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Optimal Tradeoff Between H_2 Performance and Tracking Accuracy in Servocompensator Synthesis

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The problem of optimal H_2 disturbance rejection while tracking uncertain constant or sinusoidal reference commands is considered. The internal model principle is used to ensure that the tracking error approaches zero asymptotically. Necessary conditions are given for controllers that minimize an H_2 disturbance rejection cost plus a worst-case integral square tracking error for transient tracking performance. The necessary conditions provide expressions for the gradients of the cost with respect to each of the control gains. These expressions are then used in a quasi-Newton gradient search algorithm to determine the optimal feedback gains. Numerical examples demonstrate the tradeoff between the two competing objectives in the cost function.

I. Introduction

THE servomechanism problem, in which certain system outputs are required to follow specified reference commands such as steps, ramps, sinusoids, or polynomial functions of time, has received considerable attention by researchers. For a system to achieve asymptotic tracking, the controller must contain an internal model of the exogenous dynamics that produce the reference command.[1] Furthermore, asymptotic tracking of commands in several feedback loops requires that the exogenous dynamics be replicated in each loop. A compensator is used to stabilize the augmented system consisting of the plant and the internal model. The combination of internal model and stabilizing controller is referred to as a servocompensator. A classical example of a servocompensator is the case of constant reference commands, in which an integrator provides a model of the exogenous dynamics. In this case, a type-1 controller constructed by including an integrator tracks constant reference commands with zero steady-state error.

Because a controller that achieves asymptotic tracking consists of both an internal model and a stabilizing controller, there is considerable freedom in the design of such controllers. This design freedom can be used to meet additional objectives such as pole placement, time and frequency response criteria, or optimization of a performance criterion.[2] One control objective of particular interest is disturbance rejection via minimization of an H_2 norm. Unfortunately, the problem of minimizing the H_2 norm of a closed-loop system while achieving asymptotic tracking of reference commands is not straightforward. Since the internal models for reference commands such as steps, ramps, and sinusoids have imaginary axis eigenvalues, these modes are not observable by the performance variables used in the H_2 cost function when the internal model is augmented with the plant.

The problem of suboptimal H_2 control with asymptotic tracking of constant and sinusoidal reference commands was addressed by augmenting the plant with the appropriate internal model and finding the gains that stabilize the augmented system.[3] By using control gains parameterized by a scalar parameter, the closed-loop H_2 norm can be made arbitrarily close to the optimal H_2 cost by reducing the scalar parameter.

Although the use of an internal model addresses the steady-state tracking problem, transient tracking performance is also of interest. Integral square tracking error measures the transient tracking error and thus the effectiveness of the controller in following reference commands. In addition, including integral square error in the cost function allows the tradeoff between H_2 disturbance rejection and transient tracking performance by varying the relative weights. Previously, the integral square error was considered and feedforward gains were used to minimize it.[4,5] However, it was assumed that the feedback gains are already given, having been found to meet some other criterion. In addition, it was assumed that the reference command is completely known a priori.

While previous research chose controllers primarily to achieve zero steady-state tracking error and to stabilize the augmented plant,[3] the goal of the present paper is to determine controllers that achieve better transient tracking performance for the same H_2 cost for constant reference commands whose magnitudes are unknown and for sinusoidal reference commands of known frequency and unknown amplitudes and phases. To do this, necessary conditions are given for the problem of minimizing a cost function consisting of an H_2 cost plus a worst-case integral square tracking error. These necessary conditions provide analytical expressions for the gradients of the cost with respect to each of the control gains. These expressions are then used by a gradient optimization algorithm to find the control gains that minimize the cost function. Finally, the tradeoff between H_2 performance and integral square tracking accuracy is demonstrated numerically.

II. Problem Formulation

Consider the plant model

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t) \tag{1}$$

$$y(t) = Cx(t) + D_2 w(t) \tag{2}$$

$$z(t) = E_1 x(t) + E_2 u(t) \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the plant state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^q$ is a stochastic disturbance, $z(t) \in \mathbb{R}^p$ is the performance variable, $y(t) \in \mathbb{R}^l$ is the measured output, (A, B) is controllable, and (C, A) is observable. Partitioning $y(t)$ as

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

where $y_1(t) \in \mathbb{R}^{l_1}$, the control objective is to have $y_1(t)$ follow a reference command $r(t)$ such that the expected value of the tracking error

$$e(t) \triangleq y_1(t) - r(t) \tag{4}$$

approaches zero asymptotically. In addition, we wish to minimize the H_2 norm of the closed-loop transfer function between $w(t)$ and $z(t)$ as well as the integral square tracking error $\int_0^\infty \mathbb{E}[e(t)]^T M\mathbb{E}[e(t)]\,dt$, where M is a nonnegative-definite matrix.

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In this paper, two types of reference signals $r(t)$ will be considered, namely, constant reference commands and sinusoidal reference commands. For the case of constant reference commands, we assume that each element r_i of the vector $r(t)$ is uncertain, that is,

$$r(t) = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{l_1} \end{bmatrix} \tag{5}$$

where the elements r_i are uncertain. For the case of sinusoidal reference commands, we assume that each element $r_i(t)$ of the vector $r(t)$ consists of a sinusoid whose frequency ω is known but whose amplitude and phase are uncertain, that is,

$$r(t) = \begin{bmatrix} r_1 \sin(\omega t + \phi_1) \\ r_2 \sin(\omega t + \phi_2) \\ \vdots \\ r_{l_1} \sin(\omega t + \phi_{l_1}) \end{bmatrix}$$

where the amplitudes r_i and the phases ϕ_i are uncertain. The more general case in which the components of $r(t)$ have different frequencies ω_i can also be considered. However, this generalization complicates the development and is deferred to a later paper.

