WEAK AND STRONG VIBRATION LOCALIZATION IN DISORDERED STRUCTURES: 
A STATISTICAL INVESTIGATION

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Abstract

A statistical investigation of the effects of disorder on the dynamics of one-dimensional nearly periodic structures is presented. The problem of vibration propagation from a local source of excitation is considered. While for the ordered infinite system there exists a frequency passband for which the vibration propagates without attenuation, the introduction of disorder results in an exponential decay of the amplitude for all excitation frequencies. Analytical expressions for the localization factors (the exponential decay constants) are obtained in the two limiting cases of weak and strong internal coupling, and the degree of localization is shown to depend upon the disorder to coupling ratio and the excitation frequency. Both modal and wave propagation descriptions are used. The perturbation results are verified by Monte Carlo simulations. The phenomena of weak and strong localization are evidenced, and the latter is shown to be of significant importance in structural dynamics.

1. Introduction

Mode localization in disordered structures has received significant attention in the structural dynamics community lately (see references 1–11 or the review paper by Ibrahim12). This phenomenon, originally predicted by Anderson13 in solid state physics, occurs in repetitive structures whose periodicity is broken by small structural irregularities arising from unavoidable manufacturing and material tolerances. Under conditions of weak internal coupling for the structure of interest, such small disorder has drastic effects on its dynamical properties, by localizing the vibration modes to a small geometric region of the structure and confining the vibrational energy close to the source of excitation. The localization phenomenon is illustrated in Fig. 1, which shows the free vibration modes of ordered and disordered chains of single-degree-of-freedom (DOF) oscillators in a weak coupling case. The collective, or extended, oscillations of the tuned system become strongly localized when small disorder is introduced. In an average sense the decay of the vibration amplitude is exponential, the decay constant being referred to as the localization factor.

Localization may occur in a number of engineering structures such as blade assemblies,1,2 multi-span structures,3,4 and some large space structures.5,6 Vibration localization in structures has been evidenced by experiments as well.3,4 The consequences of localization can be either damaging—as localized vibrations lead to larger amplitudes—or beneficial—as a means of passive control of vibration propagation. For practical purposes, localization is similar to damping as it manifests itself by a spatial decay of the vibration amplitude along the structure, even though for localized vibrations the energy is confined near the source of excitation while for damping it is dissipated as it propagates. Thus, to an experimentalist unaware of disorder effects, localization could be easily mistaken for, and result in an overestimation of, damping.

Research on localization in structural dynamics has been mostly limited to deterministic analyses of the spatially localized free modes of disordered one-dimensional structures.7,8,9 This is because analytical and even numerical (Monte Carlo) statistical investigations are difficult to conduct for the mode shapes. However, a consistent statistical treatment of localization can be formulated by studying the propagation of forced vibration from a local source of excitation. This is motivated by the argument that at a given frequency the localized mode shapes feature the same (exponential) spatial decay as the forced vibration patterns do.14 To the knowledge of the author, two probabilistic investigations of localization have been carried out to date in structural dynamics, both approaching the problem from a forced excitation viewpoint. First, Hodges and Woodhouse15 applied the work of Herbert and Jones16 to calculate, by statistical perturbation methods, localization factors for a (finite) stretched string with irregularly spaced masses attached to it. However, they did not perform a systematic study of the dependence of the localization factor on the excitation frequency or the system parameters. Their research, though, exhibited the occurrence of two types of localization—weak and strong—depending on the magnitude of internal coupling in the structure, but they did not elaborate. Second, in a recent study, Kissel10,11 chose a wave propagation approach to calculate localization factors for several infinite one-dimensional structures. Modeling the structures by random transfer matrices, he followed the approach originally proposed by Matsuda and Ishii,12,13 and Ishii,14 who were first to apply Farrenbach’s theorem on the limiting behavior of products of random matrices. He systematically studied the dependence of the localization factor inside the passbands on the system parameters and the excitation frequency in the limiting case of weak disorder, by essentially a perturbation approach. Unfortunately, the structural models he chose (such as the spring-mass chain shown in Fig. 2) did not allow him to vary the relative magnitudes of the internal coupling and the disorder, thereby restricting his findings to weak localization, which does not affect significantly the dynamics of most engineering structures.

This paper investigates the localization phenomenon for a generic model of one-dimensional nominally periodic structures—namely the undamped chain of single-DOF oscillators shown in Fig. 3. This system is chosen because, contrary to the spring-mass models commonly studied in solid state physics and adopted by Kissel,10 it allows one to vary the relative orders of magnitude of coupling and disorder and thus to exhibit both weak and strong localization. Contrary to most previous research on localization in structures, this paper presents a statistical treatment of the structural irregularities.

The first part of the paper reviews the techniques to calculate the localization factors for a structure under forced excitation. In the second part, the ordered structure is studied, and the concepts of frequency passband and stopband are illustrated by calculating the corresponding exponential amplitude decays. Perturbation methods are utilized in the third part of the paper to derive analytical expressions for the localization factors of disordered systems in the two limiting cases of weak and strong coupling. In the fourth part, analytical results are compared to Monte Carlo simulations and the validity of the two asymptotic results is discussed. The contributions of the present study are twofold. First, results of general significance to the theory of localization in engineering structures are derived. Second, the distinction between weak and strong localization, which often seems confused in the literature on localization in structures, is clarified, and the importance of strong localization in engineering structures is evidenced relatively to that of weak localization.

2. Methodology

2.1 Equations of Motion

The undamped assembly of N, coupled, single-DOF oscillators shown in Fig. 3 is considered. The system is nearly equivalent to the chain of pendulums of Figs. 1,5,6 and the corresponding cyclic assembly has been used as a one-component mode representation of a continuously shrouded blade assembly.2 For simplicity, fixed-fixed boundary conditions are considered, even though the results are believed to hold for any nondissipative end condition. The system is excited at its left end by a simple harmonic force of frequency, $\omega$, and the propagation of steady-state vibrations along the assembly is of interest. The equations of harmonic motion are

$$\mathbf{\omega}^2 \mathbf{m} \mathbf{q} = -\mathbf{k} (2 \mathbf{m} \mathbf{u}_{i-1} \mathbf{u}_{i+1}) - \mathbf{k} \mathbf{u}_i + \mathbf{F} \mathbf{\delta}_1, \quad i = 1, \ldots, N$$

where $\mathbf{u}_i$ is the displacement amplitude of the $i$th oscillator, with $\mathbf{u}_0 = \mathbf{u}_{N+1} = 0$, $k_i$ the $i$th oscillator's stiffness, $k_1$ the coupling stiffness, $F$ the harmonic force amplitude, and $\mathbf{\delta}_1$ the Kronecker symbol. For a perfectly periodic chain, $k_1 = k$, where $k$ is the nominal oscillator's stiffness. For the disordered structure the $k_i's$ are taken to be identically and independently distributed random variables with mean $k$. A uniform disorder distribution is chosen primarily because it leads to simpler calculations for the localization factors, even though some of the results of the paper hold for any type of distribution. The methods of analysis are also valid for any type of random disorder.

