A REDUCED ORDER OBSERVER BASED CONTROLLER
DESIGN FOR H_∞-OPTIMIZATION

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ABSTRACT

In this paper the H_∞ control problem is investigated. It is well-known that for this problem, in general, we need controllers of the same dynamic order as the given system. However, in the case that the standard assumptions on two direct feedthrough matrices are not satisfied, we shown that one can find dynamic compensators of a lower dynamical order. This result can be derived by using the standard reduced order observer based controllers in the case that one or more states are measured without noise.

1. INTRODUCTION

The H_∞ control problem attracted a lot of attention in the last decade. It started with the paper [21]. After that several techniques were developed:

- **Interpolation approach:** e.g. [10]

- **Frequency domain approach:** e.g. [7]

- **Polynomial approach:** e.g. [9]

- **J-spectral factorization approach:** e.g. [8]

- **Time-domain approach:** e.g. [6]

The above list is far from complete. In our view the time-domain approach yielded the most intuitive results. Moreover, the conditions were easily checkable: there exists a stabilizing compensator which makes the H_∞ norm less than 1 if and only if there exist positive semi-definite stabilizing solutions of two algebraic Riccati equations, which satisfy a coupling condition (the spectral radius of their product should be less than 1). However, all the techniques mentioned above had one major drawback. The systems under consideration should satisfy a number of assumptions:

- The subsystem from control to the to be controlled output should not have invariant zeros on the imaginary axis and the direct feedthrough matrix of this system should be injective.

- The subsystem from disturbance to the measurement output should not have invariant zeros on the imaginary axis and the direct feed-through matrix of this system should be surjective.

Note that identical conditions were assumed in the linear quadratic Gaussian control problem. The above assumptions for the H_∞ control problem were removed in [16], [17], [18], and [19]. In this paper we will assume that the conditions on the invariant zeros are still satisfied but we do not make assumptions on the direct feedthrough matrices. This will be called the singular case (contrary to the regular case).

In general (even without any assumptions) it turns out that if we can find a stabilizing controller which makes the H_∞ norm less than 1 (a so-called suitable controller) then we can always find a suitable controller of McMillan degree n (where n is the McMillan degree of our system). Moreover this controller has the standard form of an observer interconnected with a state feedback. However, in the regular case, the direct feedthrough matrix from the disturbance to the measurement output is surjective and hence we cannot observe any states directly: the measurement of each state is perturbed by the disturbance. On the other hand, in the singular case, we can measure, say p, states directly without any disturbances. In principle it then suffices to built a observer for the remaining $n - p$ states which would yield a controller of McMillan degree $n - p$. In this paper we will formalize the above.

In section 2 we will give the problem formulation. Then, in section 3, we will present a preliminary factorization needed in the construction of the controller. Finally, in section 4, we will present our main result and give a constructive method to derive a suitable controller of the required McMillan degree. We conclude in section 5 with some concluding remarks.

2. PROBLEM STATEMENT

Consider the following system

$$E : \begin{cases} x = Ax + Bu + Ew, \\ y = C_1 x \quad\quad + D_1 w, \\ z = C_2 x + D_2 u, \end{cases} \tag{2.1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the control input, $w \in \Re^\ell$ is the unknown disturbance, $y \in \Re^p$ is the measured output and $z \in \Re^q$ is the controlled output. The following assumptions are made:

(a) (A,B,C_2,D_2) has no invariant zeros on the $j\omega$ axis, and

(b) (A,E,C_1,D_1) has no invariant zeros on the jw axis.

Remember that invariant zeros are points in the complex plane where the Rosenbrock system matrix loses rank. Throughout this paper we will assume that there exists a suitable controller, i.e. a stabilizing controller which makes the H_∞ norm strictly less than 1. The goal of this paper is to show existence of and design a reduced order observer based controller of order $n - \text{rank}[C_1, D_1] + \text{rank}(D_1)$ using the measured output y such that the closed-loop system is internally stable and the closed-loop transfer function from the controlled output z to disturbance input w has H_∞-norm less than 1.

