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STUDIES IN RADAR CROSS SECTIONS XXIII

A VARIATIONAL SOLUTION TO THE PROBLEM OF SCALAR SCATTERING BY A PROLATE SPHEROID

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STUDIES IN RADAR CROSS SECTIONS

I Scattering by a Prolate Spheroid, by F. V. Schultz (UMM-42, March 1950), W-33(038)-ac-14222, UNCLASSIFIED.

II The Zeros of the Associated Legendre Functions $P^m_n(\mu')$ of Non-Integral Degree, by K. M. Siegel, D. M. Brown, H. E. Hunter, H. A. Alperin, and C. W. Quillen (UMM-82, April 1951), W-33(038)-ac-14222, UNCLASSIFIED.

III Scattering by a Cone, by K. M. Siegel and H. A. Alperin (UMM-87 January 1952), AF-30(602)-9, UNCLASSIFIED.

IV Comparison Between Theory and Experiment of the Cross Section of a Cone, by K. M. Siegel, H. A. Alperin, J. W. Crispin, Jr., H. E. Hunter, R. E. Kleinman, W. C. Orthwein, and C. E. Schensted (UMM-92, February 1953), AF-30(602)-9, UNCLASSIFIED.

V An Examination of Bistatic Early Warning Radars, by K. M. Siegel, (UMM-98, August 1952) W-33(038)-ac-14222, SECRET.

VI Cross Sections of Corner Reflectors and Other Multiple Scatterers at Microwave Frequencies, by R. R. Bonkowski, C. R. Lubitz, and C. E. Schensted (UMM-106, October 1953), AF-30(602)-9, SECRET - UNCLASSIFIED when Appendix is removed.

VII Summary of Radar Cross Section Studies Under Project Wizard, by K. M. Siegel, J. W. Crispin, Jr., and R. E. Kleinman (UMM-108, November 1952), W-33(038)-ac-14222, SECRET.

VIII Theoretical Cross Section as a Function of Separation Angle Between Transmitter and Receiver at Small Wavelengths, by K. M. Siegel, H. A. Alperin, R. R. Bonkowski, J. W. Crispin, Jr., A. L. Maffett, C. E. Schensted, and I. V. Schensted (UMM-115, October 1953), W-33(038)-ac-14222, UNCLASSIFIED.

IX Electromagnetic Scattering by an Oblate Spheroid, by L. M. Rauch (UMM-116, October 1953), AF-30(602)-9, UNCLASSIFIED.

X Scattering of Electromagnetic Waves by Spheres, by H. Weil, M. L. Barasch, and T. A. Kaplan (2255-20-T, July 1956), AF-30(602)-1070, UNCLASSIFIED.

XI The Numerical Determination of the Radar Cross Section of a Prolate Spheroid, by K. M. Siegel, B. H. Gere, I. Marx, and F. B. Sleator (UMM-126, December 1953), AF-30(602)-9, UNCLASSIFIED.

XII Summary of Radar Cross Section Studies Under Project MIRO, by K. M. Siegel, M. E. Anderson, R. R. Bonkowski, and W. C. Orthwein (UMM-127, December 1953), AF-30(602)-9, SECRET.

XIII Description of a Dynamic Measurement Program, by K. M. Siegel and J. M. Wolf (UMM-128, May 1954), W-33(038)-ac-14222, CONFIDENTIAL.

XIV Radar Cross Section of a Ballistic Missile, by K. M. Siegel, M. L. Barasch, J. W. Crispin, Jr., W. C. Orthwein, I. V. Schensted, and H. Weil (UMM-134, Sept. 1954), W-33(038)-ac-14222, SECRET.

XV Radar Cross Sections of B-47 and B-52 Aircraft, by C. E. Schensted, J. W. Crispin, Jr., and K. M. Siegel (2260-1-T, August 1954), AF-33(616)-2531, CONFIDENTIAL.

XVI Microwave Reflection Characteristics of Buildings, by H. Weil, R. R. Bonkowski, T. A. Kaplan, and M. Leichter (2255-12-T, May 1955), AF-30(602)-1070, SECRET.

XVII Complete Scattering Matrices and Circular Polarization Cross Sections for the B-47 Aircraft at S-band, by A. L. Maffett, M. L. Barasch, W. E. Burdick, R. F. Goodrich, W. C. Orthwein, C. E. Schensted, and K. M. Siegel (2260-6-T, June 1955), AF-33(616)-2531, CONFIDENTIAL.

XVIII Airborne Passive Measures and Countermeasures, by K. M. Siegel, M. L. Barasch, J. W. Crispin, Jr., R. F. Goodrich, A. H. Halpin, A. L. Maffett, W. C. Orthwein, C. E. Schensted, and C. J. Titus (2260-29-F, January 1956), AF-33(616)-2531, SECRET.

XIX Radar Cross Section of a Ballistic Missile - II, by K. M. Siegel, M. L. Barasch, H. Brysk, J. W. Crispin, Jr., T. B. Curtz, and T. A. Kaplan (2428-3-T, January 1956), AF-04(645)-33, SECRET.

XX Radar Cross Section of Aircraft and Missiles, by K. M. Siegel, W. E. Burdick, J. W. Crispin, Jr., and S. Chapman (WR-31-J, 1 March 1956), SECRET.

XXI Radar Cross Section of a Ballistic Missile - III, by K. M. Siegel, H. Brysk, J. W. Crispin, Jr., and R. E. Kleinman (2428-19-T, October 1956) AF-04(645)-33, SECRET.

XXII Elementary Slot Radiators, R. F. Goodrich, A. L. Maffett, N. Reitlinger, C. E. Schensted, and K. M. Siegel, (2472-13-T, November 1956), AF 33(038)-28634; HAC-PO L-265165-F31, UNCLASSIFIED.

XXIII A Variational Solution to the Problem of Scalar Scattering by a Prolate Spheroid, by F. B. Sleator (2591-1-T, March 1957), AF 19(604)-1949, UNCLASSIFIED.

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PREFACE

This paper is the twenty-third in a series growing out of studies of radar cross-sections at the Engineering Research Institute of The University of Michigan. The primary aims of this program are:

1. To show that radar cross-sections can be determined analytically.

2. A. To determine means for computing the radiation patterns from antennas by approximate techniques which determine the pattern to the accuracy required in military problems but which do not require the unique determination of exact solutions.

   B. To determine means for computing the radar cross-sections of various objects of military interest.

   (Since 2A and 2B are inter-related by the reciprocity theorem it is necessary to solve only one of these problems)

3. To demonstrate that these theoretical cross-sections and theoretically determined radiation patterns are in agreement with experimentally determined ones.

Intermediate objectives are:

1. A. To compute the exact theoretical cross-sections of various simple bodies by solution of the approximate boundary-value problems arising from electromagnetic theory.

B. Compute the exact radiation patterns from infinitesimal slots on the surface of simple shapes by the solution of appropriate boundary-value problems arising from electromagnetic theory.

(Since 1A and 1B are inter-related by the reciprocity theorem it is necessary to solve only one of these problems)

2. To examine the various approximations possible in this problem and to determine the limits of their validity and utility.

3. To find means of combining the simple-body solutions in order to determine the cross-sections of composite bodies.

4. To tabulate various formulas and functions necessary to enable such computations to be done quickly for arbitrary objects.

5. To collect, summarize, and evaluate existing experimental data.

Titles of the papers already published or presently in process of publication are listed on the preceding page.

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K. M. Siegel

CHAPTER I

INTRODUCTION

The natural limitation on the variety of problems in scattering and transmission theory which can be handled conveniently by the technique of separation of variables has led to the development of methods of essentially different character. Among these are variational methods, of which perhaps the most widely used is that developed by Schwinger and employed with considerable success by him and by numerous others in the solution of various problems in diffraction of sound and electromagnetic waves as well as quantum scattering.

Practically all the literature which has appeared so far in this field has concerned itself with problems in one and two dimensions, primarily because of the difficulty in performing the required integrations. Regarding the electromagnetic problem, Mentzer remarks in his recent book on scattering of radio waves:[1] "The formulation with three dimensional scatterers, such as spheres, leads to surface integrals which usually are completely unmanageable; the integration processes with simpler geometries are, at best, very difficult." One exception to this is a solution to the problem of the loop obtained by Kouyoumjian. However, similar statements are often found in the literature, even for the scalar case.

The present paper represents another attempted break-through into

[1]Mentzer, J. R., "Scattering and Diffraction of Radio Waves," Pergamon Press, Ltd., p. 45 (1955).

into the 'unmanageable' third dimension. It is true that the spheroidal scatterer affords perhaps the simplest geometry next to the sphere of any three-dimensional body, and the setup possesses cylindrical symmetry; nonetheless the problem is essentially three-dimensional and this result in addition to that of Kouyoumjian may help to dispel some of the pessimism noted above. The solution obtained may be of little value in itself, since the prolate spheroid has already been dealt with quite extensively by the separation technique. However, it does indicate that the integrations involved in some three-dimensional problems may be more or less manageable, and it may also shed a little more light on the value of the variational method in general.

From a mathematical standpoint, some of the procedures used here, just as in many of the papers of Schwinger and others, are not rigorous. No justification is presented for the numerous changes in order of integration, and some of the integrals which appear are at least formally divergent. However, recent work of Bouwkamp[2] has indicated that in some similar problems this formal divergence is only formal, and can be eliminated without affecting the results at all. These considerations, together with the fact that the present solution agrees with known results exactly in the limiting cases of zero eccentricity and very large wavelength and extremely well for the case of a thin spheroid in the resonance region, seem to indicate that an attempt to introduce

[2] Bouwkamp, C. J., "Diffraction Theory--A Critique of Some Recent Developments" Res. Rep. EM-50, N. Y. U. Inst. of the Math. Sciences (April 1953).

mathematical rigor into this development would not be worthwhile at present.

The principal physical quantity obtained in the present analysis is the nose-on back-scattering cross section of the rigid spheroid. It is easy to see how the results and techniques could be extended to include certain additional information. Further analytical work on the forms already obtained might also be profitable under some circumstances. Various possibilities are discussed in more detail in the final section of the paper.

The author wishes to express the deepest appreciation to certain colleagues, in particular Messrs. K. M. Siegel, C. E. Schensted, and A. H. Halpin, for many illuminating and invaluable discussions of the problem. Credit is also due Mr. H. E. Hunter of the Willow Run Laboratories for his meticulous work in computing the numerical results contained here. Finally, the utmost gratitude is accorded Prof. Wilhelm Magnus of New York University for his infallibly prompt and considered advice and assistance, without which the work could not have been completed in the appointed time.

CHAPTER 2

FORMULATION OF THE VARIATIONAL PROBLEM

We assume a rigid prolate spheroid with center at the origin, major axis of length 2a in the Z-axis, minor axis of length 2b, and a plane sound wave approaching in the negative Z direction. The following integral equation for the velocity potential at any point exterior to the scatterer can be established:[3]

$$\phi(S) = e^{ikz} - \frac{1}{4\pi}\int_{S'} \phi(S') \frac{\partial}{\partial n'} G(S,S')da' \qquad (1)$$

where S is the field point in space

S' is the point on the spheroid

$\phi(S)$ is the velocity potential at S

$G(S,S')$ is the Green's function of free space $= \frac{e^{ik\rho}}{\rho}$

($\rho \equiv$ distance from S to S')

$k = 2\pi/\lambda$ ($\lambda \equiv$ wavelength)

$\frac{\partial}{\partial n'}$ is the derivative in the direction of the exterior normal, and the integration covers the surface.

Application of the boundary condition $\left.\frac{\partial \phi}{\partial n}\right|_{S'} = 0$ to equation (1) yields

$$\left.\frac{\partial}{\partial n} e^{ikz}\right|_{S'} = \frac{1}{4\pi}\int_{S'} \phi(S') \frac{\partial^2}{\partial n\, \partial n'} G(S,S')\, da' \qquad (2)$$

[3]Sollfrey, W. "The Variational Solution of Scattering Problems," Research Report EM-11, New York Univ. Inst. of Mathematical Sciences, p.7 ff (1949).

Then it follows[4] that $\phi(S')$ is the solution of the variational problem $\delta J[\phi] = 0$, where

$$J[\phi] = \frac{\iint_S \phi(S) \frac{\partial^2}{\partial n \partial n'} G(S,S') \phi(S') \, da \, da'}{\left[\int_S \phi(S) \frac{\partial}{\partial n} e^{ikz}\Big|_S da\right]^2} \tag{3}$$

It is expedient now to introduce the prolate spheroidal coordinate system ξ, η, ϕ, which is related to the rectangular system by the formulas

$$x = F\sqrt{(\xi^2 - 1)(1 - \eta^2)} \cos\phi \equiv F\alpha\beta\cos\phi$$

$$y = F\sqrt{(\xi^2 - 1)(1 - \eta^2)} \sin\phi \equiv F\alpha\beta\sin\phi$$

$$z = F\xi\eta$$

where F is the semi-focal distance. Then if $\sqrt{\xi^2 - \eta^2} \equiv \gamma$ we have

$$da = F^2\alpha\gamma \, d\eta \, d\phi$$
$$\frac{\partial}{\partial n} = \frac{\alpha}{F\gamma}\frac{\partial}{\partial \xi} \, . \tag{4}$$

The trial function $\phi(S')$ (for S' on the scattering surface) may now be expanded in terms of the Legendre polynomials $P_\mu(\eta')$, which form a complete, orthogonal set over the interval $-1 \leq \eta' \leq 1$. Thus

$$\phi(S') = \sum_\mu A_\mu(\xi') P_\mu(\eta') \tag{5}$$

where the coefficients $A_\mu(\xi')$ are to be determined. Then the variational quantity $J[\phi]$ takes the form

[4] Sollfrey, W. "The Variational Solution of Scattering Problems," Research Report EM-11, New York Univ. Inst. of Mathematical Sciences, p. 13 (1949).

$$J[\phi] = \frac{\sum_{\mu}\sum_{\nu} A_{\mu} A_{\nu} \iint_S P_{\mu}(\eta) \frac{\partial^2}{\partial n \partial n'} G(S,S') P_{\nu}(\eta') \, da \, da'}{\left[\sum_{\mu} A_{\mu} \int_S P_{\mu}(\eta) \frac{\partial}{\partial n} e^{ikz} \Big|_S da \right]^2} \tag{6}$$

or if the integrals in the numerator and denominator are represented by $C_{\mu\nu}$ and B_{μ} respectively,

$$J[\phi] = \frac{\sum_{\mu}\sum_{\nu} A_{\mu} A_{\nu} C_{\mu\nu}}{\left[\sum_{\mu} A_{\mu} B_{\mu} \right]^2} . \tag{7}$$

It is easily shown (cf. Sollfrey, loc. cit.) that the stationary value of J is the negative reciprocal of the back-scattered amplitude, from which the back-scattering cross section is immediately obtainable.

CHAPTER 3

EVALUATION OF THE INTEGRALS $C_{\mu\nu}$

To accomplish the integrations appearing in the numerator of equation (6) it is advantageous to use the Fourier integral representation of the Green's function[5]

$$G(S,S') \equiv \frac{e^{ik\rho}}{\rho} = \frac{1}{2\pi^2}\iiint_{-\infty}^{\infty} \frac{e^{i\vec{K}\cdot\vec{\rho}}}{\vec{K}^2-k^2}\, d\vec{K} \qquad (8)$$

where $\vec{K} = (K_x, K_y, K_z)$

$d\vec{K} = dK_x \cdot dK_y \cdot dK_z$

$\vec{K}^2 = \vec{K}\cdot\vec{K}$

and $\vec{\rho} = (x-x', \; y-y', \; z-z')$.

