INFINITE HORIZON PRODUCTION PLANNING IN TIME VARYING SYSTEMS WITH CONVEX PRODUCTION AND INVENTORY COSTS

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Abstract

We consider the planning of production over the infinite horizon in a system with time varying convex production and inventory holding costs. This production lot size problem is frequently faced in industry where a forecast of the future demand must be made and a production is to be scheduled based on the forecast. Since forecasts of the future are expensive and difficult to validate, a firm would like to minimize the number of periods into the future it needs to forecast in order to make an optimal production decision today. In this paper, we first prove that under very general conditions finite horizon versions of the problem exists that lead to an optimal production level at any decision epoch. In particular, we show it suffices to solve for a horizon that exceeds the longest time interval over which it can prove profitable to carry inventory. We then develop a closed-form expression for computing such a horizon and provide a simple finite algorithm to recursively compute an infinite horizon optimal production schedule.

1 Introduction

The production lot sizing problem is a model for the control of production over a multi-period planning horizon (Denardo [1982]). It is one of the most frequently used single item deterministic inventory planning models (Federgruen and Tzur [1991]). The objective is to schedule production over the planning horizon so that demand is satisfied at minimum cost. Standard assumptions are that demand is deterministic (i.e., known in advance) and backordering is not allowed (i.e., demand cannot be satisfied by future production).

The fundamental economic tradeoff here is the balance of reductions in cost of production against corresponding increases in costs of carrying inventory. In the presence of economies of scale on the cost of production, it can prove profitable to produce more than the current periods demand and carry inventory forward to satisfy future demand, thereby lowering the average cost of production (cycle stock motive). Even in the absence of economies of scale in production costs, the future cost of production may exceed the cost of current

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production plus inventory carrying costs again leading to current production that exceeds current demand (speculative motive (Chand and Morton [1986])).

The choice of planning horizon to employ is a difficult issue since the system being modeled typically has a long but otherwise indefinite lifespan. A resolution of this problem is to utilize an infinite horizon to model the underlying long but unknown finite horizon lifespan of the system. In the general case of time-varying demand and cost, the resulting model presents a challenging problem to solve (the stationary case reduces to the classic economic lot size (ELS) model (Harris [1990])). Early efforts to solve infinite horizon versions of the problem were restricted to the case of stationary, and usually linear, production cost, although demand was allowed to be time-varying (Kunreuther and Morton [1973, 1974], Lee and Orr [1977], Modigliani and Hohn [1955], Morton [1978a,1978b], and Thompson and Sethi [1980]). The so-called dynamic lot size version of the problem where production costs are fixed-plus-linear and inventory holding costs are linear has been extensively studied in the non-stationary case. Although the recent focus has been on computational breakthroughs in solving finite horizon versions of the problem (see e.g. Aggarwal and Park [1990], Federgruen and Tzur [1991], and Wagelmans, Van Hoesel and Kolen [1989]), the properties exploited there have in some cases been used to establish conditions on finite horizon versions of the infinite horizon problem that guarantee early decision agreement with optimal decisions of the infinite horizon problem. Such a finite horizon is called a solution horizon. When the agreement does not depend on problem data beyond this solution horizon, it is also called a forecast horizon since only data over this horizon needs to be forecasted to establish infinite horizon optimal early decisions (Bes and Sethi [1988]). Although solution and forecast horizons may fail to exist here, Federgruen and Tzur [1991, 1992] provided a stopping rule that is guaranteed to be met whenever they do exist. Specifically, they exploited monotonicity of optimal cost differences to establish necessary and sufficient conditions for a horizon N to be a forecast horizon. This monotonicity condition implies monotonicity with respect to N in the last period with production. This last property has been extensively exploited to generate forecast horizon existence and discovery results for variations on the dynamic lot size problem (see e.g. Wagner and Whitin [1958], Zabel [1964], Eppen et al. [1969], Thomas [1970], Blackburn and Kunreuther [1974], Lundin and Morton [1975], Bensoussan et al [1983], Chand [1982], Chand, Sethi, and Proth [1990], and Chand, Sethi and Sorger [1989]). See also Heyman and Sobel [1984] for a general review of using policy monotonicity in homogeneous MDP problems.