We represent the reference command $r(t)$ by means of an exogenous system of the form

$$\dot{x}_r(t) = A_r x_r(t), \qquad x_r(0) = x_{r0} \tag{6}$$

$$r(t) = C_r x_r(t) \tag{7}$$

where $x_r(t) \in \mathbb{R}^{n_r}$. For the case of constant commands, let $n_r = 1$, $A_r = 0$, and $x_{r0} = 1$, so that $r(t) = C_r$, and thus the elements of C_r determine the magnitudes of the reference command components. Similarly, for sinusoidal reference commands, let $n_r = 2$,

$$A_r = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \qquad x_{r0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and let $C_r \in \mathbb{R}^{l_1 \times 2}$ be an uncertain matrix. Then, $r_i(t) = C_{r1i} \sin\omega t + C_{r2i} \cos\omega t$, where C_{r1i} and C_{r2i} are the ith elements of the first and second columns of C_r. Equivalently, $r_i(t)$ can be rewritten as $r_i(t) = r_i \sin(\omega t + \phi_i)$, where $r_i = \sqrt{(C_{r1i}^2 + C_{r2i}^2)}$ and $\phi_i = \tan^{-1}(C_{r2i}/C_{r1i})$. Conversely,

$$C_{r1i} = \frac{r_i}{\sqrt{\tan^2\phi_i + 1}}, \qquad C_{r2i} = \frac{r_i \tan\phi_i}{\sqrt{\tan^2\phi_i + 1}}$$

Hence, each component $r_i(t)$ of the reference command has uncertain amplitude and phase.

To guarantee that the expected value of the tracking error $\mathbb{E}[e(t)]$ approaches zero asymptotically, the feedback loop must contain an internal model, which is a replicated version of the exogenous dynamics (6).[1] The internal model is given in state-space form by[3]

$$\dot{x}_{sc}(t) = A_{sc} x_{sc}(t) + B_{sc} e(t) \tag{8}$$

where $x_{sc}(t) \in \mathbb{R}^{n_{sc}}$ is the servocompensator state and where A_{sc} is comprised of l_1 replications of the matrix A_r. Hence, $n_{sc} = l_1 n_r$. For a constant reference command $r(t) = C_r$, the matrices A_{sc} and B_{sc} are given by

$$A_{sc} = 0_{l_1 \times l_1}, \qquad B_{sc} = I_{l_1} \tag{9}$$

where $0_{i\times j}$ is the $i \times j$ zero matrix and I_i is the $i \times i$ identity matrix. Analogously, for a sinusoidal reference input $r(t) = C_{r1} \sin\omega t + C_{r2} \cos\omega t$, the matrices A_{sc} and B_{sc} are given by

$$A_{sc} = \begin{bmatrix} 0_{l_1\times l_1} & \omega I_{l_1} \\ -\omega I_{l_1} & 0_{l_1\times l_1} \end{bmatrix}, \qquad B_{sc} = \begin{bmatrix} 0_{l_1\times l_1} \\ I_{l_1} \end{bmatrix} \tag{10}$$

Letting $y_1(t) = C_1 x(t) + D_{21} w(t)$ where C_1 has full row rank, we can now form the augmented system

$$\dot{x}_a(t) = A_a x_a(t) + B_a u(t) + D_a w(t) + D_{ra} r(t) \tag{11}$$

where

$$x_a(t) \triangleq \begin{bmatrix} x(t) \\ x_{sc}(t) \end{bmatrix}, \qquad A_a \triangleq \begin{bmatrix} A & 0_{n\times n_{sc}} \\ B_{sc}C_1 & A_{sc} \end{bmatrix}$$

$$B_a \triangleq \begin{bmatrix} B \\ 0_{n_{sc}\times m} \end{bmatrix}, \qquad D_a \triangleq \begin{bmatrix} D_1 \\ B_{sc}D_2 \end{bmatrix}, \qquad D_{ra} \triangleq \begin{bmatrix} 0_{n\times l_1} \\ -B_{sc} \end{bmatrix}$$

The following lemma gives sufficient conditions for the pair (A_a, B_a) of the augmented system (11) to be controllable.[6]

Lemma 2.1. If

$$\operatorname{rank}\begin{bmatrix} \jmath\omega I - A & B \\ -C_1 & 0 \end{bmatrix} = n + l_1 \tag{12}$$

then the pair (A_a, B_a) is controllable.

Proof. Define

$$\Delta(\lambda) \triangleq [\lambda I - A_a \quad B_a] = \begin{bmatrix} \lambda I - A & 0_{n\times n_{sc}} & B \\ -B_{sc}C_1 & \lambda I - A_{sc} & 0 \end{bmatrix}$$

Since the pair (A, B) is controllable, $\operatorname{rank}[\lambda I - A \quad B] = n$ for all $\lambda \in \mathbb{C}$. First, suppose $\lambda \neq \jmath\omega$, in which case $\operatorname{rank}(\lambda I - A_{sc}) = n_{sc}$. Because of the $n \times n_{sc}$ zero block in $\Delta(\lambda)$, it can be seen that rank $\Delta(\lambda) = n + n_{sc}$ for all $\lambda \neq \jmath\omega$.

