Notes on a Mixed Integer Linear Programming Formulation of the Dynamic Traffic Routing Problem

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Let $G = (N, A)$ be the traffic network with node set N and link set A with origin-destination pairs $(s_i, t_i), i = 1, 2, \ldots, \ell$.

We model the problem with horizon p periods each of duration Δt as a time expanded version of G, $G(p) = (\mathcal{N}, \mathcal{A})$. Corresponding to node x of G, $G(p)$ has $p+1$ nodes $x(\tau), \tau = 0, 1, \ldots, p$; corresponding to arc (x, y) of G, $G(p)$ has arcs $(x(\tau), y(\tau + s)), 0 \leq \tau \leq p - 1$. (See Figure 1.)

Every arc $(x(\tau), y(\tau + s)) \in \mathcal{A}$ has an associated 0-1 variable $\delta(x(\tau), y(\tau + s)) = 0$ or 1 according as the total flow $f(x(\tau), y(\tau+s))$ equals 0 or is greater than 0 respectively $(x \neq y)$. A constraint associated with every arc $(x, y) \in A$ and every time τ is $\sum_{s>0} \delta(x(\tau), y(\tau+s)) \leq 1$ $(x \neq y)$. Note that vehicles are not forced to go from x to y at time τ. Let $c(x(\tau), y(\tau+s))$ be the *capacity* associated with arc $(x(\tau), y(\tau+s)) \in \mathcal{A}$ so that up to $c(x(\tau), y(\tau+s))$ vehicles could leave node x at time τ and still get to node y by time $\tau + s$. In particular, if $\sigma_{xy\tau}(v)$ is the speed attained by v vehicles entering node x at time τ along link $(x, y) \in A$ and $d(x, y)$ is the length of link (x, y), then

$$c(x(\tau), y(\tau + s)) = \sigma_{xy\tau}^{-1}(d(x, y)/s).$$

Key Assumption: The time to traverse a link (x, y) at time τ is determined by the number of vehicles already on (x, y) plus sent at time τ through x along (x, y). This time remains *fixed* independent of vehicles routed along (x, y) after time τ.

For each commodity k, S_k is the singleton set of sources, T_k is the set of sinks, and R_k is the set of other nodes respectively. In particular, if $x(\tau) \in S_k$ and $y(\tau + s) \in T_k$ for $s > 0$, then v_k trips originate at node x at time τ and terminate at node y. The source set S_k is a singleton consisting of $x(\tau)$ if the kth trip originates at node x at time τ and T_k consists of $\{y(\tau + s) | s > 0\}$ where y is the destination node. The flow $f_k(x(\tau), y(\tau + s))$ is the number of vehicles taking trip k along (x, y) at time τ in time s. The total flow $f(x(\tau), y(\tau + s))$ of vehicles along link (x, y) departing node x at time τ and arriving at node y at time $\tau + s$ is then $f(x(\tau), y(\tau + s)) = \sum_{k=1}^{K} f_k(x(\tau), y(\tau + s))$.

We set the travel time $a_k(x(\tau), y(\tau + s))$ for trip k along arc $(x(\tau), y(\tau + s))$ to be 0 if $x = y \in T_k$ (since we've arrived at the destination and can just sit at the node until the end of the horizon) and s otherwise.

This brings us to the formulation of the problem.

MILP Formulation
(System optimal criterion)

$$\min \sum_{k=1}^{K} \sum_{(x(\tau),y(\tau+s))\in\mathcal{A}} a_k(x(\tau),y(\tau+s)) f_k(x(\tau),y(\tau+s))$$

(Minimum total vehicle time)

s.t.

$$f_k(x(\tau),Y(\tau+S)) - f_k(Y(\tau-S),x(\tau)) = \begin{cases} v_k & \text{if } x(\tau)\in S_k \\ -v_k & \text{if } x(\tau)\in T_k \\ 0 & \text{if } x(\tau)\in R_k \end{cases}, \tag{1}$$

$$x(\tau)\in\mathcal{N}, \qquad k=1,2,\ldots,K$$

(Flow balance at node $x(\tau)$ for the kth O-D pair)

$$f(x(\tau),y(\tau+s)) \le c(x(\tau),y(\tau+s)) - f(x(\tau-S),y(\tau+S)), \tag{2}$$
$$(x(\tau),y(\tau+s))\in\mathcal{A},$$

(Impedance capacity constaint for each arc $(x(\tau),y(\tau+s))$)

$$f(x(\tau),y(\tau+s)) \le \delta(x(\tau),y(\tau+s))c(x(\tau),y(\tau+s)), \tag{3}$$
$$(x(\tau),y(\tau+s))\in\mathcal{A}, x\ne y$$
$$\delta(x(\tau),y(\tau+S)) \le 1, \tag{4}$$
$$x(\tau)\in\mathcal{N},(x,y)\in A, x\ne y.$$

(All vehicles that enter link $(x,y)\in A$ at time τ must exit the link at the same time $\tau+s$)

$f_k(x(\tau),y(\tau+s)) \ge 0, 1 \ge \delta(x(\tau),y(\tau+s)) \ge 0$ integer for all $(x(\tau),y(\tau+s))\in\mathcal{A}$.