In this paper, we consider the infinite horizon version of the general production lot sizing problem under diseconomies of scale in production and inventory holding costs. This convexity assumption is equivalent to the condition that marginal production and holding costs be nondecreasing. For example, this includes the case where inventory costs are linear and where a firm experiences a higher overtime rate for production exceeding the standard capacity followed by a still higher unit cost for exceeding overtime capacity through outsourcing.

The optimization problem to be solved falls within the class of doubly infinite convex programming problems. There is an extensive literature on solution and forecast horizon approaches to solving such general problems in infinite horizon optimization (see e.g. Bean and Smith [1984, 1993], Bes and Sethi [1988], and Schochetman and Smith [1989, 1992]). However a key assumption that guarantees that general purpose algorithms will successfully discover an equivalent finite horizon problem is uniqueness of an infinite horizon optimal solution. Although this condition is believed to be typically met in practice, it is difficult to verify.

In this paper, we instead explore a novel algorithmic approach for finding solution and forecast horizons that systematically exploits monotonicity of optimal *early* decisions in horizon N. This focus on early decision monotonicity, as opposed to late decision monotonicity as in the treatment of the dynamic lot size problem, leads to a closed form expression for a forecast horizon guaranteed to yield optimal early production decisions for the infinite horizon problem. As we will show, the length of the forecast horizon is the longest interval of time over which it can prove profitable to carry inventory.

The paper is organized as follows. In Section 2, we formulate the infinite horizon model of the problem. In Section 3, we prove that under very general conditions, solution horizons exist leading to finite horizon versions of the problem that yield optimal solutions to the infinite horizon problem. In section 4, we give a closed-form expression for computing a solution (indeed forecast) horizon and a simple procedure for computing an optimal infinite horizon production schedule.

2 Problem Formulation

Consider a single-product firm where a decision for production must be made at the beginning of each period $n, n = 1, 2, \cdots$. We will adopt the following notation:

Constants and functions:

$$\begin{aligned} D_n &= \text{the demand during period } n \text{ (non-negative integers)} \\ \alpha &= \text{the discount factor for the time value of money } (0 < \alpha < 1) \\ I_0 &= \text{the inventory on hand at the beginning of period 1 (integers)} \\ c_n(x) &= \text{the cost of producing } x \text{ units of the product during period } n \text{ (non-negative)} \\ h_n(x) &= \text{the cost of holding } x \text{ units of inventory ending period } n \text{ (non-negative)} \end{aligned}$$

Decision variables:

$$\begin{aligned} P_n &= \text{the production level during period } n \\ I_n &= \text{the inventory on hand at the end of period } n \end{aligned}$$

We will use the superscript (*) to denote optimality.

With the above notation, we can formulate this infinite horizon problem, labeled Q, as

$$\text{(Q) Minimize:} \quad \sum_{n=1}^{\infty} \alpha^{n-1}[c_n(P_n) + h_n(I_n)] \tag{2.1}$$

$$\text{Subject to:} \quad I_{n-1} + P_n - D_n = I_n, \quad n = 1, 2, \cdots \tag{2.2}$$

$$P_n \geq 0, \quad I_n \geq 0, \quad n = 1, 2, \cdots \tag{2.3}$$

$$P_n, I_n \text{ integer}, \quad n = 1, 2, \cdots \tag{2.4}$$

where I_0 is given. As we can see from (2.2), if we know the production levels P_n in all periods, we can determine the inventory levels I_n. Therefore, it suffices to find an optimal production schedule $P^* = P_1^*, P_2^*, P_3^*, \cdots$. Note however that this is a doubly infinite integer nonlinear programming problem and is therefore a formidable problem to solve.

3 Existence of Solution Horizons

We now investigate conditions under which a finite horizon version of the problem has an optimal first decision which is in agreement with an infinite horizon optimal first decision. If we can find an optimal infinite horizon first decision P_1^* by solving a finite horizon version of the problem, we can roll forward one period and form a new infinite horizon problem with new initial inventory $I_1^* = I_0 + P_1^* - D_1$ to obtain an optimal infinite horizon second decision for the original problem. This rolling horizon procedure can then recursively recover an optimal infinite horizon production schedule.

In this section, we formulate the N-horizon truncated version of the problem and show that optimal production levels of the N-horizon problem are increasing in N. We then identify conditions under which an N-horizon optimal nth decision, $1 \leq n \leq N$, converges as $N \to \infty$ to an infinite horizon optimal nth decision. Finally, we establish existence of a finite horizon version for solving the infinite horizon problem.

