Convergence of Finite-Dimensional Conjugate Direction and Quasi-Newton Methods for Singular Problems

B. D. Cheng and W. F. Powers, University of Michigan, Ann Arbor, Mich.
CONVERGENCE OF CONJUGATE DIRECTION AND QUASI-NEWTON METHODS
ON SINGULAR PROBLEMS: THE FINITE-DIMENSIONAL CASE

Bang-Dar Cheng and William F. Powers
University of Michigan
Ann Arbor, Michigan

Abstract

The convergence properties of the gradient, conjugate gradient, Davidon-Fletcher-Powell, and Powell's method for the singular, finite-dimensional quadratic minimization problem are developed. It is shown that for all of the methods, except the gradient method, that the minimum is obtained in at most \( m \) iterates, where \( m \) is the dimension of the range of the Hessian matrix, as opposed to \( n > m \) iterates for nonsingular problems. A class of associated nonsingular quadratic problems is defined to show that the gradient method has slower convergence on singular problems than on corresponding nonsingular approximations to the singular problems while the conjugate direction methods have more rapid convergence. This implies that slow convergence attributed to singular problems is actually a property of the gradient method as opposed to the singularity of the problem.

1. Introduction

There is widespread belief that singular optimal control problems are more difficult to compute than nonsingular problems. There is good reason for this since many researchers have experienced the slow convergence of the gradient method on a singular problem and/or the special preparations necessary to apply a Newton-type method (both shooting and function space types) to a singular problem. In fact, Johansen studied the rate of convergence of the gradient method on the singular problem and verified theoretically the poor convergence characteristics noted in practice.

In Ref. 3 a recently developed class of accelerated function-space gradient methods, known as function-space quasi-Newton methods, was shown to converge certain singular optimal control problems much more accurately than the standard gradient method. Defects in the methods with regard to storage were also eliminated and a relatively large, realistic Space Shuttle trajectory optimization problem was solved with the methods. Thus the major remaining problems associated with the methods involve theoretical questions of convergence and rate of convergence, especially on singular problems. For example, was the improved convergence reported in Ref. 3 problem dependent or applicable to more general classes of problems?

The goal of this paper is to present the results of the first part of such a theoretical study, namely the convergence of a number of algorithms on the finite-dimensional singular problem. In addition to these results being useful in their own right, they also indicate an approach to the infinite-dimensional (or optimal control) problem which will be reported in a subsequent paper.

The convergence of the gradient, conjugate gradient (CG), \( ^{5, 6} \) and Davidon-Fletcher-Powell (DFP), \( ^{7, 8} \) and Powell's \( ^{9} \) methods will be analyzed herein. For nonsingular quadratic optimization problems, convergence questions have been investigated by many authors with the following results: (i) Linear convergence for the gradient method in both finite-dimensional and function spaces \( ^{10} \). (ii) Finite-step convergence for both the CG and the DFP methods in finite-dimensional space \( ^{6, 8} \). (iii) A rate of convergence for the CG method \( ^{10, 11} \) and a convergence proof for the function-space DFP method \( ^{12, 13} \).

To date few papers have been concerned with the case of singular quadratic optimization problems. A convergence proof for a general system of \( m \) linear algebraic equations in \( m \) unknowns is implicit in Ref. 14, and as shown in Ref. 15 the necessary and sufficient condition for a very general class of iterative schemes to converge is that the linear system be positive semidefinite. Nashed and Kammerer present convergence proofs for gradient and conjugate gradient methods applied to singular linear operator equations, but no such results exist for the application of quasi-Newton methods to the singular case.

In Section 2 the singular quadratic optimization problem and algorithms are presented along with the existence conditions for the problem. The main convergence theorem is presented in Section 3 along with examples to illustrate its properties. In Section 4 an example is thoroughly analyzed to demonstrate that defects attributed to the singular problem are actually due to a defect in the gradient method, and that conjugate direction methods actually have improved convergence properties on singular problems.

2. Problem Formulation and Algorithms

---

*This research was supported by the National Science Foundation under Grant ENG 74-21618.

**Graduate Student, Department of Aerospace Engineering.

***Professor, Department of Aerospace Engineering.
Consider the problem of determining a minimum of an unconstrained function \( f \), where \( f \) has continuous partial derivatives of at least second order. Convergence and rate of convergence analyses are restricted to the neighborhood of the minimum, and thus are actually terminal convergence properties. Thus, a quadratic approximation of the problem is employed for such analyses, and thus we shall consider the problem of determining an element \( x^* \in \mathbb{R}^n \) which minimizes the quadratic function

\[
 f(x) = \frac{1}{2} <x, Q x> + <x, w> + f_0 \tag{2.1}
\]

where \( x, w \in \mathbb{R}^n, f \in \mathbb{R}, <x, w> = x^T w \) denotes the inner product in \( \mathbb{R}^n \) and \( Q \) can be assumed, without loss of generality, to be an \( n \times n \) symmetric matrix. The gradient of \( f(x) \) at the element \( x \), denoted by \( g(x) \in \mathbb{R}^n \), is

\[
 g(x) = \frac{df(x)}{dx} = Q x + w \tag{2.2}
\]

If \( Q \) is positive definite, then the minimum solution always exists, i.e., \( x^* = -Q^{-1} w \). For a general quadratic function \( f \), we need the following property.

Property 2.1: Problem (2.1) has a minimum solution \( \bar{x} \) if and only if

\[
 (i) \ Q \text{ is positive semidefinite (denoted by } Q \geq 0) \tag{2.3} \\
 (ii) w \text{ belongs to the range of } Q (\text{denoted by } w \in \text{R}(Q)) \tag{2.4}
\]

Proof: Since \( f \) is twice differentiable, if \( x \) is a minimum element, then \( g(\bar{x}) = Q \bar{x} + w = 0 \) and \( <x, \frac{d^2 f(\bar{x})}{dx^2} x> \geq 0, x \in \mathbb{R}^n \). But \( Q \bar{x} \in \text{R}(Q) \)

\[
 g(x) = 0 \text{ implies that } w = -Q\bar{x} \in \text{R} \tag{2.5}
\]

since \( \frac{d^2 f(x)}{dx^2} Q, <x, \frac{d^2 f(\bar{x})}{dx^2} x> \geq 0 \) implies that \( \frac{d^2 f(x)}{dx^2} Q, <x, \frac{d^2 f(\bar{x})}{dx^2} x> \geq 0 \), that is, \( Q \) is positive semidefinite.

