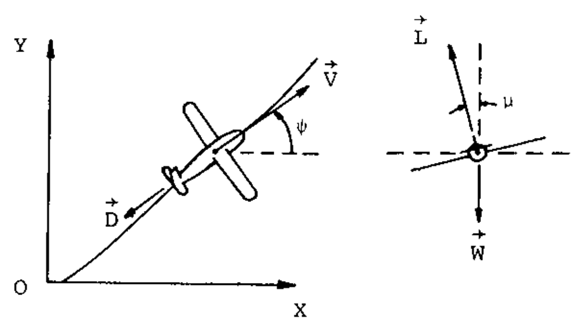

Fig.1 Geometry of Gliding in a Horizontal Plane.

$$\dot{V} = -D/m \tag{1c}$$

$$\dot{\psi} = L \sin\mu/mV \tag{1d}$$

$$L \cos\mu = W \tag{1e}$$

We shall consider a parabolic drag polar of the form

$$C_D = C_{DO} + KC_L^2 \tag{2}$$

where at subsonic speeds, C_{DO} and K are considered as constant. By the use of the dimensionless system

$$x=gX/V_0^2,\ y=gY/V_0^2,\ u=V/V_0$$
$$\theta=gt/V_0,\ \omega=2W/\rho S V_0^2 C_L^* \tag{3}$$

where V_0 is the initial speed and C_L^* is the lift coefficient for maximum lift-to-drag ratio, we obtain the set of dimensionless equations of motion:[4]

$$x' = u \cos \psi \tag{4a}$$

$$y' = u \sin \psi \tag{4b}$$

$$u' = -(u^2/2E^*\omega)\,[1+(\omega^2/u^4\cos^2\mu)] \tag{4c}$$

$$\psi' = \tan\mu/u \tag{4d}$$

$$\theta' = 1 \tag{4e}$$

where the prime denotes derivative taken with respect to the dimensionless time, θ. The last equation is introduced to treat the time as a state variable. In this system, the only performance parameter involved is the maximum lift-to-drag ratio, E^*. The dimensionless wing loading, ω, which is a constant in level flight, is a physical characteristic for a whole class of glide vehicles. It is also used to analyze the influence of the altitude on gliding performance. The constraining equation (1e) becomes in this formulation

$$\cos\mu = \omega/\lambda u^2 \tag{5}$$

where λ is the normalized lift coefficient

$$\lambda = C_L/C_L^* \tag{6}$$

Because of this constraint, the bank angle, μ, is the sole control in the dynamical system. In turning flight, there are two physical constraints on the maneuverability. The first one is the load factor

$$n = L/W = 1/\cos\mu \tag{7}$$

This value is bounded by an upper limit $n = n_{max}$ which is a physiological/structural limit. The other constraint is the maximum lift coefficient C_{Lmax}, or in normalized form, λ_{max}. Hence, the bank control belongs to the set

$$0 \le |\mu| \le \inf.\,[\cos^{-1}(1/n_{max}),\ \cos^{-1}(\omega/u^2\lambda_{max})] \tag{8}$$

Using the maximum principle, we introduce the adjoint vector $\vec{p}$ to form the Hamiltonian

$$H = p_x u \cos\psi + p_y u \sin\psi - p_u(u^2/2E^*\omega)\,[1+(\omega^2/u^4\cos^2\mu)] + p_\psi(\tan\mu/u) + p_\theta \tag{9}$$

It is known that the variational problem has the integrals:[4]

$$p_\theta = C_0 \tag{10a}$$

$$p_x = C_1 \tag{10b}$$

$$p_y = C_2 \tag{10c}$$

$$p_\psi = C_1 y - C_2 x + C_3 \tag{10d}$$

Furthermore, since the time is an ignorable coordinate, and in the maximum endurance problem the final time is not specified, we have identically for the whole duration of the flight

$$H \equiv 0 \tag{11}$$

Regarding the optimal bank angle, it suffices to consider the part of the Hamiltonian containing μ:

$$\overline{H} = (P_1 \tan\mu) + (P_2/\cos^2\mu) \tag{12}$$

where

$$P_1 = p_\psi/u \ , \quad P_2 = -(\omega p_u)/2E^* u^2 \tag{13}$$

This reduced Hamiltonian can be considered as the dot product of the two vectors (P_1, P_2) and (M_1,M_2) such that

$$M_1 = \tan\mu, \quad M_2 = 1/\cos^2\mu \tag{14}$$

When μ varies, the vector $\vec{M}$ describes a parabola

$$M_2 = 1 + M_1{}^2 \tag{15}$$

called the domain of maneuverability as shown in Fig. 2.

