AIAA JOURNAL
Vol. 30, No. 6, June 1992

Improvement of Normalization Methods for Eigenvector Derivatives

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A theoretical method is developed for improving certain calculations for eigenvector derivatives in linear systems. The subject algorithms are all forward analysis, i.e., the eigenvector changes are being driven by design parameter perturbations. The new method emphasizes proper mass normalization and is most needed when iteratively computing eigenvector perturbations within a reduced-dimension space. Previous schemes handled the normalization task adequately only for the case of small mode changes because these schemes drop certain terms from the governing normalization equation. When moderate (± 10%) mode changes are evident, as in iterative convergence difficulties or cases of moderate design variable changes, it is necessary to implement a normalization scheme that considers higher-order terms. Two roots to the governing normalization equation exist, only one of which can be used. Criteria for choice of the proper root are developed and a benchmark problem is analyzed employing the new technology. Use of this normalization is required for moderate-change algorithms and can improve existing small-change algorithms.

Nomenclature

$[D]_i$	= coefficient matrix in solution for $\{\Delta\phi\}_i$, $[K] - \lambda_i[M]$
$\mathfrak{D}$	= the discriminant, $b^2 - 4ac$ for the quadratic $a\alpha^2 + b\alpha + c = 0$
$[F]$	= matrix of $\{F\}_i$ column vectors
$\{F_i\}$	= static pseudoload driving solution for $\{\Delta\phi\}_i$ in $[D]_i\{\Delta\phi\}_i = \{F\}_i$
$[K]$	= stiffness matrix; symmetric
$[M]$	= mass matrix; symmetric
$[\mathfrak{M}]$	= modal mass matrix, $[\Phi]^T[M][\Phi]$; symmetric
α_i	= scale factor for ith $\{\Delta\phi\}$
α_i'	= real part of a complex α_i root, $\mathrm{Re}(\alpha_i)$
α_i''	= imaginary part of a complex α_i root, $\mathrm{Im}(\alpha_i)$
Δ	= perturbation symbol denoting exact change from the baseline
$[\Lambda]$	= diagonal matrix of eigenvalues arranged in decreasing mode number
λ_i	= ith eigenvalue (circular frequency squared)
$[\Phi]$	= matrix of $\{\phi\}_i$ column vectors arranged in decreasing mode number
$\{\phi\}_i$	= ith eigenvector (i.e., mode shape of structure); solution to $[K]\{\phi\}_i = \lambda_i[M]\{\phi\}_i$; normalized with appropriate $[M]$

Subscripts

exact	= exact values computed through reanalysis
Fox	= approximate values computed by Fox's modal superposition method
High	= associated with High's linear normalization method
simple	= associated with the "simple" mass normalization method
unscaled	= values not yet normalized
VMN	= associated with the vector magnitude normalization method

Superscripts

0	= baseline values
1	= perturbed values resultant from design variable change, e.g., $(\,)^1 = (\,)^0 + \Delta(\,)$

Introduction

DURING the last 20 years, there has been much research on the subject of the eigenvector derivative. Surveys and comparisons of the numerous methods for calculating these derivatives are readily available in the literature.[1,2] The papers of Fox and Kapoor[3] and Nelson[4] have provided solutions that are exact to first order (exact in the sense of the derivative, i.e., for infinitesimal design changes). A distinction should be made between the eigenvector derivative ($\partial\phi/\partial b$) and the eigenvector perturbation ($\Delta\phi$). Here, we emphasize the perturbation equations.

When dealing with complicated structures with many degrees of freedom, Fox's and Nelson's methods can be prohibitively costly, and recourse is taken to iterative methods using reduced eigenvector sets. Unfortunately, iterations on eigenvectors can be poorly behaved since the proper mass scaling of the eigenvectors varies with the perturbation to the structure and since the stability of the iterations is much worse than for eigenvalue iterations. The iterative modal method proposed by High[5] attempted to conquer these difficulties and is currently in use in MSC/NASTRAN Version 66. It is from a comparison study of Fox's, Nelson's, and High's methods that the following improvement in normalization has arisen.