Now we have formally

$$\frac{\partial^2}{\partial n \partial n'} G(S,S') \equiv \frac{\partial}{\partial n} \frac{\partial G}{\partial n'} = \frac{\alpha}{F\gamma} \frac{\partial}{\partial \xi}\left(\frac{\alpha'}{F\gamma'} \frac{\partial G}{\partial \xi'}\right) .$$

Actually this quantity must be regarded as a limit[6] as $\xi' \to \xi$, in order to avoid difficulty with the singularity at $S = S'$, and accordingly we preserve a distinction between ξ and ξ', until after the crucial integration has been performed. Furthermore

[5] Levine, H. and Schwinger, J., "Diffraction by an Aperture in an Infinite Plane Screen, I." Phys. Rev. 74, p. 961 (1948).
[6] Morse, P. M. and Feshbach, H., "Methods of Theoretical Physics," McGraw-Hill, New York, p. 1043 (1955).

$$\frac{\partial G}{\partial \xi'} = \frac{\partial G}{\partial x'}\frac{\partial x'}{\partial \xi'} + \frac{\partial G}{\partial y'}\frac{\partial y'}{\partial \xi'} + \frac{\partial G}{\partial z'}\frac{\partial z'}{\partial \xi'}$$

and

$$\frac{\partial e^{i\vec{K}\cdot\vec{\rho}}}{\partial(x',y',z')} = -\, e^{i\vec{K}\cdot\vec{\rho}} \cdot i(K_x, K_y, K_z)$$

so that, ignoring for the present the question of the legitimacy of differentiating under the integral signs, we can write

$$\frac{\partial G}{\partial \xi'} = \frac{-i}{2\pi^2}\iiint_{-\infty}^{\infty} \frac{e^{i\vec{K}\cdot\vec{\rho}}}{\vec{K}^2 - k^2}\left(K_x \frac{\partial x'}{\partial \xi'} + K_y \frac{\partial y'}{\partial \xi'} + K_z \frac{\partial z'}{\partial \xi'}\right) d\vec{K}$$

and

$$\frac{\partial^2 G}{\partial n \partial n'} = \frac{\alpha\alpha'}{F^2 \gamma\gamma' \cdot 2\pi^2}\iiint_{-\infty}^{\infty} \frac{e^{i\vec{K}\cdot\vec{\rho}}}{\vec{K}^2 - k^2}\left(K_x \frac{\partial x'}{\partial \xi'} + K_y \frac{\partial y'}{\partial \xi'} + K_z \frac{\partial z'}{\partial \xi'}\right)\cdot$$

$$\cdot\left(K_x \frac{\partial x}{\partial \xi} + K_y \frac{\partial y}{\partial \xi} + K_z \frac{\partial z}{\partial \xi}\right) d\vec{K}$$

$$= \frac{\alpha\alpha'}{2\pi^2 \gamma\gamma'}\iiint_{-\infty}^{\infty} \frac{e^{i\vec{K}\cdot\vec{\rho}}}{\vec{K}^2 - k^2}\left(K_x \frac{\beta}{\alpha}\xi \cos\phi + K_y \frac{\beta}{\alpha}\xi \sin\phi + K_z \eta\right)\cdot$$

$$\cdot\left(K_x \frac{\beta'}{\alpha'}\xi' \cos\phi' + K_y \frac{\beta'}{\alpha'}\xi' \sin\phi' + K_z \eta'\right) dK \;.$$

The divergence of this integral expression must be eliminated at a later stage. Then rearranging the integrations involved, we can write

$$C_{\mu\nu} = \frac{F^4\alpha^4}{2\pi^2}\lim_{\xi'\to\xi}\iiint_{-\infty}^{\infty}\frac{d\vec{K}}{\vec{K}^2-k^2}\iint_{-1}^{+1}\iint_{0}^{2\pi} e^{i\vec{K}\cdot\vec{\rho}}\, P_\mu(\eta)\, P_\nu(\eta')\cdot \tag{9}$$

$$\cdot\left(K_x \frac{\beta}{\alpha}\xi \cos\phi + K_y \frac{\beta}{\alpha}\xi \sin\phi + K_z\eta\right)\left(K_x \frac{\beta'}{\alpha'}\xi' \cos\phi' + K_y \frac{\beta'}{\alpha'}\xi' \sin\phi' + K_z\eta'\right) d\phi\, d\phi'\, d\eta\, d\eta'$$

or

$$C_{\mu\nu} = \frac{F^4 \alpha^4}{2\pi^2} \lim_{\xi' \to \xi} I_{\mu\nu}$$

where $I_{\mu\nu}$ represents the above integral.

It becomes convenient here to introduce the transformation

$$K_x = \frac{1}{\alpha} \; r \cos\omega \sin\psi \qquad 0 \le r \le \infty$$

$$K_y = \frac{1}{\alpha} \; r \sin\omega \sin\psi \qquad 0 \le \psi \le \pi$$

$$K_z = \frac{1}{\xi} \; r \cos\psi \qquad 0 \le \omega < 2\pi .$$

(This new coordinate system r, ψ, ω is similar to ordinary spherical coordinates but the family of spheres is replaced by a family of oblate spheroids. The lack of orthogonality is not important.) The Jacobian of this transformation is

$$\frac{\partial(K_x, K_y, K_z)}{\partial(r, \psi, \omega)} = \frac{-r^2 \sin\psi}{\xi \alpha^2} .$$

In terms of the new coordinates,

$$\vec{K} \cdot \vec{\rho} = Fr\Big[\frac{\cos\omega \sin\psi}{\alpha} (\alpha\beta\cos\phi - \alpha'\beta'\cos\phi')$$

$$+ \frac{\sin\omega\sin\psi}{\alpha} (\alpha\beta\sin\phi - \alpha'\beta'\sin\phi') + \frac{\cos\psi}{\xi} (\xi\eta - \xi'\eta')\Big]$$

$$= Fr\left\{ \frac{\sin\psi}{\alpha}\Big[\alpha\beta\cos(\omega-\phi) - \alpha'\beta'\cos(\omega-\phi')\Big] + \frac{\cos\psi}{\xi}\Big[\xi\eta - \xi'\eta'\Big]\right\}$$

$$K_x \frac{\beta}{\alpha}\xi\cos\phi + K_y\frac{\beta}{\alpha}\xi\sin\phi + K_z\eta = r\left[\frac{\beta\xi}{\alpha^2}\sin\psi\cos(\omega-\phi) + \frac{\eta}{\xi}\cos\psi\right]$$

and $\vec{K}^2 = \frac{r^2}{\alpha^2\xi^2}(\xi^2 - \cos^2\psi)$.

Substitution of these quantities into the integral above gives

$$I_{\mu\nu} = \frac{-1}{\xi\alpha^2}\int_0^{\pi}\int_0^{\infty}\int_0^{2\pi}\frac{r^2\sin\psi\, d\omega\, dr\, d\psi}{\frac{r^2}{\alpha^2\xi^2}(\xi^2-\cos^2\psi)-k^2}\int\int_{-1}^{+1} P_\mu(\eta)P_\nu(\eta')e^{iFr\cos\psi(\eta-\frac{\xi'}{\xi}\eta')}d\eta\, d\eta'$$

$$\cdot\int\int_0^{2\pi} e^{iFr\frac{\sin\psi}{\alpha}\left[\alpha\beta\cos(\omega-\phi)-\alpha'\beta'\cos(\omega-\phi')\right]}\; r^2\left[\frac{\beta\xi}{\alpha^2}\sin\psi\cos(\omega-\phi)+\frac{\eta}{\xi}\cos\psi\right]\cdot$$

$$\cdot\left[\frac{\beta'\xi'}{\alpha\alpha'}\sin\psi\cos(\omega-\phi')+\frac{\eta'}{\xi}\cos\psi\right] d\phi\, d\phi' \quad .$$

We deal first with the ϕ and ϕ' integrals, which can be resolved almost immediately as follows:

$$\int_0^{2\pi} e^{iFr\frac{\sin\psi}{\alpha}\alpha\beta\cos(\omega-\phi)}\cdot\left[\frac{\beta\xi}{\alpha^2}\sin\psi\cos(\omega-\phi)+\frac{\eta}{\xi}\cos\psi\right]d\phi$$

$$= \frac{\beta\xi}{\alpha^2}\sin\psi\int_0^{2\pi} e^{iFr\beta\sin\psi\cos(\omega-\phi)}\cos(\omega-\phi)\, d\phi$$

$$+\frac{\eta}{\xi}\cos\psi\int_0^{2\pi} e^{iFr\beta\sin\psi\cos(\omega-\phi)}\, d\phi$$

$$= \frac{2\beta\xi}{\alpha^2}\sin\psi\cdot i\pi J_1(Fr\beta\sin\psi)+\frac{2\eta}{\xi}\cos\psi\cdot\pi J_0(Fr\beta\sin\psi),$$

and similarly

$$\int_0^{2\pi} e^{-iFr\beta'\frac{\alpha'}{\alpha}\sin\psi\cos(\omega-\phi')}\left[\frac{\beta'\xi'}{\alpha\alpha'}\sin\psi\cos(\omega-\phi')+\frac{\eta'}{\xi}\cos\psi\right]d\phi'$$

$$= \frac{2\beta'\xi'}{\alpha\alpha'}\sin\psi\cdot i\pi J_1\left(-Fr\frac{\beta'\alpha'}{\alpha}\sin\psi\right)+\frac{2\eta'}{\xi}\cos\psi\cdot\pi J_0\left(-Fr\frac{\beta'\alpha'}{\alpha}\sin\psi\right) \quad .$$

Putting these expressions into $I_{\mu\nu}$ and rearranging slightly, we have

$$I_{\mu\nu} \equiv -4\xi\pi^2 \int_0^{\pi}\int_0^{\infty}\int_0^{2\pi} \frac{r^2 \sin\psi \, d\omega \, dr \, d\varphi}{\xi^2 - \cos^2\psi - \frac{k^2\xi^2\alpha^2}{r^2}} \left\{ -\frac{\xi\,\xi'}{\alpha^3\alpha'} \sin^2\psi \, I_\eta^{\,1} \right. \tag{10}$$

$$\left. + \frac{\cos^2\psi}{\xi^2} I_\eta^{\,2} + i \sin\psi\cos\psi \left[\frac{1}{\alpha^2} I_\eta^{\,3} + \frac{\xi'}{\xi\alpha\alpha'} I_\eta^{\,4} \right] \right\}$$

where

$$I_\eta^{\,1} \equiv \int\!\!\int_{-1}^{+1} P_\mu(\eta)\, P_\nu(\eta')\, e^{iFr\cos\psi(\eta - \frac{\xi'}{\xi}\eta')} \beta\beta' J_1(Fr\beta\sin\psi) J_1(-Fr\beta'\frac{\alpha'}{\alpha}\sin\psi)\, d\eta\, d\eta'$$

$$I_\eta^{\,2} \equiv \int\!\!\int_{-1}^{+1} P_\mu(\eta)\, P_\nu(\eta')\, e^{iFr\cos\psi(\eta - \frac{\xi'}{\xi}\eta')} \eta\eta' J_o(Fr\beta\sin\psi) J_o(-Fr\beta'\frac{\alpha'}{\alpha}\sin\psi)\, d\eta\, d\eta'$$

$$I_\eta^{\,3} \equiv \int\!\!\int_{-1}^{+1} P_\mu(\eta)\, P_\nu(\eta')\, e^{iFr\cos\psi(\eta - \frac{\xi'}{\xi}\eta')} \beta\eta' J_1(Fr\beta\sin\psi) J_o(-Fr\beta'\frac{\alpha'}{\alpha}\sin\psi)\, d\eta\, d\eta'$$

$$I_\eta^{\,4} \equiv \int\!\!\int_{-1}^{+1} P_\mu(\eta)\, P_\nu(\eta')\, e^{iFr\cos\psi(\eta - \frac{\xi'}{\xi}\eta')} \eta\beta' J_o(Fr\beta\sin\psi) J_1(-Fr\beta'\frac{\alpha'}{\alpha}\sin\psi)\, d\eta\, d\eta'.$$

Here the primed and unprimed integrals are separable in each case.

Consider first $I_\eta^{\,1}$. The η integral is

$$\int_{-1}^{+1} P_\mu(\eta)\, e^{iFr\cos\psi\cdot\eta} \beta \cdot J_1(Fr\beta\sin\psi)\, d\eta$$

$$= \frac{1}{2\mu+1} \int_{-1}^{1} \left[P_{\mu-1}^{1}(\eta) - P_{\mu+1}^{1}(\eta) \right] e^{iFr\cos\psi\cdot\eta} J_1(Fr\beta\sin\psi)\, d\eta.$$

We have in general[7]

$$\int_{-1}^{+1} e^{iz\eta\cos\psi} J_m(z\sqrt{1-\eta^2}\sin\psi) P_n^m(\eta)\,d\eta = i^{n-m}\sqrt{\frac{2\pi}{z}}\, P_n^m(\cos\psi) J_{n+1/2}(z) \tag{11}$$

for $0 \leq \psi \leq \pi$, so that the above integral becomes

$$\frac{1}{2\mu+1}\sqrt{\frac{2\pi}{Fr}}\left[i^{\mu-2} P^1_{\mu-1}(\cos\psi) J_{\mu-1/2}(Fr) - i^{\mu} P^1_{\mu+1}(\cos\psi) J_{\mu+3/2}(Fr)\right] .$$

In similar fashion for η' we have

$$\int_{-1}^{+1} P_\nu(\eta')\, e^{-iFr\cos\psi\frac{\xi'}{\xi}\eta'\beta'}\; J_1(-Fr\beta'\tfrac{\alpha'}{\alpha}\sin\psi)\, d\eta'$$

$$= \frac{1}{2\nu+1}\int_{-1}^{+1}\left[P^1_{\nu-1}(\eta') - P^1_{\nu+1}(\eta')\right] e^{iFr\epsilon\cos\sigma\eta'} J_1(Fr\epsilon\sin\sigma\beta')\,d\eta' \tag{12}$$

$$= \frac{-1}{2\nu+1}\sqrt{\frac{2\pi}{Fr\epsilon}}\left[\, i^{\nu-2} P^1_{\nu-1}(\cos\sigma) J_{\nu-1/2}(Fr\epsilon) - i^{\nu} P^1_{\nu+1}(\cos\sigma) J_{\nu+3/2}(Fr\epsilon)\right]$$

where $-\frac{\xi'}{\xi}\cos\psi = \epsilon\cos\sigma$ and $\frac{\alpha'}{\alpha}\sin\psi = \epsilon\sin\sigma$.