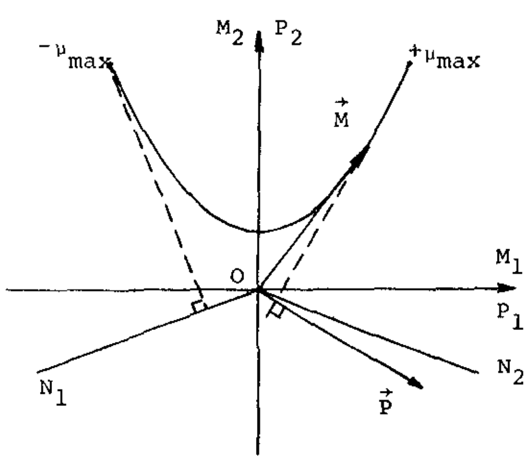

Fig. 2 Domain of Maneuverability and Optimal Bank Control.

To maximize $\overline{H}$, if the vector $\vec{P}$, with components P_1 and P_2, is inside the angle N_1ON_2, the optimal bank angle used is an interior bank angle such that at the terminus of the vector $\vec{M}$, the tangent to the parabola is orthogonal to the vector $\vec{P}$. This is expressed by

$$\frac{dM_2}{dM_1} = 2M_1 = -\frac{P_1}{P_2}$$

or

$$\tan\mu = \left(\frac{p_\psi}{p_u}\right)\left(\frac{E^* u}{\omega}\right) \tag{16}$$

When the vector $\vec{P}$ is outside the angle N_1ON_2, the bank angle is the maximum bank angle as given by the constraint (8) and it has the sign of p_ψ. In the case where $p_u < 0$ and $p_\psi \equiv 0$ for a finite time interval, there may exist a chattering control in which the bank angle rapidly switches between $-\mu_{max}$ and μ_{max}. This only occurs in some minimum time problem and in general in this maximum endurance problem, the bank angle is usually of the interior type. In the case of interior optimal bank angle, using the integrals (10) and (11) in the optimal relation (16) we have the equation

$$(k_1 y - k_2 x + k_3)\Delta^2 + 2u[1 + u(k_1 \cos\psi + k_2 \sin\psi)]\Delta$$
$$- \frac{(k_1 y - k_2 x + k_3)}{\omega^2}(\omega^2 + u^4) = 0 \tag{17}$$

where

$$\Delta = \tan\mu \tag{18}$$

and

$$k_i = C_i/C_0, \quad i = 1,2,3 \tag{19}$$

are three constants of integration. It is clear that the explicit optimal bank control can be obtained from Eq.(17), a quadratic equation in Δ. During the numerical integration for the optimal trajectory, we use the optimal bank angle obtained from Eq.(17) except that when the bank angle reaches its boundary the corresponding maximum bank angle must be used. The three constants k_1, k_2 and k_3 are to be selected such that the final and transversality conditions are identically satisfied.

Maximum Endurance Problems

As application, it is supposed that a small lifting vehicle is carried by an airplane heading directly toward a target located at the origin O. By symmetry, we can consider the initial velocity as along the X-axis. At a certain distance, the gliding vehicle is released and while the carrier is performing escaping maneuver, the small vehicle tries to reach the target at final residual speed u_f in a maximum time. Thus, we have the end conditions:

$$\begin{aligned} &\theta_0 = 0\ ,\ x_0 = \text{specified},\ y_0 = \psi_0 = 0,\ u_0 = 1 \\ &\theta_f = \max,\ x_f = y_f = 0,\ \psi_f = \text{free},\ u_f = \text{specified} \end{aligned} \tag{20}$$

For the numerical integration, we take the values

$$E^* = 20, \quad \lambda_{max} = 1.8, \quad \omega = 0.23 \qquad (21)$$

The first two values concern the aerodynamic characteristics of the gliding vehicle. They represent typical values for a whole class of vehicles which includes the example vehicle in Refs.1-2. The value of ω corresponds to the same vehicle at sea level, but it can represent a vehicle with lower wing loading flying at higher altitude. To enforce the constraint on the load factor, we can specify a value say, $n_{max}=5$. But in general, for maximum endurance problems this value is never reached since the bank angle remains in the interior of the domain of maneuverability.