Now, suppose \( w \in \text{R}(Q) \), there exist an element \( \bar{x} \) such that \( Q \bar{x} + w = 0 \). Let \( \bar{x} \) be any element in \( \mathbb{R}^n \), then \( \bar{x} = \bar{x} + y \) where \( y = x - \bar{x} \in \mathbb{R}^n \). After some calculations, \( f(\bar{x}) = f(\bar{x} + y) = \frac{1}{2} <y, Q y> + f(\bar{x}) \), or \( f(x) - f(\bar{x}) = \frac{1}{2} <y, Q y> \). Since \( Q \) is positive semidefinite, we obtain \( f(x) \geq f(\bar{x}) \), i.e., \( \bar{x} \) is a minimum solution of Problem (2.1).

Remark 2.1: If \( Q = Q^T \geq 0 \) and \( w \in \text{R}(Q) \), there exist infinitely many minimum solutions of \( f(x) \). Actually, if \( x^* \) is a minimum solution of \( f(x) \), then \( \bar{x} = x^* + x_0 N \), for any \( x_0 \in N(Q) \) (the null space of \( Q \)), are also minimum solutions of \( f(x) \), that is, \( f(\bar{x}) = f(x^*) \).

Remark 2.2: In the process of proving Property 2.1, we see that once an element \( \bar{x} \) such that \( g(\bar{x}) = Q \bar{x} + w = 0 \) is determined, then \( \bar{x} \) is a minimum solution of \( f(x) \).

The four iterative methods of interest here: the gradient, CG, DFP, and Powell's methods, will now be summarized. In general, the first three iterative schemes mentioned above attempt to generate a sequence \( \{x_i\} \) which eventually converges to a minimizing element \( x^* \), they all involve the iteration rule

\[
 x_{i+1} = x_i + \alpha_i S_i \tag{2.5}
\]

An initial element \( x_0 \in \mathbb{R}^n \) is chosen arbitrarily. At each step a direction \( S_i \) is chosen (the way this is done will define the method used) and a step size is determined such that

\[
 f(x_i + \alpha_i S_i) \leq f(x_i + \lambda S_i) \text{ for all } \lambda > 0 \tag{2.6}
\]

This leads to the condition

\[
 <g_i + \alpha_i S_i, S_i^i > = 0, \quad (g_i + \alpha_i S_i, g_i + \alpha_i S_i) > 0 \tag{2.7}
\]

In the gradient method

\[
 S_i = -g_i \text{ for all } i \tag{2.8}
\]

In both the CG and the DFP methods, the members of the sequence \( \{S_i\} \) are chosen to be \( Q \) conjugate, i.e., they satisfy

\[
 <S_i, Q S_j> = 0 \quad i \neq j \tag{2.9}
\]

In the CG method, \( S_i \) is taken as

\[
 S_i = -g_i + \frac{<g_i, g_j>}{<g_j, g_i> - <g_i, g_i>} S_{i-1} \text{ with } S_0 = -g_0 \tag{2.10}
\]

In the DFP method

\[
 S_i = -H_{i-1} g_i \tag{2.11}
\]

where

\[
 H_{i-1} = H_{i-1} + \frac{P_{i-1} - <P_{i-1}, H_{i-1} Y_{i-1}> H_{i-1} Y_{i-1}^T}{<P_{i-1}, Y_{i-1} - H_{i-1} Y_{i-1}>^2} \tag{2.12}
\]

The dyadic notation \(<x>\) is defined as \( (x^T) \)

\[
 (a < b) \in R^n \tag{2.13}
\]

and the following property is then easily verified

\[
 (a < b) e = a < b, e > a, b, e \in R^n \tag{2.14}
\]

The initial matrix \( H_0 \) is chosen to be any positive
In Powell's method, the directions of search are generated by the following four steps: Denote the starting point by \( x_0 \)

(i) Initially choose \( d_1, d_2, \ldots, d_m \) in the direction of \( n \) coordinates, that is, \( d_i = e_i \) where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \).

(ii) Given \( d_1, d_2, \ldots, d_n \), find \( t^* \) so that \( f(x) = f(x_0 + t d) \) is minimized. Define \( x_t = x_0 + t d \). Name the latter as \( d_1 \).

(iii) Generate a new direction \( d \) by \( d = x - x_t \) and replace \( d_1, \ldots, d_m \) by \( d_1, \ldots, d_m \). Re-name the latter as \( d_2 \), \( d_3 \), \ldots, \( d_m \).

(iv) Minimize \( f(x) = f(x_0 + t d) \) and replace \( x_0 \) by \( x_0 + t \). Where \( t = \text{the minimizing } t \). Take this point as \( x_0 \) of the next iteration, and go to (ii).

It can be shown that the search directions generated in Powell's method at every iteration are mutually \( Q \)-conjugate for a quadratic function (see, for instance, Ref. 20, p. 158). This property holds for \( Q \) positive semidefinite also as long as \( d \neq 0 \). We will show later that if \( d = 0 \) during some iteration, then the initial element \( x_0 \) of that iteration is indeed a minimum solution.

We shall now investigate the behavior of these algorithms on the singular quadratic problem (2.1) subject to (2.3) and (2.4).

3. The Main Convergence Theorem

Before proving the general convergence theorem, the following properties are required.

Property 3.1: Let \( Q \) be a \( n \times n \) symmetric positive semidefinite matrix (i.e., \( Q^T = Q \geq 0 \)), let \( R(Q) \), \( N(Q) \) be the range of \( Q \) and the null space of \( Q \), respectively, and \( m \) be the rank of \( Q \). Then,

\[
\begin{align*}
\text{(i)} & \quad R(Q) = \{ N(Q) \mid x \}, \quad R^n = R(Q) \oplus N(Q) \\
\text{(ii)} & \quad <x, Qx> = 0 \quad \forall x \in N(Q)
\end{align*}
\]

Proof: Part (ii) can be found in any elementary matrix book, so we need only prove (i). If \( x \not\in N(Q) \), clearly \( Qx = 0 \) and \( <x, Qx> = 0 \). Now, since \( Q \geq 0 \), \( Q^{1/2} \) exists and \( Q^{1/2} \geq 0 \). Then

\[
<q, Qx> = 0 \quad \forall x \in N(Q)
\]

which implies \( Q^{1/2} (Q^{1/2} x) = 0 \). Then \( Q^{1/2} (Q^{1/2} x) = Qx \).