Thus

$$I^1_\eta = \frac{-2\pi i^{\mu+\nu}}{(2\mu+1)(2\nu+1)Fr\sqrt{\epsilon}}\left[P^1_{\mu-1}(\cos\psi) J_{\mu-1/2}(Fr) + P^1_{\mu+1}(\cos\psi) J_{\mu+3/2}(Fr)\right] \cdot$$

$$\cdot\left[P^1_{\nu-1}(\cos\sigma) J_{\nu-1/2}(Fr\epsilon) + P^1_{\nu+1}(\cos\sigma) J_{\nu+3/2}(Fr\epsilon)\right] ,$$

and in similar fashion

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[7]Magnus, W. and Oberhettinger, F., "Functions of Mathematical Physics," Chelsea, p. 77 (1949). The integral is given in terms of Gegenbauer functions. It takes the form used here when these are converted to Legendre polynomials.

$$I_\eta^2 = \frac{-2\pi i^{\mu+\nu}}{(2\mu+1)(2\nu+1)Fr\sqrt{\epsilon}}\left[(\mu+1)P_{\mu+1}(\cos\psi)J_{\mu+3/2}(Fr)-\mu P_{\mu-1}(\cos\psi)J_{\mu-1/2}(Fr)\right]\cdot$$

$$\cdot\left[(\nu+1)P_{\nu+1}(\cos\sigma)J_{\nu+3/2}(Fr\epsilon)-\nu P_{\nu-1}(\cos\sigma)J_{\nu-1/2}(Fr\epsilon)\right],$$

$$I_\eta^3 = \frac{-2\pi i^{\mu+\nu+1}}{(2\mu+1)(2\nu+1)Fr\sqrt{\epsilon}}\left[P^1_{\mu-1}(\cos\psi)J_{\mu-1/2}(Fr)+P^1_{\mu+1}(\cos\psi)J_{\mu+3/2}(Fr)\right]\cdot$$

$$\cdot\left[(\nu+1)P_{\nu+1}(\cos\sigma)J_{\nu+3/2}(Fr\epsilon)-\nu P_{\nu-1}(\cos\sigma)J_{\nu-1/2}(Fr\epsilon)\right]$$

and

$$I_\eta^4 = \frac{2\pi i^{\mu+\nu+1}}{(2\mu+1)(2\nu+1)Fr\sqrt{\epsilon}}\left[(\mu+1)P_{\mu+1}(\cos\psi)J_{\mu+3/2}(Fr)-\mu P_{\mu-1}(\cos\psi)J_{\mu-1/2}(Fr)\right]\cdot$$

$$\cdot\left[P^1_{\nu-1}(\cos\sigma)J_{\nu-1/2}(Fr\epsilon)+P^1_{\nu+1}(\cos\sigma)J_{\nu+3/2}(Fr\epsilon)\right].$$

Putting these expressions into (10) and observing that the dependence of the integrand on ω has disappeared, we have

$$I_{\mu\nu} = \frac{-16\pi^4 i^{\mu+\nu}\xi}{(2\mu+1)(2\nu+1)F}\int_0^\pi\int_0^\infty \frac{r\sin\psi\,dr\,d\psi}{\sqrt{\epsilon}\left[\xi^2-\cos^2\psi\frac{k^2\xi^2\alpha^2}{r^2}\right]}\left\{\frac{\xi\xi'}{\alpha^3\alpha'}\sin^2\psi\left[P^1_{\mu-1}(\cos\psi)J_{\mu-1/2}(Fr)\right.\right.$$

$$\left.+P^1_{\mu+1}(\cos\psi)J_{\mu+3/2}(Fr)\right]\cdot\left[P^1_{\nu-1}(\cos\sigma)J_{\nu-1/2}(Fr\epsilon)+P^1_{\nu+1}(\cos\sigma)J_{\nu+3/2}(Fr\epsilon)\right]$$

$$-\frac{\cos^2\psi}{\xi^2}\left[(\mu+1)P_{\mu+1}(\cos\psi)J_{\mu+3/2}(Fr)-\mu P_{\mu-1}(\cos\psi)J_{\mu-1/2}(Fr)\right]\cdot$$

$$\cdot\left[(\nu+1)P_{\nu+1}(\cos\sigma)J_{\nu+3/2}(Fr\epsilon)-\nu P_{\nu-1}(\cos\sigma)J_{\nu-1/2}(Fr\epsilon)\right] \qquad (13)$$

(equation cont'd on next page)

$$+ \frac{\sin\psi\cos\psi}{\alpha^2}\left[P^1_{\mu-1}(\cos\psi)J_{\mu-1/2}(Fr) + P^1_{\mu+1}(\cos\psi)J_{\mu+3/2}(Fr)\right]\cdot$$

$$\cdot\left[(\nu+1)P_{\nu+1}(\cos\sigma)J_{\nu+3/2}(Fr\epsilon) - \nu P_{\nu-1}(\cos\sigma)J_{\nu-1/2}(Fr\epsilon)\right]$$

$$- \frac{\sin\psi\cos\psi\xi'}{\xi\alpha\alpha'}\left[(\mu+1)P_{\mu+1}(\cos\psi)J_{\mu+3/2}(Fr) - \mu P_{\mu-1}(\cos\psi)J_{\mu-1/2}(Fr)\right]\cdot$$

$$\cdot\left[P^1_{\nu-1}(\cos\sigma)J_{\nu-1/2}(Fr\epsilon) + P^1_{\nu+1}(\cos\sigma)J_{\nu+3/2}(Fr\epsilon)\right]\Bigg\} .$$

This can be rewritten as follows, considering that the r integration is to be performed first, and observing that the distinction between ξ' and ξ is essential only in the r integrals and that $\epsilon \to 1$, $\cos\sigma \to -\cos\psi$ as $\xi' \to \xi$,

$$\lim_{\xi'\to\xi} I_{\mu\nu} = \frac{-16\pi^4 i^{\mu+3\nu}\xi}{(2\mu+1)(2\nu+1)F}\int_0^{\pi}\frac{\sin\psi\, d\psi}{\xi^2-\cos^2\psi}\Bigg\{\frac{\xi^2\sin^2\psi}{\alpha^4}\Big[P^1_{\mu-1}P^1_{\nu-1}R_{11} + P^1_{\mu+1}P^1_{\nu-1}R_{21}$$

$$+ P^1_{\mu-1}P^1_{\nu+1}R_{12} + P^1_{\mu+1}P^1_{\nu+1}R_{22}\Big] + \frac{\cos^2\psi}{\xi^2}\Big[(\mu+1)(\nu+1)P_{\mu+1}P_{\nu+1}R_{22} - (\mu+1)\nu P_{\mu+1}P_{\nu-1}R_{21}$$

$$-\mu(\nu+1)P_{\mu-1}P_{\nu+1}R_{12} + \mu\nu P_{\mu-1}P_{\nu-1}R_{11}\Big] - \frac{\sin\psi\cos\psi}{\alpha^2}\Big[(\nu+1)P^1_{\mu-1}P_{\nu+1}R_{12}$$

$$-\nu P^1_{\mu-1}P_{\nu-1}R_{11} + (\nu+1)P^1_{\mu+1}P_{\nu+1}R_{22} - \nu P^1_{\mu+1}P_{\nu-1}R_{21}\Big] \qquad (14)$$

$$-\frac{\sin\psi\cos\psi}{\alpha^2}\Big[(\mu+1)P_{\mu+1}P^1_{\nu-1}R_{21} + (\mu+1)P_{\mu+1}P^1_{\nu+1}R_{22}$$

$$-\mu P_{\mu-1}P^1_{\nu-1}R_{11} - \mu P_{\mu-1}P^1_{\nu+1}R_{12}\Big]\Bigg\}$$

where

$$R_{11} = \lim_{\xi' \to \xi} \int_0^\infty J_{\mu-1/2}(Fr)\, J_{\nu-1/2}(Fr\epsilon)\, \frac{r^3\, dr}{r^2 - c^2}$$

$$R_{12} = \lim_{\xi' \to \xi} \int_0^\infty J_{\mu-1/2}(Fr)\, J_{\nu+3/2}(Fr\epsilon)\, \frac{r^3\, dr}{r^2 - c^2} \qquad (15)$$

$$R_{21} = \lim_{\xi' \to \xi} \int_0^\infty J_{\mu+3/2}(Fr)\, J_{\nu-1/2}(Fr\epsilon)\, \frac{r^3\, dr}{r^2 - c^2}$$

$$R_{22} = \lim_{\xi' \to \xi} \int_0^\infty J_{\mu+3/2}(Fr)\, J_{\nu+3/2}(Fr\epsilon)\, \frac{r^3\, dr}{r^2 - c^2}$$

and $c^2 = \dfrac{k^2 \xi^2 \alpha^2}{\xi^2 - \cos^2 \psi}$

and the argument of all the Legendre functions appearing is $\cos \psi$.

Now it may be observed that since R_{ij}, considered as a function of ψ, is even about $\psi = \pi/2$, and since $P_m^n(\cos \psi)$ is even or odd according as m-n is even or odd, the integrand in (14) is even or odd about $\psi = \pi/2$ according as $\mu + \nu$ is even or odd. Thus $I_{\mu\nu}$ vanishes if $\mu + \nu$ is odd. Furthermore since $J_n(-z) = (-1)^n J_n(z)$, it develops that the integrands in the expressions for the R_{ij} are even or odd about $r = 0$ according as $\mu + \nu$ is even or odd, and consequently only the even case need be considered. Thus we can set

$$\int_0^\infty J_{m+1/2}(Fr)\, J_{n+1/2}(Fr\epsilon)\, \frac{r^3\, dr}{r^2 - c^2} = \frac{1}{2} \int_{-\infty}^{\infty} - - - - - \, dr$$

The representation (8) used here for the Green's function is not completely defined until one specifies the manner in which the singularity at $\vec{K} = (k, 0, 0)$ is to be avoided. In the present case the proper procedure is to divert the integration path in the complex K_x-plane slightly below the point k, inasmuch as the scattered wave is to be outgoing.[8] This corresponds to a diversion below the point r = c (and above the point r = - c) in the complex r-plane.

We proceed then to evaluate the integral

$$\int_C J_{m+1/2}(Fr)\, J_{n+1/2}(Fr\epsilon)\frac{r^3\, dr}{r^2-c^2} = \frac{1}{F^2}\int_C J_{m+1/2}(z) J_{n+1/2}(\epsilon z)\, \frac{z^3\, dz}{z^2-b^2} \qquad (16)$$

where z = Fr, b = Fc, and the path C consists of the real axis from $-\infty$ to $+\infty$ but with diversions above the point z = - b and below the point z = b. This integral apparently encompasses the divergence which first appeared in the expression obtained previously for the normal derivative of the Green's function, since the integrand oscillates indefinitely without decaying as $z \to \pm\infty$ on the real axis. However, we can obtain a formal value, at least, as follows:

First let

$$\phi_n(\epsilon z) = \int^{\epsilon z} J_{n+1/2}(\zeta)\, d\zeta \qquad (17)$$

so that $\frac{\partial}{\partial \epsilon}\phi_n(\epsilon z) = z\, J_{n+1/2}(\epsilon z)$ and

[8] Morse and Feshbach, loc. cit. p. 818.

$$\int J_{m+1/2}(z) J_{n+1/2}(\epsilon z) \frac{z^3\, dz}{z^2-b^2} = \int J_{m+1/2}(z) \frac{\partial}{\partial \epsilon} \phi_n(\epsilon z) \frac{z^2\, dz}{z^2-b^2} \tag{18}$$

$$= \frac{\partial}{\partial \epsilon} \int J_{m+1/2}(z) \phi_n(\epsilon z) \frac{z^2\, dz}{z^2-b^2}$$

It is clear from the original integral equation (1) that ξ' must be less than or equal to ξ and consequently from the definition of ϵ (12) it follows that $\epsilon < 1$. Now we assume for the time being that $m < n + 3$ and write

$$J_{m+1/2}(z) = \frac{1}{2} \left[H^{(1)}_{m+1/2}(z) + H^{(2)}_{m+1/2}(z) \right] .$$

It is well known that the Hankel function $H^{(1)}_{\nu}(z)$ vanishes exponentially for large positive imaginary z, and that $H^{(2)}_{\nu}(z)$ behaves similarly for large negative imaginary z, and accordingly we should try to express the integral (18) in the following form:

$$\int_C J_{m+1/2}(z) \phi_n(\epsilon z) \frac{z^2\, dz}{z^2-b^2} = \frac{1}{2} \left[\int_{C_1} H^{(1)}_{m+1/2}(z) \phi_n(\epsilon z) \frac{z^2\, dz}{z^2-b^2} \right. \tag{19}$$

$$\left. + \int_{C_2} H^{(2)}_{m+1/2}(z) \phi_n(\epsilon z) \frac{z^2 dz}{z^2-b^2} \right]$$

where C_1 is C closed by an infinite semicircle in the upper half of the z-plane and C_2 is C closed by one in the lower half. It must first be shown that the integrals over the semicircles vanish. For this we can use Jordan's Lemma, which states that the integral of the quantity $e^{imz} f(z)$ over an infinite semicircle in the upper half plane will vanish provided that $m > 0$ and $f(Re^{i\theta}) \rightarrow 0$ uniformly in θ as $R \rightarrow \infty$, with obvious modifications to handle the lower half plane.

Consider first the behavior of $\phi_n(\epsilon z)$ for large z. Using the definition (17) in the range where ζ is large, we can employ the Hankel asymptotic form for $J_{n+1/2}(\zeta)$ to obtain

$$\phi_n(\epsilon z) \simeq \frac{1}{\sqrt{2\pi}}\left[i^{-n-1}\int^{\epsilon z} \frac{e^{i\zeta}}{\sqrt{\zeta}}\, d\zeta + i^{n+1}\int^{\epsilon z} \frac{e^{-i\zeta}}{\sqrt{\zeta}}\, d\zeta\right].$$

Setting

$$\int^{\epsilon z} \frac{e^{i\zeta}}{\sqrt{\zeta}}\, d\zeta = \int^{\infty} \frac{e^{i\zeta}}{\sqrt{\zeta}}\, d\zeta - \int_{\epsilon z}^{\infty} \frac{e^{i\zeta}}{\sqrt{\zeta}}\, d\zeta = \Lambda_1 - \int \frac{e^{i\zeta}}{\sqrt{\zeta}}\, d\zeta$$

where Λ_1 = constant, independent of z, and putting $\zeta = z + \tau$ we have

$$\int_{\epsilon z}^{\infty} \frac{e^{i\zeta}}{\sqrt{\zeta}}\, d\zeta = \frac{e^{i\epsilon z}}{\sqrt{\epsilon z}} \int_0^{\infty} \frac{e^{i\tau}}{\sqrt{1+\frac{\tau}{\epsilon z}}}\, d\tau .$$

Using operatorial symbols, we can write

$$\int f(\tau)\, d \equiv D^{-1}\left[f(\tau)\right] \quad \text{and} \quad D^{-1}\left[e^{i\tau} g(\tau)\right] = e^{i\tau}(D+i)^{-1}\left[g(\tau)\right]$$

so that

$$\int_0^{\infty} \frac{e^{i\tau}\, d\tau}{\sqrt{1+\frac{\tau}{\epsilon z}}} = \left[e^{i\tau}(D+i)^{-1}\left[\frac{1}{\sqrt{1+\frac{\tau}{\epsilon z}}}\right]\right]_0^{\infty} .$$

But

$$(D+i)^{-1} = -i\left[1 - \frac{D}{i} + \frac{D^2}{i^2} - \frac{D^3}{i^3} + \cdots\cdots\right]$$

$$= -i\left[1 + iD + i^2D^2 + i^3D^3 + \cdots\right]$$

so that

$$\int_0^\infty \frac{e^{i\tau}d\tau}{\sqrt{1+\frac{\tau}{\epsilon z}}} = -i\left[e^{i\tau}\,(1+iD+i^2D^2+ \cdots)\left[\frac{1}{\sqrt{1+\frac{\tau}{\epsilon z}}}\right]\right]_0^\infty$$

$$= -i\left[\frac{e^{i\tau}}{\sqrt{1+\frac{\tau}{\epsilon z}}}\sum_{n=0}^{\infty}\frac{i^n(2n)!}{2^{2n}n!(\epsilon z+\tau)^n}\right]_0^\infty$$

$$= \lim_{\xi\to\infty} -i\left[\frac{e^{i\xi}}{\sqrt{1+\frac{\xi}{\epsilon z}}}\sum_{n=0}^{\infty}\frac{i^n(2n)!}{2^{2n}n!(\epsilon z+\xi)^n} - \sum_{n=0}^{\infty}\frac{i^n(2n)!}{2^{2n}n!(\epsilon z)^n}\right]$$

$$\simeq i \text{ for } |z| \text{ large and } 0 < \arg z \leq \pi .$$

Similarly

$$\int^{\epsilon z}\frac{e^{-i\xi}}{\sqrt{\xi}}\,d\xi = i\left[e^{-i\xi}\,(D-i)^{-1}\left[\frac{1}{\sqrt{\xi}}\right]\right]^{\epsilon z} - \Lambda_2$$

where Λ_2 = constant, or

$$\int^{\epsilon z}\frac{e^{-i\xi}}{\sqrt{\xi}}\,d\xi = i\left[\frac{e^{-i\xi}}{\sqrt{\xi}}\sum_{n=0}^{\infty}\frac{i^n(2n)!}{2^{2n}n!\,\xi^n}\right]^{\epsilon z} - \Lambda_2$$

$$\approx \frac{ie^{-i\epsilon z}}{\sqrt{\epsilon z}} - \Lambda_2 \quad \text{for } |z| \text{ large.}$$

Then

$$\phi_n(\epsilon z) \approx \frac{1}{\sqrt{2\pi}}\left[i^{-n-1}\left(-\frac{ie^{i\epsilon z}}{\sqrt{\epsilon z}}+\Lambda_1\right)+i^{n+1}\left(\frac{ie^{-i\epsilon z}}{\sqrt{\epsilon z}}-\Lambda_2\right)\right]$$

$$\approx \frac{-i^n}{\sqrt{2\pi\epsilon z}}\left[(-1)^n e^{i\epsilon z} + e^{-i\epsilon z}\right] + \Lambda_3$$

where Δ_3 incorporates Δ_1 and Δ_2, and consequently

$$H^{(1)}_{m+1/2}(z)\phi_n(\epsilon z)\,\frac{z^2}{z^2-b^2} \approx \sqrt{\frac{2}{\pi z}}\; i^{-n-1}e^{iz}\left[\frac{-i^n}{\sqrt{2\pi\epsilon z}}\left((-1)^n e^{i\epsilon z} + e^{-i\epsilon z}\right) + \Delta_3\right]\left[1+\frac{b^2}{z^2}+ \text{---}\right]$$

$$\approx \frac{i}{\pi z\sqrt{\epsilon}}\left[(-1)^n e^{iz(1+\epsilon)} + e^{iz(1-\epsilon)}\right] + \sqrt{\frac{2}{\pi}}\; i^{-n-1}\,\frac{e^{iz}}{\sqrt{z}}\cdot\Delta_3$$

$$\approx e^{iz(1+\epsilon)}\,\frac{i(-1)^n}{\pi z\sqrt{\epsilon}} + e^{iz(1-\epsilon)}\,\frac{i}{\pi z\sqrt{\epsilon}} + e^{iz}\sqrt{\frac{2}{\pi z}}\; i^{n-1}\,\Delta_3\,. \qquad (20)$$

Each term in (20) satisfies the hypotheses of Jordan's Lemma (since $\epsilon < 1$) and therefore the integral over the semicircle is zero. The same technique can be applied to the lower semicircle, and the original integral (16) can thus be expressed in terms of the residues at the poles included in the two contours.