For the small glider to reach the target, it must be released within a zone of penetration. This zone has a limiting radius r which can be obtained by integrating the state equations with $\mu=\psi=0$. The solution is

$$r=|x_{max}|= \frac{E^*}{2}\, \omega \log\left[\frac{\lambda^2_{max}}{1+\lambda^2_{max}}\left(1+\frac{1}{\omega^2}\right)\right] \qquad (22)$$

with an absolute maximum endurance

$$\theta_{max}=\sqrt{\frac{\omega}{2}}\, E^*\left[\tan^{-1}\left(\frac{\sqrt{2\lambda_{max}}}{1-\lambda_{max}}\right)-\tan^{-1}\left(\frac{\sqrt{2\omega}}{1-\omega}\right)\right.$$

$$+\frac{1}{2}\log\left(\frac{\lambda_{max}+\sqrt{2\lambda_{max}}+1}{\lambda_{max}-\sqrt{2\lambda_{max}}+1}\right) -$$

$$\left.\frac{1}{2}\log\left(\frac{\omega+\sqrt{2\omega}+1}{\omega-\sqrt{2\omega}+1}\right)\right] \qquad (23)$$

In deriving these explicit limiting solutions, the final speed is the stall speed which corresponds to maximum lift coefficient at zero bank angle. Hence from Eq.(5),

$$u_f = \sqrt{\omega/\lambda_{max}} \qquad (24)$$

The optimal relation (17) is for the general case where the final heading is also specified. With free ψ_f, $p_\psi(\theta_f)=0$ and from Eq.(10d), $k_3=0$. Hence, the optimal problem is a two-parameter problem, with k_1 and k_2 to be selected such that at $u=u_f$, the final conditions $x_f=y_f=0$ are identically satisfied. The five optimal trajectories with points of release before and directly above the target are shown in Fig. 3.

The first trajectory is the limiting case of rectilinear flight with $\mu=\psi=0$ described above. For the next three optimal trajectories the release points are closer and closer to the target, the vehicle first banks to the right and then banks to the left to reach the target. The initial bank angle is nonzero but is increasing and then decreasing as the point of release getting closer to the target. The fifth trajectory has the release point directly above the target and the bank angle is zero initially and then to the left and finally returns to zero again at the final instant. The maximum endurances for different points of release are also given in Fig. 3. The plot of the variations of the bank angle versus the normalized time $\tau=t/t_f$ is shown in Fig. 4. Then Fig. 5 gives the variations of the normalized lift coefficient λ.

Figure 3 also presents two optimal trajectories at different points of release while the carrier is leaving the target. The turn is in the same direction as the fifth trajectory with the initial bank angle larger as the point of release getting further from the target. This is also shown in Fig. 4 for the two trajectories. The last trajectory is the limiting trajectory for the released vehicle to have sufficient speed to reach the target. This trajectory can be obtained directly by translating the origin of coordinates to the point of release and solving the problem with the end conditions

$$\begin{aligned} &\theta_0=0, \quad x_0=y_0=0, \quad \psi_0=0, \quad u_0=1 \\ &\theta_f=\max, \quad x_f=\text{free}, \quad y_f=0, \psi_f=\text{free}, \quad u=u_f \end{aligned} \qquad (25)$$

In this case, the optimal bank angle is still given by Eq.(17) but now with $k_1=0$, $k_3=k_2\, x_f$. Hence, the optimal bank angle is given by

$$k_2(x_f-x)\Delta^2+2u[1+k_2 u \sin\psi]\Delta - \frac{k_2(x_f-x)}{\omega^2}(\omega^2+u^4) = 0 \qquad (26)$$

The parameters selected for the iteration are k_2 and x_f.