For large systems, the computational time involved in the calculation of eigenvector changes can be greatly decreased by

Received Feb. 15, 1991; revision received Aug. 28, 1991; accepted for publication Sept. 9, 1991.

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reducing the dimensionality of the computational space, i.e., by truncating the full set of mode shapes to a reduced set of the most important modes. Generally, the most important modes correspond to the lowest frequencies of the structure. Many of the higher modes require such high energy for excitation that they contribute very little to the overall motion of the system. By extracting only the first few eigenpairs, the cost of the baseline analysis can be reduced dramatically.

Algorithms employed by the cited authors are based on solutions of the following first-order equation for the perturbation to the ith eigenvector:

$$[D]_i\{\Delta\phi\}_i = \{F\}_i \tag{1}$$

where

$$[D]_i = [K^0] - \lambda_i^0[M^0]$$

$$\{F\}_i = (\Delta\lambda_i[M^0] + \lambda_i^0[\Delta M] - [\Delta K])\{\phi^0\}_i$$

This equation is derived by perturbing the original eigenproblem, canceling the baseline solution, and dropping all terms of higher than first order in Δ. It can also be derived by taking derivatives of the original eigenproblem equation, with the same results.

Fox's method uses a modal superposition for $\{\Delta\phi\}_i$, solves for the coefficients, and gives exact derivatives (exact to first order in Δ) when the full set of modes is used. Nelson's method eliminates the singularity in the equation by enforcing the component of $\{\Delta\phi\}_i$ associated with the dominant component of $\{\Delta\phi_i\}$ to be zero, i.e., zero change is enforced in the largest component of the baseline eigenvector. It uses information associated with the ith mode only and therefore loses no accuracy when applied to truncated mode sets. It is also exact to first order. High's method does not use the full set of modes, but still uses Fox's method for an initial guess. Then, by isolating the $[K^0]\{\Delta\phi\}_i$ term, the equation can be used to obtain new estimates of each $\{\Delta\phi\}_i$:

$$[\Delta\Phi]^{j+1} = [K^0]^{-1}([M^0][\Delta\Phi]^j[\Lambda^0] + [F]) \tag{2}$$

For iterations of this equation to converge, according to High, mass orthogonalization and renormalization must be performed periodically on $[\Delta\Phi]$. Note that orthogonalization and normalization are separate tasks, the former handling the condition that $\{\phi^1\}_i^T[M^1]\{\phi^1\}_j = 0 (i \neq j)$ and the latter providing for $\{\phi^1\}_i^T[M^1]\{\phi^1\}_i = 1$. The work of this paper was originally intended as a "fix" for High's method since the authors observed the tendency of the iterations to diverge. Handling divergence is a task for normalization. Thus, this paper rigidly enforces normalization, but not orthogonalization (although this method does preserve orthogonality better than competing methods). Although this paper's improved normalization method was originally devised to handle runaway perturbations in the iterative modal method, it is certainly applicable to any of the methods cited earlier. High, Fox, and Nelson all use first-order normalization schemes that, although sufficient for small eigenvector changes, are quite inadequate for moderate-to-large eigenvector changes.

Theoretical Development

At this point, it will be assumed that a $\{\Delta\phi_i\}$ vector has been achieved through some nonexact means, such as High's iterative method, Fox's (first-order) method, or Nelson's (first-order) method. This eigenvector perturbation must be scaled such that the new eigenvector $\{\phi^1\}_i$ is normalized. There are two different philosophies to choose from. First, one could assume that the $\{\Delta\phi\}_i$ vector is a good vector, i.e., it points in the correct direction and it has not grown too large. In this case, normalization could be achieved simply through adding $\{\Delta\phi\}_i$ to $\{\phi^1\}_i$ and scaling the result according to whichever normalization criterion is desired (mass normalization, vector magnitude normalization, vector component normalization). Call this the simple normalization method. For the case of mass normalization, the condition is

$$(\alpha_i\{\phi^1\}_i)^T[M^1](\alpha_i\{\phi^1\}_i) = 1$$

which is easily solved for the scale factor:

$$\alpha_{i_{\text{simple}}} = \sqrt{\frac{1}{\{\phi^1\}_i^T[M^1]\{\phi^1\}_i}}$$

Unfortunately, it is quite dangerous to assume that the eigenvector perturbations are well behaved, since certain algorithms can give highly erroneous $\{\Delta\phi\}_i$ when moderate design variable changes are prescribed. The second philosophy, then, involves assuming that the $\{\Delta\phi\}_i$ vector is not necessarily correct and then resolving to preserve as much of the known baseline information as possible. This is accomplished by scaling the perturbation vector before adding it to the baseline vector instead of scaling the sum. This is the authors' philosophy. The standpoint is that although iterations should ultimately converge to good vectors, they may start off erroneously and normalization must keep the errors in check.