Property 3.2: Let \( Q = Q^T \geq 0 \) and \( \{ d_1, d_2, \ldots, d_m \} \) be a nonzero \( Q \)-conjugate set of elements in the range of \( Q \). If \( X \) minimizes problems (2.1), subject to (2.4) for all \( x \in R(Q) \), then \( g(X) = 0 \).

Proof: First note that this property is trivial if \( R(Q) = R^n \), but requires proof when \( \text{Rank}(Q) < n \). Since \( \{ d_1, d_2, \ldots, d_m \} \) are \( Q \)-conjugate and nonzero, they are also independent. Then, since \( \text{Rank}(Q) = m \), \( \{ d_1, d_2, \ldots, d_m \} \) form a basis for \( R(Q) \). For any \( x \in R(Q) \), \( c_1, c_2, \ldots, c_m \geq 0 \), implies

\[
<q, X> = \sum_{i=1}^{m} c_i d_i \quad \forall c_i \geq 0.
\]

Combining (3.2) and (3.3)

\[
g(x) = Qx + w \in R(Q) \quad \forall w \in R(Q)
\]

i.e., \( g(x) = 0 \), which completes the proof.

Theorem 1: Consider the problem (2.1) subject to (2.3) and (2.4). Let \( \{ x_k \} \) be a sequence of vectors in \( R^n \) generated by either the CG, DFP or Powell's method. Then, the sequence converges to a minimum vector \( x \), in at most \( m \) iterates, where \( x \) depends on the initial guess \( x_0 \) and \( m \) is the rank of \( Q \).

Proof: The details of the proof are presented in Appendix A. It is shown there that condition (ii) of property 3.1 guarantees that all iterates of the algorithms are well-defined, and property 3.2 leads to the finite \( m \)-step convergence result.

Remark 3.1: From the proof of the above theorem, the finite \( m \)-step convergence result holds for any algorithms which generates conjugate directions and employs an exact linear search. For example, all of Broyden's Quasi-Newton and Huang's methods possess this property.

Remark 3.2: Usually the minimum solution \( x_\infty \) depends on the initial estimate \( x_0 \). Actually, if we let \( S = \{ x \mid x \text{ is a minimum solution of } f(x) \} \), then \( S \) is a non-empty closed convex set, hence there exists \( x_\infty \in S \) such that

\[
\|x_\infty\| \leq \|x\| 
\]

for all \( x \in S \), where \( \| \cdot \| \) denotes the Euclidean norm in \( R^n \). Nash showed that the sequence \( \{ x_k, x_{k+1}, \ldots, x_m \} \) generated by the gradient method converges to \( x_\infty = x_k + (1-P)x_\infty \), where \( P \) is the projection matrix from \( R^n \) to \( R(Q) \). That is, if \( x_k = x_k + x_n \), where \( x_n \in R(Q) \) and \( x_k \in N(Q) \), then \( x_n = x_n + x_n \). Now, if \( x_k \) is in the DFP method, the search directions \( S \), in both the DFP and CG methods are linear combination of \( g_1, g_2, \ldots, g_m \). By the same procedures as in Ref. 16 we can prove that the sequence \( \{ x_1, x_2, \ldots, x_m \} \) generated by the DFP and CG methods
(with \( H_0 = I \)) converges to \( x^* = x^0 + (I - P) x \), where \( x^* \) is the unique minimum norm solution. Therefore, if we are interested in obtaining the minimum norm solution \( x \) by DFP or CG methods, the simplest way is to choose \( x = x_0 = 0 \) as the initial estimate.

**Remark 3.3**: For a nonsingular quadratic function \( f(x) = \frac{1}{2} x^T A x + b^T x + c \), where \( A = A^T > 0 \), the DFP method guarantees \( H_k > 0 \) for \( 0 < k < n \) and \( H_n = A^{-1} \) if the sequence \( \{ x_k \} \) converges in exactly \( n \) iterations. For the singular case, \( A = \lambda I \geq 0 \), \( 0 < \lambda \), \( H \) does not exist, but it is of interest to characterize the behavior of \( H_k \) in this case. The results are as follows:

**Property 3.3**: If \( \{ x_k \} \) converges at the \( k \)th iterate \((k < n)\) for the DFP method applied to a singular problem, then

\[ H_k > 0 \quad \text{and} \quad 0 < k \leq k \]

\[ H_k \geq 0 \quad \text{and} \quad k \leq k \]

\[ H_k \geq 0 \quad \text{and} \quad k \leq k \]

**Remark 3.5**: Myers showed that, for nonsingular quadratic problems in \( R^n \), the search directions generated by the DFP method and the CG method are scalar multiples of each other, provided the initial step is in the direction of steepest descent. This is true also for the singular quadratic problem. In fact, Hestenes and Stiefel showed that the search direction \( S_1 \) of the CG method can be formed as

\[ S_1 = - g_1 g_1^T \sum_{j=0}^{k-1} \frac{g_j^T g_j}{g_j^T g_j} \]

and Horowitz and Sarachik showed that the search direction \( S_1 \) of the DFP method can be written as

\[ S_1 = - H_1 g_1 g_1^T \sum_{j=0}^{k-1} \frac{g_j^T g_j}{g_j^T g_j} \]

**Equations (3.4) and (3.5)** are true for the singular quadratic case as long as \( g_k \neq 0 \). Therefore, from (3.4) and (3.5), we see that the CG and the DFP methods generate the same search directions \( S_1 \) for the singular quadratic problems. For simplicity, let \( B = [0 0 0] \), and take \( f_0 = 0 \). Then

\[ f(x) = \frac{1}{2} x^T A x + b^T x + c \]

\[ g(x) = Q x = \begin{bmatrix} x_1^2 + x_2^2 \\ x_2^2 + x_3^2 \\ x_1^2 + x_3^2 \end{bmatrix} \]

By Property 2.1, \( B \) must belong to \( R(Q) \) to insure a minimum exists, i.e., \( B \) must be of the form \( [a b c] \). For simplicity, let \( B = [0 0 0] \), and take \( f_0 = 0 \). Then

\[ f(x) = \frac{1}{2} x^T A x + b^T x + c \]

\[ g(x) = Q x = \begin{bmatrix} a \\ a \end{bmatrix} \]

Clearly, the minimum norm solution \( x^* \) is \( x = [0 0 0]^T \). Now, consider the application of the gradient, CG, DFP and Powell methods, with the same initial guess \( x = [1 1 1]^T \), to this problem.