3.1 The N-Horizon Problem

We formulate the N-horizon problem, labeled (Q(N)), corresponding to the original infinite horizon problem (Q) as:

$$\text{(Q(N)) Minimize:} \quad \sum_{n=1}^{N} \alpha^{n-1}[c_n(P_n) + h_n(I_n)] \tag{3.5}$$

$$\text{Subject to:} \quad I_{n-1} + P_n - D_n = I_n, \quad n = 1, 2, \cdots, N \tag{3.6}$$

$$P_n \geq 0, \quad I_n \geq 0, \quad n = 1, 2, \cdots, N \tag{3.7}$$

$$P_n, I_n \text{ integer}, \quad n = 1, 2, \cdots, N \tag{3.8}$$

Let S be the set of all feasible production schedules to (Q), $S(N)$ the set of feasible production schedules to (Q(N)), $P(N)$ any feasible production schedule to (Q(N)), and $I(N)$ the ending on hand inventory resulted from the production schedule $P(N)$, $N = 1, 2, \cdots$. The following lemma states that for the finite horizon problem, increasing the demands does not decrease the optimal production levels.

Lemma 1 *(Veinott [1964]) Let $P^*(N) = (P_1^*(N), P_2^*(N), \cdots, P_N^*(N))$ be an optimal solution for a vector $(D_1, D_2, \cdots, D_N)$ of demands and suppose production and inventory holding cost functions are convex. If one of these demands is increased by 1 unit, it is optimal to increase one of these production levels by 1 unit.*

The proof of this lemma can be found in Denardo [1982].

Consider now the demand profile for an $N+1$–horizon problem where $D_{N+1} = 0$. Since, without loss of optimality, we never leave positive inventory at the end of a horizon, we conclude $I_N^* = 0$ at an optimal solution if $D_{N+1} = 0$. By the principle of optimality,

$$P_1^*(N) = P_1^*(N+1) \tag{3.9}$$

with $D_{N+1} = 0$. Hence applying the lemma repeatedly, we have

$$P_1^*(N) \leq P_1^*(N+1), \quad \text{for } N = 1, 2, \cdots \tag{3.10}$$

for any D_{N+1}. Following the same argument, we also have

$$P_n^*(N) \leq P_n^*(N+1), \text{ for all } 1 \leq n \leq N, \quad N = 1, 2, \cdots. \tag{3.11}$$

Hence, we have proved the following Corollary.

Corollary 1 *$P_n^*(N)$ is monotonically increasing in N for any fixed $n, 1 \leq n \leq N$.*

3.2 Optimal Solution and Value Convergence of the N-Horizon Problems

Before we discuss convergence of optimal solutions of the N-horizon problems, we need the following notation and assumptions. Let $C(P)$ be the objective function of (Q) for any given $P \in S$ and $C^* = C(P^*)$. Also let $C(P(N); N)$ be the objective function of (Q(N)) for any given $P(N) \in S(N)$ and $C^*(N) = C(P^*(N); N)$. Furthermore, we assume,

A0: $c_n(\cdot)$ and $h_n(\cdot)$ are convex functions for all $n = 1, 2, \cdots$.

A1: $C(P') < \infty$ for some feasible production schedule $P' \in S$, i.e., there exists a finite cost feasible production schedule to (Q).

A2: $\lim_{I_n \to \infty} h_n(I_n) = \infty$, i.e., the cost of carrying inventories that increase to infinity itself increases to infinity.

A3 : $0 < \delta_n \leq c_n(P_n) - c_n(P_n - 1) \leq \gamma_n < \infty$ for all integers $P_n > 0$ and all $n = 1, 2, \cdots$, i.e., the marginal cost of production is bounded from above and bounded away from zero for all $n = 1, 2, \cdots$.

We now show $P_n^*(N)$ converges to an infinite horizon optimal nth decision $P_n^* < \infty$ for all n (Theorem 1). That is, $\lim_{N \to \infty} P_n^*(N) = P_n^*$ under the above conditions. We will achieve this first by showing that $P_n^*(N)$ converges to an infinite horizon feasible solution as $N \to \infty$ (Lemmas 2 and 3) and then that value and solution convergence holds for all $n = 1, 2, \cdots$ (Lemma 4 and Theorem 1).