Only disorder in the stiffnesses of the oscillators is considered, even though a fully disordered system would have random masses and cou-
pling stiffnesses as well. However, the effects of random oscillators' masses and stiffnesses are believed to be essentially the same, as both affect the oscillators' natural frequencies (indeed this was shown for a spring-mass system\textsuperscript{11}). Also, random coupling stiffnesses, since they do not affect the oscillators' frequencies, only lead to the weak localization studied by Kissel\textsuperscript{11}. Therefore, restricting the randomness to oscillators' stiffnesses captures all the important localization effects in a relatively simple analysis.

The equations of motion are made dimensionless by introducing \( \omega_3 = \sqrt{F/m} \), the nominal oscillator's frequency, \( \omega = \omega_3/\omega_3 \), the dimensionless excitation frequency, \( \Delta f_l = (k_i - k_l)/k_l \), the dimensionless disorder for the \( l \)th oscillator, and \( \alpha = k_i/k_l \), the ratio of the square of the normal coupling frequency to the nominal oscillator's frequency, or the dimensionless coupling, yielding

\[
\begin{pmatrix}
-u_1 \\
u_2 \\
\vdots \\
u_N
\end{pmatrix} = \begin{pmatrix}
F \\
0 \\
\vdots \\
0
\end{pmatrix} \cdot \begin{pmatrix}
(-\omega^2[I] + [A])^{-1}
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_N
\end{pmatrix}
\]

(2a)

where \( F = \frac{\alpha}{\omega^2} \), \([I]\) is the identity matrix, and \([A]\) is an \( N \times N \) tridiagonal matrix given by

\[
[A] = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & -R & 1 + \Delta f_l & 2R - \omega^2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -R & 0
\end{pmatrix}
\]

(2b)

2.2 Modal Formulation

To study vibration propagation, the amplitude at the right end is calculated when the system is driven at the left end. The approach has been originally proposed by Herbert and Jones\textsuperscript{5} and then used by Hodges and Woodhouse\textsuperscript{4} from Eq. (2), the response amplitude of the \( N \)th oscillator is

\[
\frac{u_N}{\mathcal{F}} = \frac{(-\omega^2[I] - [A])^{-1}}{\det(-\omega^2[I] + [A])}
\]

(3)

which holds only if \( \omega \) is not a natural frequency of the assembly. Remarkably, it follows from the tridiagonality of \([A]\) that the corresponding minor determinant is simply \((-R)^{N-1}\), thus

\[
\frac{u_N}{\mathcal{F}} = \frac{R^{N-1}}{\det(-\omega^2[I] + [A])}
\]

(4)

The above determinant is evaluated by modal analysis. The natural modes of the (ordered or disordered) system are determined by the eigenvalue problem

\[
(-\omega^2[I] + [A])\underline{u}_r = 0 \quad r = 1, \ldots, N
\]

(5)

where \( \omega_r \) and \( \underline{u}_r \) are the \( r \)th frequency and eigenvector. The modal matrix \([U]\), whose columns are the orthonormal eigenvectors, verifies \( [U]^T[U] = [I] \) and \( [U]^T[A][U] = \text{diag}(\omega_r^2) \), where \( T \) denotes a transpose. Applying the normal coordinate transformation yields

\[
\det(-\omega^2[I] + [A]) = \det \left( [U]^T \left( (-\omega^2[I] + [A]) \right) [U] \right) = \prod_{r=1}^{N} (\omega_r^2 - \omega^2)
\]

(6)

The expression for the (spatial) exponential decay of the amplitude is obtained by writing the modulus of the \( N \)th amplitude as \( |\mathcal{F}| = \exp(-\gamma_N \omega N) \), where \( \gamma_N \) is the exponential decay constant for a given chain of length \( N \). This yields, from Eq. (4):

\[
\gamma_N = -\frac{1}{N} \ln \left( \frac{u_N}{\mathcal{F}} \right) = -\frac{1}{N} \ln R + \frac{1}{N} \sum_{r=1}^{N} \ln (\omega_r^2 - \omega^2)
\]

(7)

This expression is exact for a finite chain with boundary conditions. It does not necessarily imply that there is an actual exponential decay of the vibration amplitude. In general to evidence such a possible decay one needs to let the system size go to infinity or to average Eq. (7) over many disordered systems. Indeed, the formula (7) is the one used to perform Monte Carlo simulations.

Of interest in the case of exponential decay of the amplitude is the limiting behavior of \( \gamma_N \) as the length of the structure, \( N \), goes to infinity. If \( \gamma_N \) admits a limit \( \gamma = \lim_{N \to \infty} \gamma_N \), then \( |\mathcal{F}| \sim \exp(-\gamma \omega N) \) for \( N \) large. In other words, the vibration amplitude decays exponentially with \( N \) at the rate \( \gamma \), referred to as the exponential decay constant (the real part of the propagation constant\textsuperscript{17}). The amplitude is then governed by an exponential envelope. From Eq. (7), the exponential decay constant (if it exists) is

\[
\gamma = -\ln R + \lim_{N \to \infty} \frac{1}{N} \sum_{r=1}^{N} \ln (\omega_r^2 - \omega^2)
\]

(8)

This expression is valid for both ordered and disordered systems. For disordered chains \( \gamma \) is not obtained by taking an average over several realizations, but by letting the size of the system become infinite. In the limiting cases for which analytical expressions of \( \gamma \) will be obtained, it will be shown that taking the limit of \( \gamma \) as \( N \to \infty \) is equivalent to averaging \( \gamma_N \) over all disordered systems, indicating that \( \gamma_N \) is a self-averaging random variable\textsuperscript{11}.

2.3 Wave Propagation Formulation

Exponential decay constants can also be calculated by a traveling wave formulation. The system of Fig. 3 is considered, but no boundary conditions are prescribed, which is equivalent to assuming that the assembly is infinite. This is a drawback as boundary conditions are usually important for engineering structures. The \( l \)th site consists of the \( l \)th oscillator and of the coupling spring at \( l \)'s right (see Fig. 4), while the state vector at the \( l \)th site is defined by \([u_l, \underline{u}_{l-1}]^T\). A two by two displacement transfer matrix is defined that relates the states at adjacent sites

\[
\begin{pmatrix}
u_{l+1} \\
\nu_{l}
\end{pmatrix} = [T_l] \begin{pmatrix}
u_{l-1} \\
\nu_{l}
\end{pmatrix}
\]

(9)

Thus the state at site \( N + 1 \) is related to that at site 1 by

\[
\begin{pmatrix}
u_{N+1} \\
\nu_{N}
\end{pmatrix} = \prod_{j=N}^{1} [T_j] \begin{pmatrix}
u_{1} \\
\nu_{0}
\end{pmatrix}
\]

(10)

For an ordered chain all transfer matrices are identical, given by

\[
[T_1] = \begin{pmatrix}
\frac{1}{\alpha} & \beta \\
\frac{\alpha}{\beta} & \frac{-\beta}{\alpha}
\end{pmatrix}
\]

(11)

For a disordered system the \([T_l]'s\) are random transfer matrices, and \([T_0]\), the transfer matrix for the ordered system, is their average matrix.