(i) Gradient Method: after straightforward calculations:

\[ x_0 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad x_1 = \frac{1}{17} \begin{bmatrix} 8 \\ -1 \\ -3 \end{bmatrix} \]

\[ g_0 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad g_1 = \frac{1}{17} \begin{bmatrix} 8 \\ -1 \\ -3 \end{bmatrix} \]

and \( x_{2n} = \frac{4}{85} x_{2n-1} \), to this problem.

(3.6)

and

\[ g_{2n} = \frac{4}{85} g_{2n-1} \]

Thus, \( x_k \rightarrow x \) and \( g_k \rightarrow 0 \) as \( k \rightarrow \infty \), but \( g_k \neq 0 \) for any finite number \( n \).

(ii) CG Method: Since the first iteration is a gradient step, the same \( x_0, x_1, g_0, g_1 \), as stated in (3.6) result and thus the search directions are

\[ S_0 = -g_0 = \begin{bmatrix} -2 \\ -8 \end{bmatrix}, \quad S_1 = \frac{18}{17} \begin{bmatrix} -1 \\ -8 \end{bmatrix} \]

(3.8)
which implies
\[ x_2 = 0, \quad g_2 = 0. \]

the CG method converges in 2 iterations. (Note again that \( \text{Rank}(Q) = 2 \).)

(iii) DFP method: with \( H = I_3 \), it follows that \( x_0, x_1, g_0, g_1 \) are the same as in (3.6) and the search directions are
\[
S_0 = \frac{1}{17} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad S_1 = \frac{1}{33} \begin{bmatrix} 8 \\ -1 \end{bmatrix},
\]
which implies
\[ x_2 = 0, \quad g_2 = 0. \]

Since the DFP method converges in 2 iterations, it is of interest to display the properties of \( H_2 \), where
\[
H_2 = \begin{bmatrix} 3 & 0 & -1 \\ 4 & 1 & 0 \\ 0 & -1 & 4 \end{bmatrix}
\]
Then,
\[
p = H_2 Q = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}
\]
is the projection matrix on \( R(Q) \), i.e., \( Px = x \), for all \( x \in R(Q) \) and \( Py = 0 \) for all \( y \in N(Q) \). Note also that \( T = I_3 - p = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) is the projection matrix on \( N(Q) \). In this example, the initial guess \( x_0 = [1 1 1]^T \), \( Tx_0 = 0 \), and therefore, \( x_1 = x_2 = 0 \) in both the CG and DFP methods. Again, this result agrees with Remark 3.2. For an arbitrary \( x_0 = [a b c]^T \), \( Tx_0 = \begin{bmatrix} a-c \\ -b \\ -c-a \end{bmatrix} \), the CG and DFP method will converge in at most 2 steps to
\[
x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

Therefore, the first conjugate direction is \( d = x_1 - x_0 = \begin{bmatrix} -2 & -1 & 0 \end{bmatrix} \). The second iteration results in
\[
x_0 = x_1 = x_2 = x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\]
Hence, the second conjugate direction is \( d = 0 \), and Powell's method converges at the second iteration to \( x^* = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \). Note that \( x \in N(Q) \) and \( g(x) = 0. \)

In the following we will consider the application of conjugate direction methods to the least squares solutions of linear algebraic equations. First we need the following definitions.

Definition 3.1: A vector \( u \in R^n \) is a least squares solution of the linear algebraic equations
\[
A x = b
\]
where \( x \in R^n, \ b \in R^m \) and \( A \) is an \( m \times n \) matrix with rank \( (A) = k, \ k \leq \min \{m, n\} \) if
\[
||Au - b|| \leq ||Ax - b|| \text{ for all } x \in R^n.
\]
The vector \( x^* \) is the least squares solution of minimum norm of (3.10) if \( x^* \) is a least squares solution of (3.10) and \( ||x^*|| \leq ||u|| \) holds for all least squares solutions \( u \) of (3.10).

Definition 3.2: The generalized inverse \( A^+ \) of \( A \) is the linear extension of \( \{ A \cap N(A^\perp) \}^{-1} \) so that its domain of definition \( D(A) \) is \( R(A) \cap R(A^\perp) = R^n \) and its null space is \( R(A^\perp) = N(ATA) \), where \( N(A) \) and \( R(A) \) are the null space of \( A \) and Range of \( A \), respectively. \( N(A^\perp) \) is the orthogonal complement of \( N(A) \), and \( A^\perp \) is the restriction of \( A \) to \( N(A^\perp) \).

The following important results have been established by many researchers (see, e.g., [23]).

Property 3.4: If \( A \) is a bounded linear transformation with closed range mapping \( X \) into \( Y \) then the least squares solution of minimum norm (LSSMN) \( x^* \) of the linear operator equation \( Ax = y, \ y \in Y \) is given by \( x^* = A^+ y \).

The linear operator \( A \) defined in (3.10) is clearly a bounded linear transformation with \( R(A) \) closed. Property 3.4 implies that the LSSMN, \( x^* \), of (3.10) is given by \( x^* = A^+ b \).

There are many papers concerned with least squares solutions of linear algebraic equations. Some of them present iterative methods to obtain a least squares solution \( x \) of (3.10) where \( x \) is the initial estimate, \( x^0 \). For example, consider the general iterative method with closed range mapping \( X \) into \( Y \), \( A \) is an \( m \times n \) matrix with rank \( (A) = k, \ k \leq \min \{m, n\} \) if
\[
||Au - b|| \leq ||Ax - b|| \text{ for all } x \in R^n.
\]

Property 3.5: Consider \( J(x) = \frac{1}{2} < A x - b, A x - b > \) or, equivalently, \( J(x) = \frac{1}{2} < x, Q x + c < x, b > \) where \( Q = A^T A \) and \( b \in R^m \) implies that \( b = b_r + b_o \), where \( b_r \in R(A) \) and \( b_o \in N(ATA) \), and
\[ \bar{b} = -A^T b = -A^T (b_r + b_o) = -(A^T b_r + t A^T b_o) = -A^T b_r \]
But \( \bar{b} \in R(A) \), which implies that there exists an \( x \) such that \( b_r = A x \), and then
\[ \bar{b} = -A^T b_r = -A^T A x = Q(-x) \in R(Q). \]
Property 3.6: $x$ is a solution of (3.10) if and only if $x$ is a least squares solution of (3.10) and $J(x) = 0$.