3. PRELIMINARY FACTORIZATION§

In this section, we recall a result from [18,19]. Let the original system (2.1) be given.
For $P \in \mathcal{R}^{n \times n}$ we define the following matrix:

$$F(P) := \begin{bmatrix} A^T P + PA + C_2^T C_2 + PEE^T P & PB + C_2^T D_2 \\ B^T P + D_2^T C_2 & D_2^T D_2 \end{bmatrix}$$

If $F(P) \geq 0$, we say that P is a solution of the quadratic matrix inequality. We also define a dual version of this quadratic matrix inequality. For any matrix $Q \in \mathcal{R}^{n \times n}$ we define the following matrix:

$$G(Q) := \begin{bmatrix} AQ + QA^T + EE^T + QC_2^T C_2 Q & QC_1^T + ED_1^T \\ C_1 Q + D_1 E^T & D_1 D_1^T \end{bmatrix}$$

If $G(Q) \geq 0$, we say that Q is a solution of the dual quadratic matrix inequality. In addition to these two matrices, we define two matrices pencils, which play dual roles:

$$L(P, s) := \begin{pmatrix} sI - A - EE^T P & -B \end{pmatrix},$$

$$M(Q, s) := \begin{pmatrix} sI - A - QC_2^T C_2 \\ -C_1 \end{pmatrix}.$$

Finally, we define the following two transfer matrices:

$$G_{ci}(s) := C_2 (sI - A)^{-1} B + D_2,$$
$$G_{di}(s) := C_1 (sI - A)^{-1} E + D_1,$$

Let $\rho(M)$ denote the spectral radius of the matrix M. By $\text{rank}_{\mathcal{R}(s)} M$ we denote the rank of M as a matrix with elements in the field of real rational functions $\mathcal{R}(s)$. We are now in a position to recall the main result from [19]:

Theorem 3.1. Consider the system (2.1). Assume that both the system (A,B,C_2,D_2) as well as the system (A,E,C_1,D_1) have no invariant zeros on the imaginary axis. Then the following two statements are equivalent:

1. For the system (2.1) there exists a time-invariant, finite-dimensional dynamic compensator Σ_{cmp} of the form (3.2) such that the resulting closed-loop system, with transfer matrix G_{cmp}, is internally stable and has H_∞ norm less than 1, i.e. $\|G_{cmp}\|_\infty < 1$.

2. There exist positive semi-definite solutions P, Q of the quadratic matrix inequalities $F(P) \geq 0$ and $G(Q) \geq 0$ satisfying $\rho(PQ) < 1$, such that the following rank conditions are satisfied

   (a) $\text{rank } F(P) = \text{rank}_{\mathcal{R}(s)} G_{ci}$,

   (b) $\text{rank } G(Q) = \text{rank}_{\mathcal{R}(s)} G_{di}$,

   (c) $\text{rank } \begin{bmatrix} L(P, s) \\ F(P) \end{bmatrix} = n + \text{rank}_{\mathcal{R}(s)} G_{ci}, \forall\ s \in \mathcal{C}^0 \cup \mathcal{C}^+$,

   (d) $\text{rank } [M(Q, s) \quad G(Q)] = n + \text{rank}_{\mathcal{R}(s)} G_{di}, \forall\ s \in \mathcal{C}^0 \cup \mathcal{C}^+$.

Note that the existence and determination of P and Q can be checked by investigating reduced order Riccati equations.

Next, we construct a new system,

$$\Sigma_{P,Q} : \begin{cases} \dot{x}_{P,Q} = A_{P,Q} x_{P,Q} + B_{P,Q} u_{P,Q} + E_{P,Q} w, \\ y_{P,Q} = C_{1,P} x_{P,Q} \quad\quad\quad + D_{P,Q} w, \\ z_{P,Q} = C_{2,P} x_{P,Q} + D_P u_{P,Q}, \end{cases} \tag{3.1}$$

where

$$F(P) = \begin{bmatrix} C_{2,P}^T \\ D_P^T \end{bmatrix} [C_{2,P} \quad D_P],$$

$$G(Q) = \begin{bmatrix} E_Q \\ D_{P,Q} \end{bmatrix} [E_Q^T \quad D_{P,Q}^T],$$

and

$$A_{P,Q} := A + EE^T P + (I - QP)^{-1} Q C_{2,P}^T C_{2,P}$$
$$B_{P,Q} := B + (I - QP)^{-1} Q C_{2,P}^T D_P$$
$$E_{P,Q} := (I - QP)^{-1} E_Q$$
$$C_{1,P} := C_1 + D_1 E^T P$$

It has been shown in [19] that this new system has the following properties:

1. $(A_{P,Q}, B_{P,Q}, C_{2,P}, D_P)$ is right invertible and minimum phase.
2. $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$ is left invertible and minimum phase.

