SET CONVERGENCE IN INFINITE HORIZON OPTIMIZATION

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Robert L. Smith
Department of Industrial & Operations Engineering
The University of Michigan
Ann Arbor, Michigan 48109

Irwin E. Schochetman
Department of Mathematical Sciences
Oakland University
Rochester, Michigan 48063

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Abstract

We consider the question of convergence of a sequence of closed (non-empty) subsets of a compact metric space in the space of all such subsets equipped with the Hausdorff metric. We obtain equivalent conditions for convergence in terms of (1) equality of lim inf and lim sup sets and (2) pointwise convergence properties of continuous set selection mappings defined on the metric space of closed subsets. We then apply these results to obtain necessary and sufficient convergence criteria for the finite horizon optimal solution sets encountered in a general infinite horizon optimization problem.

Key Words and Phrases

Compact metric space, space of closed subsets, Hausdorff metric, limit sets, selection, uniqueness point, nearest-point selection, infinite horizon optimization, finite horizon optimal strategies.

1 Introduction

In recent years a number of authors have studied various versions of infinite horizon optimization problems. Of particular interest to us are the papers by Bean and Smith [2], Bès and Sethi [3] and Schochetman and Smith [10]. In each of these cases, as well as others, we have a compact metric space (X, d) of feasible infinite horizon *strategies* and a discounted cumulative net cost function $C(x)$ defined for x in X. The problem is to find

$$\inf_{x \in X} C(x).$$

Under varying hypotheses, the infimum is attained, thereby giving rise to a non-empty, closed (hence compact) subset X^* of X consisting of optimal strategies. The finite horizon approach to approximating the elements of X^* consists of truncating the cost function $C(x)$, for each x, at finite horizon times $T > 0$, thus giving rise to the problem of finding

$$\inf_{x \in X} C(x, T),$$

for each $T > 0$, where $C(x, T)$ is the T-horizon truncated cost $C(x)$ for x in X. These infima are also attained, yielding non-empty, closed subsets of X denoted by $X^*(T), T > 0$. Consequently, we are led to the problem of defining and determining convergence of the $X^*(T)$ to X^*.

More generally, this suggests the problem of studying convergence of closed sets. Problems of this type, in different contexts, have been studied in the past by several authors. Of particular interest to us are the papers by Michael [9], Fell [6] and Effros [5]. Our motivation is to extend the results of these authors so as to give relevant answers to our infinite horizon closed set convergence problem.

In section 2 we discuss the Hausdorff metric D (corresponding to d) on the space $\mathcal{C}(X)$ of closed, non-empty subsets of X and compare the underlying metric topology on this space with two other possible topologies–the finite topology of Michael [9] and the H-topology of Fell [6]. Given a sequence $\{F_i\}$ in $\mathcal{C}(X)$, Effros [5] defines two limit sets, $\liminf F_i$ and $\limsup F_i$, for this sequence. In section 3, we characterize each of these sets in a form amenable to our problem of interest and establish various additional properties. In section 4, we show that a sequence in $\mathcal{C}(X)$ converges with respect to D if and only if its lim sup and lim inf are equal, in which case the limit of the sequence is this common set. For purposes of approximating an infinite horizon optimum, it is more important to be able to obtain a corresponding sequence of points $\{x_i\}$ from the sequence $\{F_i\}$ of sets which converges to a point x in the limit set of the sequence. This suggests the study of *selections* defined on $\mathcal{C}(X)$ and their continuity properties. Particular selections, which we call *nearest-point* selections corresponding to "uniqueness-points," play a special role. This is the subject matter of section 5. Finally, in section 6, we apply the main results of the previous sections to the closed sets $X^*(T), T > 0$, of our infinite horizon optimization problem, obtaining several equivalent conditions for the $X^*(T)$ to converge in the Hausdorff metric.

2 Topologies for Closed Subsets

Let (X, d) denote an arbitrary compact metric space and $\mathcal{C}(X)$ the set of all non-empty, closed (hence compact) subsets of X. For each x in X, the mapping $y \to d(x, y)$ is continuous on X. Thus, for each K in $\mathcal{C}(X)$, the *minimum* of $d(x, y)$, for y in K, is attained and we may define

$$d(x, K) = \min_{y \in K} d(x, y), \qquad x \in X, K \in \mathcal{C}(X).$$

Moreover, for each such K, the mapping $x \to d(x, K)$ is also continuous on X [4, Theorem 4.2]. Hence, for each C in $\mathcal{C}(X)$, the *maximum* of $d(x, K)$, for x in C, is also attained and we may therefore define

$$h(C, K) = \max_{x \in C} d(x, K), \qquad C, K \in \mathcal{C}(X).$$

Although h is *not* a metric on $\mathcal{C}(X)$ (it's not symmetric), we can obtain a metric D on $\mathcal{C}(X)$ if we define

$$D(C, K) = \max(h(C, K), h(K, C)), \qquad C, K \in \mathcal{C}(X).$$

This is the well-known Hausdorff metric on $\mathcal{C}(X)$ [4,7,8,9]; it satisfies

$$D(\{x\},\{y\}) = d(x,y), \qquad x,y \in X.$$

Moreover, $(\mathcal{C}(X), D)$ is a *compact* metric space [9, Theorem 4.2].