Proof: It is well known that $Ax = b$ has solutions if and only if $b \in R(A)$. In general, $x$ is a least squares solution of $Ax = b$ if and only if $Qx + b = 0$, or, $ATAx - ATb = 0$, since $ATb = 0$, which implies $AT(Ax - b) = 0$.

Now, $Ax - b \in R(A)$, $Ax - b \neq 0$ implies that $AT(Ax - b) \neq 0$ hence $Ax - b = 0$.

That is, $x = x + b$, where $x$ is the minimum norm solution of $Ax - b = 0$ and $n \in N(A)$. With this $x$, we have

$J(x) = \frac{1}{2} < Q, x > + \frac{1}{2} < x, b > + < b, b >$

$= \frac{1}{2} < x, Q + b > + < x, b > + < b, b >$

$= - < A(x, x) > + b > + < b, b >$

Therefore $J(x) = 0$ if and only if $b = 0$, or $b = b e \in R(A)$

Property 3.5 along with Theorem 1 and Remark 3.2 guarantee that the sequence $\{x^0, x^1, \ldots \}$ generated by either the CG or DFP method applied to the least squares problem of (3.10) converges in at most $k = \text{Rank}(A)$ steps to a least squares solution of $Ax = b$, provided the initial step each takes is in the direction of steepest descent.

Consider the singular problem (2.1) subject to (2.3) and (2.4), i.e.,

$$f(x) = \frac{1}{2} < x, Qx > + < x, w > + f_0$$

(SQP) where $x \in R^n$, $f_0 \in R$

with $Q = Q^T \geq 0$, $w \in R(\Omega)$

Define the associated nonsingular quadratic problems (ANSQP) as

$$f(\eta) = \frac{1}{2} < Q\eta x > + < x, w > + f_0$$

(ANSQP) where $Q = Q + \frac{1}{\eta} I$ for $\eta > 0$

and $x \in R^n$, $f_0 \in R(\Omega)$

The reason for introducing the ANSOP is to study the behavior of algorithms as a problem tends to singularity. There is widespread belief that singular problems are more difficult because Newton's method is not applicable and the gradient method typically exhibits very slow convergence. The goal of this section is to quantify such ideas, and to study the rate of convergence of the gradient and conjugate direction methods as a function of the degree of singularity. The following property is straightforward and presented without proof.

Property 4.1:

(i) (ANSQP) approaches (SQP) as $\eta \to \infty$.

(ii) Any eigenvector $z_i$ of $Q$ in (SQP) with corresponding eigenvalue $\lambda_i \geq 0$, $i = 1, 2, \ldots, n$, is also an eigenvector of $Q$, in (ANSQP) with corresponding eigenvalue $\beta_i = \lambda_i + \frac{1}{\eta}$.

The suitability of conjugate direction methods for singular problems is verified by the following property.

Property 4.2: If the sequence $\{x_i\}$, generated by any one of conjugate direction methods applied to the (ANSQP), converges in $k \leq n$ steps, then the sequence $\{x_i\}$, generated by the same method applied to the (SQP) with the same initial estimate $x_0$, converges in at most $k$ steps.

Proof: See Appendix B. (It is interesting to note that the proof in Appendix B involves a lemma which states that all conjugate direction methods will converge in at most $k$ steps for quadratic problems if the Hessian matrix ($Q \geq 0$) has $k$ distinct nonzero eigenvalues; this result is interesting in its own right).

Property 4.2 shows that the convergence of any conjugate direction method applied to the (SQP) is never slower (worse) than the result of the same method applied to the (ANSQP). However, the gradient method behaves in exactly the opposite way. Indeed, it has been shown that the rate of convergence of the residual error for the gradient method will be slow whenever the spread of eigenvalues for the second-variation operator is large. Furthermore, when the second-variation operator is singular, the asymptotic rate of convergence of the residual error will be zero. This result holds for the gradient method in finite-dimensional space, also. Let us consider again the example presented in Section 3 to illustrate the above results.

Example: Minimize:

$$f(x) = \frac{1}{2} < x, 0110 > + 0101 > x$$

(4.1)

with $x_0 = [1, 1, 1]^T$

$$x_1 = \frac{1}{17} [-1, 8, -1]^T$$

and, for $n = 1, 2, \ldots$

$$x_{2n} = \left( \frac{4}{85} \right)^n x_0, x_{2n+1} = \left( \frac{4}{85} \right)^n x_1$$

clearly $x_k \to x^* = 0$ as $k \to \infty$

The ratio of linear convergence, $\theta$, is defined as

$$\theta = \lim_{k \to \infty} \sup \frac{|x_{k+1} - x^*|}{|x_k - x^*|}$$

(4.2)

It is straightforward to determine $\theta$, the ratio of linear convergence of the gradient method applied to the (SQP), and
\[
\theta_s = 0.2762 \quad (4.3)
\]

\( \tau_\eta \) (ANSQP) of (4.1) is given by

\[
f_\eta(x) = \frac{1}{2} < x, \theta_\eta x > \quad (4.4)
\]

where \( \theta_\eta = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \) and \( \eta > 0 \)

with the same \( x = [1 \ 1 \ 1]^T \), the gradient method applied to the (ANSQP) (4.4) generates the sequence \( (\tau_\eta) \) with

\[
\tau_\eta = \frac{-1}{17\eta^2 + 216\eta + 55} \quad (4.5)
\]

and for \( n = 1, 2, \ldots \)

\[
\tau_\eta(n) = [F(\eta)]^n x_0, \quad x_{2n+1}(\eta) = [F(\eta)]^n x_1
\]

where \( F(\eta) = \frac{2\eta^2}{17\eta^2 + 216\eta + 55} \quad (4.6)
\]

from (4.5)

\[
\frac{dF(\eta)}{d\eta} > 0 \quad \text{for} \ \eta > 0.
\]

That is, \( F(\eta) \) is a strictly increasing function of \( \eta \) or,

\[
0 < F(\eta_1) < F(\eta_2) \quad \text{for} \ \eta_1 < \eta_2 \quad (4.7)
\]

Clearly, \( \lim_{\eta \to \infty} F(\eta) = \frac{4}{85} \quad (4.8)
\]

and

\[
\lim_{\eta \to \infty} x_k(\eta) = x(\eta) = 0 \quad \text{for any} \ \eta > 0
\]

Now, define \( h(\eta) \) as

\[
h^2(\eta) = \frac{F^2(\eta)}{||x(\eta)||^2}
\]