This paper concentrates on mass normalization, but in the interest of completeness, vector magnitude normalization and vector component normalization will be discussed in this paragraph. For *vector magnitude normalization,* the inner product of the eigenvectors yields unity:

$$\{\phi^0\}_i^T\{\phi^0\}_i = \{\phi^1\}_i^T\{\phi^1\}_i = 1$$

The current eigenvector $\{\phi^1\}_i$, is made up of the known baseline eigenvector $\{\phi^0\}_i$, and a weighted contribution from the calculated perturbation $\{\Delta\phi\}_i$, (we only scale the perturbation in order to preserve baseline information and control error). The inner product of the current eigenvector can be written as

$$(\{\phi^0\}_i + \alpha_i\{\Delta\phi\}_i)^T(\{\phi^0\}_i + \alpha_i\{\Delta\phi\}_i) = 1$$

Expanding and canceling the baseline inner product, one has

$$\alpha_i(2\{\Delta\phi\}_i^T\{\phi^0\}_i + \alpha_i\{\Delta\phi\}_i^T\{\Delta\phi\}_i) = 0$$

There are two exact solutions for the scale factor:

$$\alpha_{i_{\text{VMN}}} = 0, -\frac{2\{\Delta\phi\}_i^T\{\phi^0\}_i}{\{\Delta\phi\}_i^T\{\Delta\phi\}_i}$$

The trivial solution will usually be undesirable. For *vector component normalization,* a certain component of the eigenvector is taken to be unity. This practice is popular in modal testing. For this case, it is impossible to scale $\{\Delta\phi\}_i$ prior to adding it to $\{\phi^0\}_i$ since the only solution for α_i would be zero. Thus, we would have to revert to the first philosophy, in which the sum of $\{\Delta\phi\}_i$ and $\{\phi^0\}_i$ is scaled. Again, this procedure would not control error in $\{\Delta\phi\}_i$. Comparing vector magnitude and vector component normalization to mass normalization is not an easy task since they are very different. However, when controlling error is the goal in calculating moderate-to-large eigenvector perturbations, it will be stated that mass normalization is more desirable because 1) mass normalization is more physical since it enforces a unit kinetic energy condition, 2) vector magnitude normalization is a rigid condition that allows little freedom in the perturbations (see the following graphical representation of normalization and envision coincident circles), and 3) vector component normalization will not control error.

Returning to mass normalization, the key desire is to scale the best estimate of the current perturbed eigenvectors with respect to the current perturbed mass. For the ith eigenvector:

$$\{\phi^1\}_i^T[M^1]\{\phi^1\}_i = 1 \tag{3}$$

The correct approach when dealing with potentially erroneous eigenvector perturbations is to scale the perturbation $\{\Delta\phi\}_i$ before adding it to the baseline $\{\phi^0\}_i$:

$$(\{\phi^0\}_i + \alpha_i\{\Delta\phi\}_i)^T[M^1](\{\phi^0\}_i + \alpha_i\{\Delta\phi\}_i) = 1 \tag{4}$$

Expand, collect terms, and use the symmetry of $[M^1]$ to obtain

$$\{\phi^0\}_i^T[M^1]\{\phi^0\}_i + 2\alpha_i\{\Delta\phi\}_i^T[M^1]\{\phi^0\}_i + \alpha_i^2\{\Delta\phi\}_i^T[M^1]\{\Delta\phi\}_i = 1 \tag{5}$$