In additon to this topology on $\mathcal{C}(X)$, there is the *finite topology* of E. Michael [9] and the *H-topology* of J. M. G. Fell [6]. However, as observed in [6,9], the finite topology, the H-topology and the Hausdorff metric topology are all the same on $\mathcal{C}(X)$, since X is compact and Hausdorff. Thus, for the remainder of this paper, it will be convenient to discuss convergence in $\mathcal{C}(X)$ in terms of the Hausdorff metric D only.

3 Limit Set Results

Since $\mathcal{C}(X)$ is a metric space, it is first countable (in fact second countable), so that it suffices to consider convergence for *sequences* in $\mathcal{C}(X)$. Although we will ultimately be interested in convergence for a family of closed sets indexed by the positive reals, this poses no difficulty, since all our results are valid for such families as well. Thus, there is no harm in considering convergence for the more appropriate and convenient case of sequences in $\mathcal{C}(X)$.

Let $\{F_i\}$ be a sequence in $\mathcal{C}(X)$. As in [5], define $\liminf F_i$ (resp. $\limsup F_i$) to be the set of all x in X such that every neighborhood of x is *eventually* (resp. *frequently*) intersected by the F_i. In general,

$$\liminf F_i \subseteq \limsup F_i;$$

also $\liminf F_i$ and $\limsup F_i$ are closed in X. In fact, $\limsup F_i$ is an element of $\mathcal{C}(X)$, since it must be non-empty (the F_i are non-empty and X is compact). However, $\liminf F_i$ can be empty; for example, if $X = \{0,1\}, F_{2i} = \{0\}, F_{2i+1} = \{1\}$, then $\liminf F_i = \emptyset$.

Lemma 3.1 *Let x be an element of X. Then x belongs to $\limsup F_i$ if and only if there exists a subsequence $\{F_{i_k}\}$ of $\{F_i\}$ and a corresponding subsequence $\{x_{i_k}\}$ such that $x_{i_k} \in F_{i_k}$, all k, and $x_{i_k} \to x$, as $k \to \infty$.*

Proof: Let x be an element of $\limsup F_i$. For each $k = 1,2,\ldots$, let $B(x,1/k)$ denote the open ball of radius $1/k$ centered at x. By definition of $\limsup F_i$, there exists $i_1 \geq 1$ such that

$$F_{i_1} \cap B(x,1/1) \neq \emptyset.$$

Suppose we have chosen indices $i_1 < i_2 < \ldots < i_{k-1}$ such that

$$i_j \geq j, \qquad j = 1,\ldots,k-1,$$

and

$$F_{i_j} \cap B(x,1/j) \neq \emptyset, \qquad j = 1,\ldots,k-1.$$

Then , for $j = k$, there exists an index i_k such that $i_k \geq k$, $i_{k-1} < i_k$ and

$$F_{i_k} \cap B(x,1/k) \neq \emptyset.$$

In this way, we obtain a subsequence $\{F_{i_k}\}$ of $\{F_i\}$ having the property that

$$F_{i_k} \cap B(x,1/k) \neq \emptyset, \qquad k = 1,2\ldots.$$

For each k, let x_{i_k} be an element of $F_{i_k} \cap B(x,1/k)$. It is easy to see that $x_{i_k} \to x$, as $k \to \infty$.

Conversely, suppose $\{F_{i_k}\}$ and $\{x_{i_k}\}$ are as in the statement of the lemma. Let N be any neighborhood of x. Then there exists k_N sufficiently large such that $x_{i_k} \in N$, for $k \geq k_N$, i.e.

$$F_{i_k} \cap N \neq \emptyset, \qquad k \geq k_N.$$

Since $\{F_{i_k}\}$ is a subsequence of $\{F_i\}$, it follows that N is frequently intersected by the F_i, i.e. x belongs to $\limsup F_i$. ∎

A similar result holds for $\liminf F_i$.

Lemma 3.2 *Let x be an element of X. Then x belongs to* $\liminf F_i$ *if and only if for each i, there exists x_i in F_i such that $x_i \to x$, as $i \to \infty$.*

Proof: Suppose x is an element of $\liminf F_i$. Let $B(x, 1/k)$ be as above, for $k = 1, 2, \ldots$. As in the previous proof, we may obtain a sequence of indices

$$i_1 < i_2 < \ldots < i_k < \ldots$$

such that $i_k \geq k$ and

$$F_i \cap B(x, 1/k) \neq \emptyset, \qquad i \geq i_k,\ k = 1, 2, \ldots.$$

(Recall that $B(x, 1/k)$ is *eventually* intersected by the F_i.) For each $k = 1, 2, \ldots$, choose x_i in $F_i \cap B(x, 1/k)$, for $i_k \leq i < i_{k+1}$. For $i < i_1$, choose x_i in F_i arbitrarily. Then $\{x_i\}$ is a sequence such that $x_i \in F_i$, all i, and $x_i \to x$, as $i \to \infty$.
The reverse implication is obvious. ∎

The next lemma follows immediately from the previous two lemmas and the fact that $\liminf F_i \subseteq \limsup F_i$.