Lemma 2 *There exist finite production bounds $\bar{P}_n$, $n = 1, 2, \cdots$, so that $P_n^*(N) \leq \bar{P}_n < \infty$ for all N.*

Proof: Suppose not, then there exists some n and subsequence N_k^n, $k = 1, 2, \cdots$, such that

$$\lim_{k \to \infty} P_n^*(N_k^n) = \infty. \tag{3.12}$$

By assumption (A2),

$$\lim_{k \to \infty} h_n(I_n^*(N_k^n)) = \infty \tag{3.13}$$

where $I_n^*(N_k^n)$ is the optimal on hand inventory at the end of period n following the nth production decision $P_n^*(N_k^n)$ and hence

$$\lim_{k \to \infty} C^*(N_k^n) = \infty. \tag{3.14}$$

However,

$$C^*(N_k^n) \leq C(P'(N_k^n); N_k^n) \leq C(P') \tag{3.15}$$

where $P'(N_k^n)$ consists of the first N_k^n decisions in P' and by (A1)

$$C(P') < \infty. \tag{3.16}$$

This contradicts equation (3.14).

□

By Corollary 1, at any decision epoch, n, there exists a monotonically increasing sequence of decisions $P_n^*(N)$, $N = 1, 2, \cdots$. By Lemma 2, this sequence of values is bounded from above. Therefore, $P_n^*(N)$ must converge as N goes to infinity, i.e.,

$$\lim_{N \to \infty} P_n^*(N) = \hat{P}_n < \infty \tag{3.17}$$

exists for all $n = 1, 2, \cdots$.

It remains to show $\hat{P}$ is infinite horizon optimal.

Lemma 3 *$\hat{P} \in S$, i.e., $\hat{P}$ is infinite horizon feasible.*

Proof: Since

$$\lim_{N \to \infty} P_n^*(N) = \hat{P}_n, \quad n = 1, 2, \cdots, \tag{3.18}$$

for any $\epsilon > 0$, there exist integers $N_\epsilon(n) > 0$, $n = 1, 2, \cdots$, such that

$$|P_n^*(N) - \hat{P}_n| < \epsilon, \quad n = 1, 2, \cdots, \tag{3.19}$$

for $N \geq N_\epsilon(n)$, $n = 1, 2, \cdots$. Recall that production levels are integers. If we let $\epsilon = 1$, then there exist integers $N_1(n) > 0$, $n = 1, 2, \cdots$, such that

$$P_n^*(N) = \hat{P}_n, \quad n = 1, 2, \cdots, \tag{3.20}$$

for $N \geq N_1(n)$, $n = 1, 2, \cdots$. Therefore, at any decision epoch, n,

$$\sum_{j=1}^{n} \hat{P}_j = \sum_{j=1}^{n} P_j^*(N) \geq \left(\sum_{j=1}^{n} D_j - I_0 \right)^+ \tag{3.21}$$

for all $N > \max_{1 \leq j \leq n} \{N_1(j)\}$. Hence, $\hat{P}_n$ is feasible for the infinite horizon problem.

□

Lemma 4 $\lim_{N \to \infty} C^*(N) = C^*$, *i.e., optimal value convergence holds.*

Proof: For any feasible infinite horizon schedule $P \in S$, the first N decisions $P(N)$ are feasible to (Q(N)). Therefore, since costs are non-negative, $C^*(N) \leq C^*$. Also, for any feasible $N+1$-horizon schedule $P(N+1)$, the first N decisions $P(N)$ are N-horizon feasible. Therefore, $C^*(N)$ is increasing in N. This implies that $\lim_{N \to \infty} C^*(N)$ exists where

$$\lim_{N \to \infty} C^*(N) \leq C^* < \infty. \tag{3.22}$$

By (A1), we have $P' \in S$ and $C^* \leq C(P') < \infty$. For every N, we construct the following feasible infinite horizon production schedule $\tilde{P}$ by setting

$$\tilde{P}_n = \begin{cases} P_n^*(N) & \text{if } 1 \leq n \leq N \\ P_n' + I_{n-1}' & \text{if } n = N+1 \\ P_n' & \text{if } n > N+1 \end{cases} \tag{3.23}$$

with corresponding ending inventory

$$\tilde{I}_n = \begin{cases} I_n^*(N) & \text{if } 1 \leq n \leq N \\ I_n' & \text{if } n \geq N+1 \end{cases} \tag{3.24}$$