Next write

$$\Delta(\lambda) = \begin{bmatrix} I_n & 0 & 0 \\ 0 & B_{sc} & \lambda I - A_{sc} \end{bmatrix} \begin{bmatrix} \lambda I - A & 0 & B \\ -C_1 & 0 & 0 \\ 0 & I_{n_{sc}} & 0 \end{bmatrix}$$

For all $\lambda \in \mathbb{C}$, the rank of the first factor is $n + n_{sc}$ for A_{sc} and B_{sc} given by Eq. (10). Now, if $\lambda = \jmath\omega$, then it follows from (12) that the rank of the second factor is $n + l_1 + n_{sc}$. Now, by Sylvester's inequality,

$$(n + n_{sc}) + (n + l_1 + n_{sc}) - (n + l_1 + n_{sc}) \le \operatorname{rank}\Delta(\lambda)$$

$$\le \min(n + n_{sc}, n + l_1 + n_{sc})$$

which implies rank $\Delta(\lambda) = n + n_{sc}$ for $\lambda = \jmath\omega$. Hence, $\operatorname{rank}[\lambda I - A_a \quad B_a] = n + n_{sc}$ for all $\lambda \in \mathbb{C}$, which implies that (A_a, B_a) is controllable. □

Remark 2.1. The rank condition in Eq. (12) ensures that there are no pole-zero cancellations in the cascaded realization of the plant model and the internal model. This rank condition is a requirement for the asymptotic tracking of the reference command.

Remark 2.2. Lemma 2.1 specializes to the case of a constant reference command by letting $\omega = 0$.

Remark 2.3. The rank assumption in Eq. (12) requires that $m \ge l_1$.

Consider a dynamic compensator of the form

$$\dot{x}_c(t) = A_c x_c(t) + A_{csc} x_{sc}(t) + B_{c1} e(t) + B_{c2} y_2(t) \tag{13}$$

$$u(t) = C_c x_c(t) + C_{sc} x_{sc}(t) \tag{14}$$

where $x_c(t) \in \mathbb{R}^{n_c}$. The controller consisting of the servocompensator (8) and the dynamic compensator (13), (14) has the realization

$$G_c(s) \sim \left[\begin{array}{cc|cc} A_c & A_{csc} & B_{c1} & B_{c2} \\ 0 & A_{sc} & B_{sc} & 0 \\ \hline C_c & C_{sc} & 0 & 0 \end{array}\right] \tag{15}$$

The closed-loop system thus has the form

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t) + D_r r(t) \tag{16}$$

$$z(t) = \tilde{E}\tilde{x}(t) \tag{17}$$

$$e(t) = \tilde{C}\tilde{x}(t) - r(t) \tag{18}$$

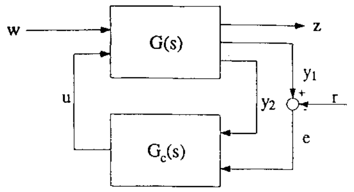

Fig. 1 Closed-loop system with reference command.

where $B_c \triangleq [B_{c1} \quad B_{c2}]$,

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_{sc}(t) \\ x_c(t) \end{bmatrix}, \qquad \tilde{A} \triangleq \begin{bmatrix} A & BC_{sc} & BC_c \\ B_{sc}C_1 & A_{sc} & 0_{n_{sc}\times n_c} \\ B_cC & A_{csc} & A_c \end{bmatrix}$$

$$\tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_{sc}D_2 \\ B_cD_2 \end{bmatrix}, \qquad D_r \triangleq \begin{bmatrix} 0_{n\times l_1} \\ -B_{sc} \\ -B_{c1} \end{bmatrix}$$

$$\tilde{E} \triangleq [\, E_1 \quad E_2C_{sc} \quad E_2C_c \,], \qquad \tilde{C} \triangleq \begin{bmatrix} C_1 & 0_{l_1\times(n_{sc}+n_c)} \end{bmatrix}$$

Since (C, A) is observable by assumption, it follows that if (A_a, B_a) is stabilizable, then a stabilizing control exists so that the closed-loop augmented system is asymptotically stable. A block diagram of the closed-loop system is shown in Fig. 1.

Lemma 2.2. Suppose the reference command $r(t) = C_r$ and assume the augmented matrix $\tilde{A}$ in Eq. (16) with internal model (9) is asymptotically stable. Then $\mathbb{E}[e(t)] \to 0$ as $t \to \infty$.

Proof. Since $\tilde{A}$ in Eq. (16) is asymptotically stable, and $r(t)$ is constant, it follows that $\mathbb{E}[\tilde{x}_\infty] \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)]$ exists and satisfies

$$0 = \tilde{A}\mathbb{E}[\tilde{x}_\infty] + D_rC_r$$

Expanding this equation in terms of its components gives

$$\begin{bmatrix} A\mathbb{E}[x_\infty] + BC_{sc}\mathbb{E}[x_{sc\infty}] + BC_c\mathbb{E}[x_{c\infty}] \\ B_{sc}C_1\mathbb{E}[x_\infty] + A_{sc}\mathbb{E}[x_{sc\infty}] \\ B_cC\mathbb{E}[x_\infty] + A_{csc}\mathbb{E}[x_{sc\infty}] + A_c\mathbb{E}[x_{c\infty}] \end{bmatrix} = \begin{bmatrix} 0 \\ B_{sc}C_r \\ B_cC_r \end{bmatrix} \tag{19}$$

where

$$\tilde{x}_\infty \triangleq \begin{bmatrix} x_\infty \\ x_{sc\infty} \\ x_{c\infty} \end{bmatrix}$$