Lemma 3.3 *The following are equivalent for the sequence $\{F_i\}$*
(i) $\liminf F_i = \limsup F_i$.
(ii) For every x in $\limsup F_i$, *there exists x_i in $F_i, i = 1, 2, \ldots$, such that $x_i \to x$, as $i \to \infty$.*

More generally, we have

Theorem 3.4 *Let $\{F_i\}$ be a sequence in $\mathcal{C}(X)$ and F an element of $\mathcal{C}(X)$. Then the following are equivalent.*
(i) $F = \liminf F_i = \limsup F_i$.
(ii) $F \supseteq \limsup F_i$ *and for every x in F, there exists x_i in $F_i, i = 1, 2, \ldots$, such that $x_i \to x$, as $i \to \infty$.*

Corollary 3.5 *If* $\limsup\{F_i\}$ *is a singleton $\{x\}$, then* $\liminf F_i = \{x\}$ *also. In this case, $x_i \to x$, as $i \to \infty$, for* all *choices of x_i in $F_i, i = 1, 2, \ldots$.*

Proof: Let $x_i \in F_i$, all i. If $x_i \not\to x$, then there exists a subsequence $\{x_{i_k}\}$ of $\{x_i\}$ and $\epsilon > 0$ such that $d(x_{i_n}, x) \geq \epsilon, n = 1, 2, \ldots$. Passing to a subsequence if necessary, we may assume there exists y in X such that $x_{i_n} \to y$, as $n \to \infty$. Therefore, $\{x_{i_n}\}$ is a sequence in the $\{F_{i_n}\}$ which is a subsequence of $\{F_i\}$, so that $y \in \limsup F_i$, which is equal to $\{x\}$ by hypothesis, i.e. $y = x$. Hence, $x_{i_n} \to x$, as $n \to \infty$ and $\{x_{i_n}\}$ is bounded away from x. Contradiction. Thus, $x_i \to x$, for *all* choices of $\{x_i\}$ in $\{F_i\}$, so that $\liminf F_i = \{x\}$ also. ∎

Before concluding this section, we establish the following useful result.

Lemma 3.6 *Let $\{F_i\}$ be a sequence in $\mathcal{C}(X)$. If $\{F_{i_j}\}$ is a subsequence of $\{F_i\}$, then*

$$\liminf_i F_i \subseteq \liminf_j F_{i_j} \subseteq \limsup_j F_{i_j} \subseteq \limsup_i F_i.$$

Proof: This follows from Lemmas 3.1 and 3.2. ∎

4 Hausdorff Metric Convergence

Recall that $(\mathcal{C}(X), D)$ is a compact metric space, where D is the Hausdorff metric. In this section we will show, amongst other things, that the conditions of Theorem 3.4 are equivalent to convergence in the Hausdorff metric.

Lemma 4.1 *Let $\{F_i\}$ be a sequence in $\mathcal{C}(X)$and F an element of $\mathcal{C}(X)$.*

(i) If $F \supseteq \limsup F_i$, then $F_i \to F$ if and only if $h(F, F_i) \to 0$.

(ii) If $F \subseteq \liminf F_i$, then $F_i \to F$ if and only if $h(F_i, F) \to 0$.

Proof: From the definition of D, it follows that $h(F_i, F) \to 0$ and $h(F, F_i) \to 0$, if $F_i \to F$, as $i \to \infty$. Thus, in each case, it suffices to verify the reverse implication.

(i) Suppose $F \supseteq \limsup F_i$. By definition of h, we have

$$h(F_i, F) = \max_{x \in F_i} d(x, F),$$

where the function $x \to d(x, F)$ is continuous and each F_i is compact. Thus, for each i, there exists x_i in F_i such that

$$h(F_i, F) = d(x_i, F), \qquad i = 1, 2, \ldots.$$

Suppose $d(x_i, F) \not\to 0$. Then there exists $\epsilon > 0$ and a subsequence $\{F_{i_j}\}$ of $\{F_i\}$ such that

$$d(x_{i_j}, F) \geq \epsilon, \qquad j = 1, 2, \ldots.$$

Since X is compact, passing to a subsequence if necessary, we may assume that there exists x in X such that $x_{i_j} \to x$, as $j \to \infty$. Since x belongs to $\limsup F_i$ by Lemma 3.1, we have that x is in F by hypothesis. Hence,

$$\begin{aligned} d(x_{i_j}, F) &= \min_{y \in F} d(x_{i_j}, y) \\ &\leq d(x_{i_j}, x), \qquad j = 1, 2, \ldots, \end{aligned}$$

which implies that $d(x_{i_j}, F)$ is eventually less than ϵ. Since this is a contradiction, it follows that $h(F_i, F) \to 0$ automatically in this case. Consequently, $h(F, F_i) \to 0$ implies $D(F_i, F) \to 0$, as $i \to \infty$.