Next wave amplitudes are introduced by a change of coordinates. Waves of frequency \( \omega \) are considered, the amplitudes of left and right traveling waves being denoted by \( L \) and \( R \), respectively. The complex amplitudes of the waves entering (resp. leaving) the site \( i \) are \( L_i \) and \( R_i \), as shown in Fig. 4. The oscillators' displacement amplitudes are related to the wave amplitudes by\textsuperscript{10,16}

\[
\begin{pmatrix}
u_i \\
\nu_{i-1}
\end{pmatrix} = [X] \begin{pmatrix} L_i \\
R_i \end{pmatrix}
\]

(12)

where \( \gamma^2 = -1 \). The first relation in (12) states that the displacement of the \( i \)th oscillator is the sum of the left and right traveling wave amplitudes. The complex \( k \) is the wave number, a function of frequency \( \omega \). As shown in the next section, \( k \) real defines a frequency passband, while \( k \) of the form \( \omega \alpha \) or \( \omega \alpha + \pi \), where \( \alpha \) is real, defines a stopband. Eqs. (9) and (12) lead to the introduction of a (mono-coupled) wave transfer matrix that relates the wave amplitudes at sites \( i \) and \( i + 1 \):

\[
\begin{pmatrix}
u_{i+1} \\
\nu_{i}
\end{pmatrix} = [W_i] \begin{pmatrix}
u_{i} \\
\nu_{i-1}
\end{pmatrix}
\]

(13)

Note that \([X]^{-1}\) exists only if \( k \neq 0 \) and \( \pi \) (corresponding to the passband edges). \([W_i]\) is a random transfer matrix of wave amplitudes for one ordered or disordered site, obtained from the displacement transfer matrix by a similarity transformation from displacement to wave coordinates. It is shown in the next section that the columns of the transformation matrix \([X]\) are the eigenvectors of the transfer matrix for the ordered chain, \([T_0]\). Hence for a periodic system \([W_j] \equiv [W_N]\), a diagonal matrix made of the eigenvalues of \([T_0]\). A general expression for \([W_i]\) can be obtained for frequencies in the passband (\( k \) real) by considering the waves entering and leaving a site in Fig. 4:
where * denotes a complex conjugate, and \( t_i \) and \( r_i \) are the (complex) transmission and reflection coefficients for the \( i \)th site, such that \( |t_i|^2 + |r_i|^2 = 1 \), since for this undamped system energy is conserved as it propagates (or, equivalently, since \( \det[T_i] = \det[W_i] = 1 \)).

For a periodic site \( [W_i] \) is diagonal, there is no reflection (\( r = 0 \)), and the eigenvalues of \([W_i]\) are \( t \) and \( \frac{1}{t} \). For frequencies in the passband (wave number not real), the transmission coefficient is real. Thus in a passband of the ordered system there is no reflection but since the magnitude of the transmission coefficient is not one there is attenuation.

For a disordered site \([W_i]\) is not diagonal because \([X]\) is the eigenvector matrix for the average, not the disordered site, and cannot diagonalize \([T_i]\). Therefore there is reflection (\( r \neq 0 \)) in the disordered system. It is this multiple scattering over many sites that results in localization.

The above formalism can be used to calculate the decay constant for an incident wave, as follows. An infinite system is considered, such that the sites \( i \) for \( i = -\infty, \ldots, 0 \) are ordered, the sites for \( i = 1, \ldots, N \) are disordered, and the ones for \( i = N + 1, \ldots, \infty \) are ordered. In other words, an \( N \)-site disordered system embedded in an otherwise ordered infinite system is considered. A wave of amplitude \( \alpha \) is incident from \( -\infty \) for \( i = -\infty, \ldots, 0 \) and \( i = N + 1, \ldots, \infty \) there is no reflection because the corresponding wave transfer matrix is diagonal, and in a passband the wave travels unattenuated. For \( i = 1, \ldots, N \), though, multiple scatterings occur, and only part of the incident wave's energy is transmitted through the disordered segment. From Eq. (13), the wave amplitudes entering and leaving the disordered segment are related by, in a frequency passband:

\[
\begin{bmatrix} L_{i+N}+1 \\ R_{i+N} \end{bmatrix} = [W_i] \begin{bmatrix} L_i \\ R_i \end{bmatrix} = \begin{bmatrix} t_i & \frac{1}{r_i} \\ r_i & t_i \end{bmatrix} \begin{bmatrix} L_i \\ R_i \end{bmatrix} = \begin{bmatrix} L_i \\ R_i \end{bmatrix} \begin{bmatrix} \frac{1}{r_i} & \frac{1}{r_i} \\ r_i & t_i \end{bmatrix}
\]

where \([W_i] = \prod_{j=1}^{n} [W_j]\) of determinant unity, is the wave transfer matrix for the \( N \) disordered sites, and \( r_i \) and \( t_i \) the corresponding transmission and reflection coefficients. Since a wave of amplitude \( \alpha \) is incident from the left and there is no reflection in the ordered segments, \( L_{i+N}+1 = 0 \) and \( R_i = \alpha \). The ratio of transmitted to incident amplitude, \( R_{i+N}/R_i \), is of interest, and can easily be shown to be equal to the transmission coefficient for the disordered segment, \( r_i \). It follows that the rate of exponential decay of the transmitted wave amplitude is

\[
\gamma_N = \frac{1}{N} \ln |R_{i+N}/\alpha| = \frac{1}{N} \ln |r_i|
\]

and the limiting value of the exponential decay constant for the infinite disordered chain is

\[
\gamma = \lim_{N \to \infty} \frac{1}{N} \ln |r_i|
\]

Thus, asymptotically, the ratio of transmitted to incident amplitude (or the modulus of the transmission coefficient) is governed by the envelope \( \exp(-\gamma N) \) (\( \exp(\gamma N) \)). Also, the transmitted energy is \( |r_i|^2 \exp(-2\gamma N) \), decaying exponentially as well. Obviously, in the passband of an ordered system, \( \gamma = 0 \) since \( \rho = 0 \) and \( |r_i| = 1 \), that is, there is pure energy transmission and no localization.

Note that \( \gamma \) is given by the limit in Eq. (17) with probability one. It is not obtained by taking an average, but by letting the size of the disordered system go to infinity, making \( \ln |r_i| \) a self-averaging random variable.

3. A Review of the Ordered Assembly

Even though finite and infinite periodic chains have been investigated extensively by wave propagation techniques (see references 17-19 for example), here the propagation constant \( \gamma \) is computed for an infinite chain over the entire frequency spectrum by the modal approach as well as by the wave method. Equivalence of the results is proven.

3.1 Modal Approach

The decay constant is given by Eq. (8), where \( \omega \) is the \( \gamma \)-th frequency of the ordered chain with fixed-fixed ends, which can be shown to be

\[
\omega^2 = 1 + 2R \left( 1 - \cos \frac{2\pi}{N+1} \right) \quad \gamma = 1, \ldots, N
\]

Substituting Eq. (18) into Eq. (8) yields, transforming the limit of the series into an integral,

\[
\gamma = -\ln R + \frac{1}{2} \int_0^1 \ln \left[ 1 + 2R(1 - \cos x) - \omega^2 \right] dx
\]

as the spectrum becomes continuous as \( N \to \infty \), the frequencies of the infinite chain lying dense in the band \([1, 1 + 4R]\), whose width decreases with the coupling. The integral (19) is always convergent, leading to a finite exponential decay for the infinite chain. Several cases are distinguished.