With the internal model matrices given by Eq. (9), the second equation in (19) reduces to $C_1\mathbb{E}[x_\infty] = C_r$, and hence, by Eq. (4), $\lim_{t\to\infty} \mathbb{E}[e(t)] = 0$. □

Remark 2.4. The internal model (9) ensures that the expected value of each component of the error decays to zero. It is essential that the exogenous dynamics be replicated l_1 times in the internal model, since a single copy of the exogenous system dynamics is not sufficient to ensure that the expected value of each of the elements of the error signal decays to zero individually. If a single copy of the exogenous system dynamics were used in the internal model, then only a linear combination of the elements of the expected value of the error vector would decay to zero. Hence, in that case, $B_{sc}\mathbb{E}[e(t)] \to 0$ as $t \to \infty$.

Remark 2.5. Although C_r is used in the proof of Lemma 2.2, the result that $\mathbb{E}[e(t)] \to 0$ does not require that C_r be known.

Lemma 2.3. Suppose the reference command $r(t) = C_{r1}\sin\omega t + C_{r2}\cos\omega t$ and assume the augmented matrix $\tilde{A}$ in Eq. (16) with internal model (10) is asymptotically stable. Then $\mathbb{E}[e(t)] \to 0$ as $t \to \infty$.

Proof. Consider the response of the closed-loop system (16) to a disturbance $r(t) = C_{r1}\sin\omega t + C_{r2}\cos\omega t$. Since $w(t)$ has zero mean, the expected value of the response of the system is

$$\mathbb{E}[\tilde{x}(t)] = \int_0^t e^{\tilde{A}(t-\tau)} D_r(C_{r1}\sin\omega\tau + C_{r2}\cos\omega\tau)\,d\tau$$

Using

$$\sin\omega t = \frac{e^{\jmath\omega t} - e^{-\jmath\omega t}}{2\jmath}, \qquad \cos\omega t = \tfrac{1}{2}(e^{\jmath\omega t} + e^{-\jmath\omega t})$$

yields

$$(\tilde{A}^2 + \omega^2 I)\mathbb{E}[\tilde{x}(t)] = (\omega e^{\tilde{A}t} - \sin\omega t\tilde{A} - \omega\cos\omega t I)D_rC_{r1} + (\tilde{A}e^{\tilde{A}t} - \cos\omega t\tilde{A} + \omega\sin\omega t I)D_rC_{r2} \tag{20}$$

Partitioning Eq. (20), taking the second component equation as was done in the proof of Lemma 2.2, noting from Eq. (10) that $A_{sc}^2 + \omega^2 I = 0$, rearranging terms, and simplifying yield

$$B_{sc}\frac{d}{dt}\mathbb{E}[e(t)] + A_{sc}B_{sc}\mathbb{E}[e(t)] = \begin{bmatrix} 0 & I_{n_{sc}} & 0 \end{bmatrix} e^{\tilde{A}t}(\omega D_rC_{r1} + \tilde{A}D_rC_{r2})$$

Taking the limit as $t \to \infty$ and noting that $\tilde{A}$ is asymptotically stable, we obtain

$$\lim_{t\to\infty}\left(B_{sc}\frac{d}{dt}\mathbb{E}[e(t)] + A_{sc}B_{sc}\mathbb{E}[e(t)]\right) = 0$$

Now, accounting for the structure of the internal model realization in Eq. (10) yields

$$\lim_{t\to\infty}\begin{bmatrix} \omega\mathbb{E}[e(t)] \\ \frac{d}{dt}\mathbb{E}[e(t)] \end{bmatrix} = 0$$

which implies $\lim_{t\to\infty}\mathbb{E}[e(t)] = 0$. □

Remark 2.6. As in the case of Lemma 2.2, the exogenous dynamics need to be replicated l_1 times in the internal model to ensure that the expected value of each element of the error signal decays to zero individually.

The following propositions provide expressions for the integral square error.

Proposition 2.1. Let $r(t) = C_r$, and suppose $\tilde{A}$ is asymptotically stable. Then the integral square error is given by

$$\int_0^\infty \mathbb{E}[e(t)]^T M\mathbb{E}[e(t)]\,dt = C_r^T D_r^T T D_r C_r \tag{21}$$

where T satisfies

$$0 = \tilde{A}^T T + T\tilde{A} + \tilde{A}^{-T}\tilde{C}^T M\tilde{C}\tilde{A}^{-1}. \tag{22}$$

Proof. It follows from Eq. (16) that

$$\mathbb{E}[\tilde{x}(t)] = \int_0^t e^{\tilde{A}(t-\tau)}D_rC_r\,d\tau = \tilde{A}^{-1}e^{\tilde{A}t}D_rC_r - \tilde{A}^{-1}D_rC_r$$

Thus, $\mathbb{E}[e(t)] = \tilde{C}\tilde{A}^{-1}e^{\tilde{A}t}D_rC_r - (\tilde{C}\tilde{A}^{-1}D_rC_r + C_r)$. Next, using Eq. (16) and since $\tilde{A}$ is asymptotically stable, $\lim_{t\to\infty}\mathbb{E}[\tilde{x}(t)] = -\tilde{A}^{-1}D_rC_r$. It follows that $\lim_{t\to\infty}\mathbb{E}[e(t)] = -(\tilde{C}\tilde{A}^{-1}D_rC_r + C_r)$. Since, by Lemma 2.2, $\mathbb{E}[e(t)] \to 0$ as $t \to \infty$, it follows that $\tilde{C}\tilde{A}^{-1}D_rC_r + C_r = 0$, and hence $\mathbb{E}[e(t)] = \tilde{C}\tilde{A}^{-1}e^{\tilde{A}t}D_rC_r$, which yields Eq. (21), where T satisfies Eq. (22). □