(ii). Now suppose $F \subseteq \liminf F_i$. This time we have

$$h(F, F_i) = \max_{y \in F} d(y, F_i),$$

so that, for each i, there exists y_i in F such that

$$h(F, F_i) = d(y_i, F_i), \qquad i = 1, 2, \ldots.$$

If $d(y_i, F_i) \not\to 0$, then there exists a subsequence $\{F_{i_j}\}$ of $\{F_i\}$ and $\epsilon > 0$ such that

$$d(y_{i_j}, F_{i_j}) \geq \epsilon, \qquad j = 1, 2, \ldots.$$

Since F is compact, passing to a subsequence if necessary, we may assume there exists y in F such that $y_{i_j} \to y$, as $j \to \infty$. But by hypothesis, we have that y is in $\liminf F_i$ necessarily. Therefore, by Lemma 3.2, there exists x_i in F_i, for each i, such that $x_i \to y$, i.e. $x_{i_j} \to y$, as $j \to \infty$. Consequently,

$$\begin{aligned} d(y_{i_j}, F_{i_j}) &= \min_{z \in F_{i_j}} d(y_{i_j}, z) \\ &\leq d(y_{i_j}, x_{i_j}) \\ &\leq d(y_{i_j}, y) + d(y, x_{i_j}), \end{aligned}$$

which implies that

$$d(y_{i_j}, F_{i_j}) \to 0, \text{ as } j \to \infty.$$

Since this is a contradiction, it follows that $h(F, F_i) \to 0$ automatically in this case. Hence, it follows that $D(F_i, F) \to 0$ if $h(F_i, F) \to 0$. ∎

Theorem 4.2 *Let $\{F_i\}$ and F be as above. The following are equivalent:*

(i) $F = \liminf F_i = \limsup F_i$.

(ii) $F_i \to F$ *in* $\mathcal{C}(X)$, *as* $i \to \infty$.

Proof: Suppose (i) is true. Then by the proof of the previous lemma, we have that $h(F_i, F) \to 0$ and $h(F, F_i) \to 0$, i.e. $D(F_i, F) \to 0$, as $i \to \infty$.
Conversely, suppose (ii) is true. We will show that $F \subseteq \liminf F_i$ and $F \supseteq \limsup F_i$. Suppose x is in F. Then, for each i, there exists x_i in F_i such that

$$\begin{aligned} d(x, x_i) &= d(x, F_i) \\ &\leq \max_{y \in F} d(y, F_i) \\ &= h(F, F_i), \qquad i = 1, 2 \ldots. \end{aligned}$$

Necessarily, $d(x, x_i) \to 0$, as $i \to \infty$ so that x is in $\liminf F_i$ by Lemma 3.2. Thus, $F \subseteq \liminf F_i$.
Now suppose x is in $\limsup F_i$ but not in F. For each y in F, there exists $\delta(y) > 0$ such that

$$B(x, \delta(y)) \cap B(y, \delta(y)) = \emptyset,$$

where $B(z, \delta)$ is the open ball of radius δ centered at z. The collection

$$\{B(y, \delta(y)) : y \in F\}$$

is an open cover of compact F. Therefore, there exist $y_1, \ldots, y_n$ in F such that

$$F \subseteq \bigcup_{j=1}^{n} B(y_j, \delta(y_j)).$$

For convenience, let $\delta_j = \delta(y_j), j = 1, \ldots, n$ and define

$$\delta = \frac{1}{2} \min\{\delta_1, \ldots, \delta_n\}.$$

Then

$$B(x, \delta) \cap (\cup_{j=1}^{n} B(y_j, \delta_j)) = \emptyset.$$

Since x is in $\limsup F_i$, by Lemma 3.1, there exists a subsequence $\{F_{i_k}\}$ of $\{F_i\}$ and a corresponding sequence $\{x_{i_k}\}$ such that $x_{i_k} \in F_{i_k}$, all k, and $x_{i_k} \to x$, as $k \to \infty$. Hence, there exists k_δ sufficiently large such that

$$x_{i_k} \in B(x, \delta),$$

i.e.

$$d(x_{i_k}, x) < \delta, \qquad k \geq k_\delta.$$

Moreover,

$$\begin{aligned} h(F_{i_k}, F) &= \max_{y \in F_{i_k}} d(y, F) \\ &\geq d(x_{i_k}, F), \qquad k = 1, 2, \ldots. \end{aligned}$$

For each k, let z_{i_k} be an element of F such that

$$d(x_{i_k}, z_{i_k}) = d(x_{i_k}, F), \qquad k = 1, 2 \ldots.$$

Necessarily, each z_{i_k} belongs to some ball $B(y_{j_k}, \delta_{j_k})$. Thus,

$$\begin{aligned} d(x, y_{j_k}) &\leq d(x, x_{i_k}) + d(x_{i_k}, z_{i_k}) + d(z_{i_k}, y_{j_k}) \\ &\leq \delta + d(x_{i_k}, z_{i_k}) + \delta_{j_k}, \end{aligned}$$

i.e.

$$d(x_{i_k}, z_{i_k}) \geq d(x, y_{j_k}) - \delta - \delta_{j_k}, \qquad k = 1, 2, \ldots.$$

But, for each $j = 1, \ldots, n$, we have

$$B(x, \delta_j) \cap B(y_j, \delta_j) = \emptyset,$$

so that

$$d(x, y_j) \geq 2\delta_j.$$

In particular, this is true for $j = j_k$. Consequently,

$$\begin{aligned} d(x_{i_k}, z_{i_k}) &\geq \delta_{j_k} - \delta, \\ &> \delta, \end{aligned}$$

since $\delta_{j_k} \geq 2\delta$, $k = 1, 2 \ldots$. Hence,

$$d(x_{i_k}, F) \geq \delta,$$

so that

$$h(F_{i_k}, F) \geq \delta, \qquad k = 1, 2, \ldots.$$

This implies that $h(F_i, F) \not\to 0$, i.e. $D(F_i, F) \not\to 0$. This is a contradiction, so that $\limsup F_i \subseteq F$. ∎

Remark The reader should note that Theorem 4.2 may be constructed from [5,6,9] by piecing together the appropriate results. We are giving an alternate proof which is direct and self-contained.