- \( 0 \leq \omega^2 < 1 \) (stopband)
  - One can show

\[
\int_0^1 \ln(\alpha + \beta \cos x) dx = \pi \ln \left( \frac{\alpha + \sqrt{\alpha^2 - \beta^2}}{2} \right) \quad \alpha > |\beta|
\]

Here \( \alpha = 1 + 2R - \omega^2 \) and \( \beta = -2R \). After some manipulations, one obtains

\[
\gamma = \cosh^{-1} \frac{1 + 2R - \omega^2}{2R} \quad 0 \leq \omega^2 < 1
\]

a strictly positive value corresponding to an attenuation of the amplitude for the infinite ordered chain. Accordingly the frequency range is referred to as a stopband. Note that \( \gamma = 0 \) as \( \omega^2 \to 1 \).

- \( \omega^2 > 1 + 4R \) (stopband)
  - The expression (20) can still be used but with \( \alpha = \omega^2 - 1 - 2R \) and \( \beta = 2R \), yielding

\[
\gamma = 2 \cosh^{-1} \frac{\sqrt{\omega^2 - 1}}{2R} \quad \omega^2 > 1 + 4R
\]

a positive quantity corresponding to a stopband as well. Also, \( \gamma \to 0 \) as \( \omega^2 \to 1 + 4R \).

- \( 1 \leq \omega^2 \leq 1 + 4R \) (passband)
  - The integral cannot be evaluated as in (20), since then \( |\alpha| \leq |\beta| \). The integral needed is

\[
\int_0^1 \ln(\cos \theta - \cos x) dx = -\pi \ln 2
\]

where \( \cos \theta = 1 + 4R - \omega^2 \), less or equal to one in absolute value. Even though there is a singularity at \( x = \theta \) (that is, when the excitation frequency coincides with one of the natural frequencies of the infinite system), the integral converges. Substituting Eq. (23) into Eq. (19) yields

\[
\gamma = 0 \quad 1 \leq \omega^2 \leq 4R
\]

This frequency range is consequently referred to as a passband, since for the infinite chain the vibration propagates without attenuation.

3.2 Wave Approach

For the ordered chain the displacement transfer matrix is \([T_i]\) given by Eq. (11), whose eigenvalues are

\[
\lambda_1, \lambda_2 = \frac{1 + 2R - \omega^2}{2R} \pm \frac{1}{2R} \sqrt{(1 - \omega^2)(1 + 4R - \omega^2)}
\]

(such that \( \lambda_1 \lambda_2 = 1 \) and whose eigenvectors are the columns of

\[
[X] = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix}
\]

The wave transfer matrix, \([W_i] = [X]^{-1} [T_i] [X]\), is diagonal of diagonal elements \( \lambda_1 \) and \( \lambda_2 \), since there is no scattering for the ordered system. Therefore the transmission coefficient for \( N \) ordered sites is simply \( r_N = 1/\lambda_2^N \), and the corresponding exponential decay constant \( \gamma = \ln |\lambda_1| \). There is no need to take the limit as \( N \to \infty \) in the ordered case, as the decay is identical for all sites.

For \( \omega^2 \) in the passband \([1, 1 + 4R]\), the eigenvalues of \([T_i]\) are complex and, letting \( \omega^2 = 1 + 4R \sin^2 k \) in Eq. (25), become \( \lambda_{1,2} = \exp(\pm 2jk) \), where \( 2k \) is the real wave number. This immediately yields \( \gamma = \ln \left| e^{2jk} \right| = 0 \), as was obtained by the modal approach.

For \( \omega^2 \geq 1 + 4R \), letting \( \omega^2 = 1 + 4R \cos^2 k \) (\( k \geq 0 \)) in Eq. (25) leads to \( \lambda_{1,2} = -e^{2jk} \), corresponding to complex wave numbers \( \pi - 2jk \).
positive decay constant is $\gamma = \ln |-e^{2ik}| = 2k$, where $k = \cos^{-1} \sqrt{\frac{4R}{2N}}$, a result identical to the one obtained by the modal approach in Eq. (22).

Finally, for $\mathcal{M} \leq 1$, one has $\lambda_1 = \frac{4k \mathcal{M} - \mathcal{S}^m}{\mathcal{S}^m} \approx \frac{4k \mathcal{M}}{2N}$, and $\lambda_2 = 2k$. The decay constant is $\gamma = \ln |\gamma|$, which, after some algebraic manipulations, can be shown to be identical to the modal result (21).

3.3 Results and Remarks

The concepts of stopband and passband are illustrated in Fig. 5, which displays the decay constant $\gamma$ versus excitation frequency for various values of coupling. In the first stopband $\gamma$ decreases to zero as the passband boundary is approached, while in the second stopband $\gamma$ becomes large as $\mathcal{M}$ increases. The passband width decreases proportionally to coupling and larger values of $\gamma$ are obtained in the stopbands as $R$ decreases, because vibration propagation becomes more difficult as the coupling between oscillators decreases. For example, for $R = 0.01$ a very small passband with sharp variation of $\gamma$ near the edges is obtained.

In general localization factors cannot be calculated in closed form, but they are applied to the analysis of the forced response. The wave formulation for a spring-mass system (different from a chain of oscillators), by an approximation to Furstenberg’s theorem. A different procedure that perturbs directly the wave transfer matrix is used here, as follows.

A classical perturbation scheme is applied to the wave formulation. The random displacement transfer matrix in Eq. (9) is written as

$$[T_r] = [T_o] + [\Delta T_r]$$

where the unperturbed system is the ordered one and the perturbation is the disorder represented by the matrix $[\Delta T_r]$. The order of the perturbation is $\mathcal{M}$. Since disorder affects very little the value of $\gamma$ in the stopbands, the analysis is restricted to excitation frequencies in the passband of the ordered system. For $\mathcal{M}^2 < 1 + 4R$, the matrix of eigenvalues of $[T_r]$ is, from Eq. (26),

$$[X] = \begin{bmatrix} 1 & e^{-2j\mathcal{M}} \\ e^{2j\mathcal{M}} & 1 \end{bmatrix}$$

where $k$ is defined by $\mathcal{M}^2 = 1 + 4R \sin^2 k$, for $0 < k < \frac{\pi}{2}$. This leads to the random wave transfer matrix $[W_r] = [X]^{-1}[T_r][X]$, which can be written as

$$[W_r] = [W_o] + [w_o]$$

where, after calculations,

$$[W_o] = \begin{bmatrix} e^{2j\mathcal{M}} & 0 \\ 0 & e^{-2j\mathcal{M}} \end{bmatrix}, [w_o] = \begin{bmatrix} -e^{2j\mathcal{M}} \sin 2k \\ 2i \sin 2k \end{bmatrix}$$

are the unperturbed and random perturbation wave transfer matrices, respectively. Note that we must impose $k \neq \frac{\pi}{2}$ (that is, exclude the passband edges), since otherwise $[T_r]$ has a twofold eigenvalue and $[X]$ is singular.