By Proposition 2.1, the minimum value of the integral square error depends on C_r, which is uncertain. For constant references, we assume that C_r belongs to the set $\mathcal{C}_r$, defined by

$$\mathcal{C}_r \triangleq \{C_r \in \mathbb{R}^{l_1}: C_rC_r^T \le V\}$$

where $V \ge 0$ is a given uncertainty bound. Thus, if $C_r \in \mathcal{C}_r$, then it follows that

$$\int_0^\infty \mathbb{E}[e(t)]^T M\mathbb{E}[e(t)]\,dt \le \operatorname{tr} D_r^T T D_r V \tag{23}$$

Proposition 2.2. Let $r(t) = C_{r1}\sin\omega t + C_{r2}\cos\omega t$, and let $\tilde{A}$ be asymptotically stable. Then the integral square error is

$$\int_0^\infty \mathbb{E}[e(t)]^T M \mathbb{E}[e(t)]\,\mathrm{d}t$$

$$= \left(\omega C_{r1}^T D_r^T + C_{r2}^T D_r^T \tilde{A}^T\right) T(\omega D_r C_{r1} + \tilde{A} D_r C_{r2}) \quad (24)$$

where T satisfies

$$0 = \tilde{A}^T T + T\tilde{A} + (\tilde{A}^2 + \omega^2 I)^{-T}\tilde{C}^T M \tilde{C}(\tilde{A}^2 + \omega^2 I)^{-1} \quad (25)$$

Proof. It follows from Eqs. (4) and (20) that

$$\mathbb{E}[e(t)] = \tilde{C}(\tilde{A}^2 + \omega^2 I)^{-1} e^{\tilde{A}t}(\omega D_r C_{r1} + \tilde{A} D_r C_{r2})$$

$$- (C_{r1}\sin\omega t + C_{r2}\cos\omega t)$$

$$- \tilde{C}(\tilde{A}^2 + \omega^2 I)^{-1}[(\tilde{A}\sin\omega t + \omega\cos\omega t I) D_r C_{r1}$$

$$+ (\tilde{A}\cos\omega t - \omega\sin\omega t I) D_r C_{r2}] \quad (26)$$

Since, by Lemma 2.3, $\mathbb{E}[e(t)] \to 0$ as $t \to \infty$, and $\tilde{A}$ is asymptotically stable, it follows that $e^{\tilde{A}t} \to 0$ as $t \to \infty$. Taking the limit of both sides of Eq. (26), it follows that the terms involving $\sin\omega t$ and $\cos\omega t$ are zero. Hence, the expected value of the error is

$$\mathbb{E}[e(t)] = \tilde{C}(\tilde{A}^2 + \omega^2 I)^{-1} e^{\tilde{A}t}(\omega D_r C_{r1} + \tilde{A} D_r C_{r2})$$

The integral square error can be written as Eq. (24) where T satisfies Eq. (25). □

By Proposition 2.2, the minimum value of the integral square error depends on C_r, which is uncertain. For sinusoidal references, we assume that C_r belongs to the set $\mathcal{C}_r$, defined by

$$\mathcal{C}_r \triangleq \left\{ C_r \in \mathbb{R}^{l_1 \times 2}\colon \begin{bmatrix} C_{r1} \\ C_{r2} \end{bmatrix}\begin{bmatrix} C_{r1} \\ C_{r2} \end{bmatrix}^T \le \begin{bmatrix} V_1 & V_{12} \\ V_{12}^T & V_2 \end{bmatrix} = V \right\} \quad (27)$$

where $V \ge 0$ is a given uncertainty bound. Thus, if $C_r \in \mathcal{C}_r$, then it follows that

$$\int_0^\infty \mathbb{E}[e(t)]^T M \mathbb{E}[e(t)]\,\mathrm{d}t \le \omega^2 \operatorname{tr} D_r^T T D_r V_1$$

$$+ \operatorname{tr} D_r^T \tilde{A}^T T \tilde{A} D_r V_2 + 2\omega \operatorname{tr} D_r^T T \tilde{A} D_r V_{12}^T \quad (28)$$

We now introduce the optimal control problem.

Optimal Robust Command-Following Problem. Given the plant dynamics (1) and the internal model dynamics (8), find control gains A_c, B_c, C_c, A_{csc}, and C_{sc} that stabilize $\tilde{A}$ and minimize

$$J(A_c, B_c, C_c, A_{\mathrm{csc}}, C_{\mathrm{sc}}) \triangleq \|T_{zw}\|_2^2$$

$$+ \max_{C_r \in \mathcal{C}_r} \int_0^\infty \mathbb{E}[e(t)]^T M \mathbb{E}[e(t)]\,\mathrm{d}t \quad (29)$$

where T_{zw} is the transfer function from $w(t)$ to $z(t)$.

III. Command-Following Problem Necessary Conditions

In this section we present the main contribution of the paper. Necessary conditions are given for the optimal robust command-following problem, for which the reference command signals are constants and sinusoids. For convenience, let X_{ij} denote the ij(th) block of X partitioned in the same manner as $\tilde{A}$.

