AIAA-83-2092
Optimal Control of Orbital Transfer Vehicles
N.X. Vinh, The Univ. of Michigan, Ann Arbor, MI

AIAA Atmospheric Flight Mechanics Conference
August 15-17, 1983/Gatlinburg, Tennessee
OPTIMAL CONTROL OF ORBITAL TRANSFER VEHICLES

Nguyen X. Vinh * **
The University of Michigan
Ann Arbor, Michigan 48109

Abstract

During the past two decades, considerable research effort has been spent to convincingly prove that the use of aerodynamic forces to assist in the orbital transfer can significantly reduce the fuel consumption as compared to the pure propulsive mode. Since in this aeroassisted mode, preliminary maneuvers in the vacuum affect the resulting performance in the atmospheric phase, and vice versa, the two, space and atmospheric maneuvers, are, to a great extent, coupled. This paper summarizes, via optimal control theory, the fundamental results in the problem of orbital transfer using combined propulsive and aerodynamic forces. For the atmospheric phase, the use of the Chapman's variables reduced the number of the physical characteristics of the vehicle and the atmosphere to a minimum and hence allows a better generalization of the results. The paper concludes with some illustrative examples.

I. Introduction

During the past two decades, considerable research effort has been spent to convincingly prove that the use of aerodynamic forces to assist in the orbital transfer can significantly reduce the fuel consumption as compared to the pure propulsive mode. An excellent review of aeroassisted orbit transfer covering an extensive literature has been presented by Walberg at the AIAA 9th Atmospheric Flight Mechanics Conference. The pioneering research has been geared toward engineering feasibility and design concept based on some basic maneuvers such as in the problems of orbital plane change, return from High Earth Orbit (HEO) to Low Earth Orbit (LEO) and planetary aero-gravity capture. Just as in the sixties, during the period of development of the theory of optimal propulsive orbit transfer, the problems investigated have covered the full range, from high-thrust to low-thrust propulsive systems, from the simple Hohmann transfer to the complex multi-impulse rendezvous problem, it is expected that in the coming years the analysis of aeroassisted orbit transfer will become more and more involved.

Basically, the use of aeroassisted transfer aims at minimizing the fuel consumption which, for a high-thrust propulsive system, is measured by the characteristic velocity, the sum of all velocity changes, produced by applications of the thrust. This is schematically represented in Fig. 1 where $S_0$ and $S_f$ denote the initial and the final state, respectively.

\[ \Delta V = \Delta V_1 + \Delta V_2 \]

Let $A$ be the space where aerodynamic force is involved. Let $O$ be the best trajectory for a pure propulsive transfer in the vacuum resulting in a total characteristic velocity $\Delta V^*$.

\[ \Delta V \leq \Delta V^* \]

As a more concrete example, for a transfer from a HEO (state $S_0$) to a LEO (state $S_f$), the direction of the trajectory in the space $A$ is such that the energy is decreasing, and furthermore if a plane change is involved (the angular distance between $S_0$ and $S_f$ is large), then this case is definitely a strong candidate for aeroassisted maneuver.

It is proposed, in this paper, to summarize via optimal control theory, the fundamental results in the problem of orbital transfer using combined propulsive and aerodynamic forces.

II. Optimal Control

The general problem is the problem of controlling the Orbital Transfer Vehicle (OTV), through the propulsive force $\mathbf{F}$, and the aerodynamic force $\mathbf{A}$, to bring it from the initial state, at time $t_0$. 

* This work was supported by the Jet Propulsion Laboratory under contract No. 072 537.

** Professor of Aerospace Engineering.

Copyright © American Institute of Aeronautics and Astronautics, Inc., 1983. All rights reserved.
with position vector \( \mathbf{r}_0 \), velocity \( \mathbf{V}_0 \) and mass \( m_0 \), to the final state \( \mathbf{r}_f, \mathbf{V}_f \) and \( m_f \) at the final time \( t_f \), such that a certain performance index, such as the final mass, is maximized (Fig. 2).

