

Localized Vibrations of Disordered Multispan Beams: Theory and Experiment

Christophe Pierre*

University of Michigan, Ann Arbor, Michigan

and

De Man Tang,† and Earl H. Dowell‡

Duke University, Durham, North Carolina

The localization of the free modes of vibration of disordered multispan beams is investigated, both theoretically and experimentally. It is shown that small deviations of the span lengths from an ideal value may have drastic effects on the dynamics of the system. Emphasis is placed on the development of a perturbation method that allows one to obtain the strongly localized modes of vibration of the disordered system without a global eigenvalue analysis of the entire system. Such a perturbation analysis is cost-effective and accurate. More importantly, it provides physical insight into the localization phenomenon, and allows one to formulate a criterion that predicts the occurrence of strongly localized modes. Also, an experiment is described which has been carried out to verify the existence of localized modes for disordered two-span beams. Theoretical and experimental results are compared in detail and excellent agreement is found, thus confirming the existence of localized modes for such weakly coupled, weakly disordered structural systems.

Nomenclature

$a_i(t)$	= generalized coordinate
\tilde{a}_i	= i th modal amplitude
A	= peak deflection ratio
c	= torsional spring constant
\tilde{c}	= $2cl/EI$
j	= unit imaginary complex number
l	= length of beam
NM	= number of component modes of Rayleigh-Ritz procedure
$\mathcal{O}(\cdot)$	= Landau notation, "of the order of"
PBW_j	= pass-band width of tuned system for j th group of modes
\overline{PBW}_j	= PBW_j/Ω_{2j}
SNF_j	= spread in natural frequencies for j th group of modes
\overline{SNF}_j	= SNF_j/Ω_{2j}
T	= kinetic energy
U	= strain energy
$w(x,t)$	= transverse deflection
$\tilde{w}(x)$	= space dependent part of transverse deflection
x	= distance along beam
\tilde{x}	= x/l
x_1	= $l/2 - \Delta l$, intermediate support location
β_1, β_2	= Lagrange multipliers
$\tilde{\beta}_1, \tilde{\beta}_2$	= amplitudes of β_1 and β_2 , respectively
$\delta\Omega$	= first order perturbation of Ω
Δl	= length deviation
$\overline{\Delta l}$	= $\Delta l/l$

θ	= angular deflection of torsional spring
$\phi_i(x)$	= $\sqrt{2/ml} \sin(i\pi x/l)$, i th component mode
ω_i	= $(i\pi)^2 \sqrt{EI/ml^4}$, i th single-span beam frequency
$\tilde{\omega}_i$	= $\omega_i/\sqrt{EI/ml^4} = (i\pi)^2$
Ω	= natural frequency of two-span beam
$\tilde{\Omega}$	= $\Omega/\sqrt{EI/ml^4}$
$\tilde{\Omega}_0$	= unperturbed value of $\tilde{\Omega}$
$\Omega_{2j-1}, \Omega_{2j}$	= first and second natural frequencies of j th group of modes
(\cdot)	= d/dt
$(\cdot)'$	= d/dx

I. Introduction

THE presence of irregularities in nominally periodic structures may localize the modes of free vibration and inhibit the propagation of vibration within the structure. This phenomenon, referred to as normal mode localization, was first predicted by Anderson in a famous paper¹ and has excited considerable interest in solid-state physics.²⁻⁴ Also, research studies in the field of structural dynamics⁵⁻¹³ have shown that some nearly periodic structures are highly sensitive to irregularities and may exhibit localized modes of vibration. These studies determined two categories of structural systems susceptible to localization:

1) Systems consisting of coupled, similar but slightly *disordered* subsystems. Typical examples include chains of coupled pendula⁵⁻⁸ and jet engine rotors,⁹ for which the physical properties vary slightly from pendulum to pendulum and from blade to blade, respectively. It was shown that strong localization occurs when the *coupling* between subsystems is *small* and that localization becomes more pronounced as the coupling decreases.

2) Structures with irregularly spaced constraints. Examples include a vibrating string with irregularly spaced masses attached⁶ and a beam or plate constrained at irregular intervals.⁷

When strong localization occurs, *small* irregularities usually due to manufacturing and material tolerances result in *dramatic* changes in the dynamics of the system. Since neglecting these irregularities may lead to completely er-

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*Assistant Professor, Department of Mechanical Engineering and Applied Mechanics. Member AIAA.

†Visiting Scholar.

‡Dean of Engineering. Fellow AIAA.

ronous results, it is particularly important to establish criteria capable of predicting the occurrence of localization.

In this paper, the strong localization of the modes of vibration of multispan beams is investigated theoretically and experimentally. Beams constrained at supposedly regular intervals are frequently encountered in structural analysis. Among numerous applications, aircraft fuselages and wings can be modeled by periodic beams. Other examples are building frames and bridges. These "periodic" structures are usually investigated by assuming ideal regularity, even though small deviations of the span lengths from an ideal value may have important effects on the free and forced response of the system.

The modes of vibration of beams simply supported at regular intervals have been studied extensively in the research literature.¹⁴⁻¹⁷ Particularly, one of the first and most important contributions was made in a well-known paper by Miles.¹⁴ Moreover, Lin and Yang¹⁸ investigated the effect of random deviations of the span lengths on the free modes of a multispan beam resting on simple supports, and showed that irregularities had a significant effect on the mode shapes. Nevertheless, their work was not concerned with the study of localization. Miles¹⁴ showed that the natural frequencies of periodic multispan beams are clustered in an infinite number of groups, or bands, with n frequencies in each band, where n is the number of spans. If, in addition to a zero deflection, torsional springs exert restoring moments at the $n-1$ intermediate constraint locations, then the width of the frequency bands diminishes as the spring constant increases. In the limit as the spring constant c goes to infinity, the beam becomes clamped at the constraint locations, and the width of the frequency bands goes to zero.

The inverse of the torsional spring constant may be viewed as a coupling parameter between spans. As $1/c \rightarrow 0$, the spans are "decoupled" because no moment can be transmitted from one span to another. For $c=0$, the beam is simply supported at the constraint locations, and the spans are strongly "coupled," since no restoring moment is exerted. Hence, depending on the value of c , the n -span beam can be regarded as a strongly or weakly coupled set of n -component systems. Moreover, the multispan beam is ordered if all the spans have the same length, and can be rendered disordered by considering slightly irregular constraint locations. Hence for large values of the spring constant and irregular spacing between supports, a multispan beam can be regarded as a *disordered chain of weakly coupled subsystems*, where each subsystem is a span. From the theory of the mode localization phenomenon,^{5,8,11} the free modes of vibration of such a system are susceptible to becoming strongly localized.