Theorem 3.1. Suppose A_c, B_c, C_c, A_{csc}, and C_{sc} solve the optimal robust command-following problem for constant reference inputs. Then there exist nonnegative-definite matrices P, Q, T, S that satisfy

$$0 = \tilde{A}^T P + P\tilde{A} + \tilde{E}^T\tilde{E} \quad (30)$$

$$0 = \tilde{A}Q + Q\tilde{A}^T + \tilde{D}\tilde{D}^T \quad (31)$$

$$0 = \tilde{A}^{2T} T\tilde{A} + \tilde{A}^T T \tilde{A}^2 + \tilde{C}^T M \tilde{C} \quad (32)$$

$$0 = \tilde{A}^2 S \tilde{A}^T + \tilde{A} S \tilde{A}^{2T} + D_r V D_r^T \quad (33)$$

$$0 = \Omega_{33}^T + \Phi_{33}^T + \Theta_{33}^T + \Psi_{33}^T \quad (34)$$

$$0 = (P_{31}D_1 + P_{32}B_{\mathrm{sc}}D_2 + P_{33}B_c D_2)D_2^T + (T_{32}B_{\mathrm{sc}} + T_{33}B_c)V$$

$$+ \left(\Omega_{13}^T + \Phi_{13}^T + \Theta_{13}^T + \Psi_{13}^T\right)C^T \quad (35)$$

$$0 = E_2^T(E_1 Q_{13} + E_2 C_{\mathrm{sc}} Q_{23} + E_2 C_c Q_{33})$$

$$+ B^T\left(\Omega_{31}^T + \Phi_{31}^T + \Theta_{31}^T + \Psi_{31}^T\right) \quad (36)$$

$$0 = \Omega_{23}^T + \Phi_{23}^T + \Theta_{23}^T + \Psi_{23}^T \quad (37)$$

$$0 = E_2^T(E_1 Q_{12} + E_2 C_{\mathrm{sc}} Q_{22} + E_2 C_c Q_{32})$$

$$+ B^T\left(\Omega_{21}^T + \Phi_{21}^T + \Theta_{21}^T + \Psi_{21}^T\right) \quad (38)$$

where $\Omega \triangleq QP$, $\Phi \triangleq S\tilde{A}^T T\tilde{A}$, $\Theta \triangleq \tilde{A}S\tilde{A}^T T$, and $\Psi \triangleq S\tilde{A}^{2T}T$.

Proof. To obtain the necessary conditions, first write the H_2 cost in the form $\operatorname{tr} P\tilde{D}\tilde{D}^T$, where P is the solution to Eq. (30). Next, write the cost (29) as

$$J(A_c, B_c, C_c, A_{\mathrm{csc}}, C_{\mathrm{sc}}) = \operatorname{tr} P\tilde{D}\tilde{D}^T + \operatorname{tr} D_r^T T D_r V \quad (39)$$

and note that Eq. (22) can be rewritten as Eq. (32). Form the Lagrangian $\mathcal{L}$ by affixing Eqs. (30) and (32) via Lagrange multipliers Q and S, respectively, to J to obtain

$$\mathcal{L} = \operatorname{tr} P\tilde{D}\tilde{D}^T + \operatorname{tr} Q(\tilde{A}^T P + P\tilde{A}\tilde{E}^T\tilde{E})$$

$$+ \operatorname{tr} T D_r V D_r^T + \operatorname{tr} S(\tilde{A}^{2T} T\tilde{A} + \tilde{A}^T T\tilde{A}^2 + \tilde{C}^T M\tilde{C}) \quad (40)$$

Setting $\frac{1}{2}(\partial\mathcal{L}/\partial A_c)$, $\frac{1}{2}(\partial\mathcal{L}/\partial B_c)$, $\frac{1}{2}(\partial\mathcal{L}/\partial C_c)$, $\frac{1}{2}(\partial\mathcal{L}/\partial A_{\mathrm{csc}})$, and $\frac{1}{2}(\partial\mathcal{L}/\partial C_{\mathrm{sc}})$ to zero yields the necessary conditions (34–38). Taking the derivatives $\partial\mathcal{L}/\partial Q$, $\partial\mathcal{L}/\partial P$, $\partial\mathcal{L}/\partial S$, and $\partial\mathcal{L}/\partial T$ and setting them equal to zero give Eqs. (30–33). □

Theorem 3.2. Suppose A_c, B_c, C_c, A_{csc}, and C_{sc} solve the optimal robust command-following problem for sinusoidal reference commands. Then there exist nonnegative-definite matrices P, Q, T, S that satisfy Eqs. (30) and (31),

$$0 = (\tilde{A}^2 + \omega^2 I)^T \tilde{A}^T T(\tilde{A}^2 + \omega^2 I)$$

$$+ (\tilde{A}^2 + \omega^2 I)^T T\tilde{A}(\tilde{A}^2 + \omega^2 I) + \tilde{C}^T M\tilde{C} \quad (41)$$

$$0 = (\tilde{A}^2 + \omega^2 I)\tilde{A}S(\tilde{A}^2 + \omega^2 I)^T + (\tilde{A}^2 + \omega^2 I)$$

$$\times S\tilde{A}^T(\tilde{A}^2 + \omega^2 I)^T + \omega^2 D_r V_1 D_r^T + \tilde{A} D_r V_2 D_r^T \tilde{A}^T$$

$$+ \omega\tilde{A} D_r V_{12}^T D_r^T + \omega D_r V_{12} D_r^T \tilde{A}^T \quad (42)$$