where I' is the ending inventory following the production schedule P'. Then

$$\begin{aligned} C(\tilde{P}) &= C^*(N) + \alpha^N[c_{N+1}(\tilde{P}_{N+1}) + h_{N+1}(\tilde{I}_{N+1})] \\ &\quad + \{C(P') - C(P'(N); N) - \alpha^N[c_{N+1}(P_{N+1}') + h_{N+1}(I_{N+1}')]\} \\ &\geq C^* \end{aligned}$$

where $P'(N)$ is the first N decisions in P'. Since $\lim_{N\to\infty} C(P'(N); N) = C(P') < \infty$ and by (A3), we have

$$\begin{aligned} &\lim_{N\to\infty} \Big\{C(P') - C(P'(N); N) - \alpha^N \left[c_{N+1}(P_{N+1}') + h_{N+1}(I_{N+1}')\right] \\ &\qquad + \alpha^N \left[c_{N+1}(\tilde{P}_{N+1}) + h_{N+1}(\tilde{I}_{N+1})\right]\Big\} \\ &= \lim_{N\to\infty} \alpha^N \left[c_{N+1}(\tilde{P}_{N+1}) - c_{N+1}(P_{N+1}') + h_{N+1}(\tilde{I}_{N+1}) - h_{N+1}(I_{N+1}')\right] \\ &= \lim_{N\to\infty} \alpha^N \left[c_{N+1}(P_{N+1}' + I_N') - c_{N+1}(P_{N+1}') + h_{N+1}(I_{N+1}') - h_{N+1}(I_{N+1}')\right] \\ &= \lim_{N\to\infty} \alpha^N \left[c_{N+1}(P_{N+1}' + I_N') - c_{N+1}(P_{N+1}')\right] \\ &\leq \lim_{N\to\infty} \alpha^N \gamma_{N+1} I_N' \\ &= 0. \end{aligned}$$

This implies

$$\lim_{N\to\infty} C^*(N) \geq C^*. \tag{3.25}$$

Combining (3.22) and (3.25), we have

$$\lim_{N\to\infty} C^*(N) = C^*. \tag{3.26}$$

□

We can now show our result that finite horizon optima monotonically converge upwards to an infinite horizon optimal solution.

Theorem 1 *The infinite horizon production schedule $P^* = \lim_{N\to\infty} P^*(N)$ where $P^*(N+1) \geq P^*(N)$ is infinite horizon optimal, i.e., monotonic optimal solution convergence holds.*

Proof: Recall that

$$\lim_{N\to\infty} P_n^*(N) = \hat{P}_n < \infty \tag{3.27}$$

for all $n = 1, 2, \cdots$, and note that

$$\sum_{n=1}^{N} \alpha^{n-1}[c_n(\hat{P}_n) + h_n(\hat{I}_n)] \geq C(P^*(N); N). \tag{3.28}$$

Therefore,

$$C(\hat{P}) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \alpha^{n-1}[c_n(\hat{P}_n) + h_n(\hat{I}_n)] \right\} \geq \lim_{N\to\infty} C(P^*(N); N) = C^*. \tag{3.29}$$

Also, for any positive integer M,

$$\begin{aligned}
&\sum_{n=1}^{M} \alpha^{n-1}[c_n(\hat{P}_n) + h_n(\hat{I}_n)] \\
&= \lim_{N\to\infty} \left\{ \sum_{n=1}^{M} \alpha^{n-1}[c_n(P_n^*(N)) + h_n(I_n^*(N))] \right\} \\
&\leq \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \alpha^{n-1}[c_n(P_n^*(N)) + h_n(I_n^*(N))] \right\} \\
&= C^*.
\end{aligned}$$

Take the limit as $M \to \infty$ on both sides of the above inequality to get

$$C(\hat{P}) = \lim_{M\to\infty} \left\{ \sum_{n=1}^{M} \alpha^{n-1}[c_n(\hat{P}_n) + h_n(\hat{I}_n)] \right\} \leq C^*. \tag{3.30}$$

Combining (3.29) and (3.30), we have

$$C(\hat{P}) = C^*.$$

From Lemma 3, $\hat{P} \in S$. Therefore, $\hat{P}$ is infinite horizon optimal.

□

Theorem 1 allows us to easily extend Veinott's monotonicity lemma to the infinite horizon case.