In Sec. II of the present paper, the modes of transverse vibration of a disordered two-span beam are investigated theoretically. It is shown that, under certain conditions, strong localization occurs. The degree of localization is dependent upon two parameters: 1) the deviation of the constraint location from the middle of the beam and 2) the value of the stiffness constant of the spring which exerts a restoring moment at the constraint location. The modes are determined by using a Rayleigh-Ritz formulation with the constraints conditions enforced by means of Lagrange multipliers.¹⁹ The method is described in subsection A. In subsection B, classical and modified *perturbation methods* are presented for the analysis of nonlocalized and localized modes, respectively. These perturbation methods provide physical insight into the mechanisms of mode localization. In subsection C results are presented and discussed. These results can be readily extended to n -span beams.

Section III presents an experiment which has been carried out to verify the existence of localized modes for disordered two-span beams. The experiment was the primary responsibility of the second author. Both free vibration natural frequencies and spatial mode shapes were measured. The experimental apparatus is described in detail in subsection A.

Subsection B presents the corresponding experimental results, along with a detailed comparison with theoretical results derived in the first part of the paper. Excellent agreement between theory and experiment is observed.

II. Theory

A. Free Vibration of a Disordered Two-Span Beam

Consider the uniform two-span beam of length l shown in Fig. 1. The beam is simply supported at both ends, and is constrained to have zero deflection at $x=x_1$. Moreover, a torsional spring of stiffness constant c exerts a restoring moment at $x=x_1$. If $x_1=l/2$, the beam is said to be tuned, or ordered; otherwise, it is mistuned, or disordered.

The equations of free bending motion are derived from Hamilton's principle, and a Rayleigh-Ritz procedure with the constraint conditions enforced by means of Lagrange multipliers is chosen.¹⁹ The transverse deflection $w(x,t)$ of the two-span beam is expanded in terms of the free modes of a single-span beam of length l pinned at both ends:

$$w(x,t) = \sum_{i=1}^{NM} a_i(t) \phi_i(x) \quad (1)$$

where a_i are the generalized coordinates, and ω_i and $\phi_i(x)$ are the natural frequencies and normalized mode shapes of the single-span beam, respectively, defined in the nomenclature. The strain and kinetic energies of the two-span beam are

$$U = \frac{1}{2} \sum_{i=1}^{NM} \omega_i^2 a_i^2 + \frac{1}{2} c \theta^2 \quad (2)$$

$$T = \frac{1}{2} \sum_{i=1}^{NM} \dot{a}_i^2 \quad (3)$$

In addition, the beam is constrained at $x=x_1$, and the two constraint equations are given by

$$f_1 \equiv \sum_{i=1}^{NM} a_i(t) \phi_i(x_1) = 0 \quad (4)$$

$$f_2 \equiv \sum_{i=1}^{NM} a_i(t) \phi_i'(x_1) - \theta = 0 \quad (5)$$

Thus the Lagrangian of the system is

$$L \equiv T - U + \beta_1 f_1 + \beta_2 f_2 \quad (6)$$

where β_1 and β_2 are the two Lagrange multipliers corresponding to the constraints in Eqs. (4) and (5).

Applying Hamilton's principle, the equations of free motion are found to be

$$\ddot{a}_i + \omega_i^2 a_i - \beta_1 \phi_i(x_1) - \beta_2 \phi_i'(x_1) = 0, \quad i = 1, \dots, NM \quad (7)$$

$$\beta_2 + c\theta = 0 \quad (8)$$

$$f_1 = 0 \quad f_2 = 0 \quad (9)$$

Assuming simple harmonic motion of natural frequency, Ω , one has

$$a_i = \bar{a}_i e^{i\Omega t}, \quad i = 1, \dots, NM; \quad \beta_k = \bar{\beta}_k e^{i\Omega t}, \quad k = 1, 2 \quad (10)$$

For $\Omega \neq \omega_i$, Eq. (7) may be written as

$$\bar{a}_i = \frac{1}{\omega_i^2 - \Omega^2} [\bar{\beta}_1 \phi_i(x_1) + \bar{\beta}_2 \phi_i'(x_1)] \quad (11)$$

Substituting the above expression of \bar{a}_i into Eqs. (8) and (9) yields, for $\Omega \neq \omega_i$

$$\beta_1 \left[\sum_{i=1}^{NM} \frac{\phi_i^2(x_1)}{\omega_i^2 - \Omega^2} \right] + \beta_2 \left[\sum_{i=1}^{NM} \frac{\phi_i(x_1)\phi_i'(x_1)}{\omega_i^2 - \Omega^2} \right] = 0 \quad (12)$$

$$\beta_1 \left[\sum_{i=1}^{NM} \frac{\phi_i(x_1)\phi_i'(x_1)}{\omega_i^2 - \Omega^2} \right] + \beta_2 \left[\frac{1}{c} + \sum_{i=1}^{NM} \frac{\phi_i'^2(x_1)}{\omega_i^2 - \Omega^2} \right] = 0 \quad (13)$$

Nonzero solutions are obtained for β_1 and β_2 if and only if the determinant of the system [Eqs. (12) and (13)] is equal to zero, yielding

$$\left[\sum_{i=1}^{NM} \frac{\sin^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}^2} \right] \left[\frac{1}{\bar{c}} + \sum_{i=1}^{NM} \frac{(i\pi)^2 \cos^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}^2} \right] - \left[\sum_{i=1}^{NM} \frac{(i\pi) \sin(i\pi\bar{x}_1) \cos(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}^2} \right]^2 = 0 \quad (14)$$

where $\bar{\omega}_i$ and $\bar{\Omega}$ are dimensionless frequencies, \bar{c} is the dimensionless spring constant, and $\bar{x}_1 = x_1/l = 1/2 - \Delta l$ is the dimensionless location of the intermediate support, where Δl is the dimensionless deviation from the middle of the beam.

Recall that Eq. (14) presupposes $\bar{\Omega} \neq \bar{\omega}_i$. It is an eigenvalue equation whose solutions are the free vibration natural frequencies $\bar{\Omega}$ of the two-span beam. For each value of $\bar{\Omega}$ solution of Eq. (14), the corresponding ratio β_1/β_2 is obtained from either Eq. (12) or Eq. (13), and the generalized coordinates amplitudes \bar{a}_i are given by Eq. (11), from which the expression of the spatial mode shape \bar{w} is readily obtained from Eq. (1).

Tuned Beam

In this case $\Delta l = 0$. The natural frequencies of a tuned beam simply supported at its middle are

$$\bar{\Omega}_k = \bar{\omega}_{k+1} = [(k+1)\pi]^2, \quad k \text{ odd} \quad (15)$$

$$\bar{\Omega}_k = \left[\frac{2k+1}{2}\pi \right]^2, \quad k \text{ even} \quad (16)$$

These natural frequencies have a pass-band character, and are placed in groups of two along the frequency axis.¹⁴ As \bar{c} increases, the first frequency of the group increases, while the second remains unchanged. For a beam clamped in the middle, the two natural frequencies of each group are equal, leading to twofold multiple eigenvalues. Hence the width of the frequency bands decreases and goes to zero as \bar{c} goes to infinity. As will be shown later, this bandwidth is one of two key parameters in determining the occurrence of localized modes.