Use $[M^1] = [M^0] + [\Delta M]$ in the first term and the fact that each $\{\phi^0\}$ is normalized with respect to $[M^0]$ to obtain

$$\{\phi^0\}_i^T[\Delta M]\{\phi^0\}_i + 2\alpha_i\{\Delta\phi\}_i^T[M^1]\{\phi^0\}_i + \alpha_i^2\{\Delta\phi\}_i^T[M^1]\{\Delta\phi\}_i = 0 \tag{6}$$

This is of the form

$$a\alpha_i^2 + b\alpha_i + c = 0 \tag{7}$$

where

$$a = \{\Delta\phi\}_i^T[M^1]\{\Delta\phi\}_i$$
$$b = 2\{\Delta\phi\}_i^T[M^1]\{\phi^0\}_i$$
$$c = \{\phi^0\}_i^T[\Delta M]\{\phi^0\}_i$$

Note that $\alpha_i = 1$ is a solution when $\{\Delta\phi\}_i^T$ is exact. Looking at Eq. (4), it is evident that the left-hand side simplifies to the left-hand side of Eq. (3) for an exact $\{\Delta\phi\}_i$ and for $\alpha_i = 1$. An extraneous root still exists, however, which can be mistaken as the correct root under certain rare circumstances (1 > extraneous $\alpha_i > 0$).

In solving for the α_i value that satisfies Eq. (6), an analyst could simplify matters by assuming small perturbations and neglecting terms of higher than first order in Δ. This is what High has done. The resulting scale factor is

$$\alpha_{i_{\text{High}}} = -\frac{\{\phi^0\}_i^T[\Delta M]\{\phi^0\}_i}{2\{\Delta\phi\}_i^T[M^0]\{\phi^0\}_i} \tag{8}$$

When large mode changes are present, however, this causes significant error and the full quadratic must be solved.

Although solving this quadratic [Eq. (7)] may seem simple at first glance, problems exist in choosing between the two roots for α_i. If the roots are real and distinct ($\mathfrak{D} > 0$), then they will both satisfy the mass normalization criterion exactly, but only one root will give the best perturbed mode shape. This can be proven by reanalyzing the perturbed system (note that with an erroneous $\{\Delta\phi\}_i$, neither root will give an exact mode shape). If, on the other hand, the discriminant is less than zero, a complex conjugate pair of α_i roots exists. This can occur because $\{\phi^0\}_i$ and $\{\phi^1\}_i$ are normalized to different mass matrices (see the following graphical representation). In the rare case that the discriminant equals zero, the roots are repeated and no choice needs to be made. Hence, an investigation has been conducted to establish criteria for choosing between two real α_i roots and for handling complex conjugate pairs when they occur.

A graphical representation of the normalization procedure will be presented to better visualize what the different α_i root possibilities convey. The unit mass normalization of an eigenvector in R^n space (n degrees of freedom) can be envisioned as a centered vector with tip on the surface of an n-dimensional ellipsoid. Any point on the surface of the ellipsoid represents a state of unit kinetic energy for the structure when assuming the motion of the ith mode. For our purposes, this will be sketched as a unit circle in R^2 space without loss of generality (the circles are clearer and demonstrate all possibilities). A baseline eigenvector $\{\phi^0\}_i$ is normalized with respect to $[M^0]$ and lies on one unit energy circle. The perturbed eigenvector $\{\phi^1\}_i$ is normalized to $[M^1]$ and lies on a different unit energy circle. If both mass matrices are nonsingular, then no motion can occur without generating kinetic energy and the origins of the circles are coincident. It will be assumed for this graphical explanation that this is the case.

If the perturbation to the system increases inertial resistance to a particular mode shape, then less displacement will be required in the perturbed eigenvector to make unit kinetic

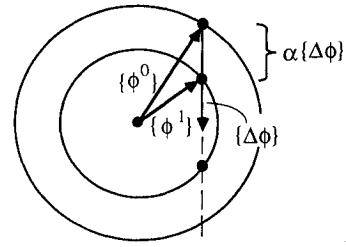

Fig. 1 Positive roots: common case; exact normalization.