Next, the product of $N$ wave transfer matrices is taken, and terms up to the first order in $\mathcal{M}$ are retained. This first-order perturbation procedure yields

$$[W_{N}r] = \prod_{i=1}^{N} ([W_o] + [w_o]) \simeq [W_o]^N \sum_{i=1}^{N} [W_o]^{N-1} [w_o] [W_o]^{N-2} \ldots [w_o] [W_o]^{N-1} + O \left( \frac{\mathcal{M}^2}{R^2} \right)$$

To obtain an approximation of the localization factor only the first diagonal element of $[W_{N}r]$, $\gamma_{N}$, is needed. One obtains some lengthy algebra:

$$\gamma_{N} = \lim_{N \to \infty} \frac{1}{N} \ln \left( 1 + \frac{1}{4R^2 \sin^2 2k} \sum_{l, m=1}^{N} \Delta f_l \Delta f_m + O \left( \frac{\mathcal{M}^2}{R^2} \right) \right)$$

Taking the modulus of $\gamma_{N}$, $\gamma$, is obtained from Eq. (17) as

$$\gamma = \lim_{N \to \infty} \frac{1}{2N} \ln \left( 1 + \frac{1}{4R^2 \sin^2 2k} \sum_{l, m=1}^{N} \Delta f_l \Delta f_m + O \left( \frac{\mathcal{M}^2}{R^2} \right) \right)$$

Note that the second-order terms in Eq. (31) may lead to second-order terms in the modulus as well, because this first-order analysis does not capture all second-order terms. However, it can readily be seen from Eq. (30) that these additional second-order terms only involve products of the form $\Delta f_l \Delta f_m$ for $l \neq m$. Approximating $\ln(1+2)$ by $2$ in Eq. (32) leads to

$$\gamma = \lim_{N \to \infty} \frac{1}{2N} \left( \frac{1}{4R^2 \sin^2 2k} \sum_{l, m=1}^{N} \Delta f_l \Delta f_m + O \left( \frac{\mathcal{M}^2}{R^2} \right) \right)$$

where the terms $O \left( \frac{\mathcal{M}^2}{R^2} \right)$ are of the form $\Delta f_l \Delta f_m$ for $l \neq m$. One has

$$\lim_{N \to \infty} \frac{1}{N} \sum_{l, m=1}^{N} \Delta f_l \Delta f_m = \sigma^2$$

because the $\Delta f_l$'s are uncorrelated and identically distributed. Hence all the terms $\Delta f_l \Delta f_m$ for $l \neq m$ do not contribute to the localization factor, which justifies the fact that a first-order perturbation analysis is sufficient. Combining Eqs. (33) and (34) leads to
\[
\gamma = \frac{\sigma^2}{8R^2 \sin^2 2k} + O\left(\frac{\sigma^4}{R^4}\right) \quad 0 < k < \frac{\pi}{2}
\]

which is a first-order approximation in the square of the standard deviation to coupling ratio, valid only for small values of \(\frac{k}{R}\). The localization factor, \(\gamma\), is always greater than zero for disordered systems, while it is zero for the corresponding ordered system. Disorder results in an attenuation of the wave amplitude as it propagates along the chain—the so-called localization, or confinement, phenomenon. Obviously the degree of localization depends only on the disorder to coupling ratio. Note that these results are for undamped systems, and thus the amplitude decay is created by disorder, not damping, resulting in a confinement, not a dissipation, of energy.

While \(\gamma\) has been obtained by taking a limit as \(N \to \infty\), it can also be calculated by taking the mathematical expectation of the logarithm of the transmission coefficient, a self-averaging variable. Eq. (31) gives

\[
\gamma = \frac{1}{2N} \ln \left(\frac{1}{N}\right)^2 \geq \frac{1}{2N} \frac{1}{4R^2 \sin^2 2k} \sum_{i=1}^{N} \Delta f_i \Delta f_m > 0
\]

which results in Eq. (35) because \(\Delta f_i \Delta f_m \geq \sigma^2 R^2\), where \(\sigma > 0\) denotes an average. Thus \(\gamma\) can be regarded either as the localization factor for an infinite chain or as the average localization factor for finite (large) chains. Also note that the assumption of uniform disorder distribution has not been used, and that Eq. (35) is valid for other distributions as well.

Next, the localization factor is written in terms of the excitation frequency by expressing the wave number \(k\) as a function of \(\omega\). A little algebra gives

\[
\gamma^{(c)} = \frac{\sigma^2}{2(\omega^2 - 1)(1 + 4R^2 - \omega^2)} 1 < \omega^2 < 1 + 4R
\]

where the superscript \((c)\) stands for a classical perturbation result, which holds in the strong coupling case. The approximation \(\gamma^{(c)}\) of \(\gamma\) deteriorates as \(\frac{\omega}{R}\) increases, and eventually the classical perturbation method fails for finite perturbations \(\frac{\omega}{R}\), that is, for weak internal coupling. Note that \(\gamma^{(c)}\) also becomes large near, and goes to infinity at, the passband edges \(\omega^2 = 1 + 1 + 4R\). Thus one may expect the perturbation expression (37) to be a poor approximation of the localization factor near the stopbands.

The variation of \(\gamma^{(c)}\) throughout the passband can be examined by introducing the parameter \(\alpha\) defined by \(\omega^2 = 1+\alpha R, 0 < \alpha < 4\). Eq. (37) can be rewritten as

\[
\gamma^{(c)} = \frac{\sigma^2}{2R} \frac{1}{2a(4-\alpha)} 0 < \alpha < 4
\]

The variation of \(\gamma^{(c)}/(\sigma^2/R^2)\) in terms of \(\alpha\) is shown in Fig. 6. The localization factor is maximum at midband \((\alpha = 2, \omega^2 = 1 + 2R)\) and becomes large near the stopbands. Eq. (38) clearly shows that localization depends only on the ratio of disorder to coupling.

Localization factors could be computed as well for \(\omega\) in the stopbands. However, as will be shown by Monte Carlo simulations, disorder only slightly modifies the already large values of \(\gamma\) in the stopbands, where strong attenuation occurs.

Finally, it is important to illustrate numerically the localization effect in the strong coupling case. Taking \(R = 1\) and \(\sigma = 10\%\) (a large disorder), the localization factor at midband \((\omega^2 = 3)\) is \(\gamma^{(c)} = 0.00125\). For an "infinite" system without boundary effects, the vibration amplitude is governed by \(e^{-\gamma N}\), and 555 sites are needed for the amplitude to decay by a factor two. Even though this effect can be regarded as impressive because it simply results from disorder and is not caused by damping, it is likely to be rather unimportant for engineering structures that rarely comprise that many substructures and for which boundary and damping effects may be more significant. In real structures unintentional disorder rarely exceeds 3-4\% and thus one may even expect smaller values of \(\gamma\). This small effect of disorder in the strong coupling case is termed weak localization. The validity of the perturbation result is checked by Monte Carlo simulations in section 5.

4.2 Large Disorder to Coupling Ratio—Strong Localization

The other limiting case, \(O\left(\frac{k}{R}\right) > 1\), is considered. Since disorder is always small, it implies that the coupling is weak. Then the classical perturbation method fails, since \(\gamma^{(c)} \to \infty\) as \(\frac{k}{R} \to 0\), and one needs to introduce a modified scheme that treats the coupling as a perturbation, while disorder is included in the unperturbed system to avoid degeneracy. The unperturbed system then consists of an assembly of uncoupled, disordered oscillators. This modified scheme has been shown to analyze the localized free modes of several disordered structures effectively, because strongly localized modes are essentially perturbations of decoupled modes. Here the method is applied to the forced vibration case. Note that the localization factor at midband has also been obtained by Hedges and Woodhouse in the weak coupling case. Here, however, it is calculated over the entire frequency spectrum.