Moreover, the following theorem has been proven in [19]:

Theorem 3.2. Let an arbitrary compensator Σ_{cmp} be given as,

$$\Sigma_{cmp} : \begin{cases} \dot{v} = A_{cmp} v + B_{cmp} y, \\ -u = C_{cmp} v + D_{cmp} y. \end{cases}$$
(3.2)

The following two statements are equivalent:

(i) The compensator Σ_{cmp} applied *to* the original system Σ as in (2.1) is internally stabilizing and the resulting closed loop transfer function from w to z has H_∞ norm less than **1**.

(ii) The compensator Σ_{cmp} applied *to* the new system $\Sigma_{P,Q}$ as in *(3.1)* is internally stabilizing and the resulting closed loop transfer function from w to $z_{P,Q}$ has H norm less than **1**.

We will show that there exists a time-invariant, finite-dimensional dynamic compensator Σ_{cmp} of the form (3.2) and with McMillan degree

$$n - \text{rank}[C_1, D_1] + \text{rank}(D_1)$$

for Σ such that the resulting closed loop system is internally stable and the closed loop transfer function from z to w has H_∞ norm less than **1**. Moreover, we give an explicit construction of such a reduced order compensator. More specific, we design a reduced order observer based control law for H_∞ -optimization problem. By the above theorem we can devote all our attention to our new system $\Sigma_{P,Q}$ and design controllers for this system.

4. REDUCED ORDER OBSERVER DESIGN

In this section, we construct explicitly a reduced order observer based controller of order

$$n - \text{rank}[C_1, D_1] + \text{rank}(D_1)$$

for $\Sigma_{P,Q}$. Note that in H_∞ control we have, like in for instance Linear Quadratic Gaussian control, a separation principle: the controllers have the structure of a state feedback interconnected with an observer.

Without loss of generality but for simplicity of presentation, we assume that the matrices $C_{1,P}$ and $D_{P,Q}$ are transformed in the following form:

$$C_{1,P} = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \quad \text{and} \quad D_{P,Q} = \begin{bmatrix} D_0 \\ 0 \end{bmatrix}. \quad (4.1)$$

Thus, the system $\Sigma_{P,Q}$ as in (3.1) can be partitioned as follows,

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{P,Q} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} w, \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \qquad + \begin{bmatrix} D_0 \\ 0 \end{bmatrix} w, \\ z_{P,Q} = C_{2,P} x_{P,Q} + D_P u_{P,Q}, \end{cases}$$
(4.2)

where $[x_1^T, x_2^T]^T = x_{P,Q}$ and $[y_1^T, y_2^T]^T = y_{P,Q}$. First we recall a well known fact in the following observation.

Observation 4.1. Given a matrix quadruple

$$(A_{P,Q}, B_{P,Q}, C_{2,P}, D_P)$$

which is minimum phase and right invertible, then for any given $\varepsilon > 0$, there exists a state feedback gain F_ε such that, $A_{P,Q} - B_{P,Q} F_\varepsilon$ is *asymptotically* stable and

$$\|[C_{2,P} - D_P F_\varepsilon][sI_n - A_{P,Q} + B_{P,Q} F_\varepsilon]^{-1}\|_\infty$$
$$< \frac{\varepsilon}{3\|E_{P,Q}\| + 1}. \quad (4.3)$$

Methods for the construction of such a F_ε can be found by dualizing the results in the appendix.