Influence of the Altitude

The present formulation has reduced the number of vehicle characteristics to a minimum. By specifying $E^*=20$, $\lambda_{max}=1.8$, we have obtained solutions for all class of vehicles with the same aerodynamic characteristics. The initial speed is rather a characteristic of the carrier. By varying V_0 we vary proportionally the endurance. Also a value $\omega=0.23$ has been selected for the numerical integration. For any specified value of V_0, by increasing the numerical value of ω, we either increase the wing loading W/S, or for a prescribed wing loading, this corresponds to an increase in the altitude. We now consider the influence of this parameter on performance.

First, by Eq.(22), the radius of penetration is a function of ω as shown in

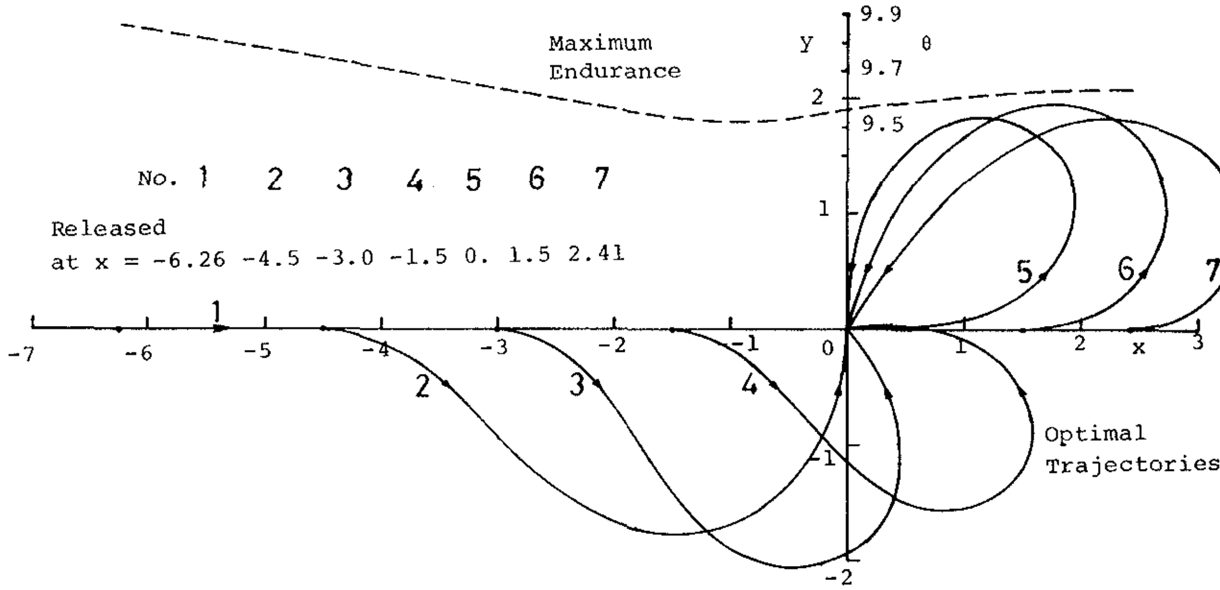

Fig. 3 Maximum Endurances and Optimal Trajectories for Different Points of Release.