Modal Approach

The localization factor is calculated from Eq. (8), the natural frequencies being approximated by the following first-order perturbation analysis. The system matrix \([A]\) is decomposed as \([A] = [A^{(m)}] + [u^{(m)}]\), where

\[
[A^{(m)}] = \text{diag}(1 + \Delta f_i) \quad [u^{(m)}] = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & -R & 2R & -R & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

are the modified unperturbed and perturbation matrices. Perturbation theory for the eigenvalue problem can be applied because the unperturbed eigenvalues are split by random disorder, and one can easily show that the first-order perturbed natural frequencies are \(\omega^{(m)} = 1 + \Delta f_i + 2R, \text{for} \ i = 1, \ldots, N\). This leads to the approximation of the localization factor

\[
\gamma^{(m)} = -\ln R + \lim_{N \to \infty} \frac{1}{N} \sum_{r=1}^{N} \ln \left|1 + \Delta f_i + 2R - \omega^2\right|
\]

where \([m]\) refers to the modified perturbation scheme for weak coupling. Obviously the limit in Eq. (40) is the expected value of the logarithmic expression, hence

\[
\gamma^{(m)} = -\ln R + \int_{-\infty}^{+\infty} p(\Delta f) \ln \left|1 + 2R + \Delta f - \omega^2\right| d(\Delta f)
\]

where \(p\) is the probability density function of the disorder. This is easiest to evaluate for a uniform distribution, even though it could be calculated in other cases as well. One obtains

\[
\gamma^{(m)} = -\ln R + \frac{1}{2W} \int_{-W}^{+W} \ln \left|1 + 2R + \Delta f - \omega^2\right| d(\Delta f)
\]

Eq. (42) could also be obtained by considering the average decay \(\gamma^{(m)}\) for finite assemblies:

\[
\gamma^{(m)} = \frac{1}{N} \sum_{r=1}^{N} \ln \left|1 + \Delta f_i + 2R - \omega^2\right|
\]

Hence Eq. (42) can be regarded either as the localization factor for an infinite chain or as the average factor for finite systems. After integration, it yields

\[
\gamma^{(m)} = \frac{1}{2W} \int_{-W}^{+W} \ln \left|1 + 2R + \Delta f - \omega^2\right| d(\Delta f)
\]

Since only first-order coupling terms were retained in the natural frequencies, the above expression is essentially a first-order result in the coupling to disorder ratio. This approximate formula is valid for frequencies in the passband and stopbands. It is of interest to consider the order of magnitude of \(\gamma^{(m)}\). At midband, \(\omega^2 = 1 + 2R\), Eq. (44) can be shown to reduce to

\[
\gamma^{(m)}_{\text{midband}} = \frac{\sigma^2}{R} + \ln \sqrt{1 + O\left(\frac{R^2}{\omega^2}\right)}
\]

which shows that the localization factor depends only upon the coupling to disorder ratio. Since the perturbation parameter is \(\frac{R}{\omega}\), the approximation \(\gamma^{(m)}\) of \(\gamma\) is valid for \(\frac{R}{\omega}\) first-order and improves as this ratio
decreases. Eq. (45) shows that as $\beta \rightarrow 0$, $\gamma^{(m)} \rightarrow \infty$. This makes sense because $\beta = 0$ oscillators to the modified unperturbed system consisting of uncoupled oscillators, for which there is total attenuation from one site to another, i.e., $\gamma \rightarrow \infty$. Localization is also seen to become more pronounced as the (small) coupling decreases. A typical value is obtained for $R = 0.01$ and $\sigma = 0.3$, given by $\gamma^{(m)}_{\text{band}} = 0.648$. This value is several orders of magnitude larger than the typical value obtained for the (weak) localization factor in the strong coupling case. A quick calculation reveals that, in average sense, the vibration amplitude of the fourth oscillator is $|y_4| \sim 0.75$, that is, only 0.75% of the vibrational energy is transmitted to the fourth oscillator! For the ordered system the value of $\gamma$ at midband is zero and 100% of the energy is transmitted. This dramatic phenomenon created by small disorder in the weak coupling case is referred to as strong localization.

Wave Approach

An approximation of the localization factor can also be obtained through the use of the wave transfer matrix. Again the basic idea is to treat the coupling as a perturbation. For small coupling to disorder ratio one can write $[T] = [T^o] + [\Delta T^{(m)}]$, where, from Eq. (9):

$$[T^{(m)}] = \begin{bmatrix} 1 + \frac{\Delta f}{R} - \frac{\omega^2}{\mathcal{A}} & 0 \\ 0 & \mathcal{A}^{(m)} \end{bmatrix}$$

(46)

Since the unperturbed matrix is diagonal, no transformation is necessary to calculate the wave transfer matrix of the perturbed system, given by $[W^{(m)}] = [T]$. A first-order approximation in the perturbation parameter, $\Delta T^{(m)}$, is obtained for the wave transfer matrix for $N$ sites:

$$[W_N] = \prod_{i=1}^{N} [T^{(m)}] + [\Delta T^{(m)}]$$

$$\approx \prod_{i=1}^{N} [T^{(m)}] + \sum_{i=1}^{N} \left( \prod_{i=1}^{N} [T^{(m)}] \right) [\Delta T^{(m)}] \left( \prod_{i=1}^{N} [T^{(m)}] \right)$$

(47)

where the first diagonal element of that matrix is needed to calculate $\gamma$. After a little algebra:

$$\langle V_N \rangle_{11} = \left( \prod_{i=1}^{N} 1 + \frac{\Delta f}{R} - \frac{\omega^2}{\mathcal{A}} \right) \left( 1 + 2 \sum_{i=1}^{N} \frac{R}{1 + \Delta f - \omega^2} \right)$$

(48)

and the approximation of the localization factor is, from Eq. (16)

$$\gamma^{(m)} = \frac{1}{N} \sum_{i=1}^{N} \ln \left| 1 + \frac{\Delta f}{R} - \frac{\omega^2}{\mathcal{A}} \right| + \frac{1}{N} \ln \left| 1 + 2 \sum_{i=1}^{N} \frac{R}{1 + \Delta f - \omega^2} \right|$$

(49)

which, after expanding the second logarithm to the first-order and taking the limit as $N \rightarrow \infty$ (or equivalently, taking the average for a finite chain), leads to

$$\gamma^{(m)} = -\ln R - \frac{1}{2W} \int_{-W}^{W} \ln(1 + z - \omega^2) dz = \frac{R}{W} - \frac{1}{2W} \int_{-W}^{W} \frac{dx}{1 + x - \omega^2}$$

(50)

Finally, after integration,

$$\gamma^{(m)} = -\ln R - 1 + \frac{2R + W - \omega^2}{2W} \ln(1 + W - \omega^2) - \frac{1 + 2R + W - \omega^2}{2W} \ln(1 + W - \omega^2)$$

(51)

which is similar, but not identical, to Eq. (44) obtained by the modal approach. Eq. (51) contains only first-order terms in $\frac{R}{W}$, while Eq. (44) has some higher-order terms as well. Expanding the modal result to retain only first-order terms in $\frac{R}{W}$, though, one can show easily that the wave result is obtained. Therefore, strictly to the first-order in $\frac{R}{W}$, both approaches are equivalent. This can also be seen from the integral formulations, as expanding the logarithm in Eq. (42) to the first-order in $\frac{R}{W}$ gives Eq. (50). Comparison of the two results to Monte Carlo simulations showed that the modal expression gives better estimates of the localization factor. Thus in the rest of the paper Eq. (44) is used for $\gamma$ in the weak coupling case. Note, however, that the two results are close, as for $R = 0.01$ and $W = 0.1$, $\gamma^{(m)} = 1.30$ and $\gamma^{(m)} = 1.28$, the agreement further improving as $\frac{R}{W}$ decreases.