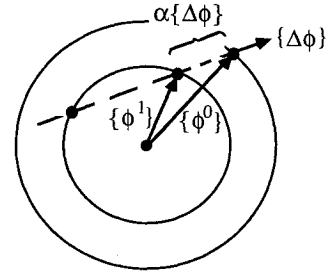

Fig. 2 Negative roots: rare case; exact normalization.

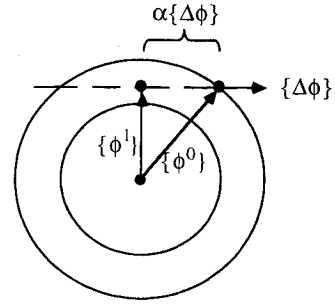

Fig. 3 Complex conjugate roots: common case; approximate normalization.

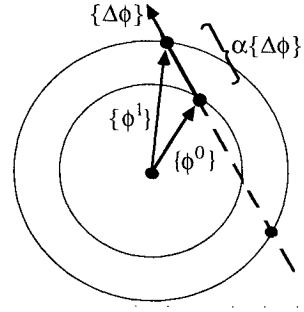

Fig. 4 Mixed roots: common case; exact normalization.

energy and the circle for $\{\phi^1\}_i$ will be smaller than that for $\{\phi^0\}_i$. Likewise, if the perturbation to the system decreases the inertial resistance to a mode shape, the $\{\phi^1\}_i$ circle will be larger than that for $\{\phi^0\}_i$ because more motion will be required to make unit kinetic energy. The relative size of these circles determines which α_i root combinations are possible. In the former case (increase in inertial resistance), pairs of positive real roots, negative real roots, and complex conjugate roots are all possible. Which of these three possibilities occurs is dependent on the orientation of the $\{\Delta\phi\}_i$ vector (Figs. 1-3). In the latter case (decrease in inertial resistance), only pairs of mixed real roots are possible (Fig. 4). In the sketches, dashed lines represent possible scalings of the vector.

Real, Repeated Roots

When the rare case of repeated real roots occurs ($\mathfrak{D} = 0$), it means that the $\{\Delta\phi\}_i$ vector is tangent to the $\{\phi^1\}_i$ energy circle. For this to happen, the $\{\phi^1\}_i$ circle must be smaller than or coincident to the $\{\phi^0\}_i$ circle. The resultant $\{\phi^1\}_i$ vector will be orthogonal to $\{\Delta\phi\}_i$. For this degenerate case, choose either root.

Real, Distinct Roots

When the roots are real and distinct ($\mathfrak{D} > 0$), the signs of the roots dictate which choice for α_i is proper (the proper choices are shown in Figs. 1, 2, and 4). For a pair of positive roots, calculations have shown that the smaller root gives the correct mode shape; the larger root is always extraneous. When the α_i roots are both negative, the $\{\Delta\phi\}_i$ vector is pointing in a completely erroneous direction and serious convergence difficulties are indicated. The smaller magnitude root should be chosen in order to control the error. When the roots are of mixed sign, the choice is more complicated since there are two competing factors influencing the decision. Usually, smaller magnitude roots are better and positive roots are better, and so a pair of mixed roots in which the positive root is much larger in magnitude than the negative root presents a problem. Two different criteria will be established for this mixed root case. When the problem is well-behaved (as in direct solutions by Fox's or Nelson's method or in iterative solutions that are near convergence), we choose the positive root since the negative root contradicts the information provided by the $\{\Delta\phi\}_i$ vector, i.e., it is known that the $\{\Delta\phi\}_i$ vector points in the correct direction for the well-behaved case. When the problem is poorly behaved (as in Fox's method applied to a truncated set or in iterative methods having convergence problems), we choose the smaller magnitude root in order to control the error.

Complex, Conjugate Roots

When the roots are complex conjugates ($\mathfrak{D} < 0$), use of the real portion as the scale factor is proposed (see Fig. 3). This choice will not exactly satisfy the normalization criteria, but since complex mode shapes are not appropriate in normal mode analysis, it is the best available approximation. We will attempt to gain some insight on the effect of neglecting the imaginary part of the root through two different analyses.