This clearly shows that, by using state feedback control we can control the system arbitrarily well. Remains our main concern of building a reduced-order observer. The idea is that we only need to build a controller for x_2. Our techniques are based on the method discussed in [1, section 7.2]. The differential equation for x_1 is given by

$$\dot{x}_1 = A_{22} x_1 + \begin{bmatrix} A_{21} & B_2 \end{bmatrix} \begin{bmatrix} y_1 \\ u_{P,Q} \end{bmatrix} + E_2 w$$

where $(y_1, u_{P,Q})$ are known. Observations of x_2 are made via y_1 and:

$$\tilde{y} = A_{12} x_2 + E_1 w = \dot{y}_1 - A_{11} x_1 - B_1 u_{P,Q} \quad (4.4)$$

If we do not worry about the differentiation for the moment we note that we have to build an observer for

the following system:

$$
\Sigma_r : \begin{cases}
\dot{x}_2 = A_{22}x_2 + [A_{21}, B_2]\begin{bmatrix} y_1 \\ u_{P,Q} \end{bmatrix} + E_2 w, \\
\begin{bmatrix} y_0 \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix} x_2 + \begin{bmatrix} D_0 \\ E_1 \end{bmatrix} w.
\end{cases} \tag{4.5}
$$

Note that a system is of minimum phase and left invertible if and only if the Rosenbrock system matrix is left-invertible for all s in the closed right half plane. Using the properties of $\Sigma_{P,Q}$ it is then straightforward to show that the system defined by (4.5) satisfies the following properties:

Lemma 4.1. *For* the system E,, we denote the subsystem from w *to* (yo,$\tilde{y}$) by Σ_{re}. Then we have

1. Σ_{re} is (non-) minimum phase iff $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$ *is (non-) minimum phase.*
2. Σ_{re} is detectable iff $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$ is detectable.
3. Invariant zeros of Σ_{re} are the same **as** those of $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$.
4. Orders of infinite zeros of Σ_{re} are reduced by one *from those of* $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$.
5. Σ_{re} is left invertible iff $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$ is left invertible.'

Proof : See [4].

Next, we build a full-order observer for the system Σ_r defined by (4.5). Using the results of the appendix we find that for all $\varepsilon > 0$ there exists a matrix K_ε such that:

$$
\left\| \left[A_{22} - K_\varepsilon \begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix} \right]^{-1} \left[E_2 - K_\varepsilon \begin{pmatrix} D_0 \\ E_1 \end{pmatrix} \right] \right\|_\infty
$$
$$
< \min \left\{ \frac{\varepsilon}{3\|D_2 F_\varepsilon\| + 1}, \frac{\|E\|}{\|B F_\varepsilon\| + 1} \right\} \tag{4.6}
$$

Using (4.4) we find the following observer related to the observer gain K_ε:

$$
\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + A_{21}y_1 + B_2 u_{P,Q}
$$
$$
+K_\varepsilon \left(\begin{bmatrix} y_0 \\ \dot{y}_1 - A_{11}x_1 - B_1 u_{P,Q} \end{bmatrix} - \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix} \hat{x}_2 \right),
$$
$$
\hat{x}_{P,Q} = \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{x}_2 + \begin{bmatrix} I \\ 0 \end{bmatrix} y_1.
$$

We factorize $IC, = [K_{\varepsilon 0} \quad K_{\varepsilon 1}]$, compatible with the sizes of (yo, $\tilde{y}$). Then, using the change of variables $v := \hat{x}_2 - K_{\varepsilon 1}\tilde{y}$ results in a proper observer. This yields the observer we are going to apply to the system $\Sigma_{P,Q}$. Finally, interconnection with the state feedback gain F_ε

defined in observation **4.1** yields the following reduced order observer based controller for $\Sigma_{P,Q}$,

$$
\Sigma_{rcmp} : \begin{cases}
\dot{v} = (A_{22} - K_{\varepsilon 0}C_{02} - K_{\varepsilon 1}A_{12})v \\
\qquad +(B_2 - K_{\varepsilon,1}B_1)u_{P,Q} \\
\qquad +[K_{\varepsilon 0}, \ A_{21} - K_{\varepsilon 1}A_{11} \\
\qquad +(A_{22} - K_{\varepsilon 0}C_{02} - K_{\varepsilon 1}A_{12})K_{\varepsilon,1}]\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}, \\
u_{P,Q} = -F_\varepsilon \begin{bmatrix} 0 \\ I_{n-p+m_0} \end{bmatrix} v \\
\qquad -F_\varepsilon \begin{bmatrix} 0 & I_{p-m_0} \\ 0 & K_{\varepsilon 1} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.
\end{cases} \tag{4.7}
$$