(Variable restrictions)

Note that for $\delta(x(\tau), y(\tau+s)) = 1$, constraint (3) is redundant in the presence of (2) and hence is only binding if $\delta(x(\tau), y(\tau+s)) = 0$ in which case $f(x(\tau), y(\tau+s)) = 0$ so that no vehicles are allowed to travel over arc (x, y) at time τ in time s. Indeed, if $\delta(x(\tau), y(\tau+S)) = 0$ for $x \neq y$, no vehicles take arc (x, y) out of x at time τ. They either take another arc (x, y') or they hold at node x. Note also that $S \equiv \{s | s > 0\}$ and $Y \equiv \{z \in N | (x, z) \in A$ for some $x \in N$ or $(z, y) \in A$ for some $y \in N\}$.

Convention: Substituting a capital letter Z for small letter z in $f(z)$ requires summing $f(z)$ over all $z \in Z$, i.e.,

$$f(Z) \equiv \sum_{z \in Z} f(z).$$

Remarks:

1. We have a MILP since removing the integer variable restriction yields a LP. This is the contribution of the formulation. The MILP may be solved by the Branch-and-Bound algorithm. Most codes automatically branch to exploit the "multiple choice" structure of constraint (4). Since there is a 0-1 integer variable associated with every arc in $G(p)$, there are $O(|N|^2 p^2)$ in all. However, by binary encoding the arc number $(x(\tau), y(\tau+s))$ taken by vehicles leaving node x at time τ for y, we can reduce this to $O(|N|^2 p \log p)$. For a micro-level transportation network, there are at most 4 arcs per node so that there are at most $(4|N|\, p \log_2 p)$ 0-1 integer variables. For example, for a horizon of 10 periods, we get at most $(120|N|)$ 0-1 variables or 12,000 for a 100-node network. What is the largest transportation network that we can solve? How good is the solution to the LP relaxation?

2. Since the arc costs $a_k(x(\tau), y(\tau+s))$ depend on the O-D pair k only through the corresponding destination and not the origin node or time, we may as in the dynamical system model redefine $f_k(x(\tau), y(\tau+s))$ as the number of vehicles going to destination t_k along (x, y) departing node x at time τ in time s, $k = 1, 2, \ldots, \ell$. This considerably reduces the number of continuous variables in the formulation.

3. Since the set of feasible solutions is closed and bounded, and the objective function is continuous, there exist a system optimal dynamic routing: $\delta^*(\cdot), f_k^*(\cdot), k = 1, 2, \ldots, K$.

4. Given $\delta^*(\cdot)$, $f_k^*(\cdot), k = 1, 2, \ldots, K$ is an optimal solution to a (nontrivial?) linear program. For example, $f_k^*(\cdot), k = 1, 2, \ldots, K$ is without loss of generality an extreme point solution. Properties?

5. Remark 3 suggests an iterative heuristic. Clearly, δ^* (or $f_k^*, k = 1, 2, \ldots, K$) yields a fixed point of the iterative method by Remark 3 that given $\delta = \delta^*, f_k = f_k^*$ is LP optimal.

6. Is it possible for $\delta(x(\tau), y(\tau+s)) > 0$ in the optimal *relaxed* LP for more than two values of $s > 0$ for strictly convex impedance functions?

7. It seems clear that it may be optimal to wait at a node other than a destination, i.e. that the optimal flow $f_k^*(x(\tau), x(\tau+1)) > 0$ for $x(\tau) \notin T_k$. (Note however that all waiting can be eliminated by setting $a_k(x(\tau), y(\tau+s)) = \infty$.) Does the optimal flow have to satisfy $f^*(x(\tau), y(\tau+s)) f^*(x(\tau'), y(\tau+s')) = 0$ for $\tau' < \tau$ and $s' > s$? That is, is the consistency condition necessarily satisfied that you cannot arrive at the end of a link sooner by arriving at the beginning of the link later?

8. To insure accuracy of the impedance model, we could without loss of optimality require that if $f^*(x(\tau), y(\tau+s)) > 0$, then

$$c(x(\tau), y(\tau+s-1)) < f^*(x(\tau), y(\tau+s)) \leq c(x(\tau), y(\tau+s)). \tag{5}$$

Otherwise all vehicles could feasibly take arc $(x(\tau), y(\tau+s-1))$ say and get to node y one period earlier. However, if it were optimal to arrive at node y at time $\tau+s$, they could wait at node y for one period so that (5) is not forced. We could insure (5) by adding the constaint:

$$\begin{aligned}\delta(x(\tau), y(\tau+s)) c(x(\tau), y(\tau+s-1)) &< f^*(x(\tau), y(\tau+s)) \\ &\leq \delta(x(\tau), y(\tau+s)) c(x(\tau), y(\tau+s)).\end{aligned} \tag{6}$$

9. If we define $\bar{S} \equiv \{s | s \geq 0\}$, constraint (2) simplifies to

$$f(x(\tau - \bar{S}), y(\tau + S)) \leq c(x(\tau), y(\tau+s)).$$

This formally suggests several alternative impedance models. Is there a more natural one than (2)?