First, we may rewrite Eq. (6) as a quadratic function that is to be driven to zero through proper α_i choice. When neglecting the imaginary part of the root, exact equality of the real function with zero is impossible, so there will be a real error in the function:

$$U(\alpha_i) = a\alpha_i^2 + b\alpha_i + c = \epsilon$$

To minimize the error $|\epsilon|$, we set the derivative of the function equal to zero:

$$\frac{dU}{d\alpha_i} = 0 = 2a\alpha_i + b$$

Thus,

$$\alpha_i = -\frac{b}{2a}$$

minimizes the error. This corresponds with the real part of the root from the quadratic formula. Note that this solution will not maximize the error due to the nature of a quadratic form with no real roots. Viewed graphically, the quadratic function is parabolic and, since there are no zero crossings, must be either concave up above the zero axis or concave down below the zero axis. Only one extremum can exist and it will correspond to the minimum distance to the zero axis. Viewed in the context of the unit energy sketches (see Fig. 3), choosing the real part of the root minimizes the distance from the center point to the line of possible $\{\Delta\phi\}_i$ scalings.

The second analysis will attempt to track terms in the governing equations by splitting a complex α_i root into its real and imaginary portions. The root

$$\alpha_i = \alpha_i' + i\alpha_i'' \tag{9}$$

was found exactly from Eq. (6) as part of a complex conjugate pair. Substituting Eq. (9) into Eq. (6), one obtains

$$\{\phi^0\}_i^T[\Delta M]\{\phi^0\}_i + 2(\alpha_i' + i\alpha_i'')\{\Delta\phi\}_i^T[M^1]\{\phi^0\}_i + (\alpha_i'^2 + 2i\alpha_i'\alpha_i'' - \alpha_i''^2)\{\Delta\phi\}_i^T[M^1]\{\Delta\phi\}_i = 0 \tag{10}$$

This equation may be decoupled into real and imaginary equations, both of which are exactly satisfied by the root already found. Looking at the real equation,

$$\{\phi^0\}_i^T[\Delta M]\{\phi^0\}_i + 2\alpha_i'\{\Delta\phi\}_i^T[M^1]\{\phi^0\}_i + (\alpha_i'^2 - \alpha_i''^2)\{\Delta\phi\}_i^T[M^1]\{\Delta\phi\}_i = 0 \tag{11}$$

it is evident that one only loses one term (the $\alpha_i''^2$ term) in neglecting the imaginary part of the root. This term is in fact the error ϵ from the first analysis. Specifically, $\epsilon = \alpha_i''^2\{\Delta\phi\}_i^T[M^1]\{\Delta\phi\}_i$. Thus, the normalization used in the complex root case is still much improved over the cited linear normalization approach [Eq. (8)], since Eq. (11) has the perturbed mass matrix $[M^1]$ instead of $[M^0]$ in the second term and retains the $\alpha_i'^2$ portion of the third term.

Summary of Root Selection

The criteria developed for choosing the proper α_i root are summarized in Table 1, where the symbols in parentheses denote the signs of the root pairs.

Benchmark Problem: Cantilever Beam in Bending

A simple five-element cantilever beam model (Fig. 5) is proposed as a benchmark problem because analytical results are readily available. The material constants are

$$E = 2.0684 \times 10^5 \text{ MPa}, \quad \nu = 0.3$$

$$\rho = 7.8334 \times 10^{-9} \text{ N s}^2/\text{mm}^4$$

The motion is constrained to allow only xz bending, which leaves a total of 10 degrees of freedom (z displacement and y rotation—w and Θ_y—at each of five nodes).

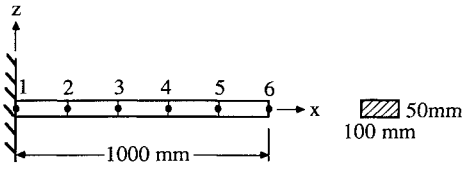

Fig. 5 Cantilever beam modeled with five elements.