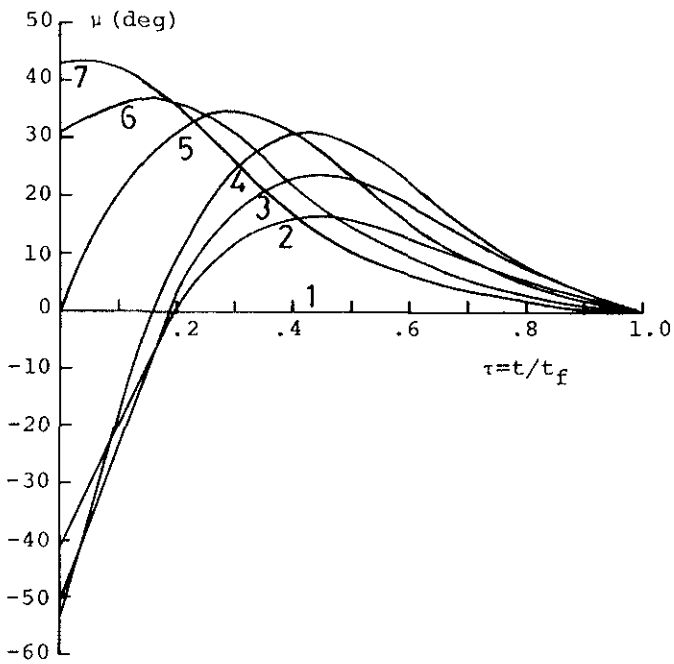

Fig. 4 Variations of the Bank Angle for the Various Optimal Trajectories.

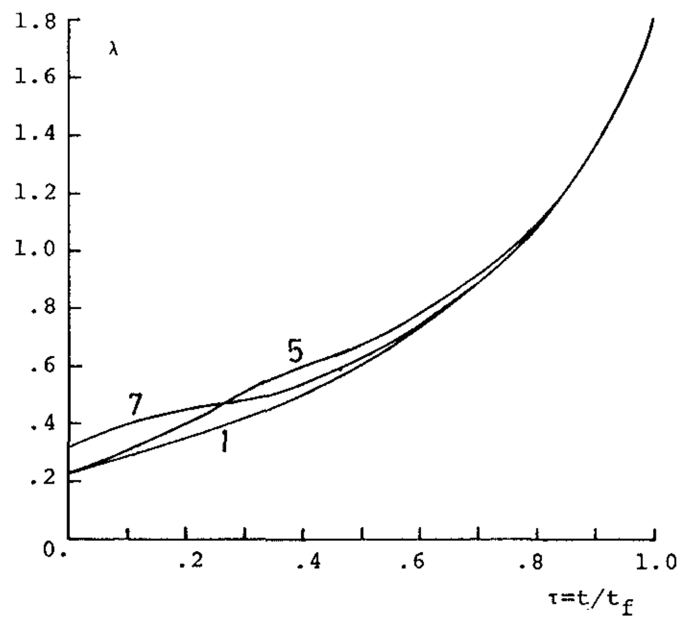

Fig. 5 Variations of the Normalized Lift Coefficient for the Various Optimal Trajectories.

Fig.6. It is maximized when

$$\frac{2}{1+\omega^2} = \log\left[\frac{\lambda^2_{max}}{1+\lambda^2_{max}}\left(1+\frac{1}{\omega^2}\right)\right] \qquad (27)$$

It is shown in Ref.6 that this equation has a unique solution for ω, that is for the altitude. With $\lambda_{max}=1.8$, the solution is $\omega=0.396151$, and hence if $\omega_o=0.23$ denotes the sea level, this corresponds to the optimum density ratio, $\omega_o/\omega=\rho/\rho_o=0.580587$. Based on the standard atmosphere, the optimum altitude for largest radius of penetration is h=5321 m. On the other hand, using the optimum value of ω, and keeping the altitude unchanged, say at sea level, this is translated into the best value of wing loading, W/S, for deepest penetration.

Likewise, for a prescribed initial speed V_o, once the aerodynamic characteristics E^* and λ_{max} are given, the absolute maximum endurance, as given by Eq.(23), is a function of ω. This time is maximized when

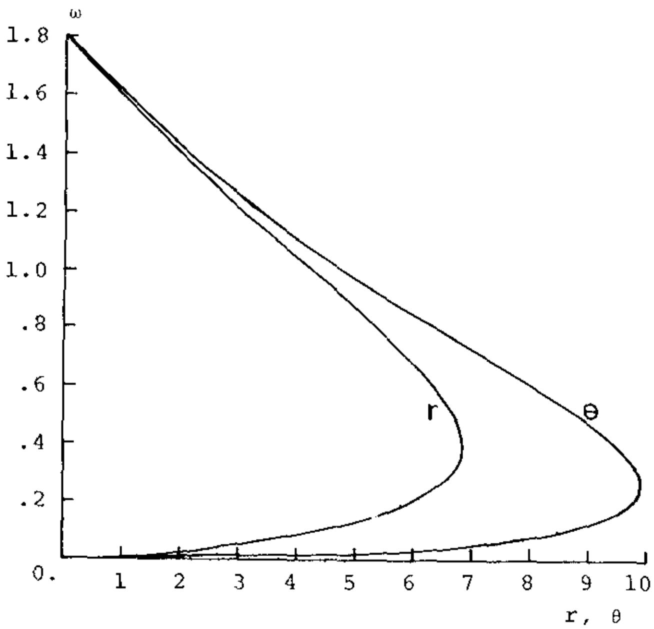

Fig. 6 Endurance and Radius of Penetration as Functions of Altitude.