Some standard algebraic manipulations show that the closed loop transfer matrix from w to $x_{P,Q} - \hat{x}_{P,Q}$ is equal to $G_1(s) := \begin{bmatrix} 0 & G_{err}^{\mathrm{T}}(s) \end{bmatrix}^{\mathrm{T}}$ where,

$$
G_{err}(s) := \left[A_{22} - K_\varepsilon \begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix} \right]^{-1} \left[E_2 - K_\varepsilon \begin{pmatrix} D_0 \\ E_1 \end{pmatrix} \right]
$$

Note that G_{err} is the transfer matrix from w to $x_2 - \hat{x}_2$ when we apply the observer we designed for Σ_r to Σ_r. Therefore, the above shows that our observer for $\Sigma_{P,Q}$ is equally good **as** our observer for Σ_r in any sense, in particular with respect to the H_∞ norm from w to the error. We define G_2 by

$$
G_2(s) := [C_{2,P} - D_P F_\varepsilon][sI_n - A_{P,Q} + B_{P,Q}F_\varepsilon]^{-1}
$$

By (4.3) and (4.6) we find H, norm bounds on G_1 and G_2 respectively. If we apply our reduced order controller Σ_{rcmp} to $\Sigma_{P,Q}$ then the closed loop transfer matrix is equal to:

$$
G_{cl}(s) := G_1(s)E - G_1(s)BFG_2(s) - D_2FG_2(s)
$$

Using the H_∞ norm bounds on G_1 and G_2 we find that $\|G_{cl}\|_\infty < \varepsilon$. Moreover by writing down a state space realization with state space $(x_2 - \hat{x}_2, x)$ for the interconnection $\Sigma_{rcmp} \times \Sigma_{P,Q}$ we immediately note that the closed loop system is asymptotically stable.

Thus we have shown the following theorem:

Theorem 4.1. Let Σ be given by *(2.1)* and define $\Sigma_{P,Q}$ by *(3.1)* and factorize it in the form *(4.2)*. Design feedback and observer gains by *(4.3)* and *(4.6)* respectively for $\varepsilon < 1$. Then the controller defined by *(4.7)* applied *to* Σ is internally stabilizing and the H, norm of the closed loop transfer matrix from w *to* z is strictly less than $\varepsilon < 1$.

Remark 4.1. *In* the case that the given system Σ is regular *(i.e.* in additions *to* the assumptions (a) and

(b), the feedthrough matrices D_1 and D_2 are surjective and injective, respectively), then the controller *(4.7)* reduced *to* the well-known'full order observer based control design for the regular H, -optimization of *[6]*.

5. CONCLUSION

In this paper we presented a technique of finding stabilizing controllers of a dynamical order lower than the dynamical order of the plant which make the H_∞ norm of the closed loop system strictly less than 1. If p states of a system with McMillan degree n are measured without noise, then we find a compensator with McMillan degree n $-p$.

We think that the technique presented in this paper is quite general and can for instance also be applied to the linear quadratic Gaussian control problem .in the case that states are measured without noise.

Acknowledgement

Appendix: Design Algorithms

For economy of notation, in this appendix we will consider the following system Σ_a,

$$\Sigma_a : \begin{cases} \tilde{x} = A\tilde{x} + B\tilde{u}, \\ \tilde{y} = C\tilde{x} + DQ, \end{cases} \quad \text{(A.1)}$$

where $x \in \Re^n$, $u \in \Re^m$ and $y \in \Re^p$. The goal of this appendix is to introduce two algorithms of designing a parameterized gain matrix $K_\epsilon(\sigma)$ such that **for** all $\sigma > \sigma^* > 0$, $A - K_\epsilon(\sigma)C$ is asymptotically stable and

$$\|[sI_n - A + K_\epsilon(\sigma)C]^{-1}[B - K_\epsilon(\sigma)D]\|_\infty < \epsilon$$

for any given $\epsilon > 0$, under the assumptions that Σ_a is left invertible and of minimum phase. Without loss of generality but for simplicity of presentation, we assume that matrices $[C, D]$ and $[B^T, D^T]^T$ are of maximal rank and matrix D is in the form of

$$D = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix},$$

where m_0 is the rank of D. Thus, Σ_a can be partitioned 'asfollows,

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + [B_0, B_1]\begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \end{bmatrix}, \\ \begin{bmatrix} \tilde{y}_0 \\ \tilde{y}_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\tilde{x} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \end{bmatrix}, \end{cases} \quad \text{(A.2)}$$

where Bo, B_1, C_0 and C_1 are the matrices of appropriate dimensions. First we have the following simple observation.