It can be shown, by the same procedure for \( F(\eta) \), that for \( 0 < \eta_1 < \eta_2 \)

\[
0 < ||x_1(\eta_1)|| < ||x_1(\eta_2)|| \quad (4.9)
\]

\[
0 < ||h(\eta_1)|| < ||h(\eta_2)|| \quad (4.10)
\]

and

\[
\lim_{\eta \to \infty} ||x_1(\eta)||^2 = \frac{66}{289} = ||x_1||^2 \quad (4.11)
\]

Define \( \theta(\eta) \), the ratio of linear convergence of the gradient method applied to the (ANSQP) (4.4)

\[
\theta(\eta) = \lim_{k \to \infty} \sup_{k \to \infty} \frac{||x_{k+1}(\eta)|| - x(\eta)||}{||x_k(\eta)|| - x(\eta)||}
\]

Straightforward application of conditions (4.5) to (4.11) imply that, for \( 0 < \eta_1 < \eta_2 \)

\[
0 < \theta(\eta_1) < 0 (\eta_2)
\]

and,

\[
\lim_{\eta \to \infty} \theta(\eta) = \theta_s \quad (4.12)
\]

The results in (4.12) show that, as \( \eta > 0 \) increases to infinity, i.e., the (ANSQP) approaches the (SQP), the ratio of linear convergence \( \theta(\eta) \) of the gradient method applied to the (ANSQP) strictly increases to \( \theta_s \), the ratio of linear convergence of the method applied to the (SQP). In nomenclatural terms, this implies that as the problem becomes more singular, the performance of the gradient method deteriorates, whereas the performance of conjugate direction methods improves. This indicates that difficulties attributed to singular problems are actually due to defects in the two main classical methods: the gradient method (as the above analysis shows) and Newton’s method, which is not applicable to the SQP in its standard form.

5. Concluding Remarks

In finite-dimensional space conjugate direction methods determine the inherent lower-dimensionality of the SQP, and converge in at most \( m \) steps, where \( m \) is the rank of the \( n \times n \) Hessian matrix \( Q \). Furthermore, the rate of convergence on the SQP is better than or equal to the rate of the same conjugate direction method applied to associated nonsingular quadratic problems. The gradient method has exactly opposite properties. This shows explicitly that the slow convergence of the gradient method applied to a singular quadratic problem is due to the method and not the problem.

Similar properties hold for the conjugate gradient and quasi-Newton methods applied to singular quadratic optimal control problems, and these results are in preparation.

Appendix A. Proof of Theorem 1

First consider the CG and the DFP methods. Given \( x_0 \), if \( g(x_0) = 0 \), \( x \) is a minimum. Thus, assume \( g(x_0) \neq 0 \), since \( g(x_0) \in R(Q) \) (by Property 3.1 (ii)), \( <Q^T g, Q g> > 0 \). In the CG method, \( S = -H g \), so that \( <S^T x, x> > 0 \). In the DFP method, \( S = -H g \), where \( H \) is any positive definite matrix, and thus \( <S^T x, x> = <H g, x> > 0 \). If \( H g = 0 \), then \( <H g, x> = 0 \) for all \( x \in R(Q) \). But, with \( x = g \), \( H g = 0 \) implies \( g = 0 \), contradicting the assumption \( g \neq 0 \). Hence we conclude that in both the CG and the DFP methods

\[
<S^T, Q S> > 0 \quad \text{if} \ g \neq 0
\]

This condition insures that the algorithms are well-defined from one iterate to another, i.e., the linear search requires

\[
<\hat{g}, S> = 0 \quad \text{and} \quad a = \frac{<S, g>}{<S, Q S>} > 0.
\]

Now, consider \( g = g(x_0) \); if \( g = 0 \) we are done, so assume \( g \neq 0 \). In the CG method:
In the DFP method:
\[ S_1 = -g_1^T g_1 > S_0 \]

By the construction of the CG and the DFP methods, we have
\[ <S_o, Q S_1 > = 0 \]

Note that in the CG method, \( S_o \) is the linear combination of \( g_o, g_i, \) with \( g_o \neq 0, g_i \neq 0, g_i \in R(Q) \) and \( <g_o, g_i> \neq 0 \). This implies that \( S_1 \neq 0 \) and belongs to \( R(Q) \), hence

\[ <S_1, Q S_1 > > 0 \]

In the DFP method, it has been proved that \( \{H_k\} \) is a sequence of positive definite symmetric matrices if \( H_k = H_k^T > 0 \) for the nonsingular case. This result can be carried over directly to the singular case as long as \( g_i \neq 0 \), i.e., the sequence \( \{H_1, H_2, \ldots, H_k\} \) of matrices is a finite sequence of positive definite symmetric matrices if \( H_k = H_k^T > 0 \) and \( g_i \neq 0 \) for \( i = 1, 2, \ldots, k \). Now, with \( H_k > 0 \) and \( g_i \neq 0 \in R(Q) \), by condition (ii) of Property 3.1, we obtain
\[ <S_0, Q S_1 > = <H_1 g_1, Q H_1 g_1 > \]

We proceed by introduction. Suppose after \( k \) iterations with \( g_k \neq 0 \), we have the following properties:
\[ <S_k, Q S_k > = 0 \quad i \neq j, 0 \leq i, j \leq k \]  
\[ <S_k, Q S_k > > 0, \quad i < k \]  
\[ <g_k, s_i > = 0 \quad i < k \]

The condition (A.2) allows the \( k+1 \) iteration to be well-defined. Assume \( g_k \neq 0 \); we shall show that (A.2) hold for \( k+1 \). Note that (A.1) is true by the construction of the CG and DFP methods, and (A.3) has been proved in Ref. 20 and can be carried over to the singular case as long as \( g_k \neq 0 \). Therefore, we need only prove (A.2).