5. Comparison of Analytical and Monte Carlo Results

This section’s purpose is twofold: to analyze the dependence of the localization factor upon frequency, coupling, and disorder, and to verify the validity of the analytical results by numerical simulations. There are two classes of Monte Carlo simulations. First, one can choose the modal formulation for a finite system of size $N$ with boundary conditions, and average the values of $\gamma_N$ obtained by Eq. (7) over a number of disordered systems, yielding the estimate $\gamma = \langle \gamma_N \rangle$. Second, the wave formulation can be adopted, the transmission coefficient $\tau_N$ being computed for a finite disordered segment but without boundary conditions by multiplying $N$ transfer matrices $[T]$ and applying the similarity transformation defined by $[X]$ to the resulting matrix. The estimate of the localization factor is then the average, over a number of realizations of the disordered segment, of $\gamma_N$ given by Eq. (16). The basic difference between the two techniques is that one accounts for boundary conditions, while the other does not. Because we are interested in engineering structures, the modal approach that simulates an actual finite structure with boundary conditions was chosen whenever possible, the limitations of this choice being discussed later. Another question that arises is how many realizations (ensembles) should be considered, and how large each system should be. These considerations are primarily governed by the computer cost. Monte Carlo simulations by the modal approach were found to be several orders of magnitude more expensive than those by the wave approach. This is because eigenvalue problems of size $N$ must be solved by the modal approach, where $N$ is the number of realizations, while the wave approach only demands the multiplication of $N$ matrices. To keep the computer cost reasonable, 300 realizations of systems of size 60 were considered at most for a modal simulation. For the expensive wave simulation no such limitations were necessary, and two configurations were chosen: 1000 realizations for 300 disordered sites, or 6000 realizations for a single site. The rationale for choosing the simulation procedure is discussed below.

Fig. 7 displays the variation of the localization factor in the passband, by classical perturbation method (Eq. (37)) and Monte Carlo simulation, for $R = 1$ and $\sigma = 1$. Note the excellent agreement between the theoretical and numerical results over the entire passband. Also, observe that $\gamma$ is larger near the stopbands, but that its magnitude is very small throughout the passband, indeed orders of magnitude smaller than $\gamma$ for the tuned system in the stopbands (which precludes us from displaying $\gamma$ in the stopbands).

Figs. 8 and 9 are for a larger disorder to coupling ratio equal to 0.1. Fig. 8 is for a 10% disorder and $R = 1$, while Fig. 9 is for a 1% disorder and $R = 0.1$. Almost identical values are obtained for $\gamma$, which is expected because the degree of localization depends only on the disorder to coupling ratio. From Eq. (37), $\gamma$ is 100 times greater than that for the tuned system in the stopbands, which precludes us from displaying $\gamma$ in the stopbands.

As the ratio of disorder to coupling increases, the approximation of $\gamma$ becomes worse for a given frequency, while as the ratio decreases, the approximation becomes better. It approaches an edge sufficiently, though, a discrepancy always occurs between the analytical and Monte Carlo result. Finally, Figs. 7-9 show that for such small disorder to coupling ratios the effect of disorder is merely to extend the stopbands a little, as $\gamma$ inside the passband is quite small.

A wave formulation was chosen for the Monte Carlo simulations in Figs. 7-9. Even though a modal calculation would have been preferred, it was found that the very small value of $\gamma$ for small disorder (less than 0.004) could not be obtained. Cost considerations limited the size of the systems considered to 60, and a rapid calculation shows that this leads to a (maximum) amplitude decay of 22% between the two ends. It was found, though, that boundary conditions influenced the value of $\gamma$ very significantly, and no meaningful result could be obtained over 200 ensembles. The values of $\gamma$ obtained by the modal simulation ranged approximately from 0.05 to 0.03, without any definite pattern, and the variance of the localization factor was much larger than the mean. This is not surprising when one recalls that for a 50-DOF ordered system $\gamma$'s of magnitude 0.01 were obtained in the passband instead of zero! Therefore very small decays cannot be simulated with reasonable...
cost for finite structures. Note that this is not a failure of the modal approach to analyze weak localization. Rather, it means that the concept of weak localization is irrelevant for a number of finite structures, because very weak disorders influence the amplitude decay as much as and sometimes more than, the disorder. Therefore, the results of Figs. 7-9 are only for infinite structures (or at least such that $N$ is much larger than the localization scale $\frac{d}{L}$). Based on the experience acquired from Monte Carlo simulations, we estimate that a localization factor of 0.004 would require a structure of at least 600 sites in order for the expected localization effect to take place in a given disordered structure. This remark shows that not only the average of the localization factor is of interest, but its variance as well. For we are interested in the behavior of individual disordered structures as much as in that of the average of an ensemble of structures. Localization effects are meaningful only if they occur for typical disordered systems, that is, only if the variance of $\gamma$ is much smaller than the mean. Finally, it was found that the modal approach could be used for 60-DOF systems only if $\gamma > 0.98$. Otherwise the wave method had to be used.

A few more words are in order concerning the wave simulation results. As noted by Kissel, we found that taking the average of the logarithm of the transmission coefficient for one disordered site led to excellent results. It should be pointed out, though, that this is not valid when dealing with larger localization factors ($\gamma > 0.05$), and thus in general this simplified approach should not be a substitute for complete simulations performed by multiplying many transfer matrices.

Fig. 10 displays the localization factor in terms of frequency for a disorder to coupling ratio of one. Here the larger values of $\gamma$ allow us to display its variation in the stopband. One observes that the agreement between the classical perturbation result and the simulation is still fair in the middle of the passband, but deteriorates very rapidly as $\Omega$ moves toward the edges of the passband, where the perturbation results overpredict $\gamma$ significantly. For larger disorder to coupling ratios the classical perturbation results would deteriorate further. In the stopbands $\gamma$ is very close to the decay of the ordered chain, justifying the fact that perturbation expressions were obtained only in the passband. The Monte-Carlo results were obtained by the modal approach for systems of size $N = 1000$, and very good agreement was observed with the wave simulation results performed on 1000 ensembles of 300 matrices. Therefore, for structures made of 60 sites with 10% disorder and excited at midband, an average decay of 99.8% of the vibration amplitude from one end to the other can be expected.