Corollary 4.3 *Suppose F_i is a singleton $\{x_i\}$, for i sufficiently large. If x is an element of X, then $x_i \to x$ if and only if $\limsup F_i = \liminf F_i = \{x\}$.*

Proof: This follows from the fact that

$$D(\{x_i\}, \{x\}) = d(x_i, x), \qquad i = 1, 2, \ldots.$$

Corollary 4.4 *Let $\{F_i\}$ be an arbitrary sequence in $\mathcal{C}(X)$. Any accumulation point F of $\{F_i\}$ (i.e. any limit F of a subsequence of $\{F_i\}$) satisfies*

$$\liminf F_i \subseteq F \subseteq \limsup F_i.$$

Proof: If $\{F_{i_k}\}$ is a subsequence of $\{F_i\}$ such that $F_{i_k} \to F$, then, by the theorem,

$$F = \liminf F_{i_k} = \limsup F_{i_k}.$$

Now apply Lemma 3.6. ∎

5 Selections and Point Convergence

In this section, we consider the convergence of F_i to F versus the convergence of some x_i in F_i to some x in F. Roughly speaking, we will see that convergence of the F_i to F is equivalent to convergence of the x_i to x for *smooth* choices of the x_i and x.

In this regard, we define a *selection* on $\mathcal{C}(X)$ to be a mapping $S : \mathcal{C}(X) \rightarrow X$ such that

$$S(F) \in F, \qquad F \in \mathcal{C}(X).$$

Note that selections need *not* be continuous as is the case in [9]. Our objective will be to equate convergence of F_i to F with convergence of $S(F_i)$ to $S(F)$. before we can do this, we need two more concepts.

Let p be an arbitrary point in X. For each F in $\mathcal{C}(X)$, there exits at least one x in F such that

$$d(p, x) = d(p, F).$$

If, for each F in $\mathcal{C}(X)$, we define $S_p(F)$ to be *any* such x in F, then we will call S_p a *nearest-point selection defined by p*. Of course, there exists more than one such selection in general for a given p in X. Now fix F in $\mathcal{C}(X)$. If p is such that there exists a *unique* x in F as above, then we will say that p is a *uniqueness point* for F (relative to d). In this case,

$$d(p, x) < d(p, y),$$

for all y in F different from x. Let F^1 denote the *uniqueness set* for F, i.e. the set of all uniqueness points for F. If $p \in F^1$ and S_p is a nearest-point selection defined by p, then $S_p(F)$ is uniquely determined in F. In general,

$$F \subseteq F^1 \subseteq X$$

and $\emptyset \neq F^1 \neq X$. Note also that F^1 being equal to X is a generalization of F being a singleton.

Lemma 5.1 *Let F be an element of $\mathcal{C}(X)$ and p a point in F^1. If S_p is a nearest-point selection defined by p, then S_p is continuous at F.*

Proof: Suppose $F_i \rightarrow F$ and $S_p(F_i) \not\rightarrow S_p(F)$. Then there exists a subsequence $\{F_{i_k}\}$ of $\{F_i\}$ and $\epsilon > 0$ such that

$$d(S_p(F_{i_k}), S_p(F)) \geq \epsilon, \qquad k = 1, 2, \ldots.$$

But $\{S_p(F_{i_k})\}$ is a sequence in compact X. Thus, passing to a subsequence if necessary, we may assume there exists z in X such that $S_p(F_{i_k}) \rightarrow z$, as $k \rightarrow \infty$. Since S_p is a selection, it follows that z is in $\limsup F_i$, which is in turn contained in F by hypothesis (Theorem 4.2). We then have that

$$\begin{aligned} \epsilon &\leq d(S_p(F_{i_k}), S_p(F)) \\ &\leq d(S_p(F_{i_k}), z) + d(z, S_p(F)), \end{aligned}$$

where

$$d(S_p(F_{i_k}), z) \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Therefore,

$$d(z, S_p(F)) \geq \epsilon,$$

also, i.e. z and $S_p(F)$ are distinct elements of F. Consequently,

$$d(p, S_p(F)) < d(p, z),$$

since $p \in F^1$. Define

$$\eta = \frac{1}{2}[d(p, z) - d(p, S_p(F))].$$

Then

$$d(p,z) - d(p,S_p(F)) > \eta,$$

so that

$$d(p,S_p(F)) < d(p,z) - \eta.$$

Since the function $y \rightarrow d(p,y)$ is continuous on X, there exists an open ball $B(S_p(F),\delta_\eta)$ of radius $0 < \delta_\eta < \eta$ centered at $S_p(F)$ such that

$$d(p,y) < d(p,z) - \eta, \qquad y \in B(S_p(F),\delta_\eta).$$

We claim that $D(F_{i_k},F) \geq \delta_\eta$ if k is such that

$$F_{i_k} \cap B(S_p(F),\delta_\eta) = \emptyset.$$

To see this, observe that

$$d(S_p(F),w) \geq \delta_\eta, \qquad w \in F_{i_k},$$

i.e.

$$d(S_p(F),F_{i_k}) \geq \delta_\eta,$$

so that

$$h(F,F_{i_k}) \geq \delta_\eta.$$

Consequently,

$$D(F_{i_k},F) \geq \delta_\eta,$$

for such k.