Mistuned Beam

If, for a given value of \bar{c} , a mistuning Δl is introduced, the two frequencies of a group move apart: the width of the frequency bands increases with Δl . This behavior is shown in Fig. 2, which represents the first and second natural frequencies (first group of modes) in terms of Δl for various values of \bar{c} . It should be noted that, for relatively large values of Δl such as 0.07, the band character of the natural frequencies is lost.

Convergence

The convergence of the Rayleigh-Ritz procedure with the number of component modes NM has been checked by considering a mistuned beam clamped at \bar{x}_1 , hence defined by $\Delta l \neq 0$ and $\bar{c} \rightarrow \infty$. In this case the natural modes are those of the two hinged-clamped spans of lengths $1/2 - \Delta l$ and $1/2 + \Delta l$, and thus the exact mode shapes should have exactly a zero

deflection over one of the spans. It was found that a large number of component modes must be considered in order to achieve good convergence. Typically, the mode shapes were almost perfectly flat over one of the spans if $NM \geq 1000$. In the subsequent calculations, 1000 component modes were used. Convergence was also checked for higher modes: until the 25th mode at least, nearly zero deflection in one of the spans was obtained if 1000 or more component modes were used.

This rather slow convergence can be explained by noticing that the series $\sum_{i=1}^{NM} (1/i^4)$ and $\sum_{i=1}^{NM} (1/i^2)$ are involved in the eigenvalue Eq. (14). Although the former series quickly converges, the latter is slowly convergent, thus the Rayleigh-Ritz expansion adopted here slowly converges to the exact solution. However, since this is a linear calculation, the computer cost remains reasonably low. Moreover, if the number of component modes is large enough, very accurate results are obtained, and higher modes can be calculated as easily and accurately as lower ones.

B. Perturbation Analyses

It is much easier to calculate the modes of a tuned two-span beam than of a mistuned one, because the eigenvalue equation is significantly simpler in the former case. This is characteristic of nearly periodic structures with small irregularities: when the structure is disordered, its periodicity properties are lost, and investigating its modes of vibration requires a computational effort much greater than for the associated periodic system. Hence the idea, for small irregularities, of performing a perturbation analysis.

Classical Perturbation Analysis

The unperturbed system consists of the tuned beam. It is perturbed by moving the constraint by a distance Δl . A perturbation analysis can be readily defined from Eq. (14) by expanding the terms $\sin(i\pi\bar{x}_1)$ and $\cos(i\pi\bar{x}_1)$ in terms of Δl to the first or second order, hence obtaining the corresponding natural frequency perturbations $\delta\bar{\Omega}$ and $\delta^2\bar{\Omega}$. This perturba-

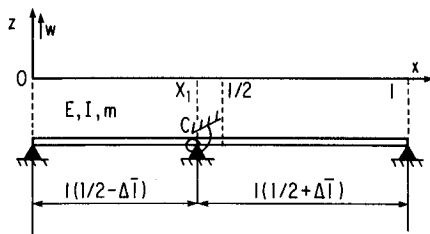


Fig. 1 Geometry of disordered two-span beam.

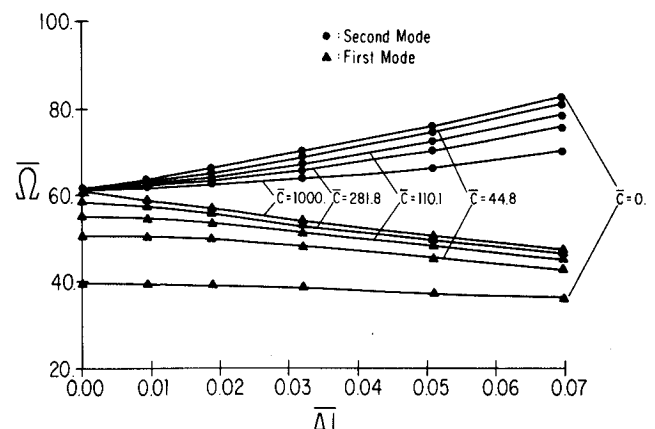


Fig. 2 Natural frequencies of the first group of mode vs mistuning Δl , for various values of \bar{c} .

tion analysis is straightforward and will not be presented in detail. The method has been previously used by Lin and Yang¹⁸ for a beam on simple supports. Note that this approach presupposes that the term $1/\bar{c}$ in Eq. (14) is not small, but has a finite or large value. For $1/\bar{c}$ finite, the modes of vibration of the mistuned system are perturbations of (hence are very similar to) the modes of the tuned system, and are certainly not localized. Thus this case is not of great interest to the present study.

Modified Perturbation Analysis

Herein, small values of $1/\bar{c}$ are considered. If, when the system is mistuned, *only* Δl is considered as a perturbation, then one can expect qualitatively erroneous results: *all* the small parameters (not only Δl , but also $1/\bar{c}$) must be treated as perturbations. The unperturbed system would then be characterized by $\Delta l=0$ and $1/\bar{c}=0$, defining a beam clamped at its middle. The perturbations would consist of Δl and $1/\bar{c}$, leading to a mistuned, "almost" clamped beam. The natural frequencies of the unperturbed system are then repeated, with one twofold multiple eigenvalue in each group. The corresponding mode shapes are defined by any linear combination of a left-span hinged-clamped mode and of a right-span clamped-hinged mode, since the eigenfunctions associated with each double natural frequency span a space of dimension two. In order to perform a perturbation analysis, one must first determine the unique set of two unperturbed mode shapes from which the modes of the perturbed system are continuously obtained. It can be shown²⁰ that this is equivalent to solving the eigenvalue problem for the modes of interest, hence rendering this perturbation procedure ineffective. The conclusion is that one must avoid multiple eigenvalues for the unperturbed system.

This is achieved by introducing some mistuning in the unperturbed system. The unperturbed state is then defined by $1/\bar{c}=0$ and $\bar{x}_1 = 1/2 - \Delta l$. It consists of a mistuned two-span beam clamped at the constraint location. The only perturbation parameter is $1/\bar{c}$. This perturbation method is referred to as modified perturbation method (MPM), and is similar to the one developed in Ref. 8 for a disordered chain of weakly coupled pendula. Since the unperturbed beam is mistuned, its eigenvalues are simple. Also, since it is clamped at $\bar{x} = \bar{x}_1$, its natural modes are the ones of hinged-clamped beams of lengths $1/2 - \Delta l$ and $1/2 + \Delta l$. Note that these unperturbed modes are *decoupled*, that is, they have a zero deflection over one of the two spans, depending on the mode number. When the system is perturbed by $1/\bar{c}$, the modes cease to be decoupled in the left or right spans, and have nonzero deflection in both spans. However, since $1/\bar{c}$ is small, they are perturbations of the decoupled modes, hence are characterized by a deflection which is much larger in one span than in the other one: the modes are *strongly localized*. It is remarkable that one is able to predict whether the modes are localized or not, just by considering perturbations of the eigenvalue Eq. (14).