The motion of the vehicle, considered as a point mass with varying mass, flying in a general gravitational force field and subject to aerodynamic force and thrusting force, is governed by the equations with standard notation

\[
\begin{align*}
\frac{d\mathbf{r}}{dt} &= \mathbf{V} \\
\frac{d\mathbf{V}}{dt} &= \frac{1}{m} (\mathbf{F} + \mathbf{G}) + \frac{\mathbf{V}}{t} \\
\frac{dm}{dt} &= -\frac{C}{g_0} \mathbf{V}
\end{align*}
\]

(2)

If we choose the operating mode such that the thrusting phase is always in the vacuum, \( \mathbf{A} = 0 \), and the aerodynamic maneuver is performed with power off, \( \mathbf{P} = 0 \), then only one vector control, \( \mathbf{P} \) or \( \mathbf{A} \), is active at any given time and we can derive independently the optimal control law for \( \mathbf{P} \) and \( \mathbf{A} \). This is typical for an aeroassisted mission profile. The results can be easily modified to account for a small variation of the mass in the case where a continuous thrust is applied to balance the aerodynamic drag for a turn inside the upper atmosphere.

In the case where the propulsion system is inactive when \( \mathbf{A} \neq 0 \), then \( m = \text{constant} \), and from system (2), we consider the Hamiltonian

\[
H = \mathbf{p}_V \cdot \mathbf{V} + \mathbf{p}_A \cdot (\mathbf{a} + \mathbf{g})
\]

(12)

where \( \mathbf{a} \) is the acceleration due to the aerodynamic force. Since the aerodynamic force \( \mathbf{A} \) is decomposed into a drag force \( \mathbf{D} \), opposite to the velocity \( \mathbf{V} \), and a lift force \( \mathbf{L} \) orthogonal to it, to maximize the Hamiltonian, the lift force has to be rotated such that the three vectors \( \mathbf{A}, \mathbf{P}_V \) and \( \mathbf{L} \) are co-planar, that is

\[
(\mathbf{V} \times \mathbf{P}_V) \cdot \mathbf{A} = 0
\]

(13)

This gives the optimal bank angle and we have the situation in Fig. 3.
useful integrals made to paper by phase and the aerodynamic control phase the most
where the constant arc ample, m prescribed.

is if the gravitational acceleration is central
and only depends on the radial distance, and if we have

similar dependence for the atmospheric density and engine performance, then we have the vector integral

\[ \vec{A} = \vec{\tau} \times \vec{p}_T + \vec{\tau} \times \vec{p}_V \]  \hspace{1cm} (17)

where \( \vec{A} \) is a constant vector.

Furthermore, for flight in the vacuum, along a coast arc, \( T = 0 \), the motion is Keplerian, the equations of motion can be integrated, and the equations for the trajectory is obtained in closed form. By canonical transformation, the adjoint vectors are also obtained. For the time-free case, \( H^* = 0 \), we have the solution\(^5\)

\[ \vec{p}_V = \lambda_1 \vec{v} + \lambda_2 \times \vec{r} + (\vec{r} \times \vec{v}) \times \lambda_3 + \vec{a} \times (\vec{v} \times \lambda_3^*) \]  \hspace{1cm} (18)

where \( \lambda_1 \) is a constant and \( \lambda_2, \lambda_3 \) are two constant vectors. Since \( d\vec{p}_V/dt = -\partial H^*/\partial \vec{v} = -\vec{p}_r \), we deduce

\[ \vec{p}_r = -\lambda_1 \vec{v} + \vec{v} \times \lambda_2^* + \vec{v} \times (\lambda_3 \times \vec{v}) \times \vec{g} \times (\vec{r} \times \lambda_3^*) \]  \hspace{1cm} (19)

where

\[ \vec{g} = -\mu \frac{\vec{r}}{r^3} \]  \hspace{1cm} (20)

is the gravitational acceleration for a Newtonian central force field. By forming the cross products in Eq. (17), we have a relation between the constant vectors \( \vec{A} \) and \( \lambda_1^* \).