By our hypothesis, $F_{i_k} \rightarrow F$, so that there exists k_η sufficiently large such that

$$D(F_{i_k},F) < \delta_\eta, \qquad k \geq k_\eta.$$

By the previous claim, we must have that

$$F_{i_k} \cap B(S_p(F),\delta_\eta) \neq \emptyset, \qquad k \geq k_\eta.$$

For each $k \geq k_\eta$, let x_{i_k} be an element of $F_{i_k} \cap B(S_p(F),\delta_\eta)$, so that

$$d(S_p(F),x_{i_k}) < \delta_\eta.$$

For such k, we have:

$$\begin{aligned} d(p,S_p(F_{i_k})) &\leq d(p,x_{i_k}) \\ &\leq d(p,S_p(F)) + d(S_p(F),x_{i_k}) \\ &< d(p,S_p(F)) + \delta_\eta \\ &< d(p,S_p(F)) + \eta. \end{aligned}$$

But

$$d(p,S_p(F)) + \eta = d(p,z) - \eta,$$

so that

$$d(p,S_p(F_{i_k})) < d(p,z) - \eta, \qquad k \geq k_\eta.$$

Also,

$$d(p,S_p(F_{i_k})) \rightarrow d(p,z), \text{ as } k \rightarrow \infty,$$

since

$$S_p(F_{i_k}) \rightarrow z, \text{ as } k \rightarrow \infty.$$

Consequently,

$$d(p,z) \leq d(p,z) - \eta,$$

which is a contradiction since $\eta > 0$. Hence, $S_p(F_i) \rightarrow S_p(F)$ and S_p is continuous at F. ∎

Theorem 5.2 *Let $\{F_i\}$ be a sequence in $\mathcal{C}(X)$ and F an element of $\mathcal{C}(X)$. Then the following are equivalent.*

(i) $F_i \rightarrow F$, as $i \rightarrow \infty$.

(ii) $F \supseteq \limsup F_i$ and $S_p(F_i) \rightarrow S_p(F)$, as $i \rightarrow \infty$, for all nearest-point selections S_p on $\mathcal{C}(X)$ defined by p in F^1.

Proof: Suppose (i) is true. Then, for each p in F^1, S_p is continuous at F by Lemma 5.1, so that $S_p(F_i) \rightarrow S_p(F)$, as $i \rightarrow \infty$. The rest of (ii) follows from Theorem 4.2.

Now suppose (ii) is true and it is not the case that $F_i \rightarrow F$. Since $F \supseteq \limsup F_i$, by Lemma 4.1, it must be true that $h(F, F_i) \not\rightarrow 0$. Thus, there exits $\epsilon > 0$ and a subsequence $\{F_{i_k}\}$ of $\{F_i\}$ such that

$$h(F, F_{i_k}) \geq \epsilon, \qquad k = 1, 2, \ldots.$$

Let x_{i_k} be an element of F for which

$$h(F, F_{i_k}) = d(x_{i_k}, F_{i_k}) \geq \epsilon, \qquad k = 1, 2, \ldots.$$

Passing to a subsequence if necessary, we may assume that there exists x in F such that $x_{i_k} \rightarrow x$, as $k \rightarrow \infty$. Consequently, $x \in F^1$. If S_x denotes any nearest-point selection defined by x, then, by our hypothesis, we have that $S_x(F_i) \rightarrow S_x(F)$. Since $\{S_x(F_{i_k})\}$ is a subsequence of $\{S_x(F_i)\}$, it follows that

$$S_x(F_{i_k}) \rightarrow S_x(F), \text{ as } k \rightarrow \infty,$$

where

$$S_x(F_{i_k}) \in F_{i_k}, \text{ all } k,$$

and

$$x = S_x(F) \in F.$$

Therefore,

$$\begin{aligned} d(x_{i_k}, F_{i_k}) &= \min_{y \in F_{i_k}} d(x_{i_k}, y) \\ &\leq d(x_{i_k}, S_x(F_{i_k})) \\ &\leq d(x_{i_k}, x) + d(x, S_x(F_{i_k})), \end{aligned}$$

where

$$d(x_{i_k}, x) \rightarrow 0, \text{ as } k \rightarrow \infty,$$

and

$$d(x, S_x(F_{i_k})) = d(S_x(F), S_x(F_{i_k})) \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Consequently,

$$d(x_{i_k}, F_{i_k}) \rightarrow 0, \text{ as } k \rightarrow \infty,$$

which is a contradiction. Hence, $F_i \rightarrow F$, as $i \rightarrow \infty$. ∎

Corollary 5.3 *The following are equivalent:*

(i) $F_i \to F$, *as* $i \to \infty$.

(ii) $F \supseteq \limsup F_i$ *and* $S(F_i) \to S(F)$, *for* all *selections* S *on* $\mathcal{C}(X)$ *which are continuous at* F.

Corollary 5.4 *If* $F^1 = X$, *then the following are equivalent:*

(i) $F_i \to F$, *as* $i \to \infty$.

(ii) $S_p(F_i) \to S_p(F)$, *as* $i \to \infty$, *for all nearest-point selections* S_p *on* $\mathcal{C}(X)$ *defined by* p *in* X.