Let $\bar{\Omega}_0$ be a natural frequency of the unperturbed system. The system is perturbed by replacing the clamped condition by a spring of high stiffness \bar{c} , and the natural frequencies $\bar{\Omega}$ of the perturbed system are such that

$$\bar{\Omega} = \bar{\Omega}_0 + \delta\bar{\Omega} + \mathcal{O}(1/\bar{c}^2) \quad (17)$$

where $\delta\bar{\Omega}$ is a first order perturbation in $1/\bar{c}$. Substituting this first order expansion into Eq. (14) and expanding to the first order yields

$$\delta\bar{\Omega} = \frac{1}{\bar{c}} \frac{-1}{2\bar{\Omega}_0\alpha_{NM}} \sum_{i=1}^{NM} \frac{\sin^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}_0^2} \quad (18a)$$

where

$$\begin{aligned} \alpha_{NM} = & \sum_{i=1}^{NM} \frac{\sin^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}_0^2} \sum_{i=1}^{NM} \frac{(i\pi)^2 \cos^2(i\pi\bar{x}_1)}{[\bar{\omega}_i^2 - \bar{\Omega}_0^2]^2} \\ & + \sum_{i=1}^{NM} \frac{\sin^2(i\pi\bar{x}_1)}{[\bar{\omega}_i^2 - \bar{\Omega}_0^2]^2} \sum_{i=1}^{NM} \frac{(i\pi)^2 \cos^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}_0^2} \\ & - 2 \sum_{i=1}^{NM} \frac{(i\pi) \sin(i\pi\bar{x}_1) \cos(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}_0^2} \sum_{i=1}^{NM} \frac{(i\pi) \sin(i\pi\bar{x}_1) \cos(i\pi\bar{x}_1)}{[\bar{\omega}_i^2 - \bar{\Omega}_0^2]^2} \end{aligned} \quad (18b)$$

The corresponding perturbed mode shape is readily calculated by substituting the perturbed value of $\bar{\Omega}$ into Eqs. (1) and (11-13). Note that a second-order perturbation analysis can also be easily developed. Two important characteristics of perturbation methods are retained by the present analysis. First, the method is cost effective. Second, strongly localized modes are predicted for small values of $1/\bar{c}$ if the beam is mistuned: perturbation methods provide physical insight into mode localization.

C. Results and Discussion

Results

For given values of Δl and \bar{c} , the eigenvalue equation [Eq. (14)] is solved by a standard bisection technique. The bisection process converges rapidly. Typically, 20 to 35 iterations are necessary to obtain natural frequencies converged up to the 10th decimal place. This kind of accuracy is required because very small variations in the natural frequencies may result in significant variations in the mode shapes, since a large number of component modes are considered. The accuracy of the Rayleigh-Ritz procedure and of the bisection process was checked against known results, and in all cases excellent agreement was observed. Even though the number of component modes considered is very large, the CPU time necessary was not excessive. Unless otherwise stated, the following results were obtained by solving directly Eq. (14), not by perturbation analysis.

Figure 3 shows the lower two modes of a tuned beam ($\Delta l=0$) such that $\bar{c}=1000$. One observes that the modes are collective, as opposed to localized: the magnitude of the deflection is the same in each span. Figure 4 displays the lower two modes of a mistuned beam such that $\Delta l=0.01$, for the same $\bar{c}=1000$. One clearly sees that the peak deflection is much larger in one span than in the other one: slight mistuning is sufficient to localize strongly the natural modes. The localized modes are perturbations of the "decoupled" modes corresponding to $1/\bar{c}=0$. Since the system is *mistuned*, these decoupled modes correspond to *simple* eigenvalues. On the other hand, the modes of the *tuned* system such that $1/\bar{c}=0$ correspond to *twofold multiple* eigenvalues, and perturbed modes for small $1/\bar{c}$ do not vary continuously from individual decoupled modes, giving rise to collective modes. The results shown in Fig. 4 were obtained by both the exact method and the modified perturbation method. The agreement is excellent, confirming the fact that the MPM is suitable for the analysis of strongly localized modes. If the spring constant \bar{c} increases, the mode shapes become even more strongly localized. This is observed in Fig. 5, which displays the lower two modes for $\bar{c}=5000$, and for the same mistuning parameter $\Delta l=0.01$. In the limit $\bar{c} \rightarrow \infty$, the modes of the mistuned beam tend to become decoupled. On the other hand, for larger values of $1/\bar{c}$, the modes are only partially localized, and to the limit $\bar{c} \rightarrow 0$, the modes of the (simply supported) mistuned beam are not localized. For $\Delta l=0.01$ and $\bar{c}=0$, it was found that there is only a slight difference between the peak deflections in each span. In this case, the mode shapes are no longer perturbations of decoupled modes, but are perturbations of the collective

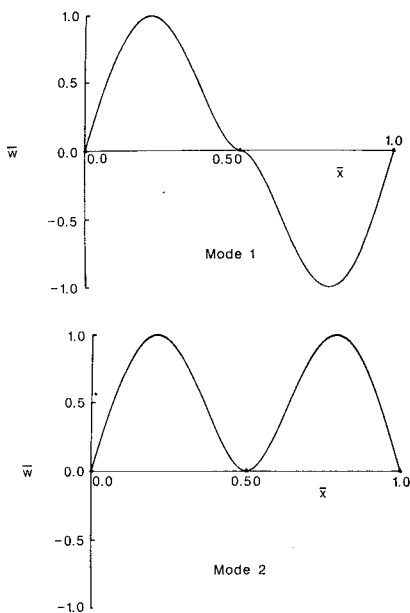


Fig. 3 Lower two mode shapes for a tuned two-span beam ($\Delta l=0$) for $\bar{c}=1000$.

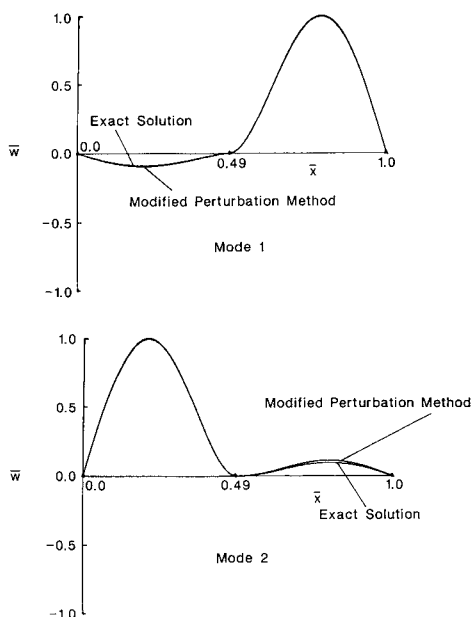


Fig. 4 Lower two mode shapes for a mistuned two-span beam, for $\Delta l=0.01$ and $\bar{c}=1000$, by exact method and modified perturbation method.

modes of the tuned beam, which are shown in Fig. 3. Hence the classical perturbation method (defined for large values of $1/\bar{c}$) would be suitable for this analysis.