Fig. 11 shows the localization factor in a weak coupling case, for a disorder to coupling ratio equal to three. Note that the Monte Carlo results compare rather well with the modified perturbation results, considering that the disorder to coupling ratio is not large. As expected, the classical perturbation analysis gives extremely large values of $\gamma$ and yields erroneous results. Also, the localization factor for the disordered system varies little over the passband, and in the stopbands $\gamma$ is close to the tuned system's decay. Interestingly enough, in the stopband the decay for the disordered system is smaller than that for the tuned system, hence disorder decreases slightly the attenuation in the stopband! In the passband, however, the zero value of $\gamma$ for the tuned system becomes nearly 0.7 when small disorder is introduced, corresponding to a severe attenuation of the amplitude in this weak coupling case. It should be noted that the Monte Carlo results were obtained by simulating the modal formulation and agreed very well with the wave simulation results obtained by multiplying transfer matrices. The approach suggested by Kissel to average over only one disordered site led to severe underestimates of $\gamma$, and it was concluded that it should not be used for large decays.

Fig. 12 is for a larger disorder to coupling ratio equal to ten. Excellent agreement between the Monte Carlo and modified perturbation results (44) is observed. The classical perturbation results are so large that they could not be shown. If the plot were extended to include more of the stopbands it would show that the localization factor for tuned and mistuned systems approach each other as the distance to the passband increases. Note that the very large value of the localization factor, nearly equal to 1.9, leads to a drastic decay over a few oscillators—the so-called strong localization. Also, $\gamma$ is almost constant over the frequency range shown, a very different behavior from that observed for weak localization, where the localization factor is largest in the vicinity of the stopbands. In fact, Fig. 12 shows that the introduction of disorder practically eliminates the passband. Again, Monte Carlo results were obtained by simulating systems with boundary conditions.

Finally, Fig. 13 shows a typical variation of the localization factor in terms of the disorder to coupling ratio, for a midband excitation. Monte Carlo simulation and classical and modified perturbation results are shown. As $\frac{G}{G_0}$ increases from zero to approximately one, the simulation and classical perturbation results are in excellent agreement and correspond to the phenomenon of weak localization. This agreement deteriorates as the disorder to coupling ratio further increases, and while the classical perturbation results diverge from the numerical solution, the latter approaches the modified perturbation results. For $\frac{G}{G_0} > 6$ good agreement is observed between the Monte Carlo and the modified perturbation results, corresponding to strong localization. The agreement would improve as the disorder to coupling ratio further increases.

The region where neither perturbation scheme gives accurate results (approximately 1.5 < $\frac{G}{G_0} < 3$) corresponds to the transition between weak and strong localization, for which significant effects of disorder begin to occur. These intermediate cases require a numerical simulation of the effects of disorder. Finally, note that the Monte Carlo results are a little jagged because of cost limitations in the simulations. Smoother results could be obtained, but the resulting additional expense was not deemed worthwhile.

6. On Weak and Strong Localization

Weak localization occurs in systems with strong (finite) internal coupling and weak disorder, for which the amplitude decay per site is very small. Even though of considerable academic interest, weak localization is of little concern to the structural dynamicist, based on the following three arguments. First, even for an "infinite" structure without boundaries, the decay per site is very small. For example, a localization factor of 0.005 corresponds to a localization length of 200 sites, and 140 substructures are needed for the wave amplitude to decay by a factor two. Typically, the localization effect is significant for chains consisting of several hundreds or thousands of sites, making it relevant to systems like chains and lattices studied in physics. Second, Monte Carlo simulations have shown that boundary conditions can affect the very small values of the localization factors significantly, and in many cases obscure any noticeable effect of disorder. This happens when the size of the structure is not much larger than the localization length. Third, it is believed that the light damping that exists in all structures provides spatial amplitude decays that are at least comparable in magnitude to those created by disorder. This will be studied in a future paper. In engineering, the typical number of component systems making up a periodic structure is much smaller than in solid state physics (say less than one hundred), therefore making weak localization of little interest. A notable exception, however, could be truss-type structures that may consist of many components.

Strong localization, on the other hand, occurs in weakly coupled, weakly disordered systems, and is a drastic phenomenon: for a strongly disordered system, only a few oscillators (say a few percent) interact in the motion of the structure. This phenomenon is doubtlessly the most relevant to engineering structures. Indeed, past research on localization in structures has mainly focused on strong localization effects, while most work in physics has been concerned with weak localization.

This important distinction seems to be unclear in the structural dynamics literature, as it is rarely stated clearly which type of localization—weak or strong—is being studied. For instance, a recent statistical study of localization in a spring-mass chain could not exhibit the role of coupling or the strong localization behavior, and therefore was only concerned with weak localization. This is because the equations of motion for such a system can be reduced to a form in which the coupling does not appear, making the spring-mass chain fundamentally different from the chain of coupled oscillators studied here.

7. Conclusions

The effects of disorder on vibration propagation in chains of coupled oscillators have been studied by a probabilistic approach. The phenomenon of weak and strong localization have been shown to occur for strong and weak coupling between oscillators, respectively. Asymptotic expressions have been derived for the localization factors in these two limiting cases and have been confirmed by the Monte Carlo simulations. While weak localization phenomena are probably of little concern to the structural dynamicist, small disorder has been shown to cause a drastic spatial attenuation of the vibration amplitude in the weak coupling case.
References


Fig. 1. First three mode shapes of ten-DOF pendulum system. The natural frequency of the nominal pendulum is ten times the coupling frequency.

Fig. 2. Spring-mass chain.
Fig. 3. Two equivalent representations of an assembly of coupled oscillators.

Fig. 4. Site representation for wave formulation.

Fig. 5. Exponential decay constant versus excitation frequency for ordered chains of oscillators with various internal couplings.

Fig. 6. Variation of the localization factor in the frequency passband for a disordered system, by the classical perturbation method.

Fig. 7. Localization factor in the frequency passband, for $\sigma = 1\%$ and $R = 1$, by classical perturbation method (−) and Monte Carlo simulations (+).
Fig. 8. Localization factor for $\sigma = 10\%$ and $R = 1$, by classical perturbation method (−) and Monte Carlo simulations (+).

Fig. 9. Localization factor for $\sigma = 1\%$ and $R = 0.1$, by classical perturbation method (−) and Monte Carlo simulations (+).

Fig. 10. Localization factor in frequency passband and stopbands, for $\sigma = 10\%$ and $R = 0.1$, by classical perturbation method (−) and Monte Carlo simulations (+). The decay constant for the tuned system is also shown (−−−).

Fig. 11. Localization factor in frequency passband and stopbands, for $\sigma = 3\%$ and $R = 0.01$, by modified perturbation method (−), Monte Carlo simulations (+), and classical perturbation method (−−−). The decay constant for the tuned system is shown (−−−).
Fig. 12. Localization factor for $\sigma = 10\%$ and $R = 0.01$, by modified perturbation method (−) and Monte Carlo simulations (+). The decay constant for the tuned system is shown (---).

Fig. 13. Localization factor versus disorder to coupling ratio, at midband frequency $\omega^2 = 1+2R$, by Monte Carlo simulations (−−), classical perturbation method (−− −), and modified perturbation method (−− −).