$$0 = \Omega_{33}^T + \Phi_{33}^T + \Theta_{33}^T + \Gamma_{33}^T + \Psi_{33}^T + \Pi_{33}^T + \Delta_{33}^T + \Lambda_{33}^T \quad (43)$$

$$0 = (P_{31}D_1 + P_{32}B_{\mathrm{sc}}D_2 + P_{33}B_c D_2)D_2^T$$

$$+ \omega^2(T_{32}B_{\mathrm{sc}} + T_{33}B_c)V_1 + \left[(\tilde{A}^T T\tilde{A})_{32}B_{\mathrm{sc}}\right.$$

$$\left. + (\tilde{A}^T T\tilde{A})_{33}B_c\right]V_2 + \omega[(T\tilde{A})_{32}B_{\mathrm{sc}} + (T\tilde{A})_{33}B_c]V_{12}^T$$

$$+ \omega\left[(\tilde{A}^T T)_{32}B_{\mathrm{sc}} + (\tilde{A}^T T)_{33}B_c\right]V_{12} + \left(\Omega_{13}^T + \Phi_{13}^T + \Theta_{13}^T\right.$$

$$\left. + \Gamma_{13}^T + \Psi_{13}^T + \Pi_{13}^T + \Delta_{13}^T + \Lambda_{13}^T\right)C^T \quad (44)$$

$$0 = E_2^T(E_1 Q_{13} + E_2 C_{\mathrm{sc}} Q_{23} + E_2 C_c Q_{33}) + B^T\left(\Omega_{31}^T + \Phi_{31}^T\right.$$

$$\left. + \Theta_{31}^T + \Gamma_{31}^T + \Psi_{31}^T + \Pi_{31}^T + \Delta_{31}^T + \Lambda_{31}^T\right) \quad (45)$$

$$0 = \Omega_{23}^T + \Phi_{23}^T + \Theta_{23}^T + \Gamma_{23}^T + \Psi_{23}^T + \Pi_{23}^T + \Delta_{23}^T + \Lambda_{23}^T \quad (46)$$

$$0 = E_2^T(E_1 Q_{12} + E_2 C_{\mathrm{sc}} Q_{22} + E_2 C_c Q_{32}) + B^T\left(\Omega_{21}^T + \Phi_{21}^T\right.$$

$$\left. + \Theta_{21}^T + \Gamma_{21}^T + \Psi_{21}^T + \Pi_{21}^T + \Delta_{21}^T + \Lambda_{21}^T\right) \quad (47)$$

where $\Phi \triangleq S(\tilde{A}^2 + \omega^2 I)^T T\tilde{A}^2$, $\Theta \triangleq \tilde{A}S(\tilde{A}^2 + \omega^2 I)^T T\tilde{A}$, $\Gamma \triangleq (\tilde{A}^2 + \omega^2 I)S(\tilde{A}^2 + \omega^2 I)^T T$, $\Omega \triangleq QP$, $\Psi \triangleq \tilde{A}S(\tilde{A}^2 + \omega^2 I)^T\tilde{A}^T T$, $\Pi \triangleq S(\tilde{A}^2 + \omega^2 I)^T\tilde{A}^T T\tilde{A}$, $\Delta \triangleq \omega D_r V_{12}^T D_r^T$, and $\Lambda \triangleq D_r V_2 D_r^T \tilde{A}T$.

Proof. To obtain the necessary conditions, first write the H_2 cost in the form $\mathrm{tr}\, P\tilde{D}\tilde{D}^T$, as in the proof of Theorem 3.1. Next, write the cost J from Eq. (29) as

$$J(A_c, B_c, C_c, A_{csc}, C_{sc}) = \mathrm{tr}\, P\tilde{D}\tilde{D}^T + \omega^2 \mathrm{tr}\, D_r^T T D_r V_1 + \mathrm{tr}\, D_r^T \tilde{A}^T T \tilde{A} D_r V_2 + 2\omega\, \mathrm{tr}\, D_r^T T \tilde{A} D_r V_{12}^T \tag{48}$$

and note that Eq. (25) can be rewritten as Eq. (41). Form the Lagrangian $\mathcal{L}$ by affixing Eqs. (30) and (41) via Lagrange multipliers Q and S, respectively, to J to obtain

$$\begin{aligned}\mathcal{L} = {} & \mathrm{tr}\, P\tilde{D}\tilde{D}^T + \mathrm{tr}\, Q(\tilde{A}^T P + P\tilde{A} + \tilde{E}^T\tilde{E}) \\ & + \omega^2\, \mathrm{tr}\, D_r^T T D_r V_1 + \mathrm{tr}\, D_r^T \tilde{A}^T T \tilde{A} D_r V_2 + 2\omega\, \mathrm{tr}\, D_r^T T \tilde{A} D_r V_{12}^T \\ & + \mathrm{tr}\, S[(\tilde{A}^2 + \omega^2 I)^T \tilde{A}^T T(\tilde{A}^2 + \omega^2 I) \\ & + (\tilde{A}^2 + \omega^2 I)^T T\tilde{A}(\tilde{A}^2 + \omega^2 I) + \tilde{C}^T M \tilde{C}]\end{aligned} \tag{49}$$