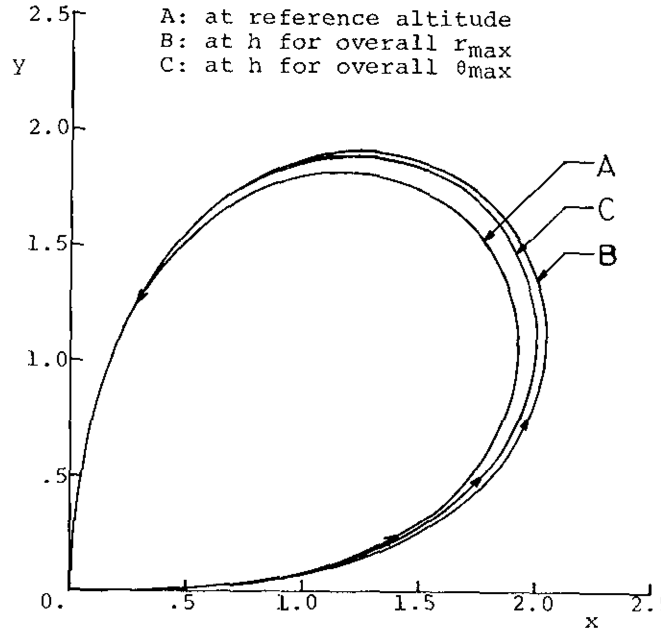

Fig. 7 Three Optimal Return Trajectories at Different Levels of Altitude.

$$\tan^{-1}\left(\frac{\sqrt{2\lambda_{max}}}{1-\lambda_{max}}\right)-\tan^{-1}\left(\frac{\sqrt{2\omega}}{1-\omega}\right) + \frac{1}{2}\log\left(\frac{\lambda_{max}+\sqrt{2\lambda_{max}}+1}{\lambda_{max}-\sqrt{2\lambda_{max}}+1}\right) - \frac{1}{2}\log\left(\frac{\omega+\sqrt{2\omega}+1}{\omega-\sqrt{2\omega}+1}\right) - \frac{2\sqrt{2\omega}}{1+\omega^2} = 0 \tag{28}$$

We consider this equation, written as $f(\omega)=0$. At the altitude level $\omega=\lambda_{max}$, we have the condition at the ceiling $u_f=u_o=1$. There is no possibility for gliding at constant altitude and the endurance is zero. Furthermore, $f(\lambda_{max})<0$. As the altitude decreases, the time first increases and then below a certain altitude it decreases since $f(\varepsilon)>0$ for very small value of ω. Hence, it is clear that there exists a value of ω satisfying Eq.(28) for overall maximum endurance. With $\lambda_{max}=1.8$, the solution is $\omega=.264489$, corresponds to a density ratio $\rho/\rho_o=.869601$ and an altitude h=1432 m.

Figure 7 plots the optimal trajectories for a release over the target and a return to the origin at three different altitude levels. Trajectory A corresponds to the case of the reference altitude which is the sea level for the vehicle considered in Refs.1-2. Trajectory B corresponds to the case where the radius of penetration is largest and trajectory C is generated at the altitude where the flight time is maximized.

Conclusions

The optimal trajectories for maximum endurance gliding in a horizontal plane are obtained by the application of Pontryagin's maximum principle. The formulation reduces to a minimum the number of aerodynamic characteristics involved and hence the results apply to a whole class of vehicles. The influence of altitude and wing loading on performance is discussed. It is shown that there exist an altitude(wing loading) for maximum radius of penetration and another one for overall maximum time of flight.

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