Theorem 2 *Suppose assumptions (A0) through (A3) hold. Let P^* be an optimal solution for a vector $(D_1, D_2, \cdots)$ of demands and suppose production and inventory holding cost functions are convex. If one of these demands is increased by 1 unit, it is optimal to increase one of these production levels by 1 unit.*

Proof: Let $\tilde{P}^*$ be the optimal infinite horizon schedule for a vector $(D_1, \cdots, D_j + 1, \cdots)$ of demands for some j. We know from Theorem 1 that for any integer $n > 0$, there exists an integer N_n such that

$$P_n^*(N) = P_n^*$$

and

$$\tilde{P}_n^*(N) = \tilde{P}_n^*$$

for all $N \geq N_n$. By Lemma 1,

$$\tilde{P}_n^*(N) \geq P_n^*(N). \tag{3.31}$$

Therefore,

$$\tilde{P}_n^* \geq P_n^*. \tag{3.32}$$

□

The result of Theorem 1, (3.17) is called **optimal solution convergence** and that of Lemma 4, (3.26) is called **optimal value convergence**. Result (3.26) is analogous to the method of successive approximations applied to homogeneous MDP problems (Denardo [1982]). The latter may be viewed as equivalent to solving successively longer horizon problems as we iterate (the initial guess of value function is seen here as a terminal value at the end of horizon).

Optimal value convergence (3.26) says that for N large enough, the corresponding optimal N-horizon plan $P^*(N)$ achieves a cost arbitrarily close to that achieved by an optimal infinite horizon solution P^*, i.e., $P^*(N)$ and P^* are close in value. But $P^*(N)$ is not an infinite horizon feasible solution. Optimal value convergence is therefore of limited use, approximating infinite horizon optimal cost, but not solutions, while it is the latter we need to implement. Still we may at times be able to extend $P^*(N)$ feasibly over the infinite horizon at small cost to achieve an infinite horizon feasible solution with nearly the same cost as P^*. The solution convergence result of Theorem 1 is however far more powerful since policies and not just costs are arbitrarily well approximated by sufficiently long finite horizon optimal solutions. In fact, the approximation to early decisions is without error in this case as we note in the next subsection.

3.3 Solution Horizons for Solving the Infinite Horizon Problem

By Theorem 1,

$$\lim_{N \to \infty} P_n^*(N) = P_n^*, \quad n = 1, 2, \cdots. \tag{3.33}$$

where P^* is an infinite horizon optimum. This implies that for any $\epsilon > 0$, there exists a horizon, $N_\epsilon(n)$ such that

$$|P_n^*(N) - P_n^*| < \epsilon, \quad \text{for all } \ N \geq N_\epsilon(n). \tag{3.34}$$

Let $\epsilon = 1$. Then

$$|P_n^*(N) - P_n^*| < 1 \tag{3.35}$$

so that

$$P_n^*(N) = P_n^* \tag{3.36}$$

for all $N \geq N_1(n)$. In particular,

$$P_1^*(N) = P_1^*, \text{ for all } N \geq N_1^* \tag{3.37}$$

so that $N_1^* = N_1(1)$ is a *solution horizon.* That is, there exists a finite horizon N_1^* sufficiently distant that an optimal finite production lot size for any horizon that long or longer yields an infinite horizon optimal first period production lot size.

By forward dynamic programming, let $f_n(i)$ be the present value of the optimal cost from period 1 to period n with ending inventory level i in period n, where $i \geq \left(I_0 - \sum_{j=1}^{n} D_j\right)^+$. Then

$$f_n(i) = \min_{0 \leq P_n \leq D_n + i} \{\alpha^{n-1}[c_n(P_n) + h_n(i)] + f_{n-1}(i + D_n - P_n)\}$$

where $f_0(i) = 0$ for $i = I_0$ and ∞ otherwise. If we knew the value of the solution horizon N_1^*, we could then solve for $f_{N_1^*}(0)$ to get

$$P_1^* = P_1^*(N_1^*).$$

From optimal solution convergence (3.17), we conclude that large horizon optimal nth period production lot sizes yield nth period optimal infinite horizon production lot sizes. We can then recursively find $(P_1^*, P_2^*, \cdots) = P^*$ with zero error. We turn to the computation of solution (and forecast) horizons in section 4.

4 Computing Solution and Forecast Horizons

We have shown in the previous section that there exists a solution horizon N_n^* such that

$$P_n^*(N) = P_n^*, \text{ for all } N \geq N_n^*$$

at any decision epoch n. In this section, we seek to find a method to compute solution horizons for all $n = 1, 2, \cdots$ and a corresponding simple algorithm to compute the optimal infinite horizon solution P_n^* for all n.