In order to investigate systematically the effect of the mistuning parameter Δl and of the spring constant \bar{c} , it is suitable to adopt a compact representation of the modes. The degree of localization of a mode can be characterized by the ratio A of the peak deflection in one span to the peak deflection in the other span, such that the numerator of this ratio corresponds to the span with the smaller peak deflection

$$A = A_s/A_l \tag{19}$$

where A_s and A_l are the peak deflections in each span, such that $A_s \leq A_l$. Note that the ratio A takes values ranging be-

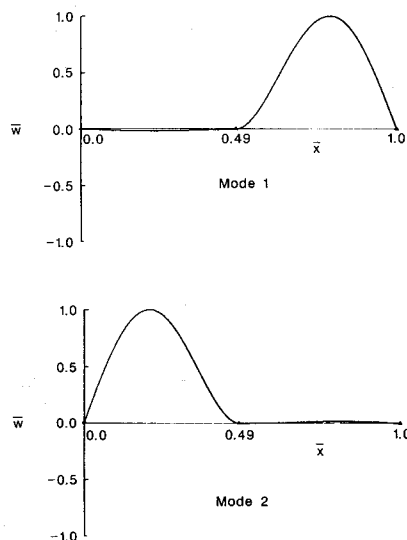


Fig. 5 Lower two mode shapes for a mistuned two-span beam, for $\Delta l=0.01$ and $\bar{c}=5000$.

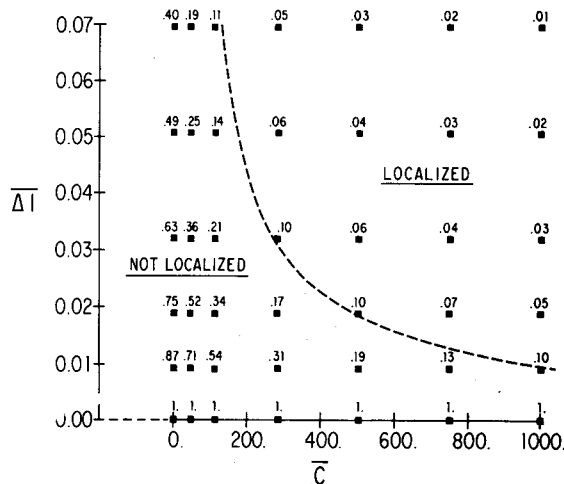


Fig. 6 Values of $|A|$ in the $(\bar{c}, \Delta l)$ plane, for the first group of modes.

tween -1 and $+1$. The smaller the absolute value of A , the more localized the corresponding mode. For decoupled modes, $A=0$. For a tuned beam, no matter how large \bar{c} is, $A = \pm 1$, depending on the mode number.

Figure 6 displays values of $|A|$ in the $(\bar{c}, \Delta l)$ plane, for the first group of modes. To fix ideas, localization is said to occur if the absolute value of the peak ratio A is less than 10%. Note that for a given \bar{c} , A decreases as Δl increases, hence the mode localization becomes more pronounced as the amount of mistuning is increased. In the limit $\bar{c} \rightarrow \infty$, the modes are localized for an arbitrarily small, but nonzero, mistuning. Also, for a given mistuning Δl , localization becomes more pronounced as \bar{c} increases. The larger Δl , the smaller the threshold value of \bar{c} necessary to give rise to localized modes. However, strong localization does not occur for $\bar{c} < 110$, even for relatively large values of mistuning Δl such as 0.07. In particular, the lower modes of a beam simply supported at the constraint location do not become strongly localized. Even if the value 0.07 seems to be small, the reader should bear in mind that this study is conducted within the context of small perturbations, and that $\Delta l=0.07$ corresponds to a 14% deviation of the length of the individual spans, a fairly large value.

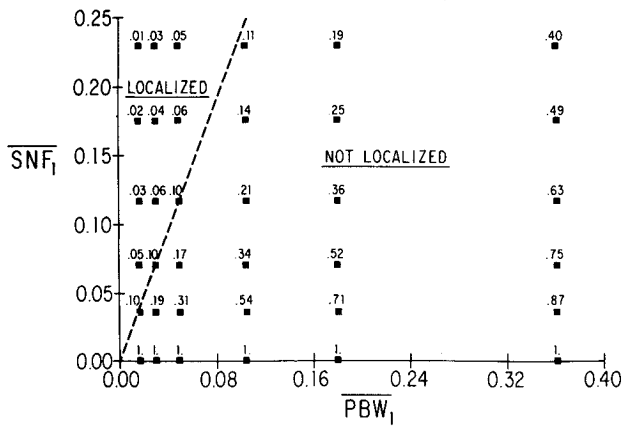


Fig. 7 Values of $|A|$ in the $(\overline{PBW}_1, \overline{SNF}_1)$ plane, for the first group of modes.

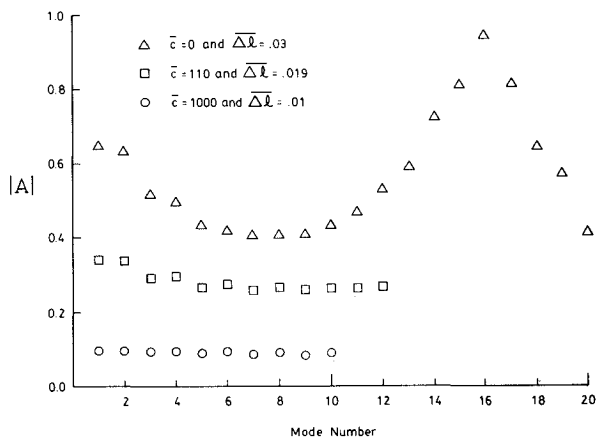


Fig. 8 Variation of $|A|$ in terms of mode number.

An approximate boundary of localization, corresponding to $|A| = 10\%$, is represented on Fig. 6 by a dotted line. From numerical results, one can show that, for various small values of $1/\bar{c}$ and $\Delta\bar{l}$ such that the product $\bar{c}\Delta\bar{l}$ is the same, the peak ratio A remains approximately constant. This can be seen in Fig. 6, as the localization boundary is similar to a hyperbola of equation $\bar{c} = \text{const}/\Delta\bar{l}$. The product $\bar{c}\Delta\bar{l}$ is in fact a disorder to coupling ratio $\Delta\bar{l}/(1/\bar{c})$, and the degree of localization seems to depend only on the value of this ratio. This result is similar to the one obtained in Refs. 5 and 8 for a chain of coupled pendula. Nevertheless, in Ref. 8, this was shown analytically, whereas in the present study, one is required to investigate numerically the dependence of A upon $\Delta\bar{l}$ and \bar{c} .