In addition, we have a scalar integral

\[ 2 \cdot \vec{r} \cdot \vec{p}_r - \vec{v} \cdot \vec{p}_V = B \]  \hspace{1cm} (21)

### III. Integrals of the Motion

For the integration of the canonical system (11), the initial state \( \vec{x}_0 \) is prescribed at \( t_0 \) while the final state \( \vec{x}_f \) is usually partially prescribed. This leads to a certain transversality conditions involving \( \vec{x}_f \) and \( \vec{p}_f \). The simplest method is to choose an initial value \( \vec{p}_f \) for the integration and adjust it to ultimately satisfy the final and transversality conditions. However, \( \vec{x}_f \) and \( \vec{p}_f \) are extremely sensitive with respect to \( \vec{p}_f \) and effort has been made to develop direct numerical techniques for the computation of the optimal trajectory. The survey paper by A. Miele contains 147 references on the subject. To alleviate the computational effort, one may search for some exact and explicit relations between the variables \( x, p \) and \( t \). Such relations are called integrals of the motion. For example, \( m = \) constant is an integral along a coast arc while \( \vec{p}_v + \vec{p}_r = 0 \) is an integral along an intermediate-thrust arc. For both the thrust control phase and the aerodynamic control phase the most useful integrals are the following.

If the gravitational field is time invariant, we have the Hamiltonian integral

\[ H^* = c_0 \]  \hspace{1cm} (16)

where the constant \( c_0 \) is zero if the final time is not prescribed.

If the problem has a spherical symmetry, that is if the gravitational acceleration is central and only depends on the radial distance, and if we have

\[ \begin{align*}
S_1^2 + T_1^2 + W_1^2 & = 1, \\
S_2^2 + T_2^2 + W_2^2 & = 1
\end{align*} \]  \hspace{1cm} (22)

where \( S_1, T_1, W_1 \) and \( S_2, T_2, W_2 \) are the direction cosines of the impulse \( I_1 \). By eliminating the constants involved, Marchal has obtained the explicit relations\(^5\)

\[ 1 + \cos \omega V = \frac{(1 - \cos \Delta) (1 - 2 S_1^2 S_2^2 + 2 S_1^2 T_2^2 + 2 S_1^2 W_2^2)}{1 - (S_1^2 + T_2^2 + W_2^2) \sin \Delta (S_1^2 + T_2^2) \cos \Delta - W_2^2} \]  \hspace{1cm} (23)
and
\[ 1 + \cos v = \frac{1 - 2S_z^2 - S_z S_y - 0(S_z + S_y)T_z}{1 + (S_z + S_y)T_z} \]

where \( v \) and \( v \) are the true anomalies, along the transfer orbit with eccentricity \( e \), and

\[ \Delta = v - v_1, \quad \theta = \tan \left( \frac{\Delta}{2} \right) \]

with \( \Delta \) being the transfer angle.

Another useful relation is

\[ \theta^3(T_z - T_y)(S_z + S_y)^2 + \theta^2(S_z - S_y) \left[ 3 - 2S_z^2 - 2S_y^2 - S_z \right] 
- 3T_z - W \cdot W = \theta \left[ 2T_z - 2T_y + 3T_z S_z^2 - 3T_y S_z \right] 
- 3T_z S_z^2 + (S_z - S_y) \left[ 1 - 2S_z^2 - 2S_y^2 + 3S_z S_y - T_z W \cdot W \right] \]

\[ = 0 \]

These relations are the optimal switching relations, and their usefulness will be displayed in the last section.

![Fig. 4. Optimal switching.](image)

V. Equations for Atmospheric Flight

While in orbital maneuver in the vacuum, especially for the case of time-free, impulsive transfer, the theory is very complete and has explicit solutions for many cases of practical interest, the theory for optimal maneuver in the upper atmosphere using lift and drag modulation is still in the development stage. Although qualitatively the results obtained by various authors on the same type of problem, such as the problem of synergetic plane change, or the problem of aeroassisted return from a geosynchronous orbit (GEO) to a LEO, they are similar and hence, corroborating on each other, it is still difficult from a vast literature surveyed by Walberg to obtain a definite conclusion for each specific problem due to the fact that, for atmospheric flight, the atmospheric density is not precisely known, and furthermore each OTV considered has its own physical characteristics.