To find a solution horizon N_n^*, we need to slightly strengthen assumption (A3) to include an upper bound on marginal production costs, i.e. we require that

$$\sup_{n \geq 1} \left\{ \lim_{P_n \to \infty} [c_n(P_n) - c_n(P_n - 1)] \right\} = \sup_{n \geq 1} \{\gamma_n\} = \gamma < \infty. \tag{4.38}$$

Now consider $P_1^*(N)$ as N increases. By Corollary 1,

$$P_1^*(N + 1) \geq P_1^*(N). \tag{4.39}$$

Therefore, the optimal first decision either remains the same or increases as N increases. Suppose

$$P_1^*(N + 1) > P_1^*(N). \tag{4.40}$$

Since moreover

$$P_n^*(N + 1) \geq P_n^*(N), \text{ for all } 1 \leq n \leq N \tag{4.41}$$

by Corollary 1, an additional unit of inventory is produced in period 1 and held for N periods to satisfy a unit of demand in period $N+1$. Evidently, by (4.40) it is then less costly to satisfy a unit of demand in period $N+1$ by production in period 1 then by production in later periods. It follows that if N_1^* were a bound on the greatest number of periods it were economic to carry inventory, then $P_1^*(N)$ cannot increase and must remain constant for all $N \geq N_1^*$. Hence *a solution horizon is provided by the largest period of time it can prove profitable to hold a unit of inventory produced in period 1.* For example, if inventory turns over four times a year, N_1^* would correspond to a horizon of 3 months. In this case, solving for optimal first production decisions for a planning horizon of 3 months or beyond would yield an infinite horizon optimal production decision for period 1.

We turn now to deriving a formula for the computation of N_1^*. By convexity of the production and inventory costs, the production cost of one more unit in the first period is

$$c_1(P_1^*(N)+1) - c_1(P_1^*(N)) \geq c_1(1) \tag{4.42}$$

and the inventory cost for carrying one unit to period $N+1$ is bounded from below by $\sum_{n=1}^{N} \alpha^{n-1} h_n(1)$. Since producing one unit in period 1 and holding it for N periods to satisfy one unit of demand in period $N+1$ is by hypothesis optimal (i.e., it is more expensive to produce one more unit in period $N+1$) and by (A3) the marginal production cost is bounded from above, we have

$$c_1(1) + \sum_{n=1}^{N} \alpha^{n-1} h_n(1) \leq \alpha^N \gamma_N. \tag{4.43}$$

Let

$$\mathcal{N} = \left\{ N | \sum_{n=1}^{N} \alpha^{n-1} h_n(1) > \alpha^N \gamma_N - c_1(1) \right\}. \tag{4.44}$$

Then any $N \in \mathcal{N}$ is a solution horizon for the first decision, since it is prohibited to produce in period 1 to satisfy demand in period N or beyond. Let

$$\underline{N}_1 = \min \{N | N \in \mathcal{N}\}. \tag{4.45}$$

By (A3), $\mathcal{N}$ is non-empty and therefore $\underline{N}_1 < \infty$. Note that $\underline{N}_1$ is also a *forecast horizon* since its value and hence $P_1^*(\underline{N}_1) = P_1^*$ is independent of demand D_n for $n > \underline{N}_1$ (in fact, its value is independent of all demand). That is, *we need only forecast demand through period $\underline{N}_1 < \infty$ in order to determine P_1^*.*

Now let us compute a forecast horizon. If we let

$$\inf_{n \geq 1} \{h_n(1)\} = \sigma,$$

then it is obvious that $\sigma \geq 0$ and

$$\sum_{n=1}^{N} \alpha^{n-1} h_n(1) \geq \frac{1-\alpha^N}{1-\alpha} \sigma. \tag{4.46}$$