Discussion

There is a strong analogy between the two-span beam and the system of two coupled pendula studied in Ref. 8. The two pendulum system consists of two coupled single DOF oscillators, each of them being characterized by an individual natural frequency. The amount of coupling is governed by the value of the spring constant k , and mistuning is achieved by changing slightly the individual natural frequencies of the pendula. Similarly, for the two-span beam, the coupling between spans is determined by the inverse of the torsional spring constant. If $1/\bar{c} = 0$, the spans are "decoupled," the same way the two pendula are decoupled for $k = 0$. Each of the individual spans possesses an infinity of natural frequencies, which are for hinged-clamped boundary conditions. Hence if the beam is tuned (respectively mistuned), the two spans have identical (respectively different) individual

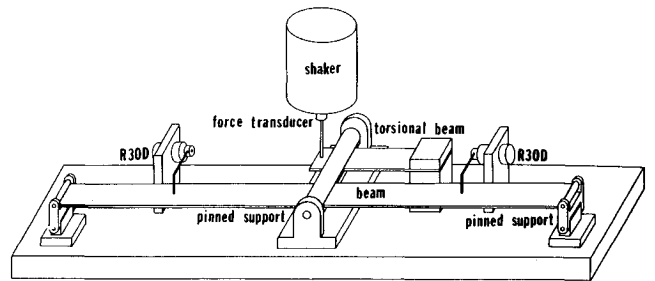


Fig. 9 Experimental apparatus.

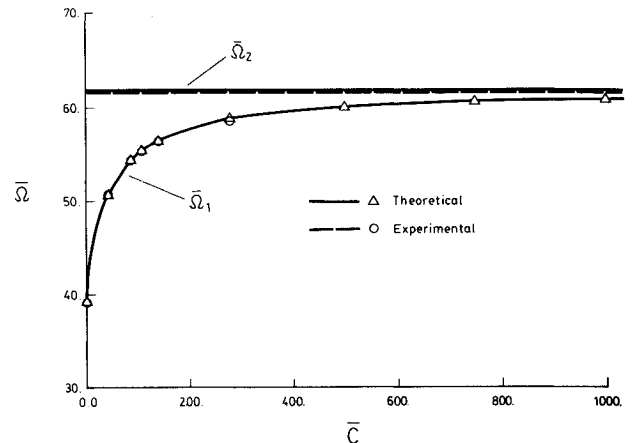


Fig. 10 Natural frequencies of the first group of modes vs \bar{c} , for a tuned two-span beam: theoretical and experimental results.

natural frequencies. It should also be noted that the two-span beam is a system of two coupled, infinite number of DOF oscillators, whereas the pendulum system is constituted of single DOF oscillators. However, recall that the natural frequencies of a two-span beam are distributed by groups of two, and each of these groups can be regarded as corresponding to a two-pendulum system.

It has been shown in Refs. 5 and 8 that the modes of the pendulum system are strongly localized for small values of mistuning and coupling. Similarly, for the two-span beam, strong localization occurs for small values of $\Delta\bar{l}$ and $1/\bar{c}$. For the pendulum system, localized modes are perturbations of decoupled oscillations; for the two-span beam, they are perturbations of "decoupled" hinged-clamped modes. It has also been shown in Ref. 8 that strong localization does not occur for strong coupling between pendula. Similarly, the modes of a two-span beam are not localized for finite or large values of $1/\bar{c}$. In particular, strong localization does not occur for $\bar{c} = 0$, even for relatively large values of $\Delta\bar{l}$ such as 0.07. Moreover, it is clearly seen in Fig. 6 that the effect of mistuning on the peak ratio is not drastic for $\bar{c} = 0$, but rather slowly increasing with $\Delta\bar{l}$. On the other hand, for larger values of \bar{c} , a rapid change of A in terms of $\Delta\bar{l}$ is observed: strong localization occurs. To conclude, the theory of the mode localization phenomenon is for small departure from ideal regularity. However, for larger values of mistuning, even though the modes do not become strongly localized, significant changes can also be observed. For instance, in the case $\bar{c} = 0$ and $\Delta\bar{l} = 0.07$, one observes from Fig. 6 that the peak ratio A of the first group of modes is 0.4, significantly different from the tuned case. Although not relevant to the study of strongly localized modes, these changes may be of interest and of potential importance to the designer. It should also be mentioned that the analogy between the two-pendulum system and the two-span beam can be readily generalized to an n -pendulum system and an

n -span beam, for any n , suggesting that localized vibrations also occur for multispan beams.

In order to understand thoroughly the physical mechanisms of localization for multispan beams, the criterion formulated in Ref. 8 is now considered. This criterion states that a disordered chain of coupled oscillators is susceptible to having strongly localized modes if the natural frequencies of the corresponding tuned, or periodic, system are distributed in groups, and if the widths of these pass-bands are *small* relative to the values of the frequencies belonging to the pass-bands. Localization may occur for such systems if a characteristic spread (due to mistuning) in the individual frequencies of the component subsystems is small, and of the order of, or larger than the pass-band width of the ordered system

$$PBW \leq \mathcal{O}(SNF) \tag{20}$$

of the corresponding tuned, or periodic, system are distributed in groups, and if the widths of these pass-bands are small. Note that for a tuned, or periodic system, $SNF=0$.

As mentioned previously, the natural frequencies of a tuned multispan beam have a pass-band character.¹⁴ Denoting the pass-band width of the j th pair of modes, which contains the $(2j-1)$ th and $2j$ th modes, by PBW_j , one can write

$$PBW_j(\bar{c}) = \sqrt{\frac{EI}{mI^4}} [\bar{\Omega}_{2j} - \bar{\Omega}_{2j-1}(\bar{c})] \tag{21}$$

where the natural frequency of the $2j$ th mode is given by Eq. (16). Since $\bar{\Omega}_{2j-1}(\bar{c})$ increases with \bar{c} , $PBW_j(\bar{c})$ decreases as \bar{c} increases. For $1/\bar{c}$ small, $PBW_j(\bar{c})$ is approximately a linear function of $1/\bar{c}$, and in the limit $1/\bar{c} \rightarrow 0$, PBW_j goes to zero. Hence, small values of the "coupling" $1/\bar{c}$ mean small pass-band width of the ordered system.

The other variable that needs to be defined is the SNF . As previously stated, the beam is decoupled if $1/\bar{c}=0$, its natural frequencies being the ones of the two individual hinged-clamped spans. The spread resulting from mistuning can be written as

$$SNF_j(\bar{\Delta}l) = \sqrt{\frac{EI}{mI^4}} \bar{\Omega}_{2j} \left| 1 - \frac{1}{(1 \pm 2\bar{\Delta}l)^2} \right| \approx \sqrt{\frac{EI}{mI^4}} \bar{\Omega}_{2j} 4 |\bar{\Delta}l| \tag{22}$$

Hence, for small mistuning, SNF_j is proportional to the amount of mistuning.