To have a unified approach, we have suggested using Chapman's variables

\[ Z = \frac{\rho SC_y}{2m} \sqrt{\frac{\pi}{\rho}} \quad u = \frac{V^2}{gr} \]

These equations of motion with the variable \( \Delta = C_L \cdot \cos \sigma \), \( \Omega = C_L \cdot \sin \sigma \)

Then, with a Newtonian gravitational field and a locally exponential atmosphere, that is with the differential variation

\[ \frac{d\phi}{d\tau} = \beta (\tau) \rho \, d\tau \]

and with the spherical coordinates as shown in Fig. 5 for the position and velocity, we have the universal dimensionless equations of motion

\[ \frac{dZ}{ds} = -k^2 Z \tan \gamma \]

\[ \frac{du}{ds} = -k^2 u \left( 1 + \cos^2 \gamma \right) - (2u \tan \gamma) \]

\[ \frac{dv}{ds} = -k^2 \Delta + 1 - u \]

\[ \frac{dp}{ds} = \cos \psi \]

\[ \frac{d\psi}{ds} = -k^2 \Omega - \cos \psi \tan \phi \]

In these equations, the only physical characteristic of the vehicle is its maximum lift-to-drag ratio, \( E^* \), and the nature of the atmosphere is specified by the constant value \( k^2 = \beta \), called Chapman's atmospheric parameter. For the Earth's atmosphere, we have the value \( k^2 = 900 \).

Introducing the adjoint variables \( P \), we form the Hamiltonian.
H = -k^2 Z pZ tan \gamma - p_u \left[ \frac{kZu(1+\Lambda^2+2\Omega^2)}{E^x \cos \gamma} + (2-u) tan \gamma \right] \\
+ p_y \left[ \frac{k\Lambda}{\cos \gamma} \right] + p_0 \frac{\cos \phi}{\cos \psi} + p_\psi \sin \psi \\
+ p_\psi \left[ \frac{kZ \Omega}{2 \cos \psi} - \cos \psi \tan \phi \right] 
\tag{34}

Hence, as shown in the general vector formulation above, the optimal aerodynamic control is function of the adjoint variables \( p_y, p_0 \), and \( p_\psi \), components of the primer vector \( p_y \). By writing the adjoint equations, \( \frac{dp}{dt} = -\frac{\partial H}{\partial x} \), it is easy to verify the following integrals
\begin{align*}
E p_y = c_0 , \\
E p_0 = c_1 , \\
p_y = c_2 \sin \theta - c_3 \cos \theta, \\
p_\psi = c_4 \sin \phi + (c_5 \cos \theta + c_6 \sin \theta) \cos \phi 
\end{align*} 
\tag{36}

The last three integrals can be deduced from the primer vector integral \( \Lambda \), as given in Eq. (17), by canonical transformation. It remains the integration of the equations for \( p_z, p_u, \) and \( p_\psi \). Explicitly, we have
\begin{align*}
\frac{dp_z}{ds} &= k^2 p_z \tan \gamma + \frac{ku(1+\Lambda^2+\Omega^2)}{E^x \cos \gamma} \\
&= k \Lambda \cos \gamma - \frac{p_y}{\cos \gamma} \\
\frac{dp_u}{ds} &= p_u \left[ \frac{kZu(1+\Lambda^2+2\Omega^2)}{E^x \cos \gamma} - \tan \gamma \right] - \frac{p_y}{u^2} \\
\frac{dp_\psi}{ds} &= \frac{1}{\cos^2 \gamma} \left\{ k^2 Z p_z + p_u \left[ \frac{kZu(1+\Lambda^2+2\Omega^2)\sin \gamma}{E^x} \right] \\
&+ (2-u) \right] - kp_\psi Z \Lambda \sin \gamma - 2kp_\psi Z \Omega \tan \gamma \right\} 
\end{align*} 
\tag{37}

Because of the Hamiltonian integral, we can delete one of these adjoint equations, but in practice to generate an optimal trajectory, for a specified initial condition, we can integrate the six state equations (33) and the three adjoint equations (37) while using the Hamiltonian integral as a check for the accuracy of the numerical integration. The three adjoint variables \( p_y, p_0 \), and \( p_\psi \) are, of course, given by the exact integrals (36).

The optimal equations derived in this section can be used to obtain the solution to any unconstrained reentry problem, and as a special case the problem of plane change in an aeroassisted orbital maneuver.