Since the functions $H^{(1,2)}_{m+1/2}(z)\,J_{n+1/2}(\epsilon z)$ have poles of order $m - n$ at $z = 0$, it is clear that as long as $m < n + 3$, the integrands $H^{(1,2)}_{m+1/2}(z)\,J_{n+1/2}(\epsilon z)\,\frac{z^3}{z^2-b^2}$ will be regular at $z = 0$, and the only singularities appearing will be simple poles at $z = \pm\, b$, one of which is included in the upper contour and the other in the lower. Thus we have

$$\int_C J_{m+1/2}(z)\phi_n(\epsilon z)\frac{z^2dz}{z^2-b^2} = \frac{i\pi}{2}\left[H^{(1)}_{m+1/2}(b)\phi_n(\epsilon b)\cdot b + H^{(2)}_{m+1/2}(-b)\phi_n(-\epsilon b)\cdot b\right]$$

and from (18)

$$\int_C J_{m+1/2}(z)J_{n+1/2}(\epsilon z)\frac{z^3dz}{z^2-b^2} = \frac{i\pi b^2}{2}\left[H^{(1)}_{m+1/2}(b)J_{n+1/2}(\epsilon b) - H^{(2)}_{m+1/2}(-b)J_{n+1/2}(-\epsilon b)\right].$$

But $H^{(2)}_{m+1/2}(-b)\, J_{n+1/2}(-\epsilon b) = -H^{(1)}_{m+1/2}(b)\, J_{n+1/2}(\epsilon b)$ for m+n even, so that

$$\int_C J_{m+1/2}(z) J_{n+1/2}(\epsilon z) \frac{z^3 dz}{z^2-b^2} = i\pi b^2\, H^{(1)}_{m+1/2}(b)\, J_{n+1/2}(\epsilon b). \qquad (21)$$

As for the question of what happens when $m > n+3$, examination of equation (14) shows that m takes only the values $\mu - 1$ and $\mu + 1$, and n only the values $\nu - 1$, $\nu + 1$, so that if $\mu \leq \nu + 2$, this situation will not occur. Furthermore it is apparent from the form of the quantity $C_{\mu\nu}$ given in equation (9) that in the limit as $\xi' \to \xi$ this must be symmetrical in μ and ν, so that the situation for $\mu > \nu + 2$ need not concern us. The case $\mu > \nu$ could be dealt with in similar fashion if desired, but the development would be slightly more complicated.

When the expression (21) is substituted in the formulas (15) and the limit process carried out, there result the expressions

$$R_{11} = \frac{\pi i c^2}{2} H^{(1)}_{\mu-1/2}(Fc)\, J_{\nu-1/2}(Fc)$$

$$R_{12} = \frac{\pi i c^2}{2} H^{(1)}_{\mu-1/2}(Fc)\, J_{\nu+3/2}(Fc)$$

$$R_{21} = \frac{\pi i c^2}{2} H^{(1)}_{\mu+3/2}(Fc)\, J_{\nu-1/2}(Fc)$$

$$R_{22} = \frac{\pi i c^2}{2} H^{(1)}_{\mu+3/2}(Fc)\, J_{\nu+3/2}(Fc)$$

and when these are substituted into (14), it develops that the bulk of the integrand can be separated into two factors, one of which depends only on μ, ξ, k, ψ and the other only on ν, ξ, k, ψ. Thus

$$\lim_{\xi' \to \xi} I_{\mu\nu} = \frac{-8\pi^5 i^{\mu+3\nu+1} k^2 \xi}{F\alpha^2(2\mu+1)(2\nu+1)} \int_0^{\pi} \frac{\sin\psi \, d\psi}{(\xi^2-\cos^2\psi)^2} \Gamma_\mu(\xi,k,\psi)\Delta_\mu(\xi,k,\psi) \tag{22}$$

where

$$\Gamma_n(\xi,k,\psi) \equiv nH^{(1)}_{n-1/2}\left(\frac{kF\xi\alpha}{\sqrt{\xi^2-\cos^2\psi}}\right)\left[\cos\psi\, P_{n-1}(\cos\psi) - \xi^2 P_n(\cos\psi)\right]$$

$$- (n+1)H^{(1)}_{n+3/2}\left(\frac{kF\xi\alpha}{\sqrt{\xi^2-\cos^2\psi}}\right)\left[\cos\psi\, P_{n+1}(\cos\psi) - \xi^2 P_n(\cos\psi)\right] \tag{23}$$

$$\Delta_n(\xi,k,\psi) \equiv nJ_{n-1/2}\left(\frac{kF\xi\alpha}{\sqrt{\xi^2-\cos^2\psi}}\right)\left[\cos\psi\, P_{n-1}(\cos\psi) - \xi^2 P_n(\cos\psi)\right]$$

$$-(n+1)J_{n+3/2}\left(\frac{kF\xi\alpha}{\sqrt{\xi^2-\cos^2\psi}}\right)\left[\cos\psi\, P_{n+1}(\cos\psi) - \xi^2 P_n(\cos\psi)\right]$$

for n = μ, ν. These expressions can be put into slightly less cumbersome form by introducing the spherical Bessel functions of Sommerfeld and using standard recurrence relations for these and the Legendre polynomials. Letting

$$\rho \equiv \frac{kF\xi\alpha}{\sqrt{\xi^2-\cos^2\psi}}$$

$$J_{n+1/2}(\rho) = \sqrt{\frac{2\rho}{\pi}}\, \psi_n(\rho), \quad H^{(1)}_{n+1/2}(\rho) = \sqrt{\frac{2\rho}{\pi}}\, \zeta_n(\rho)$$

we can write

$$\Gamma_n(\xi,k,\psi) = (2n+1)\sqrt{\frac{2\rho}{\pi}}\left[\frac{n}{\rho}(\cos\psi\, P_{n-1} - \xi^2 P_n)\zeta_n(\rho) + (\xi^2-\cos^2\psi)P_n \zeta_{n+1}(\rho)\right]$$

$$\equiv (2n+1)\sqrt{\frac{2\rho}{\pi}}\, \Gamma_n'(\xi,k,\psi)$$

with a corresponding form for $\Delta_n(\xi,k,\psi)$, and $C_{\mu\nu}$ becomes

$$C_{\mu\nu} = -8\pi^2 F^4 \xi^2 \alpha^3 k^3 \, i^{\mu+3\nu+1} \int_0^{\pi} \frac{\sin\psi \, d\psi}{(\xi^2-\cos^2\psi)^{5/2}} \Gamma'_{\mu}(\xi, k, \psi) \Delta'_{\nu}(\xi, k, \psi) \, . \tag{24}$$

It is of course possible by expanding the products $\Gamma_\mu \Delta_\nu$ and applying recurrence relations for the Legendre polynomials to write $C_{\mu\nu}$ as a linear combination of integrals of the form

$$\int_0^{\pi} P_m(\cos\psi) P_n(\cos\psi) \left\{ \begin{matrix} J_{r+1/2}(\rho) \\ N_{r+1/2}(\rho) \end{matrix} \right\} J_{s+1/2}(\rho) \frac{\sin\psi \, d\psi}{(\xi^2-\cos^2\psi)^2}$$

and since both the Legendre and Bessel functions involved are expressible in closed form, the integrands here can be decomposed into expressions involving only elementary functions. Recurrence relations among the various terms can then be obtained, and the only integrations remaining are a few initial values of fairly simple form. However the number of terms involved for any moderate values of μ and ν is so large that it was judged more economical for actual computation to use the form (24) and perform the integration by numerical methods.

Alternatively, since the integrand in (24) can be expressed entirely in terms of Bessel and Legendre functions of order μ and ν and derivatives of these with respect to $\cos\psi$, some thought has been given to the possibility of performing partial integrations in order at least to reduce the complexity of the remaining integral. It is not immediately clear that anything can be gained by this approach, however, and to date no thorough investigation has been made.

CHAPTER 4

EVALUATION OF THE INTEGRALS B_μ

Referring to equations (6) and (7), we have

$$B_\mu \equiv \int_S P_\mu(\eta) \frac{\partial}{\partial n} e^{ikz} \Big|_S da$$

$$= \int_0^{2\pi} \int_{-1}^{+1} P_\mu(\eta) \frac{\alpha}{F\gamma} \frac{\partial}{\partial \xi} - e^{ikF\xi\eta} \Big|_S F^2 \alpha \gamma \, d\eta \, d\phi$$

$$= 2\pi i \alpha^2 F^2 k \int_{-1}^{+1} P_\mu(\eta) \, e^{ikF\xi\eta} \eta \, d\eta$$

or setting $kF\xi = ka$, $\eta = \cos\theta$

$$B_\mu = 2\pi i \alpha^2 F^2 k \int_0^\pi P_\mu(\cos\theta) e^{ika\cos\theta} \sin\theta \cos\theta \, d\theta.$$

Using a recurrence relation to eliminate $\cos\theta$, we get

$$B_\mu = \frac{2\pi i \alpha^2 F^2 k}{(2\mu+1)} \Big\{ (\mu+1) \int_0^\pi P_{\mu+1}(\cos\theta) \, e^{ika\cos\theta} \sin\theta \, d\theta$$

$$+ \mu \int_0^\pi P_{\mu-1}(\cos\theta) \, e^{ika\cos\theta} \sin\theta \, d\theta. \Big\}$$

The integrals here are of the form given in equation (11), with $m = 0$, $\eta = \cos\theta$, $z = ka$, and $\psi = 0$. Thus

$$B_\mu = \frac{2\pi i \alpha^2 F^2 k}{(2\mu+1)} \left\{ (\mu+1) i^{\mu+1} \sqrt{\frac{2\pi}{ka}}\, J_{\mu+3/2}(ka) + \mu i^{-1} \sqrt{\frac{2\pi}{ka}}\, J_{\mu-1/2}(ka) \right\}$$

$$= \frac{(2\pi F)^{3/2} i^\mu \alpha^2 \sqrt{k}}{(2\mu+1)\sqrt{\xi}} \left[\mu J_{\mu-1/2}(ka) - (\mu+1)\, J_{\mu+3/2}(ka) \right] .$$

In terms of the spherical Bessel functions used in the preceding section, this becomes

$$B_\mu = 4\pi F^2 \alpha^2 i^\mu k \frac{d}{d(ka)} \psi_\mu(ka). \qquad (25)$$

CHAPTER 5

DETERMINATION OF THE SCATTERING CROSS SECTION

Once values have been obtained for the quantities $C_{\mu\nu}$ and B_μ over a sufficient range of the indices, the problem of finding the back-scattering cross section is relatively trivial. The stationary value of J [ϕ] is found by setting the derivative of J with respect to each A_μ equal to zero. This operation yields the system of equations[9]

$$\sum_{\mu=0}^{\infty} A_\mu \left[\frac{C_{\mu\nu}}{B_\nu} - J_o B_\mu \right] = 0 \qquad \text{for all } \nu \;, \tag{26}$$

where J_o is the stationary value of J [ϕ]. Existence of a solution of this set of homogeneous linear equations in the unknowns A_μ requires that the determinant of the coefficients vanish, i.e.,

$$\left| \frac{C_{\mu\nu}}{B_\nu} - J_o B_\mu \right| = 0 \qquad \text{or} \qquad \left| \frac{C_{\mu\nu}}{B_\mu B_\nu} - J_o \right| = 0 \tag{27}$$

These are linear equations in J_o, the solution of which can be written in the form

$$J_o = \frac{\left| a_{\mu\nu} \right|}{\sum_\mu \sum_\nu A_{\mu\nu}} \tag{28}$$

where

$$a_{\mu\nu} \equiv \frac{C_{\mu\nu}}{B_\mu B_\nu}$$

$$A_{\mu\nu} \equiv \text{cofactor of } a_{\mu\nu} \;.$$

[9]Cf. Sollfrey, loc. cit. p. 15.

Or we can write

$$\frac{1}{J_o} = \frac{\left| a_{\mu\nu} + C \right|}{C \left| a_{\mu\nu} \right|} - \frac{1}{C} \tag{29}$$

where C is any constant, which is perhaps a more convenient form. Furthermore the fact that $a_{\mu\nu} = 0$ for $\mu + \nu$ odd means that the coefficients of even and odd index are completely independent, so that if we let $a_{\mu\nu} \equiv \alpha_{\mu\nu}$ for the odd case and $a_{\mu\nu} \equiv \beta_{\mu\nu}$ for the even, we have

$$\frac{1}{J_o} = \frac{\left| \alpha_{\mu\nu} + C_1 \right|}{C_1 \left| \alpha_{\mu\nu} \right|} + \frac{\left| \beta_{\mu\nu} + C_2 \right|}{C_2 \left| \beta_{\mu\nu} \right|} - \left(\frac{1}{C_1} + \frac{1}{C_2} \right) \tag{30}$$

where C_1 and C_2 are arbitrary constants. Once a value is obtained for J_o, the system (26) can be solved if desired for the coefficients A_μ. The back-scattering cross section, however, is given directly by J_o, which is inversely proportional to the back-scattered amplitude, as remarked in Chapter 2. Specifically, if σ is the total scalar back-scattering cross section, then

$$\sigma = 4\pi \left| \frac{1}{J_o} \right|^2 . \tag{31}$$

The above solution may also be obtained without resorting to variational language. The method is due to Galerkin and has been shown by Jones[10] to be exactly equivalent to the variational approach. If the expansion (5) is substituted into the integrand of equation (2), and both sides are then multiplied by $P_\nu(\eta)$ and integrated over the spheroid

[10] Jones, D. S., "A Critique of the Variational Method in Scattering Problems," IRE Trans. Vol. AP-4, No. 3 (July 1956).

(in the manner of the above development) there results the system

$$\sum_{\mu=0}^{\infty} A_\mu C_{\mu\nu} = 4\pi B_\nu , \qquad \nu = 0, 1, 2 \text{ ---- } \infty , \tag{32}$$

The solution $\{A_\mu\}$ of this system is equivalent (within a normalization constant) to that of the system (26). The value of J_o is immediately obtainable from this system by application of the stationary condition

$$J_o B_\nu \sum_\mu A_\mu B_\mu = \sum_\mu A_\mu C_{\mu\nu} \qquad \text{for all } \nu$$

derived from equation (7). Thus

$$J_o = \frac{4\pi}{\sum_\mu A_\mu B_\mu}$$

and

$$\sigma = \frac{1}{4\pi} \left| \sum_\mu A_\mu B_\mu \right|^2 . \tag{33}$$

This expression is probably more convenient for analytical and computation purposes than those derived above, and is used in the developments which follow.