(iii) $S(F_i) \to S(F)$, *as* $i \to \infty$, *for all selections* S *on* $\mathcal{C}(X)$ *which are continuous at* F.

Example 5.5 Let (X, d) be an arbitrary compact metric space having at least two distinct elements. Suppose $x_i \to x$ and $y_i \to y$ in X, as $i \to \infty$, where $x \neq y$. Define

$$F_i = \{x_i, y_i\}, \qquad i = 1, 2, \ldots,$$

and

$$F = \{x, y\}.$$

Then it is easy to see that $F \subseteq \liminf F_i$ and $F \supseteq \limsup F_i$, so that they are equal. Hence, $F_i \to F$ in $\mathcal{C}(X)$, as $i \to \infty$. Moreover,

$$F^1 = \{z \in X : d(x, z) \neq d(y, z)\}.$$

The following example of an infinite, non-linear mathematical program is much more interesting.

Example 5.6 Let X denote the product of countably many copies of the unit interval $[-1, 1]$. The compact product topology is metrizable by the metric given by

$$d(x, y) = \textstyle\sum_{i=1}^{\infty} |x_i - y_i| / 2^i, \qquad x, y \in X,$$

where $x = (x_i), y = (y_i)$. Let $0 < \alpha < 1$ and consider the following infinite horizon mathematical program:

(MP) $$\max \textstyle\sum_{i=1}^{\infty} \alpha^i x_i^2$$

subject to

$$|x_i| \leq 1, \qquad i = 1, 2, \ldots.$$

Let X^* denote the set of optimal solutions to (MP). It is easy to see that

$$X^* = \{x \in X : x_i = \pm 1, \text{ all } i\},$$

so that X^* is uncountable. One interpretation of this is that for every discount factor α, there exist uncountably many infinite horizon optima for this problem. Now let N be a positive integer and consider the following N-horizon modification of (MP):

$(\text{MP})_N$ $$\max \sum_{i=1}^{N} \alpha^i x_i^2$$

subject to

$$|x_i| \leq 1, \qquad i = 1, 2, \ldots.$$

If X_N^* denotes the set of optimal solutions to $(\mathrm{MP})_N$, then it is obvious that

$$X_N^* = \{x \in X : x_i = \pm 1, 1 \leq i \leq N\}, \qquad N = 1, 2, \ldots.$$

Thus,

$$X_1 \supset X_2 \supset X_3 \supset \ldots \supset X_N \supset \ldots \supset X^*.$$

We leave it to the reader to verify that

$$X^* = \liminf X_N^* = \limsup X_N^*,$$

so that $X_N^* \rightarrow X^*$ in $\mathcal{C}(X)$, as $N \rightarrow \infty$. In fact, one may verify that

$$D(X_N^*, X^*) \leq 2^{1-N}, \qquad N = 1, 2, \ldots.$$

The uniqueness set of X^* is given by

$$(X^*)^1 = \{x \in X : x_i \neq 0, \text{ all } i\}.$$

Let p be an element of $(X^*)^1$. Then a nearest-point selection S_p defined by p satisfies:

$$S_p(X^*)_i = \begin{cases} 1, & \text{if } p_i > 0, \\ -1, & \text{if } p_i < 0, \end{cases}$$

and

$$S_p(X_N^*)_i = \begin{cases} 1, & \text{if } 1 \leq i \leq N \text{ and } p_i > 0, \\ -1, & \text{if } 1 \leq i \leq N \text{ and } p_i < 0, \\ x_i, & \text{if } i > N. \end{cases}$$

Note that $S_p(X_N^*)$ is the *unique* element of X_N^* having the property that

$$d(p, X_N^*) = d(p, S_p(X_N^*)), \qquad N = 1, 2, \ldots.$$

Of course,

$$S_p(X_N^*) \rightarrow S_p(X^*), \text{ as } N \rightarrow \infty,$$

as required by Theorem 5.2.

6 Application to Infinite Horizon Optimization

We are now ready to apply the main results of the previous sections to our general infinite horizon optimization problem studied in [10]. We refer the reader to this reference for the details of what follows.

As observed in section 1, we have a compact metric space (X, d) of feasible infinite horizon strategies (or solutions), a closed, non-empty subset X^* of X consisting of the optimal infinite horizon solutions and, for each $T > 0$, a closed, non-empty subset $X^*(T)$ of X consisting of the T-horizon optimal solutions. In [10], we observed that only the elements of $\limsup X^*(T)$ could be approximated by elements of the $X^*(T), T > 0$. Accordingly, we defined $X^*(\infty)$ to be $\limsup X^*(T)$. More generally, we defined a finite-horizon solution *algorithm* to be a mapping $T \rightarrow A(T)$ on the positive real numbers for which $A(T)$ is a non-empty, closed subset of $X^*(T), T > 0$. For each such A, we analogously defined $A(\infty)$ to be $\limsup A(T)$, so that $A(\infty)$ is a closed, non-empty subset of $X^*(\infty)$. In particular, if

$$A(T) = X^*(T), \qquad T > 0,$$

so that $A(\infty) = X^*(\infty)$, then we called A the *maximal* algorithm and $X^*(\infty)$ the set of all *algorithmically* optimal infinite horizon solutions. If $A(T)$ is a singleton $\{x^*_A(T)\}$ in $X^*(T)$, all $T > 0$, then we called A a *simple* algorithm.