For example, in the case of optimal aerodynamic turning we have the initial condition at entry
\begin{align*}
s &= 0, \\
Z_e, u_e, \gamma_e, \theta_e = \phi_e = \psi_e &= 0 \tag{38}
\end{align*}

The speed \( u_e \) and flight path angle \( \gamma \) result from the preliminary space maneuver. The value \( Z \) is evaluated at the top of the sensible atmosphere. It is proposed to use lift and bank modulation to achieve a maximum plane change \( \delta \), and hence we use the performance index
\begin{align*}
J &= \cos \delta f = \cos \phi f \cos \psi f \tag{39}
\end{align*}

At the final, exit point, we require that
\begin{align*}
Z_f &= Z_e, \\
u_f &= \text{prescribed}, \gamma_f &= \text{free} \tag{40}
\end{align*}

Since the final time and the final longitude are not prescribed, we have the transversality conditions
\begin{align*}
c_0 &= 0, \\
c_1 &= 0 \tag{41}
\end{align*}

To start the integration, it requires selecting the values for the constants \( c_0, c_1 \), in the expressions for \( p_y \) and \( p_\psi \), and the three initial values for \( p_z, p_u \), and \( p_\psi \). But, since in the Hamiltonian and in the adjoint equations, the adjoint variables appear either linearly, or in the form of a ratio, through \( \Lambda \) and \( \Omega \) as shown in Eq. (35), we can use one of the constants, say \( c_4 \), as a normalizing factor. This is achieved by dividing all the equations by \( c_4 \) and using the rescaled adjoints \( p_\psi = \frac{p_\psi}{c_4} \), \( c_2 = \frac{c_2}{c_4} \), \( c_3 = \frac{c_3}{c_4} \). To simplify the notation, in the following discussion, we shall omit the bar in the variables. Then, we have to estimate four constants, but because of the Hamiltonian integral, only three parameters are truly independent. The problem of sensitivity in selecting these constants can be alleviated by an educated guess based on the knowledge of the control as follows.

From Eq. (36), at the initial time, we have
\begin{align*}
p_\psi (0) &= -c_3, \\
p_\psi (0) &= 1 \tag{42}
\end{align*}

Next, we notice that
\begin{align*}
\Lambda^2 + \Omega^2 &= \left( \frac{E^x}{2Zu p_y} \right)^2 \left( p_y^2 + \frac{p_\psi^2}{\cos^2 \gamma} \right) \tag{43}
\end{align*}

Then, a correct guess of the initial bank angle, \( \sigma \),
and normalized lift coefficient, \( C_L / C_D \), will provide a good estimate of \( p(0) \), and \( p_L \). Furthermore, for short range, \( c_3 \) is constant and if the final latitude is free the constant is zero. Hence \( c_3 \) is a small constant.

In summary, we have three parameters to be adjusted such that three final and transversality conditions are identically satisfied. The first condition is that when \( Z = Z_f = Z_0 \), the speed is equal to the prescribed speed \( v_f \). The second condition is that since \( v_f \) is free

\[
P_{v_f} = 0
\]

This condition implies that \( c_4 = 90^\circ \).

For the final transversality condition, based on the performance index (39), we have

\[
P_{\phi_f} = \frac{\partial J}{\partial \phi_f} = \sin \phi_f \cos \psi_f = c_2 \sin \theta_f - c_3 \cos \theta_f
\]

\[
P_{\psi_f} = \frac{\partial J}{\partial \psi_f} = \cos \phi_f \sin \psi_f = (c_2 \cos \theta_f + c_3 \sin \theta_f)
\]

\[
\cos \phi_f
\]

Taking the ratio of these equations and using \( c_3 \) for the ratio \( c_2 / c_3 \), we have the transversality condition which must be identically satisfied at the final time

\[
\tan \psi_f = \frac{1 + c_3 \tan \theta_f}{\tan \theta_f - c_3} \sin \phi_f
\]

VI. Examples of Aeroassisted Transfer

As illustrative examples, we shall consider the following two problems.

**Planar Rotation of Orbit**

It is proposed to rotate, with minimum fuel consumption, the line of apses of an orbit by an angle \( 2\alpha \) (Fig. 6).