CHAPTER 6

VERIFICATION OF RESULTS

The lack of mathematical rigor in some of the preceding analysis makes it imperative that some sort of check be obtained on the validity of the results. The most obvious means to this end is to examine the behavior of the solution in the extremes of wavelength and eccentricity, where the correct solutions are well known.

Considering first the eccentricity, we can see at once that as this becomes infinite the nose-on scattering cross section should vanish for any finite ka , and little information is to be gained on the forms in question. We examine rather the case of vanishing eccentricity, i.e., where the spheroid becomes a sphere. This transformation is accomplished by letting $\xi \to \infty$ and $F \to 0$ in such a way that the product $\xi F \to a$, the radius of the sphere. Geometrically this implies that the major axis of the spheroid remains fixed and the minor axis is increased until the two are equal. From equation (23) and following it is apparent at once that $\rho \to ka$ and that the terms $\cos\psi\, P_{n+1}$ become negligible in comparison to the terms $\xi^2 P_n$. Thus, since $\cos^2\psi$ becomes negligible compared to ξ^2, the integral in (22) reduces to

$$\int_0^{\pi} P_\mu(\cos\psi)\, P_\nu(\cos\psi)\, \sin\psi\, d\psi \cdot \left[\mu H^{(1)}_{\mu-1/2}(ka) - (\mu+1) H^{(1)}_{\mu+3/2}(ka)\right] \cdot \left[\nu J_{\nu-1/2}(ka) - (\nu+1) J_{\nu+3/2}(ka)\right]$$

(equation con'd on next page)

$$= \delta_{\mu\nu} \frac{2\cdot(2\mu+1)(2\nu+1)}{(2\mu+1)\,\pi}\, 2ka \frac{d}{d(ka)} \zeta_\mu(ka) \frac{d}{d(ka)} \psi_\nu(ka) \qquad (34)$$

$$= \begin{cases} 0 & \text{for } \mu \neq \nu \\ \dfrac{4ka}{\pi}(2\mu+1) \dfrac{d}{d(ka)} \zeta_\mu(ka) \dfrac{d}{d(ka)} \psi_\mu(ka) & \text{for } \mu = \nu\,. \end{cases}$$

Equation (24) then yields

$$C = \begin{cases} 0 & \text{for } \mu \neq \nu \\ \dfrac{-16\pi^2 i k^3 a^4}{(2\mu+1)} \dfrac{d}{d(ka)} \zeta_\mu(ka) \dfrac{d}{d(ka)} \psi_\mu(ka) & \text{for } \mu = \nu \end{cases} \qquad (35)$$

and equation (25) immediately becomes

$$B_\mu = 4\pi a^2 i^\mu k \frac{d}{d(ka)} \psi_\mu(ka). \qquad (36)$$

The velocity potential at a large distance R from the scatterer in the direction of the approaching plane wave can be written in the form

$$\phi(R) = e^{ikz} + f(\pi) \frac{e^{ikR}}{R}$$

and as stated earlier, $f(\pi)$ is equal to the negative reciprocal of J_o, the stationary value of J. Referring to equation (32) we have for the sphere

$$A_\nu\, C_{\nu\nu} = 4\pi B_\nu \qquad \text{or} \qquad A_\mu = \frac{4\pi B_\mu}{C_{\mu\mu}}\,,$$

and

$$f(\pi) = -\frac{1}{4\pi}\sum_{\mu} A_\mu B_\mu = -\sum_{\mu}\frac{B_\mu^{\,2}}{C_{\mu\mu}} \quad . \tag{37}$$

Substitution of (35) and (36) here yields

$$f(\pi) = \frac{i}{k}\sum_{\mu}(-1)^\mu (2\mu+1) \frac{d}{d(ka)}\psi_\mu(ka) \Big/ \frac{d}{d(ka)}\zeta_\mu(ka). \tag{38}$$

The classical solution for the sphere is given by Sommerfeld[11] in the form

$$V = -\sum_{\mu} i^\mu (2\mu+1)P_\mu(\cos\theta)\,\zeta_\mu(kr)\,\frac{d}{d(ka)}\psi_\mu(ka) \Big/ \frac{d}{d(ka)}\zeta_\mu(ka)$$

where V is the velocity potential of the scattered wave at the point r, θ. If this is restricted to give the back-scattered field at a large distance R, it becomes

$$V = \frac{ie^{ikR}}{kR}\sum_{\mu}(-1)^\mu(2\mu+1)\,\frac{d}{d(ka)}\psi_\mu(ka) \Big/ \frac{d}{d(ka)}\zeta_\mu(ka),$$

and multiplication of this by $R\,e^{-ikR}$ to obtain $f(\pi)$ renders it identical to equation (38). The variational solution is thus shown to be correct in the limit of vanishing eccentricity.

We consider next the extremes of wavelength as compared to the dimensions of the scatterer. The relation with the sphere solution exhibited above makes it quite apparent that the present solution should be most practical in the region of large wavelength, and we should

[11] Sommerfeld, A., "Partial Differential Equations in Physics," Academic Press, p. 164 (1949).

certainly be able to compare it with the result obtained by Rayleigh for this region. To this end we write the back-scattered amplitude as in equation (37)

$$f(\pi) = -\frac{1}{4\pi}\sum_{\mu} A_{\mu}\ B_{\mu}$$

and expand the quantities A_{μ} and B_{μ} in powers of k. The terms of order less than or equal to k^2 should then give the Rayleigh result. Setting

$$B_{\mu} = \sum_{j} b_j^{\mu}\, k^j$$
$$A_{\mu} = \sum_{j} a_j^{\mu}\, k^j \ , \tag{39}$$

we obtain immediately, to order k^2,

$$f(\pi) \approx -\frac{1}{4\pi}\Big\{ a_o^o b_o^o + a_o^1 b_o^1 + a_o^2 b_o^2 + k\Big[a_o^o b_1^o + a_1^o b_o^o + a_o^1 b_1^1 + a_1^1 b_o^1 + a_o^2 b_1^2 + a_1^2 b_o^2\Big] + k^2\Big[a_o^o b_2^o + a_1^o b_1^o + a_2^o b_o^o + a_o^1 b_2^1 + a_1^1 b_1^1 + a_2^1 b_o^1 + a_o^2 b_2^2 + a_1^2 b_1^2 + a_2^2 b_o^2\Big]\Big\} . \tag{40}$$

The quantities b_j^i can be obtained easily from (25) by substituting the power series expansion for the Bessel functions. We have, to order k^2,

$$B_o \approx - (2\pi F)^{3/2} \alpha^2 \sqrt{\frac{k}{\xi}} \left(\frac{kF\xi}{2}\right)^{3/2} \frac{1}{\Gamma(5/2)}$$

$$B_1 \approx 1/3 (2\pi F)^{3/2} i \alpha^2 \sqrt{\frac{k}{\xi}} \left(\frac{kF\xi}{2}\right)^{1/2} \frac{1}{\Gamma(3/2)}$$

$$B_2 \approx - 1/5 (2\pi F)^{3/2} \alpha^2 \sqrt{\frac{k}{\xi}} \left(\frac{kF\xi}{2}\right)^{3/2} \frac{1}{\Gamma(5/2)} \quad .$$

These yield

$$\begin{aligned} b_o^o &= b_1^o = 0, \qquad b_2^o = - 4/3 \pi F^3 \alpha^2 \xi \\ b_o^1 &= b_2^1 = 0, \qquad b_1^1 = 4/3\, \pi i F^2 \alpha^2 \\ b_o^2 &= b_1^2 = 0, \qquad b_2^2 = - 8/15 \pi F^3 \alpha^2 \xi \quad . \end{aligned} \tag{41}$$

To find the a_j^i we refer to the linear system (32), substitute power series expansions for A_μ, $C_{\mu\nu}$, and B, and equate coefficients of like powers of k. Thus if we put

$$C_{\mu\nu} = \sum_j C_j^{\mu\nu} k^{|\nu-\mu| + j}$$

and remember that $C_{\mu\nu} = 0$ for $\mu + \nu$ odd, there result presently the equations

$$\begin{aligned} &C_o^{oo} a_o^o = 4\pi b_o^o \\ &C_o^{oo} a_1^o + C_1^{oo} a_o^o = 4\pi b_1^o \\ &C_o^{oo} a_2^o + C_1^{oo} a_1^o + C_2^{oo} a_o^o + C_o^{2o} a_o^2 = 4\pi b_2^o \\ &C_o^{11} a_o^1 = 4\pi b_o^1 \\ &C_o^{11} a_1^1 + C_1^{11} a_o^1 = 4\pi b_1^1 \\ &C_o^{22} a_o^2 = 4\pi b_o^2 \quad . \end{aligned} \tag{42}$$

Referring now to formulas (23) and (24) it is clear at once that $C_o^{\infty} = C_1^{\infty} = 0$, while $C_o^{22} \neq 0$, and it develops easily that

$$C_2^{\infty} = \frac{8\pi^3 F^3 \alpha^2 \xi}{\Gamma(5/2)\Gamma(-1/2)} = -16/3\pi^2 F^3 \alpha^2 \xi .$$

We find also by the obvious procedure that

$$C_o^{11} = \frac{-8\pi^2 F\alpha^2}{3\xi}\left\{ 3\int_{-1}^{+1} \frac{\eta^4 d\eta}{\xi^2 - \eta^2} - (2\xi^2+1)\int_{-1}^{+1} \frac{\eta^2 d\eta}{\xi^2-\eta^2} \right\}$$

$$= \frac{16\pi^2 F\alpha^2}{3\xi}\left[\xi^2 - \frac{\xi}{2}\alpha^2 \log \frac{\xi+1}{\xi-1} \right]$$

Applying these relations and those in (41) to (42), we find that

$$C_2^{oo} a_o^o + C_o^{2o} a_o^2 = 4\pi b_2^o$$

$$C_o^{11} a_o^1 = 0$$

$$C_o^{11} a_1^1 + C_1^{11} a_o^1 = 4\pi b_1^1$$

$$C_o^{22} a_o^2 = 0$$

which reduces immediately to

$$C_2^{oo} a_o^o = 4\pi b_2^o$$
$$C_o^{11} a_1^1 = 4\pi b_1^1$$

or

$$a_o^o = \frac{4\pi \cdot -4\pi F^3 \alpha^2 \xi \cdot 3}{3 \cdot -16\pi^2 F^3 \quad 2} = 1$$

$$a_1^1 = \frac{4\pi \cdot 4\pi i F^2 \alpha^2 \cdot 3\xi}{3 \cdot 16\pi^2 F \alpha^2 \left[\xi^2 - \frac{\xi}{2}\alpha^2 \log \frac{\xi+1}{\xi-1}\right]} = \frac{iF\xi}{\left[\xi^2 - \frac{\xi}{2}\alpha^2 \log \frac{\xi+1}{\xi-1}\right]} .$$

Equation (40) now reduces to

$$f(\pi) \approx -\frac{k^2}{4\pi}\left[a_o^o\, b_2^o + a_1^1\, b_1^1\right]$$

$$\approx k^2\, \frac{F^3\alpha^2\xi}{3}\left\{\frac{\xi^2\left(1 - \frac{\xi}{2}\alpha^2 \log \frac{\xi+1}{\xi-1}\right)}{\xi^2 - \frac{\xi}{2}\alpha^2 \log \frac{\xi+1}{\xi-1}}\right\}. \tag{43}$$

In his original work on scattering by small obstacles, Rayleigh gives formulas for the case of a plane sound wave incident on a prolate spheroid which can be written as follows:[12]

$$f(\pi) = \frac{\pi T}{\lambda^2}\left[\frac{2-L}{1-L}\right] \tag{44}$$

where

$$L \equiv \left(\frac{1}{e^2} - 1\right)\left[\frac{1}{2e} \log \frac{1+e}{1-e} - 1\right]$$

$$T \equiv 4/3\,\pi a^3(1 - e^2).$$

Upon substitution of the relations $e \equiv \frac{1}{\xi}$, $a \equiv F\xi$, and $\lambda \equiv 2\pi/k$, equation (44) becomes identical to (43), and the variational solution

[12] Rayleigh (Strutt, J. W.) "On the Incidence of Aerial and Electromagnetic Waves on Small Obstacles," Phil. Mag., Vol. 44, p. 28 (1897).

is thus shown to be in agreement with the Rayleigh result.

In the limit of small wavelength the analysis is more difficult, and the geometrical optics result has not been obtained from the variational solution. It becomes apparent, however, that the situation is similar to that which prevails in the case of the sphere, in that the number of terms used in the series in (33) must be of the order of ka. This can be shown by the following analysis.

We examine first the behavior of the quantity $C_{\mu\nu}$ as ka becomes large in comparison to μ and ν. In this range we can use the Hankel asymptotic forms for the Bessel and Hankel functions appearing in the expressions for Γ_μ and Δ_ν (equation (23)). Thus

$$\Gamma_\mu(\xi, k, \psi) \approx \left(\frac{2}{\pi\rho}\right)^{1/2} e^{i\rho} \cdot i^{-\mu} \left[\mu(\cos\psi P_{\mu-1} - \xi^2 P_\mu) - (\mu+1)(\cos\psi P_{\mu+1} - \xi^2 P_\mu)\right]$$

$$\simeq -\left(\frac{2}{\pi\rho}\right)^{1/2} e^{i\rho}\, i^{-\mu}(2\mu+1) P_\mu(\cos\psi)\,(\xi^2 - \cos^2\psi),$$

$$\Delta_\nu(\xi, k, \psi) \approx \left(\frac{2}{\pi\rho}\right)^{1/2} \left[\nu(\cos\psi P_{\nu-1} - \xi^2 P_\nu) - (\nu+1)(\cos\psi P_{\nu+1} - \xi^2 P_\nu)\right] \cdot \frac{1}{2}\left[i^{-\nu} e^{i\rho} + i^{\nu} e^{-i\rho}\right]$$

$$\simeq -\left(\frac{2}{\pi\rho}\right)^{1/2} (2\nu+1) P_\nu(\cos\psi)(\xi^2 - \cos^2\psi) \cdot \frac{1}{2}\left[i^{-\nu} e^{i\rho} + i^{\nu} e^{-i\rho}\right]$$

and consequently

$$C_{\mu\nu} \simeq -4\pi^2 F^2 \alpha\, i^{\mu+3\nu+1} k \cdot \int_0^\pi P_\mu(\cos\psi) P_\nu(\cos\psi) \cdot \left[i^{\mu-\nu} e^{2i\rho} + i^{\nu-\mu}\right] (\xi^2 - \cos^2\psi)^{1/2} \sin\psi \, d\psi. \qquad (45)$$

An approximate value for the first term of the integral when ka is large can be obtained by means of the stationary phase formula. Letting

$$\cos\psi \equiv \eta$$

$$\frac{2\alpha}{\sqrt{\xi^2 - \cos^2\psi}} \equiv \Psi(\eta)$$

$$P_\mu(\cos\psi)\, P_\nu(\cos\psi)\sqrt{\xi^2 - \cos^2\psi} \equiv \chi(\eta)$$

we find that if η_o is the stationary phase point, defined by the relation $\Psi'(\eta_o) = 0$, then

$$\int_0^\pi e^{2i\rho} P_\mu(\cos\psi) P_\nu(\cos\psi)\sqrt{\xi^2 - \cos^2\psi}\ \sin\psi\, d\psi$$

$$\equiv \int_{-1}^{+1} e^{ika\Psi(\eta)}\, \chi(\eta)\, d\eta.$$

$$\approx \left(\frac{2\pi}{ka\left|\Psi''(\eta_o)\right|}\right)^{1/2} \chi(\eta_o)\, e^{ika\Psi(\eta_o) + i\frac{\pi}{4}\,\mathrm{sgn}\,\Psi''(\eta_o)}$$

$$\approx \begin{cases} \xi^2\sqrt{\dfrac{\pi}{kF\alpha}}\; e^{i(2kF\alpha + \frac{\pi}{4})}\, i^{\mu+\nu}\, \dfrac{\mu!\,\nu!}{2^{\mu+\nu}\left(\frac{\mu}{2}!\,\frac{\nu}{2}!\right)^2} & \text{for } \mu, \nu \text{ even} \\ 0 & \text{for } \mu, \nu \text{ odd} \end{cases} \qquad (46)$$

The second term in the integral becomes

$$i^{\nu-\mu}\int_{-1}^{+1} P_\mu(\eta)\, P_\nu(\eta)\, \sqrt{\xi^2-\eta^2}\; d\eta\,, \tag{47}$$

which is independent of k, so that the dependence of $C_{\mu\nu}$ on k has the form

$$C_{\mu\nu} \approx -4\pi^2 F\alpha\, i^{\mu+3\nu+1} \cdot \left\{ \frac{1}{2}\left[1+(-1)^\mu\right] \sigma_{\mu\nu}(F,\xi)\sqrt{k}\; e^{2ikF\alpha} + \tau_{\mu\nu}(F,\xi)\, k \right\} \tag{48}$$

for ka large.