The following discussion investigates the ability of the criterion [Eq. (20)] to predict localized modes. This paragraph is concerned with the first group of modes, corresponding to $j=1$. Figure 7 displays the absolute value of the peak ratio $|A|$ in the (PBW_1, SNF_1) plane, for the first group of modes. Note that PBW_1 and SNF_1 have been nondimensionalized by $\bar{\Omega}_2$, as defined in the nomenclature. It is observed that localization occurs when SNF_1 and PBW_1 are both small. Moreover, with the definition of localization $|A| \leq 10\%$, the modes are localized in the region approximately defined by $PBW_1 \leq 0.42 SNF_1$, the localization boundary being given by $PBW_1 \approx 0.42 SNF_1$. Note that this boundary is dependent upon the definition chosen for localization: stronger or weaker requirements for localization to occur would result in a quantitatively different, but qualitatively similar boundary. It should also be noted that, from numerical results, the degree of localization $|A|$ seems to be only dependent upon the ratio PBW_1/SNF_1 . Since the localization region shown in Fig. 7 is consistent with the criterion [Eq. (20)], the latter has the ability to predict the occurrence of localization for the first group of modes.

Finally, it is of interest to investigate the localization of higher groups of modes. From Eq. (22), it is clear that SNF_j remains constant when j increases. Also, for a given \bar{c} ,

PBW_j decreases as j increases, and goes to zero in the limit $j \rightarrow \infty$ [for example this can be seen from Eqs. (15) and (16) for $\bar{c}=0$]. Thus for any given \bar{c} (even small), and a given $\bar{\Delta}l$, there exists a group number j^* such that PBW_j is smaller than SNF_j , for any $j > j^*$. According to Eq. (20), this would mean that, for any \bar{c} and $\bar{\Delta}l$, no matter how small, there always exists a threshold value j^* such that higher groups of modes are localized. However, preliminary results do not seem to confirm this hypothesis.

Figure 8 shows the variation of the peak ratio $|A|$ in terms of the mode number, for various values of \bar{c} and $\bar{\Delta}l$. For $\bar{c}=1000$ and $\bar{\Delta}l=0.01$, localization occurs in the first group of modes. Higher modes are still localized, but no more strongly than the first two modes. As a matter of fact, the peak ratio remains almost constant when the mode number increases. For $\bar{c}=110$ and $\bar{\Delta}l=0.019$, the first group of modes is not localized, and the peak ratio decreases only slightly in the higher modes, from 0.33 for the first group to a plateau value of 0.26 for the sixth group. Finally, for $\bar{c}=0$ and $\bar{\Delta}l=0.03$, the peak ratio decreases significantly from 0.64 for the first group to 0.42 for the fifth group. However, the modes do not become localized. Moreover, after the 10th mode, $|A|$ increases to reach 0.93 in the eighth group of modes, and goes back to 0.49 for the 20th mode. In this case, PBW_j decreases monotonically and one can show that according to Eq. (20) localization ought to occur in the eighth or ninth group of modes. However, it does not. Two hypotheses can be formulated from the study of these few representative cases:

1) If the modes of the first group are not localized, it seems that localization will not occur for the modes of higher groups either.

2) If the first two modes are localized, then higher modes are also localized, but no more strongly than in the first group.

This suggests that higher modes do not behave significantly differently than lower modes with respect to localization. Hypothesis 2 is a reassuring result, since it states that localization does not disappear in higher modes. Hypothesis 1 is, of course, in a sense disappointing. These preliminary results suggest that the criterion [Eq. (20)] *cannot* be used *independently* of the physical system to which it applies: the most important condition for strong localization to occur is to have a weakly coupled system, that is to have $1/\bar{c}$ small. If this requirement is met, then the criterion [Eq. (20)] can be applied effectively to determine the minimum value of mistuning $\bar{\Delta}l$ necessary to obtain strongly localized modes. Finally, one ought to mention that, even though the higher modes do not become strongly localized for $\bar{c}=0$, mistuning may have a significant effect, since for $\bar{\Delta}l=0.03$ the peak ratio of the seventh mode is 0.40.

III. Experiment

Few experimental studies of localized vibrations have been conducted to date. Hodges and Woodhouse⁶ carried out an experiment to demonstrate localization, and found satisfactory agreement with the theoretical predictions. The system used in the experiment was a stretched string with irregularly spaced masses attached to it. Also, the research study by Craig et al.¹³ evidenced strong discrepancies between experimental and analytical results for a system of two weakly coupled beams. It was shown in Ref. 13 that these discrepancies were due to small physical dissimilarities between the two beams. The present authors believe that the system studied in Ref. 13 did indeed exhibit strongly localized modes.

A. Experimental Apparatus

The modes of free vibration of a two-span beam were investigated experimentally. The vibration tests were performed on a spring steel beam resting on three supports. The beam

was pinned at both ends. In addition, a third support with variable torsional stiffness was located near the middle of the beam, but could be moved to various locations. The experimental apparatus is shown in Fig. 9. The geometric dimensions of the specimen beam were 53 cm (length) and $0.0635 \times 1.015 \text{ cm}^2$ (cross section). The variable torsional stiffness of the intermediate support was created by a pinned-clamped beam, the distance between the pinned point and the clamped one being varied to adjust the torsional stiffness. The torsional beam was parallel to the specimen beam (see Fig. 9).

The possibility of dynamic interaction between the specimen and torsional beams had to be considered. The frequency range of the first and second modes of the specimen beam was 19–40 Hz. The dynamic torsional stiffness of the torsional beam was measured near the pinned point when removing the specimen beam. It was found that the torsional stiffness remained essentially constant in the frequency range of interest (19–40 Hz). In general, when the fundamental natural frequency of the torsional beam is two to three times higher than the one of the specimen beam, the dynamic torsional stiffness does not vary significantly, and thus can be considered to be constant. In our experimental apparatus, the torsional beam frequency was more than ten times higher than the specimen beam frequency.

The major equipment components for the vibration test were as follows:

- 1) A sine generator provided a sweeping sinusoidal signal.
- 2) A minishaker (B&K 4810) and a power amplifier (B&K 2706) were used to excite the beam. In order to reduce the effect of the additional mass of the joint components between the shaker and the beam on the vibration characteristics of the specimen beam, the driving point was located near the pinned end of the torsional beam (see Fig. 9). Thus a pure excitation torque was applied to the specimen beam near its intermediate support, without inducing any appreciable added mass effect. Because of the small mass of the specimen beam, this effect could have been potentially very important.
- 3) A force transducer (B&K 8200) and two rotary-variable-differential transformers (R30D) were used to measure the exciting force and transverse beam displacement, respectively. The displacement transducers had only a small added mass.
- 4) The charge amplifier was a portable conditioning amplifier (B&K 2635) which provided high-voltage output sensitivity of the force. Digital voltage meters were used to record all signals and to analyze natural frequencies and mode shapes.

In addition to the effect of added mass, there were two other important considerations in the design of the experiment. The first one was concerned with minimizing the effect of the additional constraint due to jointing the R30D transformers with the specimen beam. In order to avoid additional stiffness constraint when large-amplitude vibration occurs, the contact between the needle and the beam had to be sufficiently flexible. This requirement was met by using a flexible needle with a pinned end.