![Fig. 6. Planar rotation of orbit.](image)

The optimal solution is either by two impulses or via parabolic orbits. For a two-impulse transfer, the solution is obviously symmetric as shown in the Figure. The given terminal orbits are defined by their common conic parameter \( p \) and eccentricity \( E \). The unknown elements are the corresponding parameter \( p \) and \( \alpha \) for the transfer orbit. If \( r \) is the radius at the impulses \( I_1 \) or \( I_2 \), it is defined by the true anomalies \( \nu_1 \) on the transfer orbit, and \( \nu_2 \) on the initial orbit. We shall use the notation in section IV. In particular, if \( \Delta \) is the transfer angle between impulses

\[
\theta = \tan \frac{\Delta}{2}, \quad \frac{\Delta}{2} + \nu_1 = \pi, \quad \tan \nu_1 = -\theta
\]

We define the new variable

\[
x = \frac{\Delta}{\theta} = \frac{1}{\theta} \cos \nu_1
\]

Then

\[
e \cos \nu_1 = \frac{1-x}{x}, \quad e \sin \nu_1 = \frac{(x-1) \theta}{x}
\]

Next, since \( r = p \left[ 1 + E \cos(\alpha + \nu_1) \right] \), we have

\[
E \cos(\alpha + \nu_1) = \frac{v^2}{x} - 1
\]

where

\[
y = \sqrt{\frac{p}{E}} (51)
\]

is another new variable. We now take benefit of the optimal switching relations in section IV. Let \( \delta \) be the angle between the \( T \)-axis and the impulse \( I_1 \). By definition

\[
T_1 = \cos \delta, \quad S_1 = \sin \delta
\]

Because of the symmetry between \( I_1 \) and \( I_2 \), we have

\[
T_2 = -T_1, \quad S_2 = S_1, \quad \alpha = 0, \quad \nu_2 = -\nu_1
\]

Using these relations in the optimal condition (26), we have

\[
(0 \tan \delta - 1)[(1 + 20^2) \tan \delta - (1 - 20^2) \sin \delta] = 0
\]

The first factor is a parasite solution and from the second factor we have

\[
\tan 2 \delta = \frac{20}{1 + 20^2}
\]

which is an interesting relation between the thrust angle and the transfer angle. On the other hand, if we use the symmetric condition in the optimal switching condition (23), we have

\[
\frac{1}{x} = \frac{\theta^2 (1 - 2 \tan^2 \delta - 2 \theta \tan \delta)}{1 - 2 \theta \tan \delta + \theta^2 \tan^2 \delta}
\]

Using the condition (55) for simplification, we obtain

\[
0 \tan \delta = \frac{1}{x+1}, \quad 0 = (x+2) \tan \delta
\]

Finally, if \( \vec{V}^+ \) and \( \vec{V}^- \) are the velocity vectors before and after the application of the impulse, we have

\[
\vec{V}^+ + \vec{V}^- = \vec{V}^4
\]

By projecting this vector relation into the \( S \) and \( T \) axes, we have

\[
S_1 \Delta V + \sqrt{\frac{\mu}{p}} E \sin (\alpha + \nu_1) = \sqrt{\frac{\mu}{p}} \sin \nu_1
\]

\[
T_1 \Delta V + \sqrt{\frac{\mu p}{r}} = \sqrt{\frac{\mu p}{r}}
\]

By eliminating \( \Delta V \), we have
Fig. 7. Transfer between noncoplanar circular orbits.

In the case from LEO to GEO the optimal transfer is by two impulses, via a generalized Hohmann orbit, with plane change made at both impulses.
The switching relations in section IV are given with respect to this transfer orbit. The impulses are perpendicular to the position vectors and hence \( S_1 = S_2 = 0 \). If \( \delta \) is the thrust angle with respect to the \( T \) axis, then, by definition

\[
T = \cos \delta, \quad W = \sin \delta, \quad T_2 = \cos \delta_2, \quad W_2 = \sin \delta_2
\]