Use of the same asymptotic forms for the Bessel functions in the expression for B_μ (see Chapter 4) gives the result

$$\begin{aligned} B_\mu &\approx \frac{2\pi F\alpha^2}{\xi}\left[e^{ika} + (-1)^\mu e^{-ika}\right] \\ &\approx 4\pi F\,\frac{\alpha^2}{\xi}\begin{cases} \cos ka & \text{for } \mu \text{ even} \\ i \sin ka & \text{for } \mu \text{ odd.} \end{cases} \end{aligned} \tag{49}$$

For large k, the second term on the right in equation (48) dominates, and combining (45), (46), (47), (48), and (49) we can write the linear system (32) approximately as

$$\begin{aligned} \sum_{\mu \text{ even}} A_\mu \tau_{\mu\nu} &\approx \frac{4i\alpha \cos ka}{ka} & \nu \text{ even} \\ \sum_{\mu \text{ odd}} A_\mu \tau_{\mu\nu} &\approx \frac{4\alpha \sin ka}{ka} & \nu \text{ odd.} \end{aligned} \tag{50}$$

Application of Cramer's rule to these systems gives

$$A_\mu \approx \frac{4i\alpha \cos ka}{ka\,|\tau_{\mu\nu}|}\sum_\nu \tau'_{\mu\nu} \qquad \mu \text{ even}$$

$$A_\mu \approx \frac{-4\alpha \sin ka}{ka\,|\tau_{\mu\nu}|}\sum_\nu \tau'_{\mu\nu} \qquad \mu \text{ odd}$$

where $\tau'_{\mu\nu}$ is the cofactor of $\tau_{\mu\nu}$ and $|\tau_{\mu\nu}|$ is the determinant.

Then

$$A_\mu B_\mu \approx \frac{16\pi i \alpha^3 F \cos^2 ka}{\xi ka \; |\tau_{\mu\nu}|} \sum_\nu \tau'_{\mu\nu} \qquad \mu \text{ even}$$

$$A_\mu B_\mu \quad \frac{-16\pi i \alpha^3 F \sin^2 ka}{\xi ka \; |\tau_{\mu\nu}|} \sum_\nu \tau'_{\mu\nu} \qquad \mu \text{ odd.}$$

Assuming the quantities $\sum_\nu \tau'_{\mu\nu}/|\tau_{\mu\nu}|$ are bounded, each term of the series (33) with $\mu << ka$ is of order 1/ka. Therefore any reasonable approximation to the true cross section would require at least approximately ka of these terms, as in the case of the sphere. The dominant terms in this region of the spectrum may be those with $\mu \approx ka$. In order to obtain the approximate values of these terms, the Debye asymptotic forms for the Bessel functions might be utilized; however these lead to much more awkward integrals than those in equation (45), and no further attempts have been made in this direction.

CHAPTER 7

CONVERGENCE OF THE SOLUTION

The principal question remaining is that of the convergence of the series in equation (33). It will be shown that at least for small enough values of ka, this series must be absolutely convergent. Some of the estimates used in the following are rather rough, and error terms are in general ignored. To make the proof absolutely rigorous a more careful analysis of these error terms would be necessary. However this would materially complicate the already tedious development, and we therefore limit ourselves to what might be called a strong plausibility argument.

We consider first the behavior of the quantities B_μ (equation (25)) as μ increases with ka fixed. The asymptotic form of the Bessel function $\psi_n(z)$ for z fixed and n large is easily found to be[13]

$$\psi_n(z) \approx \frac{e^{n+1/2}\, z^n}{2^{n+3/2}(n+1/2)^{n+1}} , \tag{51}$$

which can be differentiated with respect to z to give

$$\frac{d}{dz}\psi_n(z) \approx \frac{e^{n+1/2}\, n\, z^{n-1}}{2^{n+3/2}(n+1/2)^{n+1}} . \tag{52}$$

Using this form in equation (25) immediately shows that for fixed ka, the quantities $|B_\mu|$ ultimately die out as $\left(\frac{eka}{2\mu}\right)^\mu$. It is clear then,

[13] Cf. Watson, G. N., "Theory of Bessel Functions," Cambridge, p. 225 (1952).

that as long as the A_μ do not diverge too rapidly, the series $\sum_\mu A_\mu B_\mu$ will converge absolutely for any finite value of ka. Due to the complicated form which still prevails for the coefficients $C_{\mu\nu}$ it is difficult to obtain a direct proof of the boundedness of the quantities $|A_\mu|$; however, a constructive proof of the existence of a set of A_μ which are bounded in absolute value and satisfy the system (32) can be given by the following line of argument.

We assume for the moment that the first N values of $|A_\mu|$ are bounded, where N is a number which is large with respect to ka and unity. Discarding the first N equations temporarily and transposing the products $A_\mu C_{\mu\nu}$ for $\mu \leq N$ to the right hand sides of the remaining equations, we can show that the resulting system of equations in the A_μ for $\mu > N$ has, for some range of ka, a solution of which each member is bounded in absolute value and which can be determined in the limit by the usual method of truncation. Furthermore we can show that when this solution is substituted in the first N equations of the original system, the resulting system can in general be solved for the A_μ with $\mu \leq N$. The result is the unique bounded solution to the original system (32), whose existence guarantees the convergence of the series in (33).

The first step is to show that there exists a number N such that if $|A_\mu|$ is bounded for all $\mu \leq N$, then the system

$$\sum_{\mu=N+1}^{\infty} A_\mu C_{\mu\nu} = 4\pi B_\nu - \sum_{\mu=0}^{N} A_\mu C_{\mu\nu} , \qquad \nu = N+1, \text{------} \infty \tag{53}$$

has a solution $\{A_\mu\}_{\mu > N}$ expressible linearly in terms of the set $\{A_\mu\}_{\mu \leq N}$ such that $|A_\mu| \leq M$ for some $M < \infty$ and all $\mu > N$. To accomplish this we make use of the following theorem, due to Pellet and Wintner:[14]

Given the system $X_\nu - \psi_\nu = C_\nu$, $\nu = 1, 2, \ldots \infty$, where

$$\psi_\nu \equiv \sum_{\mu=1}^{\infty} a_{\mu\nu} X_\mu ,$$

if the quantities $|C_\nu|$ are bounded and the coefficients $a_{\mu\nu}$ are subject to the condition

$$S_\nu \equiv \sum_{\mu=1}^{\infty} |a_{\mu\nu}| < 1, \qquad \nu = 1, 2, \ldots \infty, \tag{54}$$

then the X_ν exist and are equal to the limiting form as $m \to \infty$ of the $X_\nu^{(m)}$ determined by solving the reduced system

$$X_\nu^{(m)} - \sum_{\mu=1}^{m} a_{\mu\nu} X_\mu^{(m)} = C_\nu, \quad \nu = 1, 2, \ldots m. \tag{55}$$

We first translate the notation of the theorem into that of the system (53), after dividing each equation of the latter by the corresponding quantity $C_{\nu\nu}$ for convenience. This entails the relations

$$\begin{aligned} X_\nu &\equiv A_{N+\nu} \\ a_{\mu\nu} &\equiv - C_{N+\nu, N+\mu} / C_{N+\nu, N+\nu}, \qquad \nu \neq \mu \\ a_{\nu\nu} &\equiv 0 \\ C_\nu &\equiv 4\pi B_{N+\nu}/C_{N+\nu, N+\nu} - \sum_{\mu=0}^{N} A_\mu C_{N+\nu, \mu}/C_{N+\nu, N+\nu} \end{aligned} \tag{56}$$

14 Cf. Davis, H. T., "Theory of Linear Operators," Principia Press, p. 130, (1936).

$$\psi_\nu \equiv -\sum_{\mu \neq \nu} C_{N+\nu, N+\mu} A_{N+\mu} / C_{N+\nu, N+\nu}$$

$$S_\nu \equiv \sum_{\mu \neq \nu} \left| C_{N+\nu, N+\mu} / C_{N+\nu, N+\nu} \right| .$$

The boundedness of $|C_\nu|$ follows immediately from the assumption on the $|A_\mu|$ for $\mu < N$ and from the forms given previously for B_μ and $C_{\mu\nu}$. It remains to show that $S_\nu < 1$ for $\nu = 1, 2, \ldots \infty$.

Referring to equation (24) we can write the real and imaginary parts of $C_{\mu\nu}$ for the range $\nu < \mu$ as

$$C^R_{\mu\nu} \equiv \text{Re}\, C_{\mu\nu} = 8\pi^2 \xi F^3 \alpha^2 (-1)^{\frac{\nu+3\mu}{2}} k^2 \int_{-1}^{+1} \frac{\rho \, d\eta}{(\xi^2 - \eta^2)^2} \cdot$$

$$\cdot \left[\frac{\mu+1}{\rho} \eta P_{\mu+1}(\eta) \psi_\mu(\rho) + \xi^2 P_\mu(\eta) \psi'_\mu(\rho) \right] \left[\frac{\nu+1}{\rho} \eta P_{\nu+1}(\eta) \eta_\nu(\rho) + \xi^2 P_\nu(\eta) \eta'_\nu(\rho) \right] \tag{57}$$

and

$$C^I_{\mu\nu} \equiv I_m C_{\mu\nu} = -8\pi^2 \xi F^3 \alpha^2 (-1)^{\frac{\nu+3\mu}{2}} k^2 \int_{-1}^{+1} \frac{\rho \, d\eta}{(\xi^2 - \eta^2)^2} \cdot$$

$$\cdot \left[\frac{\mu+1}{\rho} \eta P_{\mu+1}(\eta) \psi_\mu(\rho) + \xi^2 P_\mu(\eta) \psi'_\mu(\rho) \right] \left[\frac{\nu+1}{\rho} \eta P_{\nu+1}(\eta) \psi_\nu(\rho) + \xi^2 P_\nu(\eta) \psi'_\nu(\rho) \right] \tag{58}$$

Here $\eta_\nu(\rho)$ is the spherical Neumann function $\sqrt{\frac{\pi}{2\rho}}\, N_{\nu+1/2}(\rho)$, and the quantity $\rho \equiv \frac{ka\alpha}{\sqrt{\xi^2 - \eta^2}}$ ranges between kb and ka (b is the semi-minor axis of the spheroid) as η goes from -1 to $+1$, so that if N is large compared to ka and unity, then for all $\mu, \nu > N$ we can use the asymptotic forms (51), (52), together with the corresponding ones for

$\eta_n(z)$ and $\eta_n'(z)$, namely

$$\eta_n(z) \approx \frac{(-1)^{2n+1}\, 2^{n-1/2} (n+1/2)^n}{e^{n+1/2}\, z^{n+1}} \quad , \eta_n'(z) \approx \frac{(-1)^{2n}\, 2^{n-1/2} (n+1/2)^n (n+1)}{e^{n+1/2}\, z^{n+2}} \tag{59}$$

to obtain estimates for $C^R_{\mu\nu}$ and $C^I_{\mu\nu}$. After some manipulation we arrive at the expressions

$$C^R_{\mu\nu} \approx \frac{-2\pi^2 F(-1)^{\frac{\nu+3\mu}{2}} \left(\frac{eka\alpha}{2}\right)^{\mu-\nu} (\nu+1/2)^{\nu} (\nu+1)}{\xi(\mu+1/2)^{\mu+1}} \cdot \tag{60}$$

$$\cdot \int_{-1}^{+1} \left[\eta P_{\nu+1}(\eta) - \xi^2 P_{\nu}(\eta)\right] \left[(\mu+1)\eta P_{\mu+1}(\eta) + \mu\xi^2 P_{\mu}(\eta)\right] \frac{d\eta}{(\xi^2-\eta^2)^{\frac{\mu-\nu+2}{2}}}$$

$$C^I_{\mu\nu} \approx \frac{-\pi^2 F^2 \alpha\, ek(-1)^{\frac{\nu+3\mu}{2}} \left(\frac{eka\alpha}{2}\right)^{\mu+\nu}}{} \cdot \tag{61}$$

$$\cdot \int_{-1}^{+1} \left[(\mu+1)\eta P_{\mu+1}(\eta) + \mu\xi^2 P_{\mu}(\eta)\right] \left[(\nu+1)\eta P_{\nu+1}(\eta) + \nu\xi^2 P_{\nu}(\eta)\right] \frac{d\eta}{(\xi^2-\eta^2)^{\frac{\mu+\nu+3}{2}}}.$$

The integrals here can be roughly bounded from above by substituting the least upper bounds of the factors in the numerators of the integrands, and the greatest lower bounds of their denominators. The following inequalities result:*

*Here and hereafter in this chapter the symbol "$\leq$" should in general be read "less than or approximately equal to."

$$\left| C^R_{\mu\nu} \right| \leq \frac{8\pi^2 F(\xi^2+1)^2 \left(\frac{eka}{2}\right)^{\mu-\nu} (\nu+1)^{\nu+1} (\mu+1)}{\xi \alpha^2 (\mu+1/2)^{\mu+1}} \tag{62}$$

$$\left| C^I_{\mu\nu} \right| \leq \frac{\pi^2 F^2 ek(\xi^2+1)^2 \left(\frac{eka}{2}\right)^{\mu+\nu} (\mu+1)(\nu+1)}{\alpha^2 (\mu+1/2)^{\mu+1} (\nu+1/2)^{\nu+1}} \quad . \tag{63}$$

Bounds for the summation of these quantities from $\mu = \nu+1$ to ∞ may be obtained with the aid of the relations

$$\sum_{\mu=\nu+1}^{\infty} \left(\frac{eka}{2\mu+1}\right)^{\mu} \frac{\mu+1}{\mu+1/2} \leq \frac{2\nu+4}{2\nu+3} \sum_{\mu=\nu+1}^{\infty} \left(\frac{eka}{2\nu+3}\right)^{\mu} \leq \frac{4}{3} \sum_{\mu=\nu+1}^{\infty} \left(\frac{eka}{2\nu+3}\right)^{\mu}$$

$$\leq \frac{4}{3} \left(\frac{eka}{2}\right)^{\nu+1} \frac{1}{(\nu+1)^{\nu+1} \left[1 - \frac{eka}{2\nu+3}\right]} \quad .$$

The resulting expressions are

$$\sum_{\mu=\nu+1}^{\infty} \left| C^R_{\mu\nu} \right| \leq \frac{16\pi^2 F(\xi^2+1)^2 \, eka}{3\xi\alpha^2 \left[1 - \frac{eka}{2\nu+3}\right]} \tag{64}$$

and

$$\sum_{\mu=\nu+1}^{\infty} \left| C^I_{\mu\nu} \right| \leq \frac{4\pi^2 F(\xi^2+1)^2 \left(\frac{eka}{2\nu+1}\right)^{2\nu+1}}{3\xi\alpha^2 \left[1 - \frac{eka}{2\nu+3}\right]} \quad . \tag{65}$$