Table 1 Criteria for choice of proper α_i

$\mathfrak{D}=0$ (real, repeated)	$\mathfrak{D}>0$ (real, distinct)	$\mathfrak{D}<0$ (complex, conjugate)
choose either root	(+, +): choose smallest magnitude (−, −): choose smallest magnitude (+, −): well behaved: positive poorly behaved: smallest magnitude	choose real part of pair

Table 2 Quality indices (design variable set 1, full modal set)

	First mode		Second mode	
	Norm. q.i.[a]	Acc. q.i.[b]	Norm. q.i.[a]	Acc. q.i.[b]
Minimum α root	1.0000	1.0000	1.0000	1.0000
Maximum α root	1.0000	0.2596	1.0000	0.6770
High's linear	0.9977	0.9988	1.0005	1.0002
Simple mass	1.0000	1.0000	1.0000	1.0000

[a]Normalization quality index. [b]Accuracy quality index.

Element stiffness and mass matrices are assembled and used to find the baseline eigenpairs through the use of a MATLAB program. This program then truncates the sets to the first n eigenvalues and eigenvectors (an actual large system analysis would extract only the necessary number of eigenpairs). The truncated set of eigenvectors is then used as a basis for the subsequent modal superposition analysis (herein called Fox's method). Upon prescription of design variable changes, one computes new stiffness and mass matrices and Fox's method is performed to get an initial estimate for the eigenpair perturbations. This initial estimate is generally greatly in error due to the truncated set, and an iteration scheme similar to that proposed by High must be performed. (Note that this approach should be faster than the standard Nelson's method for large systems.)

Full Modal Set

For the initial testing of the quadratic α_i equation, all 10 of the eigenpairs were retained. By doing this, Fox's method could give eigenvector changes that were exact to first order and each resulting normalized $\{\phi^1\}_{i_{Fox}}$ could be compared with the normalized exact eigenvector $\{\phi^1\}_{i_{exact}}$. Figures 6 and 7 show the difference between choosing the correct α_i root and the extraneous root for the first two bending modes of the cantilever beam. The exact perturbed mode shape $\{\phi^1\}_{i_{exact}}$ was computed through reanalysis, and the approximate $\{\phi^1\}_{i_{Fox}}$ perturbed mode shapes were achieved after scaling $\{\Delta\phi\}_i$ by the minimum or maximum magnitude α_i root. The minimum magnitude root is seen to be the correct choice.

For a quantitative evaluation, an accuracy quality index can be defined as $\{\phi^1\}_{i_{Fox}}[M^1]\{\phi^1\}_{i_{exact}}$ and a normalization quality index can be defined as $\{\phi^1\}_{i_{Fox}}[M^1]\{\phi^1\}_{i_{Fox}}$. The former measures how closely the $\{\phi^1\}_{i_{Fox}}$ vector approximates the exact perturbed mode shape, and the latter measures how well the normalization task is performed. Any deviation from a unit value in either index shows error in the $\{\phi^1\}_{i_{Fox}}$ approximation resulting from an α_i root choice.

The quality indices for the modes graphed in Figs. 6 and 7, along with indices for High's linear normalization and the simple mass normalization, are shown in Table 2.

The important conclusion is that, although both α_i roots normalize exactly, only the proper root (here the minimum magnitude root) will preserve accuracy. The linear and simple normalizations are competitive when the eigenvector perturbations are well behaved (full modal set). These results came from beam element thickness changes (changes in the design variables) of +19, −2.9, −9.0, −6.3, and −8.3% for the five elements. Call these design variable scaling set 1. This set corresponds to case 1, subcase 3 of the cantilever beam work of Kim et al.[6] and gives real and distinct α_i roots for the $\{\Delta\phi\}_i$ from Fox's method.

For another design variable scaling, specifically −19, +2.9, +9.0, +6.3, +8.3% (design variable scaling set 2), complex α_i roots were evident in the normalization of the fourth through eighth modes. These modes are instrumental in illustrating the use of the real portion of the complex conjugate α_i roots. The fourth mode (see Fig. 8) has α_i roots of $1.1503 \pm 0.2357i$, and so 1.1503 was used in the scaling of $\{\Delta\phi\}_4$. It was chosen because its plot shows the largest mode perturbation from the baseline of the four modes with imaginary α_i. The seventh mode (see Fig. 9) has α_i roots of $0.5737 \pm 0.7923i$, and so 0.5737 was used to scale $\{\Delta\phi\}_7$. This mode was chosen since its α_i roots have the largest imaginary part relative to the real part of the four subject modes. The quality indices for these modes are shown in Table 3.