By (4.38),

$$\alpha^N \gamma - c_1(1) \geq \alpha^N \gamma_N - c_1(1). \tag{4.47}$$

Hence, if

$$\frac{1-\alpha^N}{1-\alpha}\sigma > \alpha^N\gamma - c_1(1) \tag{4.48}$$

or

$$N > \log_\alpha \left\{ \frac{(1-\alpha)c_1(1)+\sigma}{(1-\alpha)\gamma+\sigma} \right\}, \tag{4.49}$$

then

$$\sum_{n=1}^{N} \alpha^{n-1} h_n(1) > \alpha^N \gamma_N - c_1(1). \tag{4.50}$$

Hence, if we set

$$N_1^* = \left\lceil \log_\alpha \left\{ \frac{(1-\alpha)c_1(1)+\sigma}{(1-\alpha)\gamma+\sigma} \right\} \right\rceil \tag{4.51}$$

$$= \left\lceil \log_\alpha \left\{ \frac{(1-\alpha)c_1(1)+\inf_{n\geq 1}\{h_n(1)\}}{(1-\alpha)\sup_{n\geq 1}\left\{ \lim_{P_n\to\infty}[c_n(P_n)-c_n(P_n-1)] \right\} + \inf_{n\geq 1}\{h_n(1)\}} \right\} \right\rceil \tag{4.52}$$

where $\lceil X \rceil$ represents the smallest integer strictly greater than X, then

$$P_1^* = P_1^*(N_1^*) \tag{4.53}$$

is an infinite horizon first decision depending only on $D_1, D_2, \cdots, D_{N_1^*}$. That is, N_1^* *is a forecast horizon for the first production decision.* Following the same argument, we can compute the forecast horizon for the second production decision, and so on. The following is an algorithm for computing the first n infinite horizon optimal production levels for any $n = 1, 2, \cdots$.

A Finite Algorithm for Finding an Infinite Horizon Optimal Production Schedule:

1. Compute N_1^* using formula (4.52);
2. Compute $P^*(N_1^*)$ and $f_{N_1^*}(i)$ using the forward recursive dynamic programming procedure in Section 3.3.
   Let $i = 0$ to get P_1^* and $P_2^*(N_1^*)$;
   $I_1^* = I_0 + P_1^* - D_1$;
3. $j = N_1^* + 1$;
   $k = 2$;
   Compute N_k^*;
4. While $k \leq n$ do begin
   Repeat:
      Compute $f_j(i)$ using $f_{j-1}(i)$;
      Let $i = 0$ and choose $P_k^*(j)$ and $P_{k+1}^*(j)$ so that $P_j^* \geq P_k^*(j-1)$;
      $j = j + 1$;
   until $j = N_k^*$;

r	v	u				
		1.2	1.4	1.6	1.8	2
0.2	0.2	1	2	3	4	5
0.2	0.1	2	4	6	8	10
0.2	0.05	4	8	12	16	20
0.1	0.2	1	2	3	4	5
0.1	0.1	2	4	6	8	10
0.1	0.05	4	8	12	16	20
0.05	0.2	1	2	3	4	5
0.05	0.1	2	4	6	8	10
0.05	0.05	4	8	12	16	20

Table 1: The forecast horizon in days for the first infinite horizon optimal production level

```
    k = k + 1;
    Compute N*_k;
  end;
```

Note that N_1^* is independent of all demands as well as the production and inventory costs. It only depends on the value of bounds on the inventory and marginal production costs. To get a feeling for the magnitude of our forecast horizon, we look at some examples.

In the simple case where production costs are stationary and linear over time,

$$\sup_{n\geq 1}\left\{\lim_{P_n\to\infty}[c_n(P_n)-c_n(P_n-1)]\right\}=c_1(1)$$

and $N_1^*=1$. In other words, as we would expect, we only need to know the demand in the first period to make the optimal first decision regardless of the inventory costs since no inventory is needed when production cost does not vary over time.

Consider now the case where the production costs are piecewise linear or even nonlinear. In this case if we set $\gamma = uc_1(1)$, $u > 1$ (i.e., the marginal production cost will not exceed $uc_1(1)$) and $\sigma = vc_1(1)$ where v is the inventory charge as the sum of a proportion of production cost, opportunity costs, taxes, insurance costs, the value loss over time (e.g., certain products have to be sold by discount), floor space rental costs, etc., then

$$N_1^*=\left\lceil \log_\alpha\left\{\frac{1-\alpha+v}{(1-\alpha)u+v}\right\}\right\rceil.$$

For various inventory charges v per day, discount factor $\alpha=\frac{1}{1+r/365}$ per day where r is the interest rate per year, we computed N_1^* for $u = 1$ to 2. The results are shown in Table 1. We chose inventory costs unusually high here to illustrate how short these forecast horizons can be. However, even in the case of moderate inventory costs, forecast horizons can be significantly reduced by a more detailed analysis using more precise cost information to provide better bounds on the minimal forecast horizon.

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