The formula corresponding to (57) for the range $\mu < \nu$ is

$$C^R_{\mu\nu} = 8\pi^2 \xi F^3 \alpha^2 (-1)^{\frac{\mu+3\nu}{2}} k^2 \int_{-1}^{+1} \left[\frac{\mu+1}{\rho} \eta P_{\mu+1} \eta_\mu(\rho) + \xi^2 P_\mu(\eta) \eta'_\mu(\rho) \right] \cdot$$

$$\cdot \left[\frac{\nu+1}{\rho} \eta P_{\nu+1}(\eta) \psi_\nu(\rho) + \xi^2 P_\nu(\eta) \psi'_\nu(\rho) \right] \frac{\rho \, d\eta}{(\xi^2 - \eta^2)^2}$$

and the expression for $C^I_{\mu\nu}$ is the same as before. Using the previous estimates and procedures, we obtain, to order $1/\nu$,

$$\sum_{\mu=0}^{\nu-1} \left| C^R_{\mu\nu} \right| \le \frac{4\pi^2 F(\xi^2+1)^2 eka\,\nu}{\xi \alpha^2 (2\nu+1)} \tag{66}$$

and

$$\sum_{\mu=0}^{\nu-1} \left| C^I_{\mu\nu} \right| \le \frac{2\pi^2 F(\xi^2+1)^2 (\nu+1) eka}{\xi \alpha^2 (\nu+1/2)} \left(\frac{eka}{2\nu+1} \right)^\nu . \tag{67}$$

Combining (64), (65), (66) and (67) we can thus write

$$\sum_{\substack{\mu=0 \\ \mu \ne \nu}}^{\infty} \left| C^R_{\mu\nu} \right| + \left| C^I_{\mu\nu} \right| \le \frac{\pi^2 F(\xi^2+1)^2 eka}{\xi \alpha^2} \left[\frac{22}{3} + O\left(\frac{1}{\nu}\right) \right] . \tag{68}$$

An estimate is now required for $C^R_{\nu\nu}$. This can be obtained from equation (60). When μ is set equal to ν, the resulting integral can be evaluated exactly with the aid of the formula

$$\int_{-1}^{+1} (z-x)^{-1} P_m(x)\, dx = 2 P_m(z) Q_n(z)$$

for $m \le n$, $|z| > 1$.[15] The resulting expression for $C^R_{\nu\nu}$ is

15 Cf. Erdelyi, et. al., "Tables of Integral Transforms" Vol. 2, Bateman Manuscript Project, McGraw-Hill, p. 278 (1954).

$$C^R_{\nu\nu} \approx \frac{4\pi^2 F(\nu+1)}{\xi(\nu+1/2)} \left\{ \frac{(\nu+1)(\nu+2)}{2\nu+3} P_{\nu+1}(\xi) Q_{\nu+2}(\xi) + \left[\frac{(\nu+1)^2}{2\nu+3} - \xi^2 \right] P_\nu(\xi)\, Q_{\nu+1}(\xi) - \nu\xi^3 P_\nu(\xi) Q_\nu(\xi) \right\} . \qquad (69)$$

For m large and $\xi > 3/(2\sqrt{2})$, the Legendre functions $P_m(\xi)$ and $Q_m(\xi)$ may be approximated to order 1/m by the formulas[16]

$$P_m(\xi) \approx \frac{\Gamma(m+1/2)\, e^{(m+1)\cosh^{-1}\xi}}{\sqrt{\pi}\,\Gamma(m+1)\,(e^{2\cosh^{-1}\xi} - 1)^{1/2}}$$

$$Q_m(\xi) \approx \frac{\sqrt{\pi}\,\Gamma(m+1) e^{-(m+1)\cosh^{-1}\xi}}{\Gamma(m+3/2)(1 - e^{-2\cosh^{-1}\xi})^{1/2}} .$$

Substituting these into (69) we obtain, after some simplification,

$$C^R_{\nu\nu} \approx \frac{2\pi^2 F}{\xi\alpha\,(\xi+\alpha)} \left[2\xi^4 + 2\xi^3\alpha - 1 \right] + O(\tfrac{1}{\nu}) . \qquad (70)$$

Returning now to the theorem quoted above, the condition (54) requires, according to (56), that

$$S_\nu = \sum_{\substack{\mu=N+1 \\ \mu\neq\nu}}^{\infty} \left| \frac{C_{\mu\nu}}{C_{\nu\nu}} \right| < 1 \qquad \text{for all } \nu > N. \qquad (71)$$

Using the triangle inequality we can write

[16]Cf. Magnus and Oberhettinger, loc. cit., p. 73

$$S_\nu = \left(C_{\nu\nu}^{R\,2} + C_{\nu\nu}^{I\,2}\right)^{-1/2} \sum_{\substack{\mu=N+1 \\ \mu\neq\nu}}^{\infty} \left(C_{\nu\nu}^{R\,2} + C_{\mu\nu}^{I\,2}\right)^{1/2}$$

$$\leq \left|C_{\nu\nu}^{R}\right|^{-1} \sum_{\substack{\mu=N+1 \\ \mu\neq\nu}}^{\infty} \left|C_{\mu\nu}^{R}\right| + \left|C_{\mu\nu}^{I}\right|. \tag{72}$$

(Here the quantity $C_{\nu\nu}^{I}$ is ignored since its order of magnitude is clearly less than that of the errors in the other approximations.) Finally, combining (68), (70) and (72), we arrive at the relation

$$S_\nu \leq \frac{11\ eka(\xi^2 + 1)^2(\xi + \alpha)}{3\alpha(2\xi^4 + 2\xi^3\alpha - 1)} \qquad \text{for all} \quad \nu > N, \tag{73}$$

so that for sufficiently small values of ka (and for $\xi > 3/(2\sqrt{2})$) the system (53) must have a solution $\left\{A_\mu\right\}_{\mu > N}$ obtainable as a limit by means of the truncation technique.

Construction of the remaining set $\left\{A_\mu\right\}_{\mu \leq N}$ is accomplished by substituting the previous set into the first N equations of the system (32) and solving the resulting N x N system. To show that this is possible we must prove that the infinite series which appear as coefficients of the A_μ converge. First consider the series

$$\sum_{\mu=N+1}^{\infty} C_{\mu\nu} A_\mu , \qquad \nu \leq N. \tag{74}$$

Use of the approximations (51) and (52) in equation (57) yields the expression

$$\left| C^R_{\mu\nu} \right| \leq \frac{8\pi^2 F\sqrt{e}\, k^2 a^2(\mu\xi^2 + \mu + 1)}{\sqrt{2}\,\xi\,\alpha^2(2\mu+1)} \left(\frac{eka}{2\mu+1}\right)^{\mu} \cdot$$

$$\cdot \int_{-1}^{+1} \left[\frac{\mu+1}{\rho}\,\eta\, P_{\nu+1}(\eta)\,\eta_\nu(\rho) + \xi^2 P_\nu(\eta)\,\eta'(\rho) \right] d\eta,$$

with a similar formula obtaining for $\left| C^I_{\mu\nu} \right|$. It is obvious that the integrals here are bounded in absolute value for any $\nu \leq N$, and consequently it follows at once that the quantities $\left|C_{\mu\nu}\right|$ die out with increasing μ at such a rate that for any set of bounded $\left|A_\mu\right|$ the series (74) converges absolutely. Since the coefficients of the A_μ for $\mu \leq N$ are formed by rearrangement of these series, it follows that they must exist, and the system possesses a solution provided their determinant does not vanish. The latter is a function of k, a, and ξ, and while it is conceivable that it might have zeroes in one or another of these parameters, it cannot vanish identically. This completes the argument.

As remarked at the beginning of this chapter, the error terms in the estimates used here are not taken into account and the proof may not be considered rigorous until this is done. The criterion for convergence obtainable from equation (73) is probably of little value, due to the rough character of some of the inequalities used. However, in view of the rates of which the critical quantities die out with increasing index it seems clear that the approach outlined here could be made to yield a rigorous proof of the convergence and a more significant criterion for the range of ka over which it holds.

CHAPTER 8

NUMERICAL RESULTS

On the basis of the forms developed in the preceding chapters a value was computed for the nose-on back-scattering cross section of a particular spheroid at a single wavelength. In order to obtain a comparison with the exact solution, parameter values were chosen for which the latter had previously been determined.[17] The axis ratio of the spheroid was taken as 10:1, and the wavelength ratio ka was given the value 1.40, which is very near the location of the first maximum in the curve of cross section vs. ka. The integrals in equation (24) were evaluated by means of Simpson's rule, using intervals of $.5^{\circ}$ in the range $0 \leq \psi < 10^{\circ}$ and 2.5° in the range $10 \leq \psi \leq 90^{\circ}$. (The smaller intervals near the origin were necessary because with the value of ξ very near unity, the denominator of the integrand is small in this region and the value of the integrand in general rises quite sharply. This effect would be less pronounced for a fatter spheroid). The linear system (32) was then solved, under the usual truncation assumptions, as an $N \times N$ system, and the order N was given several values in order to obtain some indication of the convergence rate. Values of the scattering cross section σ were computed from equation (33) and divided by the geometric

17 Siegel, K. M., et al., "Theoretical and Numerical Determination of the Radar Cross Section of a Prolate Spheroid" IRE Trans. Vol. AP. 4, No. 3, July 1956, p.266.

optics result $\pi b^4/a^2$ for convenience in comparing with the known solution. The value of the latter at the point in question, to five significant figures, is

$$\frac{\sigma}{\pi b^4/a^2} = 1.1022$$

The following table contains the values computed from the variational result, listed as a function of the order N of the linear system.

$N =$	1	2	3	4	5
$\frac{\sigma}{\pi b^4/a^2} =$	.7337	1.833	1.025	1.110	1.105

The fifth order result is seen to agree with the exact answer within about .3% at this point.

Some of the intermediate quantities used in obtaining these figures may also be of interest and are tabulated here for the sake of completeness. We list first the coefficients $C_{\mu\nu}$ and B_ν, removing certain common factors for convenience:

μ	ν	$C_{\mu\nu} \cdot [16\pi^2 k^2 a^3 \alpha^2 / \xi^2]^{-1}$ Re	Im	ν	B_ν / a
0	0	$2.68085 \cdot 10^{-3}$	$-3.40193 \cdot 10^{-1}$	0	$-5.33925 \cdot 10^{-3}$
0	2	$9.36340 \cdot 10^{-5}$	$-6.38906 \cdot 10^{-2}$	1	$2.22699 \cdot 10^{-3} i$
0	4	$1.58590 \cdot 10^{-6}$	$-1.37306 \cdot 10^{-4}$	2	$-1.93905 \cdot 10^{-3}$
2	2	$7.13970 \cdot 10^{-5}$	$2.46674 \cdot 10^{-1}$	3	$-6.50311 \cdot 10^{-4} i$
2	4	$1.85214 \cdot 10^{-6}$	$-1.60819 \cdot 10^{-2}$	4	$1.41940 \cdot 10^{-4}$
4	4	$1.11642 \cdot 10^{-7}$	$4.74239 \cdot 10^{-1}$		
1	1	$1.45043 \cdot 10^{-4}$	$1.01764 \cdot 10^{-1}$		
1	3	$4.66476 \cdot 10^{-6}$	$-2.57312 \cdot 10^{-2}$		
3	3	$3.53351 \cdot 10^{-6}$	$3.69719 \cdot 10^{-1}$		

These quantities yield the following values of the A_μ as the solution to the fifth order linear system:

μ	Re A_μ	Im A_μ
0	$6.92536 \cdot 10^{-1}$	$-5.44415 \cdot 10^{-3}$
1	$-1.69158 \cdot 10^{-3}$	$1.15989 \cdot$
2	$-5.82917 \cdot 10^{-1}$	$9.76788 \cdot 10^{-4}$
3	$1.31725 \cdot 10^{-4}$	$-1.70612 \cdot 10^{-1}$
4	$3.52111 \cdot 10^{-2}$	$-3.00564 \cdot 10^{-5}$

Substitution of these quantities in equation (5) gives the potential distribution on the scattering surface.

Several possibilities might be considered in an attempt to improve the accuracy of the variational result. The discrepancy between this and the exact answer must be attributable primarily to three factors: 1) round-off error in the numerical quantities, 2) approximations inherent in numerical integration, and 3) truncation of the series. The first of these appears to be most significant in the present case. The accuracy employed throughout the computations was six decimal places, and a fifth-order answer was also computed using intervals of twice the above specified lengths in the integration process. This yielded a value of 1.1025 for the quantity in question, which is somewhat more accurate than the value obtained with the shorter intervals, indicating that the point of diminishing returns had already been passed in the direction of refining the integration intervals. Regarding the third factor, it seems that the successive orders of approximation form an oscillating sequence, and since the fifth-order answer is between the fourth and the correct value, it is to be expected that a sixth-order result, employing the same decimal accuracy, might be worse than the fifth.

The problem of maintaining greater accuracy in the numerical quantities might be troublesome for hand computation, due to limitations in the accuracy of available tables of the special functions and the necessity for complicated interpolation methods; however for a large scale computing machine it should not prove difficult.

CHAPTER 9

CONCLUSIONS

As remarked previously, some of the procedures employed in the foregoing are of extremely doubtful mathematical character and are justified here only by the results they yield. It would not be hard to reformulate the problem in mathematically rigorous fashion, after the manner of Bouwkamp and others, but it seems likely that the resulting integral forms might be even more difficult to handle than those incurred in the present approach, and since the latter apparently gives the correct result there is little reason to change it at this stage. As for the risks involved in proceeding in the above manner, they are perhaps better left unexamined.

There is little doubt that the solution obtained here is correct, but the question of why it is correct might bear considerable discussion. One is led to the conclusion that, although it is not obvious at first glance, the various operations of differentiation, integration, and passing to limits have actually been performed in their proper sequence. Some of the formal expressions used are thus incorrect, or at least misleading, but given proper (or mathematically improper) interpretation and handling, they can be made to yield a valid result.

From the standpoint of accuracy and economy the present form of the variational solution still leaves something to be desired. For cases where the available tables of spheroidal coefficients do not apply, it is probably considerably superior to the wave-function solution, at least for hand computation, and the best available estimates indicate that it may be competitive even where these tables are useful for the latter. However it must be admitted that the numerical evaluation of the remaining integrals is tedious when done by hand, and the cross section depends very sensitively on the values of these integrals. Further analysis of the forms derived here might produce some means of facilitating the computation process or even eliminating the numerical integrations entirely. For example, the power series expansions of the Bessel and Hankel functions appearing in equation (23), together with the explicit representations of the Legendre polynomials, leave only elementary integrals to be evaluated, and the resulting forms might be handled fairly simply by a computing machine. Alternatively something might be gained through integration by parts, a certain amount of which is possible after suitable manipulation of the integrands.

For values of ka up to about 1.0 (for the 10:1 spheroid) an excellent approximation to the cross section is given by the three-term power series, described in the appendix which follows.

The expressions given there yield the cross section of an arbitrary spheroid in this region of the spectrum much more easily than do the variational forms. It seems possible that one more term in this series might give a good approximation to the first maximum in the curve of cross section vs. ka, but the algebra involved in obtaining this would be rather formidable, and since the series is expected to diverge somewhere in the near vicinity of this maximum there is considerable uncertainty about the value of the result.

Another factor in the practicality of the variational solution is of course the rate of convergence. Nothing specific has been determined about this yet except that in the case computed it seems to be comparable to that of the wave-function solution. At higher values of ka the convergence would almost certainly be slower, though it is not obvious how fast the rate changes. This probably depends on the eccentricity of the spheroid in some manner which is difficult to predict.

Although the back-scattering cross section is the only physical quantity computed in the foregoing, it appears that more information could be obtained without too much trouble. The values of the A_μ listed in the preceding chapter yield immediately the potential layer on the scattering surface, through equation (5). Furthermore once these are known, the integration of equation (1) to give the scattered field at any point in space should be feasible.

APPENDIX

POWER SERIES SOLUTION

The development in Chapter 5 indicates that the variational forms may be used in deriving a power series representation for the scattered field. The procedure and results in the case of the circular aperture problem have been discussed by Magnus.[18] For the case of an electromagnetic wave striking an ellipsoid, the first two non-vanishing coefficients have been derived by Stevenson[19] without reference to any variational expressions. Whether the latter offer any material advantage in deriving these and subsequent coefficients in the scalar problem is not immediately clear. At any rate they have been utilized to obtain the second and third coefficients for the prolate spheroid, and the results are given here. The derivation proceeds along the lines described in Chapter 5. It is straightforward but tedious, and the details, which are contained in an unpublished memorandum (2591-509-M, 11 June 1957), will not be included here.