10. Suppose we replace the objective function by

$$\min \sum_{k=1}^{K} \sum_{(x(\tau), y(\tau+s)) \in \mathcal{A}} \sum_{s'=1}^{s} a_k(x(\tau), y(\tau+s')) c(x(\tau), y(\tau+s')) \delta(x(\tau), y(\tau+s))$$

(Minimum average vehicle time)

Note that the objective function remains linear. Do we get a *user optimal* formulation? That is, does an optimal solution have the property that a) all routes taken by the vehicles departing origin s_i for destination t_i at time τ experience the same trip times, and b) any vehicle would experience a greater trip time if it took an alternate route? If so, given an optimal link loading δ^* for this formulation, can it be shown that a single pass of the Dynamic Router recovers the associated optimal routing $f_k^*, k = 1, 2, \ldots, K$? (Assume here as elsewhere that the Δt period error is negligible.)

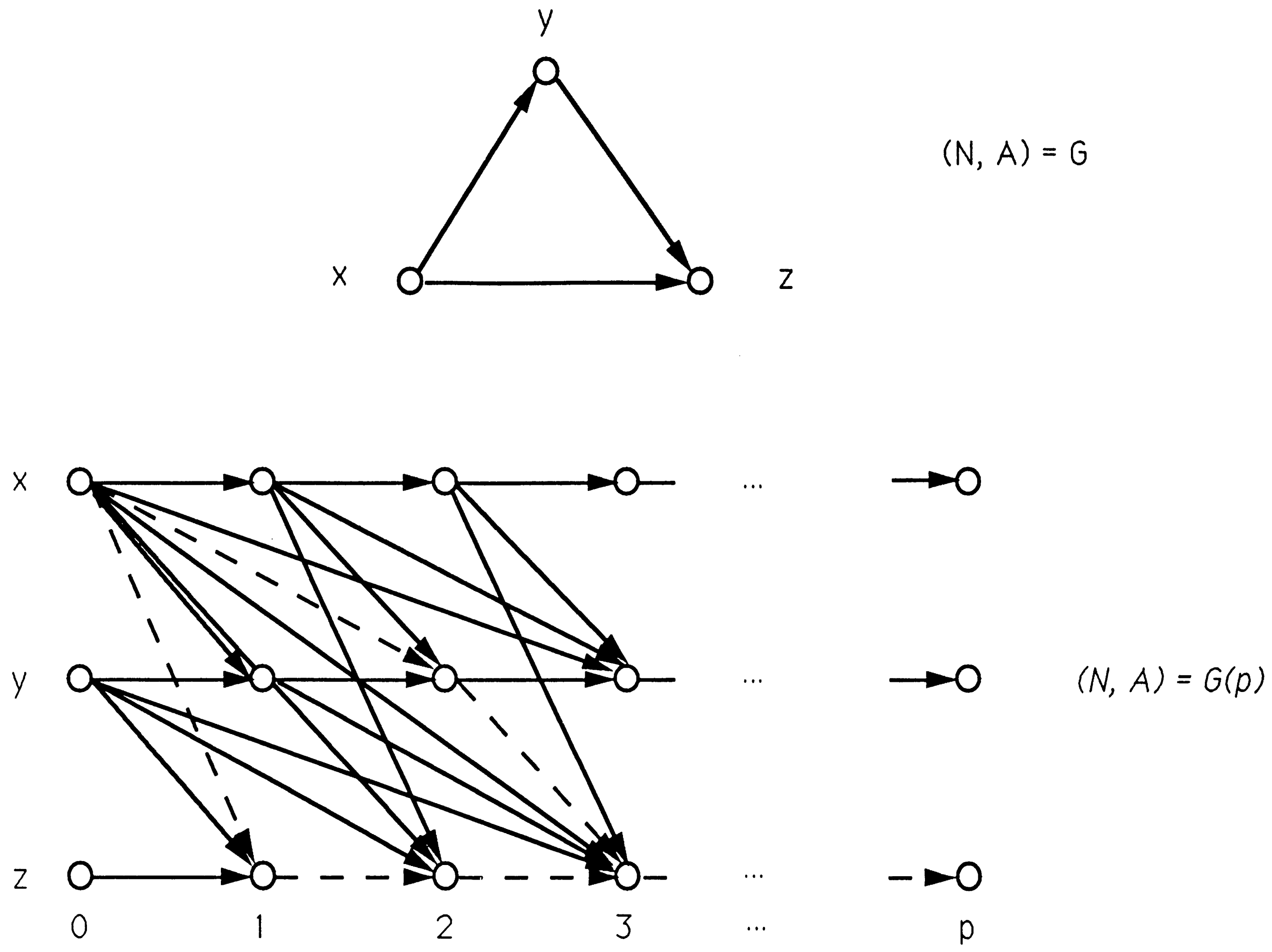

Figure 1 - Triangle network with two alternate routes