The next \( Q \)-conjugate direction \( S_{k+1} \) of the CG method is
\[ S_{k+1} = -\frac{<g_{k+1}, g_{k+1}>}{<g_k, g_k>} S_k \]

It is easy to show that
\[ S_{k+1} = -\frac{\sum_{i=0}^{k} <g_{k+1}, g_i>}{<g_i, g_i>} g_i \]

i.e., \( S_{k+1} \) is a linear combination of \( \{g_0, g_1, \ldots, g_k\} \). It is well known that (see, e.g., Ref. 20)
\[ <g_i, g_j> = 0 \quad i \neq j, 0 \leq i, j \leq k+1 \]

Again, these results are true for the singular case as long as \( g_i \neq 0, i = 0, 1, \ldots, k+1 \). Therefore, \( S_{k+1} \neq 0 \) and \( S_{k+1} \in R(Q) \). This implies (by Property 3.1(ii)) that, for the CG method,

\[ <S_{k+1}, Q S_{k+1} > > 0 \]

In the DFP method, \( S_{k+1} = H_{k+1} g_{k+1} \), and we know that \( H_k = H_k^T > 0 \), and \( S_{k+1} = H_{k+1} g_{k+1} \). If \( <S_{k+1}, Q S_{k+1} > = 0 \), then\[ S_{k+1} = H_{k+1} g_{k+1} \in R(Q), \] i.e., \( <S_{k+1}, Q S_{k+1} > = 0 \). For the necessary contradiction, again pick \( x = g_{k+1} \in R(Q) \). Then, \( H_{k+1} g_{k+1} = 0 \), which contradicts the assumption \( g_{k+1} \neq 0 \). Therefore, (A.2) is true for \( k+1 \).

Consider the iteration number \( k = m-1 \), where \( m \) is the rank of \( Q \). If \( g_{k+1} = 0 \), we are done; if \( g_{k+1} \neq 0 \), from (A.3) we have \( <g_{k+1}, S_k > = 0 \) (i = 0, 1, \ldots, m-1). This means that \( x_m \) minimizes \( f(x) \) over the subspace spanned by \( \{S_0, S_1, \ldots, S_{m-1}\} \), a basis of \( R(Q) \). Let \( x = x_m + z \) \((x_m \in R(Q)) \) such that \( f(x_m) = f(x_m + z) = f(x_m) \), it follows that \( x_m + z \in R(Q) \) minimizes \( f(x) \) for all \( x \in R(Q) \). From Property 3.2, we conclude that \( g_{k+1} = g(x_m) = (x_m + z) = g(x_m) = 0 \), i.e., the CG and the DFP methods converge in at most \( m \) steps.

For Powell's method, the proof is as follows. Given \( x_o \), the method generates a \( Q \)-conjugate direction \( d = x_o - x_o \), the directions \( d_1, \ldots, d_n \) are replaced by \( d_1, \ldots, d_n, d \), and then defined as \( d_1, \ldots, d_n \). Suppose after \( k \) iterations that the last \( k \) directions of \( \{d_1, d_2, \ldots, d_n\} \) are nonzero \( Q \)-conjugate vectors, i.e., \( d_n \neq 0 \), \( d_{n-k+1} \neq 0 \), \( d_{n-k} \neq 0 \), \( d_{n-k+1} \neq 0 \), 

The starting point \( x_o \) for the \( (k+1) \)th iteration is the minimum of \( f(x) \) in the subspace spanned by these \( Q \)-conjugate directions. If, on the \( (k+1) \)th iteration the new direction \( d = x_o - x_o = 0 \), then (as will be shown) the starting point \( x_o \) at this iteration is a minimum of \( f(x) \), i.e., \( f(x_o) = 0 \).

First note that \( x_k \) is determined by \( x_k = x_{k+1} + t_d \) where \( t \) is chosen such that \( f(x_k + t_d) = \min f(x_k + t_d) \), for \( R(Q) \). Therefore, \( t = 0 \), i.e., \( x_k = x_{k+1} \). This can happen if and only if \( <g_k, x_{k+1}> = 0 \). This implies that

(i) \( g(x_{k+1}) = 0 \), (ii) \( d \in R(Q) \), (iii) \( d \in R(Q) \) and

\[ d \perp g(x_{k+1}) \]

If (i) is true, then \( x_{k+1} \) is a minimum. Thus, assume \( g(x_{k+1}) \neq 0 \). Note that (ii) and (iii) are mutually exclusive. Equation (ii) shows that the new directions \( \{d_1, d_2, \ldots, d_n\} \) for the next iteration at least span the range of \( Q \). (Recall that the initial \( d_1, d_2, \ldots, d_n \) form a basis for \( R^n \). On the other hand, if (iii) is true, we have \( g(x_{k+1} - d_j) = 0 \) for \( j \geq 1 \) but the renamed directions \( d_1, d_2, \ldots, d_n \) for the next iteration span the same subspace as \( \{d_1, d_2, \ldots, d_n\} \). It follows that if at the \( (k+1) \)th iteration the new direction \( d = x_o - x_o \) is a zero vector, then the nth vector minimizes \( f(x) \) over the space spanned by \( \{d_1, d_2, \ldots, d_n\} \), which either contains the range of \( Q \) or contains
some part of the range of \( Q \), say \( \{ e_i, e_j, \ldots, e_m \} \) with \( i < j \) for \( 1 \leq i < j \leq m \) (where \( \{ e_i, e_j, \ldots, e_m \} \) form a \( \{ i, j \} \) for the range of \( Q \)). Both situations lead to the same condition that \( g(x_k) \perp R(Q_i) \), i.e., \( g(x_k) \in R(Q_i) \) for some \( i \). Since \( g(x_k) = Q x_k \) for some \( i \), then \( g(x_k) \) and \( Q \) are \( m \) minimum solutions of \( f(x) \). Thus, we have proved that if \( d = x_k - x_{k-1} \) at the \( k-1 \) iteration, \( k + 1 \leq m \), then the \( x_k \) of that iteration is a minimum solution.

Now suppose that after \( m \) iterations \( m \) \( Q \)-conjugate directions have been generated, i.e., \( d_1, d_2, \ldots, d_m \) are nonzero \( Q \)-conjugate directions. Then, the starting point \( x_0 \) for the \( (m+1) \)th iteration is the minimum of \( f(x) \) in the subspace spanned by the \( m \) \( Q \)-conjugate vectors. Now, for each \( d_i \), there exist unique \( p_i \) and \( z_i \) with \( P_iQ(R(Q_i) \cap \{ p_i, z_i \}) \) such that

\[
d_i = p_i + z_i \quad \text{for } i = m+1, \ldots, n.
\]

and

\[
d_i Q d_j = c_{i,j} < z_i, Q d_j > = c_{i,j} Q < d_i, Q d_j >.
\]

Since \( d_1 Q d_j = 0 \) for \( 1 \leq j \leq i \leq n \) and \( d_1 Q d_j \) for \( i \neq j \)

\[
\text{we have } c_{i,j} = 0 \quad \text{for } i \neq j \leq n \quad \text{and } c_{i,j} = 0 \quad \text{for } i \neq j \leq n.
\]

That is, \( \{ p_{n+1}, p_{n+2}, \ldots, p_n \} \) are nonzero \( Q \)-conjugate directions in \( R(Q_i) \). Therefore, the starting point \( x_0 \) for the \( (m+1) \)th iteration is a minimum of \( f(x) \) on the subspace spanned by \( (d_{n+1}, \ldots, d_n) \) which contains \( \{ p_{n+1}, \ldots, p_n \} \). \( Q \)-conjugate directions in \( R(Q) \). This means that, by Property 3.2, \( g(x_0) = 0 \), and \( x_0 \) is a minimum solution of \( f(x) \) in \( R^n \).