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[18] Magnus, W. "Infinite Matrices Associated with Diffraction by an Aperture." Research Report EM-32, New York Univ., Math. Research Group, 1957.

[19] Stevenson, A. F. "Electromagnetic Scattering by an Ellipsoid in the Third Approximation," Jour. Applied Physics. Vol. 24, No. 9, Sept. 1953.

We are concerned with an expression for the scattered field $f(\pi)$ of the form

$$f(\pi) = \sum_n R_n k^n . \qquad (75)$$

It was shown earlier that the coefficients of index 0 and 1 vanish identically. Stevenson has shown that in the electromagnetic case at least the coefficient R_3 also vanishes. It develops that this is also true in the scalar problem, and that here R_5 vanishes as well, though R_7 apparently does not. We will limit ourselves to determination of expressions for R_4 and R_6.

When the power series expansions of the quantities A_μ, B_ν, and $C_{\mu\nu}$ are substituted into the linear system (32) and the expression (37) for the scattered field, it is easily shown that the n^{th} coefficient in the series (75) can be written as an inner product

$$R_n = - \frac{b^2}{2\sqrt{a}} \overrightarrow{A_n} \circ \overrightarrow{B_n} \qquad (76)$$

where $\overrightarrow{A_n}$ and $\overrightarrow{B_n}$ are vectors whose components are proportional to certain coefficients in the series (39) for A_μ and B_μ respectively, and that furthermore

$$\overrightarrow{A_n} = - \frac{2}{a^{3/2}} C_n^{-1} \overrightarrow{B_n} \qquad (77)$$

where C_n^{-1} is the inverse of a matrix C_n whose elements are proportional to certain coefficients in the series for $C_{\mu\nu}$.

Combining (76) and (77) we can write R_n as a quadratic form

$$R_n = \left(\frac{b}{a}\right)^2 \overrightarrow{B_n} \circ C_n^{-1}\overrightarrow{B_n} . \qquad (78)$$

It can also be shown that the dimension of $\overrightarrow{A_n}$ and $\overrightarrow{B_n}$ should in general be $\frac{1}{4}(n+2)^2$ for n even, or $\frac{1}{4}(n+1)(n+3)$ for n odd, though in the present problem a number of the components vanish, so that the dimension is actually less than the specified value in each case considered. Moreover, because of the vanishing of $C_{\mu\nu}$ for $\mu+\nu$ odd, it develops at once that the matrix C_n is the direct sum of two submatrices, one deriving from the even values of μ and ν and the other from the odd, so that the transformation is considerably simplified.

For the computation of R_4 and R_6 the essential forms are as follows:

The quantities B_ν and $C_{\mu\nu}$ are expanded in the power series

$$B_\nu = \frac{2\pi b^2}{\sqrt{a}} \sum_{r=0}^{\infty} b_r^\nu k^r$$

$$C_{\mu\nu} = -4\pi^2 b^2 a \sum_{s=0}^{\infty} c_s^{\mu\nu} k^{|\nu-\mu|+s} \qquad (79)$$

(This notation differs slightly from that in Chapter 5 in that for the sake of economy we have removed common factors from the coefficients b_r^ν and $c_s^{\mu\nu}$.) The vector B_4 may then be written

$$\overrightarrow{B_4} = (b_4^0, b_2^0, b_2^2, b_3^1, b_1^1)$$

and the corresponding matrix C_4 is

$$C_4 = \begin{pmatrix} c_4^{00} & c_2^{00} & c_0^{02} \\ c_2^{00} & 0 & 0 \\ c_0^{02} & 0 & c_0^{22} \end{pmatrix} \dot{+} \begin{pmatrix} c_2^{11} & c_0^{11} \\ c_0^{11} & 0 \end{pmatrix}$$

Inversion of this matrix and substitution in (78) gives finally

$$R_4 = \frac{b^2}{a^2}\left\{\frac{(b_2^2)^2}{c_0^{22}} - 2\,\frac{b_2^0 b_2^2 c_0^{02}}{c_2^{00}c_0^{22}} + 2\,\frac{b_2^0 b_4^0}{c_2^{00}} + \frac{(b_2^0)^2\left[(c_0^{02})^2 - c_4^{00}c_0^{22}\right]}{(c_2^{00})^2 c_0^{22}} + \frac{2b_1^1 b_3^1}{c_0^{11}} - \frac{(b_1^1)^2\ c_2^{11}}{(c_0^{11})^2}\right\}. \tag{80}$$

Similarly we can write

$$\vec{B} \equiv (b_2^0,\ b_2^2,\ b_4^0,\ b_4^2,\ b_6^0,\ b_1^1,\ b_3^1,\ b_3^3,\ b_5^1)$$

and

$$C_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_2^{00} \\ 0 & 0 & 0 & c_0^{22} & c_0^{02} \\ 0 & 0 & c_2^{00} & c_0^{02} & c_4^{00} \\ 0 & c_0^{22} & c_0^{02} & c_2^{22} & c_2^{02} \\ c_2^{00} & c_0^{02} & c_4^{00} & c_2^{02} & c_6^{00} \end{pmatrix} \dot{+} \begin{pmatrix} 0 & 0 & 0 & c_0^{11} \\ 0 & c_0^{11} & 0 & c_2^{11} \\ 0 & 0 & c_0^{33} & c_0^{13} \\ c_0^{11} & c_2^{11} & c_0^{13} & c_4^{11} \end{pmatrix}$$

which yield

$$R_6 = \frac{b^2}{a^2}\left[\frac{1}{(c^{00})^3(c^{22})^2}\left\{(b_2^0)^2\left[2\ c_2^{00}c_0^{22}c_0^{02}c_2^{02} - c_2^{00}c_6^{00}(c_0^{22})^2\right.\right.\right.$$

$$\left. - c_2^{00}c_2^{22}(c_0^{02})^2 + (c_4^{00}c_0^{22})^2 - 2\ c_4^{00}c_0^{22}(c_0^{02})^2 + (c_0^{02})^4\right]$$

$$+ 2\ b_2^0 b_2^2\left[(c_2^{00})^2 c_2^{02} c_0^{22} - c_0^{02}c_2^{00}c_4^{00}c_0^{22} + c_2^{00}(c_0^{02})^3 - (c_2^{00})^2 c_0^{02} c_2^{22}\right]$$

$$- 2\ b_2^0 b_4^0\left[c_2^{00}c_4^{00}(c_0^{22})^2 - c_2^{00}c_0^{22}(c_0^{02})^2\right] - 2\ b_2^0 b_4^2 c_0^{02} c_0^{22}(c_2^{00})^2 +$$

$$+ 2\, b_2^0 b_6^0 (c_2^{00})^2 (c_0^{22})^2 - (b_2^2)^2 \left[(c_2^{00})^3 c_2^{22} - (c_2^{00} c_0^{02})^2 \right]$$

$$- 2\, b_2^2 b_4^0 (c_2^{00})^2 c_0^{02} c_0^{22} + 2\, b_2^2 b_4^2 (c_2^{00})^3 c_0^{22} + (b_4^0)^2 (c_2^{00} c_0^{22})^2 \Big\}$$

$$- \frac{1}{(c_0^{11})^3 c_0^{33}} \Big\{ (b_1^1)^2 \left[c_0^{11} c_0^{33} c_4^{11} - c_0^{11} (c_0^{13})^2 - (c_2^{11})^2 c_0^{33} \right] - (b_3^1)^2 (c_0^{11})^2 c_0^{33}$$

$$- (b_3^3)^2 (c_0^{11})^3 + 2\, b_1^1 b_3^1 c_0^{11} c_2^{11} c_0^{33} + 2\, b_1^1 b_3^3 (c_0^{11})^2 c_0^{13} - 2\, b_1^1 b_5^1 (c_0^{11})^2 c_0^{33} \Big\} \Big] . \quad (81)$$

The values of the b_r^ν and $c_s^{\mu\nu}$ are obtainable from the forms derived in Chapters 2 and 3. The general expressions are tabulated below:

Values of b_r^ν

ν \ r	1	2	3	4	5	6
0		$-\frac{2a}{3}$		$\frac{a^3}{15}$		$-\frac{a^5}{420}$
1	$\frac{2i}{3}$		$-\frac{ia^2}{5}$		$\frac{ia^4}{84}$	
2		$-\frac{4a}{15}$		$\frac{4a^3}{105}$		
3			$\frac{2ia^2}{35}$			

μ	ν	s	Values of $c_s^{\mu\nu}$
0	0	2	$4/3$
0	0	4	$\frac{4a^2\alpha^2}{15\,\xi}\log\frac{\xi+1}{\xi-1}$
0	0	6	$-\frac{2a^4\alpha^2}{35\xi^3}\left[2\xi+\alpha^2\log\frac{\xi+1}{\xi-1}\right]$
0	2	0	$\frac{2}{15}\left[-2(3\xi^2-2)+3\xi\alpha^2\log\frac{\xi+1}{\xi-1}\right]$
0	2	2	$\frac{4a^2\alpha^2}{105\,\xi}\left[6\xi-(3\xi^2-1)\log\frac{\xi+1}{\xi-1}\right]$
2	2	0	$\frac{6\xi^2}{5a^2}\left[-2(3\xi^2-2)+3\xi\alpha^2\log\frac{\xi+1}{\xi-1}\right]$
2	2	2	$\frac{4}{105}\left[(54\xi^4-60\xi^2+11)+3\xi\alpha^2(9\xi^2-4)\log\frac{\xi+1}{\xi-1}\right]$
1	1	0	$\frac{2\xi}{3a^2}\left[-2\xi+\alpha^2\log\frac{\xi+1}{\xi-1}\right]$
1	1	2	$\frac{2}{15\xi}\left[2\xi(3\xi^2-2)-(3\xi^2-1)\alpha^2\log\frac{\xi+1}{\xi-1}\right]$
1	1	4	$\frac{a^2\alpha^2}{210\xi^3}\left[-2\xi(23\xi^2+3)+(23\xi^4-18\xi^2+3)\log\frac{\xi+1}{\xi-1}\right]$
1	3	0	$\frac{1}{70\,\xi}\left[2\xi(45\xi^4-48\xi^2+7)-(15\xi^2-1)(3\xi^2-1)\alpha^2\log\frac{\xi+1}{\xi-1}\right]$
3	3	0	$\frac{3\xi(5\xi^2-1)}{14a^2}\left[-2\xi(5\xi^2-13)+3(5\xi^2-1)\alpha^2\log\frac{\xi+1}{\xi-1}\right]$

These formulas along with equation (43) may be used to obtain the first three coefficients in the power series solution for an arbitrary spheroid. For the 10:1 spheroid the values of ξ and α are 1.0050 and .1005 respectively, and substitution of these in the above gives the following values of the R_n:

$$R_2 = 6.736\, a^3 \cdot 10^{-3}$$

$$R_4 = -2.838\, a^5 \cdot 10^{-3}$$

$$R_6 = 2.922\, a^7 \cdot 10^{-4}$$

For comparison with the exact solution it is convenient to obtain the corresponding series representation of the quantity

$$\frac{\sigma}{\pi b^4/a^2} = \sum_n S_n\,(ka)^n$$

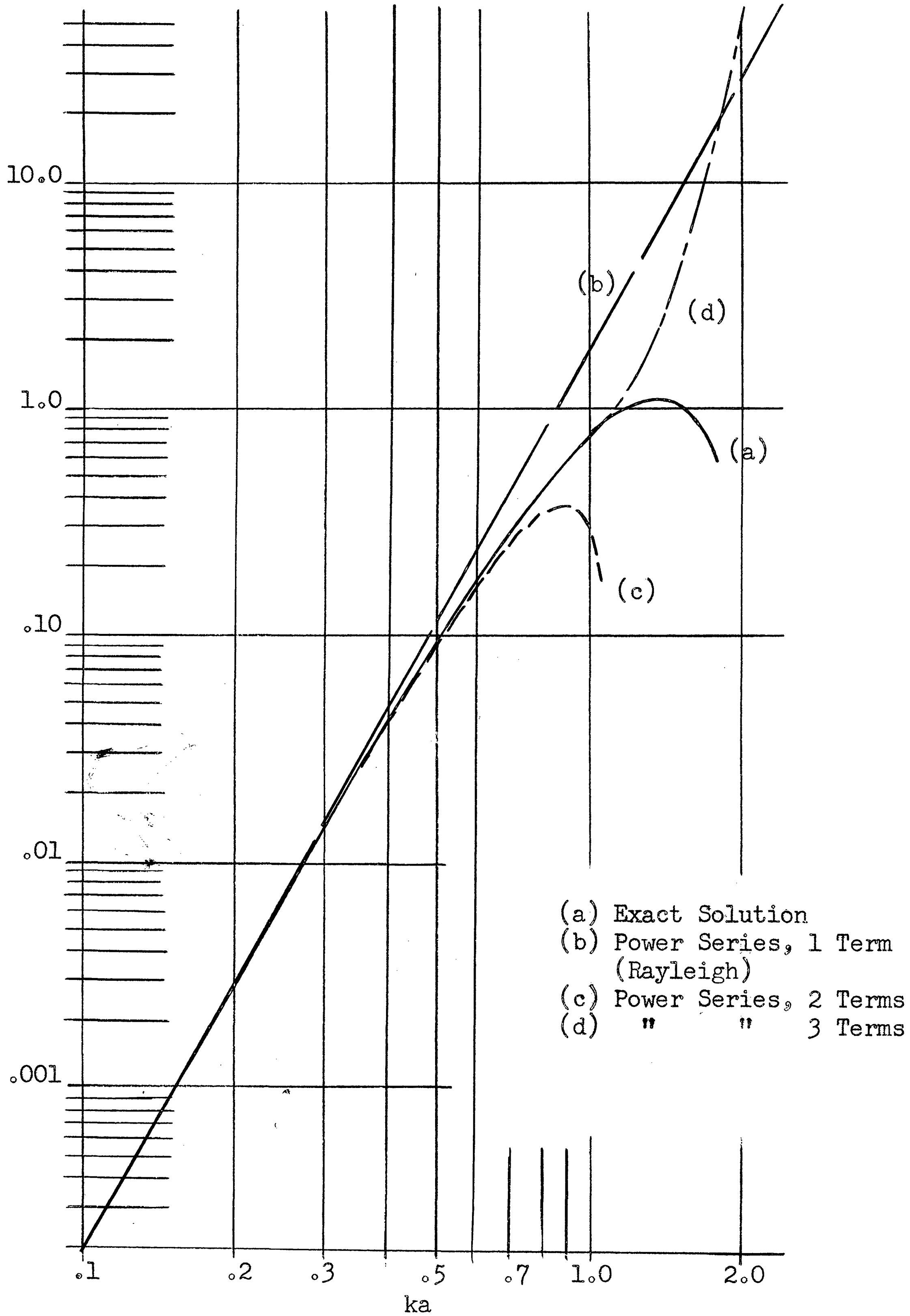
we obtain as the first three non-vanishing coefficients

$$S_4 = 4\left(\frac{a}{b}\right)^4 \left(\frac{R_2}{a^3}\right)^2 = 1.8148$$

$$S_6 = 8\left(\frac{a}{b}\right)^4 \cdot \frac{R_2}{a^3} \cdot \frac{R_4}{a^5} = -1.5296$$

$$S_8 = 4\left(\frac{a}{b}\right)^4 \left[\left(\frac{R_4}{a^5}\right)^2 + 2\,\frac{R_2}{a^3} \cdot \frac{R_6}{a^7}\right] = .4796$$

The accompanying graph shows how the exact curve is approximated by the power series representations. It appears that the three-term expression gives an excellent approximation to the correct curve out to about $ka = 1.0$. Using the three coefficients derived here it may be possible to develop an expression which gives a still better approximation in some range of $ka > 1$, but this has not yet been thoroughly investigated.



BACK-SCATTERING CROSS SECTION $\frac{\sigma}{\pi b^4/a^2}$

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