The second consideration concerned the design of the pinned end supports of the specimen beam. From a transient decay test, the critical damping ratio of the beam was found to be approximately 0.001. Thus, large amplitudes occurred near the resonance frequencies. If the horizontal displacement of the end supports were constrained, the measured frequency was found to be dependent upon the level of the excitation torque, which is characteristic of a *nonlinear* system. Thus, in the experimental apparatus, *two* degrees of freedom, namely rotation and horizontal displacement, were allowed at the pinned ends, in order to ensure the *linearity* of the system. It was then found that the natural frequencies were independent of the excitation torque level, and that even when the response amplitude was very large, the system behaved in a linear fashion.

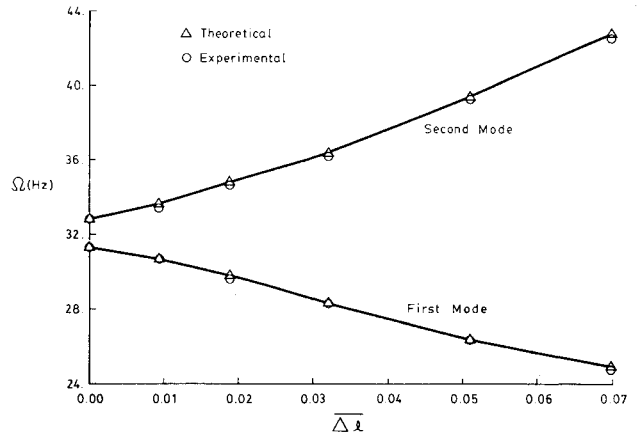


Fig. 11 Comparison of experimental and theoretical natural frequencies of the first group of modes, for $\bar{c}=281.8$.

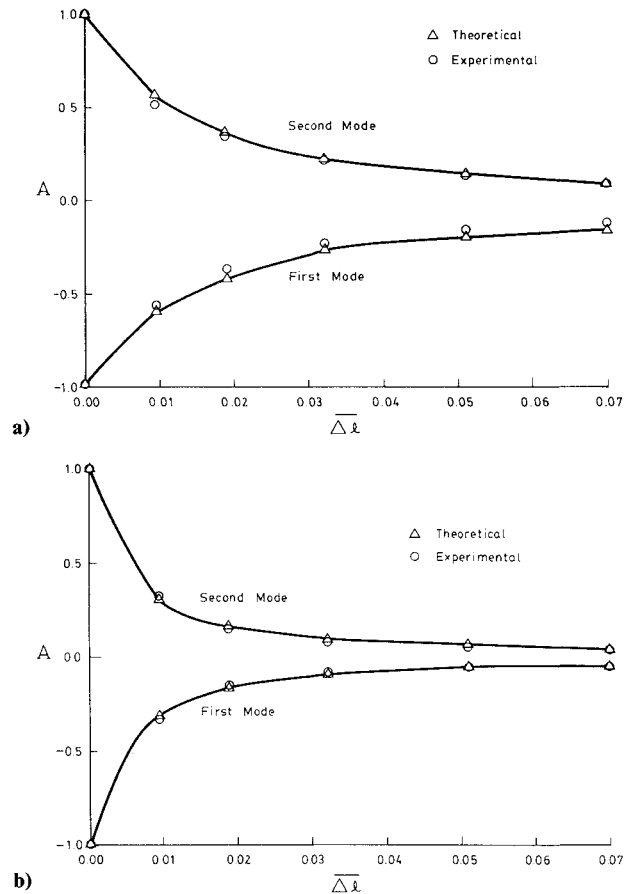


Fig. 12 Comparison of experimental and theoretical peak ratio A for the first group of modes: a) $\bar{c}=90.4$ and b) $\bar{c}=281.8$.

B. Experimental Results: Comparison with Theory

The displacement mobility concept (response displacement/excitation torque) was used to determine the natural frequencies. Figure 10 shows the dependence of the lower two natural frequencies upon torsional stiffness for a tuned beam ($\Delta l=0$), for both experimental and theoretical results. The agreement between theory and experiment is observed to be excellent. Note that the torsional stiffness was determined from a static stiffness measurement.

Figure 11 displays the comparison between theoretical and experimental natural frequencies versus mistuning Δl , for a coupling $\bar{c}=281.8$. Again, the agreement is found to be excellent. Lack of space precludes the authors from presenting

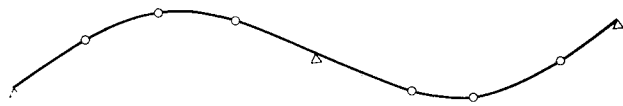


Fig. 13a First mode shape of tuned two-span beam, for $\overline{\Delta l} = 0$ and $\bar{c} = 281.8$, from measurements.



Fig. 13b First mode shape of mistuned two-span beam, for $\overline{\Delta l} = 0.02$ and $\bar{c} = 281.8$, from measurements.

results obtained for other values of the coupling \bar{c} . Nevertheless, in all cases studied, the maximum discrepancy between theoretical and experimental results was always less than 2.5%.

Very good agreement was also found between theoretical and experimental results in terms of mode shapes. This can be observed on Figs. 12a and 12b, which display the peak ratio A for the lower two modes in terms of mistuning $\overline{\Delta l}$, for values of the torsional spring constant $\bar{c} = 90.4$ and $\bar{c} = 281.8$, respectively. Peak ratios were also compared for other values of \bar{c} , but these results are not presented here. The maximum difference between theoretical and experimental data was always less than 15%. This error was mainly due to inaccuracies in the measurement of the small response amplitudes which were encountered for strongly localized modes. For at very small amplitudes the signal to noise ratio of the transducers R30D becomes smaller.

Finally, Fig. 13 shows the motion in the first mode for a torsional spring constant $\bar{c} = 281.8$. Figure 13a is for the tuned system, whereas Fig. 13b is for a slightly mistuned beam such that $\overline{\Delta l} = 2\%$. Both are obtained from displacement measurement. It is observed that the first mode of the mistuned beam is strongly localized in the second span, whereas the one of the tuned beam is collective, that is the peak deflection is the same in both spans.

IV. Concluding Remarks

The modes of vibration of disordered two-span beams subject to a restoring torsional spring moment at the intermediate support have been investigated theoretically and experimentally. The following conclusions can be drawn:

- 1) For small mistuning and large torsional spring constant, the modes of vibration become strongly localized in one of the two spans.
- 2) A modified perturbation method has been developed. It predicts strongly localized modes accurately and provides physical insight into mode localization.
- 3) For the first group of modes, strong localization occurs if the relative pass-band width of the tuned beam is of the order of, or smaller than the relative spread in the frequencies of the individual spans, and if these two quantities are both small.
- 4) From preliminary results, it is suspected that if localization does not occur in the lower two modes, then it does not occur in the higher ones either. On the other hand, if the first two modes are localized, then the higher ones are also localized.
- 5) An experiment has been carried out to verify the existence of strongly localized modes for disordered two-span beams. Excellent quantitative agreement has been found with theoretical results.

6) An immediate generalization of the present study is to investigate the localization of vibrations for n -span beams, where $n > 2$. Since disorder is usually caused by uncertainties, a statistical approach will be required. Future work is also in order concerning localized vibrations of multispan beams resting on elastic supports (leading to multicoupling between spans) and two-dimensional structures.

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