Setting $\frac{1}{2}(\partial\mathcal{L}/\partial A_c)$, $\frac{1}{2}(\partial\mathcal{L}/\partial B_c)$, $\frac{1}{2}(\partial\mathcal{L}/\partial C_c)$, $\frac{1}{2}(\partial\mathcal{L}/\partial A_{csc})$, and $\frac{1}{2}(\partial\mathcal{L}/\partial C_{sc})$ to zero gives the necessary conditions (43–47). Taking the derivatives $\partial\mathcal{L}/\partial Q$, $\partial\mathcal{L}/\partial P$, $\partial\mathcal{L}/\partial S$, and $\partial\mathcal{L}/\partial T$ and setting them equal to zero gives Eqs. (30), (31), (41), and (42). □

IV. Numerical Example

Theorems 3.1 and 3.2 give expressions for the gradient of the Lagrangian with respect to each of the control gains. For example, in Theorem 3.1, the gradients (34–38) are $\frac{1}{2}(\partial\mathcal{L}/\partial A_c)$, $\frac{1}{2}(\partial\mathcal{L}/\partial B_c)$, $\frac{1}{2}(\partial\mathcal{L}/\partial C_c)$, $\frac{1}{2}(\partial\mathcal{L}/\partial A_{csc})$, and $\frac{1}{2}(\partial\mathcal{L}/\partial C_{sc})$, respectively. Although there is no straightforward way to solve these equations directly, gradient optimization algorithms can use the to find the optimal gains. A quasi-Newton algorithm was used to find the feedback gains that minimize the cost function by approximating the inverse of the Hessian and using the gradient expressions.[7]

Consider the second-order model of a pressurized head box[2]:

$$\dot{x}(t) = \begin{bmatrix} -0.395 & 0.001145 \\ -0.011 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.03362 & 1.038 \\ 0.000966 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} w_1(t)$$

$$y(t) = [1 \quad 0]x(t) + 0.1 w_2(t), \qquad z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

For constant reference commands, a family of controllers was found by setting $V = 1$ and varying the weighting M in the tracking error cost. The two components of the cost are plotted as the solid line in Fig. 2. For comparison, a family of controllers was computed using the technique of Abedor et al.[3] by varying the scalar design parameter α. The suboptimal costs are shown by the dashed line in Fig. 2. The two sets of costs were plotted on different axes because of the large difference in the sizes of the tracking costs. Clearly, the controllers found using the technique in this paper have lower tracking costs for comparable H_2 costs.

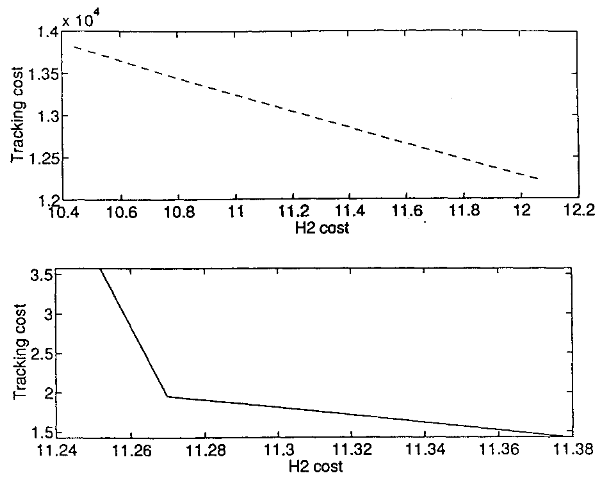

Fig. 2 Tracking cost vs H_2 cost for constant reference command (optimal, solid; suboptimal, dashed).

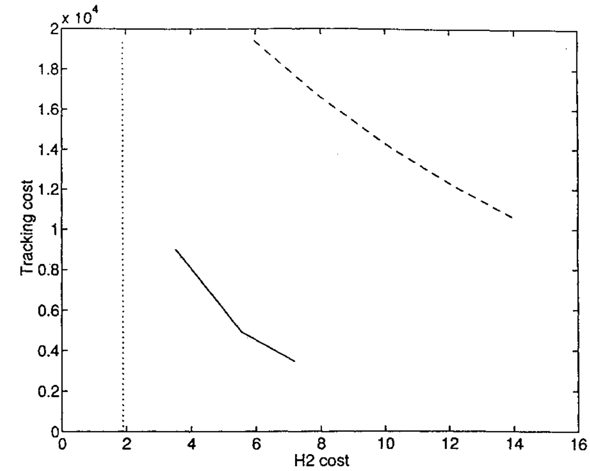

Fig. 3 Tracking cost vs H_2 cost for sinusoidal reference command (optimal, solid; suboptimal, dashed; H_2 optimal, dotted).

Two families of controllers were found in a similar fashion for a sinusoidal reference command of frequency $\frac{1}{2}\pi$ with $V_1 = 1$, $V_{12} = V_2 = 0$. The costs for these controllers are shown in Fig. 3. Again, the tracking costs for the controllers found using the technique in this paper are lower than those found using the technique of Abedor et al.[3] for comparable H_2 costs.

V. Conclusions

A technique for finding a control law that achieves asymptotic tracking of constant and sinusoidal reference commands while minimizing a cost consisting of an H_2 disturbance rejection component and an integral square error component was presented. The solution was derived by writing the integral square error in terms of the solution to a Lyapunov equation and attaching the H_2 and integral square error Lyapunov equations to the cost via matrix Lagrange multipliers. Necessary conditions were obtained as the gradients of the Lagrangian with respect to each of the control gains. Controllers satisfying the necessary conditions provide better transient tracking performance for a given level of H_2 disturbance attenuation than all other controllers that achieve asymptotic tracking. Hence, controllers satisfying the necessary conditions provide the optimal tradeoff between the two components of the cost function.

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