**Appendix B: Proof of Property 4.2**

From Theorem 1 in Section 3, there is nothing to prove if \( k \leq m \), where \( m \) is the rank of \( Q \). Hence, assume \( k > m \). It was shown in Ref. 28 that for the DFP method applied to non-singular quadratic problems, we find the \( (m+1) \)th direction \( x_{k+1} \) to the gradient at \( x_k \) byProperty 2. Then, assume \( \lambda_1, \lambda_2, \ldots, \lambda_k \) be the \( k \) distinct positive eigenvalues of \( A \) corresponding to the same eigenvalue \( \lambda \) for \( i = 1, 2, \ldots, k \). Then clearly \( \{ z_1, z_2, \ldots, z_k \} \) form a basis in \( R^n \). Given \( x, g_0 = A x_0 + w \), and there exist unique \( c_i \) such that

\[
g_0 = \sum_{i=1}^{k} c_i z_i, \quad z_i \perp y, \quad i = 1, \ldots, k.
\]

Note that: \( A y = \lambda y, y \perp \{ y_{i=1}, \ldots, k \}, < y, y_j > = 0 (i \neq j) \) (B.2).

Now, if \( c_i = c_i \) for some \( i \), then \( y \) is zero for that index \( i \). Therefore, \( g_0 \) is the sum of \( y \), \( y^2 \), \ldots, \( y^k \) and some of the \( y \) may be zero-vectors. Assume that \( g_0 \) is the sum of \( \ell \) nonzero vectors and rename them as \( \{ w(1), w(2), \ldots, w(\ell) \} \) where \( \ell \leq k \). Recall that \( A w(i) + \tau w(i) = \lambda w(i) \) for \( 1 \leq i \leq \ell \), \( \tau \) are distinct positive real numbers, the matrix \( B = \{ b_{i,j} \} \), defined by \( b_{i,j} = \{ (i,j) \} (1 \leq i, j \leq \ell) \), is a nonsingular Vandermonde matrix. This leads to the results that \( g_0, A g_0, \ldots, A^{(\ell-1)} g_0 \) are nonzero linear independent vectors, and span \( \{ g_0, A g_0, \ldots, A^{(\ell-1)} g_0 \} \) is a span of \( \{ w(1), w(2), \ldots, w(\ell) \} \) since \( w(1), \ldots, w(\ell) \) are linearly independent and \( g_0 = \sum_{i=1}^{\ell} (t_i w(i) \in R^n \).

Since \( A^T g_o = \sum_{i=1}^{\ell} (t_i w(i) \in R^n \).

Since \( \tau_1, \tau_2, \ldots, \tau_\ell \) are distinct positive real numbers, then \( \alpha_1, \alpha_2, \ldots, \alpha_\ell \) in (B.3) are all nonzero numbers. Therefore, from (B.3)

\[
g_0 = \sum_{i=1}^{\ell} \alpha_i A^i g_0 \quad \text{is spanned by} \quad \{ g_0, A g_0, \ldots, A^{(\ell-1) g_0} \}
\]

But, as previously mentioned

\[
\text{span} \{ A g_0, A^2 g_0, \ldots, A^{(\ell-1) g_0} \} \text{is spanned by} \quad \{ A g_0, A^2 g_0, \ldots, A^{(\ell-1) g_0} \}
\]

which implies \( g_0 = \sum_{i=1}^{\ell} \alpha_i AS_i \quad (1 \leq k \leq k) \), and

\[
\text{then from (B.4) } g_k = \sum_{i=1}^{\ell} \alpha_i AS_i - g_0 = 0 \quad (1 \leq k \leq k) \text{.}
\]

This implies that the DFP method converges in exactly \( k \) steps with \( k \leq k \).

With Lemma B.1 it is easy to prove Property 4.2 . Now, assume that the DFP method applied to the (ANSQP) converges at \( k < m \) steps, where \( m \) is the rank of \( Q \). From the arguments of Lemma B.1, we can write \( g_0 = \sum_{i=1}^{\ell} w_i \), where

\[
w_i \text{ is a nonzero eigenvector of } Q \text{ corresponding to}
\]
the eigenvalue $\xi_i$ with $\xi_i \neq \xi_j$ for $i \neq j$. But,
\[ g_i = Q_i x + w = Qx + w = \sum_{i=1}^{k} \xi_i Q_i^{-1} x + w = \sum_{i=1}^{k} \xi_i g_i \in R(Q) \text{and} \]
\[ Qw, i = 1, \ldots, k \] (where $\lambda_i = \xi_i^{-1}$, $i > 0$), we conclude that $g_0 = \sum_{i=1}^{k} c_i w_i$. Also from the above and the fact that $\xi_1, \xi_2, \ldots, \xi_k$ are distinct positive real values, at most one of the $\lambda_i$ may become zero and the rest of the $\lambda_i$ are distinct positive values. If $\lambda_i = \xi_i^{-1} = 0$ for a specific index $j$, then $Qw = 0$ and the corresponding coefficient $c_j$ must be zero. Let us denote $z = c_i w_i$.

To collect the nonzero vectors $z_i$, and rename the index order to form $g_0 = \sum_{i=1}^{k} z_i (r \leq k)$, where $z_i$ are the eigenvectors of $Q$ corresponding to the distinct reordered eigenvalues $\lambda_i$. Using the same arguments as in Lemma B.1 and the fact that $span \{Qg, OQg, \ldots, Q^k g\} = span \{Qz, Qz, \ldots, Qz\}$. We get $g_0 = \sum_{i=1}^{k} a_i Qz_i (r \leq k)$, or
\[ g_0 = \sum_{i=0}^{r-1} a_i Qz_i - g_0 = 0, \text{ and, thus, the DFP method applied to the (SCP) converges in} \ r \ \text{steps,} \ r \leq k.\]

References