From the velocity diagrams in Fig. 8, we have the relations

\[
\frac{\Delta V_1}{V_1} = \frac{\sin i_1}{\sin \delta_1}, \quad \frac{\Delta V_2}{V_2} = -\frac{\sin i_2}{\sin \delta_2}
\]

with the constraining relations

\[
i_1 + i_2 = i, \quad \frac{\sin(i_1 + \delta_1)}{\sin \delta_1} = \sqrt{\frac{2n}{n+1}}, \quad \frac{\sin(i_2 + \delta_2)}{\sin \delta_2} = \sqrt{\frac{2}{n+1}}
\]

For given \( n \) and \( i \), to solve for the four unknowns \( \delta_1, \delta_2, i_1 \) and \( i_2 \), we need one more relation. This is obtained by the optimal switching relations (23) and (24). Since \( V_1 = 0, V_2 = \pi, \Delta = \pi \) we deduce from these relations

\[
26(S_1 + S_2)T_2 = (1+e)[14T_1T_2 - W_1W_2] - 2
\]

Eliminating the factor \( 0(S_1 + S_2) \), which obviously has a finite limit, and noticing that \( e = (n-1)/(n+1) \), we have

\[
n \sin \delta_1 + \sin \delta_2 = 0
\]

The solution is obtained by solving the Eqs. (72) and (74). By combining these equations it can be shown that

\[
\sin^2(\delta_1 + \delta_2 + i) = \frac{2n}[1 - (2n^{1/2}(n+1))] \cos(\delta_1 + \delta_2 + i) \]

\[
\frac{\sin \delta_1 \sin \delta_2}{2n+1} - 2n \cos(\delta_1 + \delta_2 + i) \]

\[
\sin^2(\delta_1 + \delta_2) = \frac{2n}{n+1} \sin^2(\delta_1 + \delta_2)
\]

For given \( n \) and \( i \), we solve for \( \delta_1 + \delta_2 \) and deduce the other elements of the transfer orbit. For the case considered, \( n = 6.278389, i = 28.5^o \), we have \( i_1 = 2.212^o, i_2 = 26.288^o, \delta_1 = 7.002^o, \delta_2 = -49.943^o \), with the resulting characteristic velocity, normalized with respect to circular speed at distance \( R \)

\[
\frac{\Delta V_{PA}}{\sqrt{\mu/R}} = 0.538068
\]

To return from GEO to LEO, for a pure propulsive mode, the reverse operation applies, and we have the same characteristic velocity.

Now, if aeroassist is considered, then for the return trajectory the following parabolic-aeroassisted mode is the absolute optimal. In this mode, an accelerative impulse is applied at GEO, to send the vehicle into a parabola. At infinity, the plane change, for any amount, is performed without fuel consumption. The vehicle then returns along a parabolic orbit grazing the atmosphere at distance \( R \). Following an orbit contraction due to drag force acting at perigee, the apogee will decrease progressively to the LEO level where an accelerative impulse is applied for circularization. The total characteristic velocity for this mode is

\[
\frac{\Delta V_{PA}}{\sqrt{\mu/R}} = \frac{1}{\sqrt{c}} \left[ \sqrt{\frac{2}{n}} - \frac{n}{n+1} - \sqrt{\frac{2}{c+1}} \right]
\]

where \( n \) and \( c \) are the ratios defined in Eq. (69). With the given radii, this normalized characteristic velocity has the value 0.171043 which is only 32% of the cost for pure propulsive maneuver. Again, we notice that this parabolic-aeroassisted mode is independent of the amount of plane change and only requires drag capacity for the OTV. If we want an immediate return from GEO, then with a decelerative impulse applied at GEO, the vehicle returns with a plane change \( i \), to enter the atmosphere at distance \( R \) with a small entry angle \( y \) and a relatively high entry speed \( u = V \sqrt{2/GR} \). The equations and method displayed in section V allows the computation of the atmospheric plane change, with exit speed sufficient for climbing to LEO altitude for circularization. In this case, we also have plane change without fuel consumption in the amount of \( i = 22.5^o \). The remaining angle \( i_1 = 6^o \) must be performed at GEO. The computation has been done with an OTV having a maximum lift-to-drag ratio \( E_k = 1.5 \). The total characteristic velocity, also normalized with respect to circular speed at distance \( R \), is now 0.2040, and hence at about 38% of the cost for pure propulsive maneuver.
References