Observation A.1. Assume that $rank(B_1) > 0$ and let $A_1 = A - B_1 C_1$. We have the following:
1. (A_1, B_1, C_1) is left invertible and of minimum phase *iff* (A, B, C, D) is left invertible and **of** minimum phase.
2. Invariant zeros of (A_1, B_1, C_1) are the same as those of (A, B, C, D).

Proof : See lemma 2.1 of [2]. ∎

In the following we present two design algorithms for the computation of $K_\epsilon(\sigma)$. The first one is the cheap control approach or ARE-based design and the second one is the asymptotic time-scale and eigenstructure assignment (ATEA) design. In ARE-based design the asymptotic behavior of the fast eigenvalues of $A - K_\epsilon(\sigma)C$ are fixed by the infinite zero structure of the system Σ_a. However in ATEA design one can assign arbitrarily the asymptotic behavior of these eigenvalues. For a detailed discussion and comparison between ARE-based and ATEA design the interested readers are referred to [12].

The Cheap Control Approach

Step 1 : Solving the following algebraic Riccati equation,

$$A_1 P + PAT - PC_1^T C_1 P + \sigma^2 B_1 B_1^T = 0, \quad \text{(A.3)}$$

for the positive solution P.
Step 2 : Calculate

$$K_1(\sigma) = PC_1^T.$$

Step 3 : Let

$$K_\epsilon(\sigma) = [B_0, K_1(\sigma)]. \quad \text{(A.4)}$$

We have the following lemma.

Lemma A.1. Consider a system Σ_a as in (A.l) which is left invertible and which is of minimum phase. Let $K_\epsilon(\sigma)$ be computed via the above algorithm. Then for any given $E > 0$, there exists a $\sigma^* > 0$ such that **for** all $\sigma > \sigma^*$, $A - K_\epsilon(\sigma)C$ is asymptotically stable and

$$\|[sI_n - A + K_\epsilon(\sigma)C]^{-1}[B - K_\epsilon(\sigma)D]\|_\infty < \epsilon$$

Proof : Since (A_1, B_1, C_1) is of minimum phase and left invertible, it is shown in Doyle and Stein [5] that $K_1(\sigma)$ calculated in the above procedure has the following properties : as $\sigma \to \infty$,

$$[sI_n - A_1 + K_1(\sigma)C_1]^{-1}B_1 \to 0 \quad \text{pointwise in } s$$

and $A_1 - K_1(\sigma)C_1$ is asymptotically stable. Hence, for any given $\epsilon > 0$ there exists $\sigma^* > 0$ such that for all $\sigma > \sigma^*$, $A_1 - K_1(\sigma)C_1 = A - K_\epsilon(\sigma)C$ is stable and

$$\|[sI_n - A_1 + K_1(\sigma)C_1]^{-1}B_1\|_\infty < \epsilon,$$

which implies

$$\|[sI_n - A + K_\epsilon(\sigma)C]^{-1}[B - K_\epsilon(\sigma)D]\|_\infty < \epsilon.$$

The ATEA Approach

We recall the following theorem first.

Theorem A.1. Under the condition that (A_1, B_1, C_1) is of minimum phase and left invertible, there exist nonsingular transformations Γ_1, Γ_2 and Γ_3 and integer indexes q_j, $j = 1$ to $m - m_0$, such that

$$\tilde{x} = \Gamma_1 x, \quad \tilde{y}_1 = \Gamma_2 y_1, \quad \tilde{u}_1 = \Gamma_3 u_1,$$

$$x = [x_a^T, x_b^T, x_{f_1}^T, x_{f_2}^T, \cdots, x_{f_{m-m_0}}^T]^T,$$

$$u_1 = [u_{11}, u_{12}, \ldots, u_{1m-m_0}]$$

$$y_1 = [y_f^T, y_b^T]^T, \quad y_f = [y_{f_1}, y_{f_2}, \cdots, y_{f_{m-m_0}}]^T$$