As before, let $(\mathcal{C}(X), D)$ denote the compact metric space of closed, non-empty subsets of X with D the Hausdorff metric corresponding to d. Then, for each algorithm A, $A(\infty)$ is an element of $\mathcal{C}(X)$and $\{A(T) : T > 0\}$ is a generalized sequence in $\mathcal{C}(X)$.

As in section 5, we are interested in selections S defined on $\mathcal{C}(X)$; in particular, we wish to concentrate on nearest-point selections S_p defined by points p in X. If A is an arbitrary algorithm, it will be convenient in this setting to denote $S_p(A(T))$ by $x^A_p(T), T > 0$, and $S_p(A(\infty))$ by x^A_p. If A is the maximal algorithm, we will write $x^*_p(T)$ and x^*_p instead. Thus, given an algorithm A and point p in X, each nearest-point selection S_p defined by p determines a simple algorithm given by

$$\{x^A_p(T) : T > 0\}.$$

In this sense, S_p provides us with a *tie-breaking rule* by selecting, for each $T > 0$, a T-horizon optimal solution from the set $A(T)$ of discovered such solutions using nearness to p as the selection criterion, and then choosing *arbitrarily* from amongst those nearest to p. We are interested in determining when a simple algorithm of the above form converges to x^A_p.

As observed in section 3, all the results of the previous sections (as well as their proofs) are valid for generalized sequences indexed by the positive reals. In this section, we apply these results to $\{A(T) : T > 0\}$ and $A(\infty)$. Since $A(\infty) = \limsup A(T)$, the following theorem is immediate from Theorems 3.4, 4.2, and 5.2.

Theorem 6.1 *Let A be an arbitrary algorithm for the general infinite horizon optimization problem studied in [10]. Then the following are equivalent:*

(i) $A(\infty) = \liminf A(T)$.

(ii) For every x^* in $A(\infty)$, there exists $X^*(T)$ in $A(T)$, all $T > 0$, such that $x^*(T) \to x^*$ in (X, d), as $T \to \infty$.

(iii) $A(T) \to A(\infty)$ in $(\mathcal{C}(X), D)$, as $T \to \infty$.

(iv) $x^A_p(T) \to x^A_p$ in (X, d), as $T \to \infty$, for all simple algorithms of the form $\{x^A_p(T) : T > 0\}$, where p is in the uniqueness set of $A(\infty)$.

In particular, this theorem is valid for the maximal algorithm.

Corollary 6.2 *The following are equivalent:*

(i) $X^*(\infty) = \liminf X^*(T)$.

(ii) For every x in $X^*(\infty)$, there exists $x^*(T)$ in $X^*(T)$, all $T > 0$, such that $x^*(T) \to x^*$ in (X, d), as $T \to \infty$.

(iii) $X^*(T) \to X^*(\infty)$ in $(\mathcal{C}(X), D)$, as $T \to \infty$.

(iv) $x^*_p(T) \to x^*_p$ in (X, d), as $T \to \infty$, for all simple algorithms of the form $\{x^*_p(T) : T > 0\}$, where p is in the uniqueness set of $X^*(\infty)$.

Corollary 6.3 *If $A(\infty)$ is a singleton $\{x^*\}$, then the conditions of Theorem 6.1 hold. Moreover, given any $\{x^*(T) : T > 0\}$ in $\{A(T) : T > 0\}$, we have that $x^*(T) \to x^*$, as $T \to \infty$.*

Proof: Apply Corollary 3.5. ∎

Remark The previous corollary is true, in particular, for A equal to the maximal algorithm, i.e. $A(T) = X^*(T), T > 0$, and $A(\infty) = X^*(\infty)$. This observation is the Planning Horizon Theorem (Theorem 6) of Bean and Smith in [2]. See also the end of section 5 of [10].

In [10], we defined an algorithm to be convergent if $A(\infty)$ is a singleton. However, in view of the previous theorem, it makes more sense to say that A is *convergent* if the equivalent conditions of Theorem 6.1 are satisfied. Therefore, given a convergent algorithm A, the problem of finding a nearest-point simple algorithm $\{x_p^A(T) : T > 0\}$ which converges to an optimal infinite horizon solution can be replaced by the problem of finding a point p in the uniqueness set $(A(\infty))^1$ of the limit set $A(\infty)$ of A. In general, this may be difficult to do. Of course, it is not if this uniqueness set is *all* of X, in which case *any* point p in X will do. Our next example gives sufficient conditions for this to happen.

Example 6.3 Suppose X is contained in a Hilbert space. If A is an algorithm having the property that $A(\infty)$ is *convex*, then for each p in X, there exists a unique point y_p in $A(\infty)$ which is nearest to p [1,p.15], i.e. the uniqueness set of $A(\infty)$ is all of X. The point y_p is called the *best approximation* to p in $A(\infty)$. A sufficient condition for $A(\infty)$ to be convex is that $A(T)$ be convex for all $T > 0$ and $A(T) \rightarrow A(\infty)$.

Such a case is important since it provides for a tie-breaking finite horizon algorithm $x_p^*(T)$ that arbitrarily well approximates an infinite horizon optimum x_p^* without the restrictive assumptions of [2,3,10] that $X^*(\infty)$ be a singleton.

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