and

$$\dot{x}_a = A_a x_a + L_{af} y_f + L_{ab} y_b,$$

$$\dot{x}_b = A_b x_b + L_{bf} y_f, \quad y_b = C_b x_b,$$

$$\dot{x}_{f_j} = A_{f_j} x_{f_j} + L_j y_f$$
$$+ B_{f_j}[u_{1j} + E_{ja} x_a + E_{jb} x_b + \sum_{\ell=1}^{m-m_0} E_{j\ell} x_{f_\ell}],$$

$$y_{f_j} = C_{f_j} x_j, \quad j = 1, 2, \cdots, m - m_0.$$

Moreover, $\lambda(A_a) \in C^-$ are the invariant zeros of (A_1, B_1, C_1); (A_b, C_b) is observable and for $j = 1$ to $m - m_0$,

$$A_{f_j} = \begin{bmatrix} 0 & I_{q_j-1} \\ 0 & 0 \end{bmatrix}, \quad B_{f_j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{f_j} = [1, 0].$$

Proof : See Sannuti and Saberi [15]. ∎

Now we are ready to introduce the ATEA design algorithm. For the sake of simplicity, we only consider ATEA design with two–time-scale structure assignment in this appendix while the details of ATEA design using multi-time-scale structure assignment can be found in Saberi and Sannuti [13].

Step 1 : Select a gain K_b such that $\lambda(A_b - K_b C_b)$ are in the desired locations in C^-.

Step 2 : For each $j = 1$ to $m - m_0$, select a gain K_{f_j} such that $\lambda(A_{f_j} - K_{f_j} C_{f_j})$ are in the desired finite location in C^-, where K_{f_j} can be partitioned as

$$K_{f_j} = [K_{j1}, K_{j2}, \cdots, K_{jq_j}]^T.$$

Let

$$\tilde{K}_{f_j}(\sigma) = [\sigma K_{j1}, \sigma^2 K_{j2}, \cdots, \sigma^{q_j} K_{jq_j}]^T.$$

Step 3 : Let

$$K_1(\sigma) = \Gamma_1 \begin{bmatrix} L_{af} & L_{ab} \\ L_{bf} & K_b \\ K_f(\sigma) + L_f & 0 \end{bmatrix} \Gamma_2^{-1},$$

where

$$K_f(\sigma) = \text{diag}[\tilde{K}_{f_1}(\sigma), \tilde{K}_{f_2}(\sigma), \cdots, \tilde{K}_{f_{m-m_0}}(\sigma)],$$

and

$$L_f = [L_1^T, L_2^T, \cdots, L_{m-m_0}^T]^T.$$

Step 4 : Let

$$K_\epsilon(\sigma) = [B_0, K_1(\sigma)]. \tag{A.5}$$

We have the following lemma which is analogous to lemms A.1.

Lemma A.2. Consider a system Σ_a as in (A.1) which is left invertible and which is of minimum phase. Let $K_\epsilon(\sigma)$ be computed via the above ATEA algorithm. Then for any given $\epsilon > 0$, there exists a $\sigma^* > 0$ such that for all $\sigma > \sigma^*$, $A - K_\epsilon(\sigma)C$ is asymptotically stable and

$$\|[sI_n - A + K_\epsilon(\sigma)C]^{-1}[B - K_\epsilon(\sigma)D]\|_\infty < \epsilon$$

Proof : The above algorithm is a special case of Saberi and Sannuti [14]. It is shown in [14] that $K_1(\sigma)$ calculated in the above ATEA procedure has the following properties : as $\sigma \to \infty$,

$$[sI_n - A_1 + K_1(\sigma)C_1]^{-1}B_1 \to 0 \quad \text{poitwise in } s$$

and $A_1 - K_1(\sigma)C_1$ is asymptotically stable. Hence, for any given $\epsilon > 0$ there exists $\sigma^* > 0$ such that for all $\sigma > \sigma^*$, $A_1 - K_1(\sigma)C_1 = A - K_\epsilon(\sigma)C$ is stable and

$$\|[sI_n - A_1 + K_1(\sigma)C_1]^{-1}B_1\|_\infty < \epsilon,$$

which implies

$$\|[sI_n - A + K_\epsilon(\sigma)C]^{-1}[B - K_\epsilon(\sigma)D]\|_\infty < \epsilon.$$

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