The quality obtained in using the real part of the α_i root is good, but not quite as good as when there are real and distinct α_i roots, since an approximation has been made in neglecting the imaginary part of the root. Again, the competing methods are just as good when the full set of modes is used.

Truncated Modal Set

With the criteria for alpha root choice established, let us now apply this new normalization technology to a solution involving a truncated set of modes, where large mode changes can easily arise at intermediate steps due to the iterative nature of the problem. For the benchmark cantilever, the first five modes are retained in the truncated set. Fox's method gives erroneous answers for the eigenvector perturbations, which are used as an initial guess in the subsequent iterations of Eq. (2). For design variable scaling set 1, the erroneous $[\Delta\Phi]$ evident at iteration 1, and the subsequent modal mass matrices for High's normalization, the simple mass normalization, and the full quadratic normalization follow (note that the $[\Delta\Phi]$

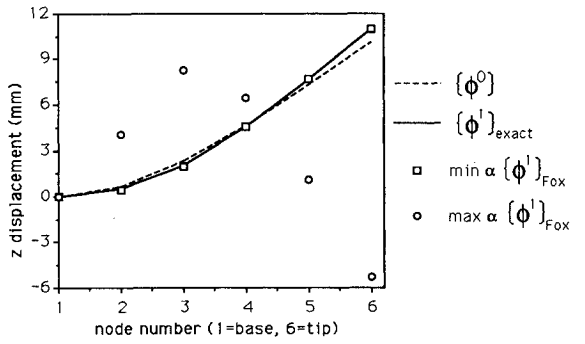

Fig. 6 First bending mode (minimum vs maximum root normalization).

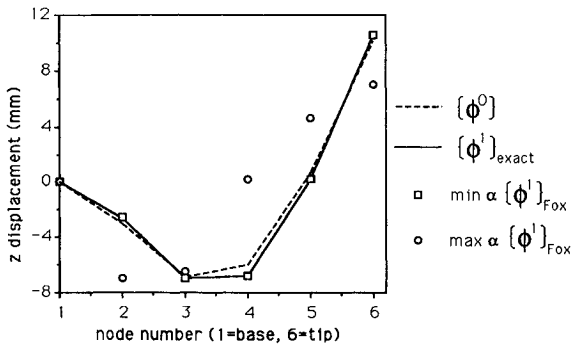

Fig. 7 Second bending mode (minimum vs maximum root normalization).

column vector components alternate between nodal displacements and rotations and that mode 5 is the first column and mode 1 is the last):

$$[\Delta\Phi]_{\text{unscaled}} =$$

$$\begin{bmatrix} 3.4650 & -0.4620 & -1.0201 & 0.5205 & -0.1897 \\ -0.0736 & 0.0391 & -0.0059 & -0.0015 & 0.0018 \\ 13.327 & -5.8881 & 0.2730 & -0.0363 & -0.3273 \\ -0.0409 & 0.0117 & -0.0028 & 0.0056 & -0.0003 \\ 25.367 & -9.2314 & -0.8336 & -0.7605 & -0.0932 \\ -0.0596 & 0.0306 & 0.0091 & 0.0007 & -0.0019 \\ 35.064 & -16.545 & -2.0368 & -0.4246 & 0.3572 \\ -0.0443 & 0.0335 & 0.0009 & -0.0037 & -0.0025 \\ 46.555 & -20.751 & -1.2949 & 0.5230 & 0.8719 \\ -0.0645 & 0.0152 & -0.0056 & -0.0051 & -0.0026 \end{bmatrix}$$

The modal mass after High's normalization is

$$[\mathfrak{M}] = \begin{bmatrix} 91.703 & -23.897 & -2.0388 & -1.0689 & 9.4398 \\ -23.897 & 7.4222 & 0.5878 & 0.1072 & -2.5240 \\ -2.0388 & 0.5878 & 1.0456 & 0.0406 & -0.2205 \\ -1.0689 & 0.1072 & 0.0406 & 0.9994 & 0.0090 \\ 9.4398 & -2.5240 & -0.2205 & 0.0090 & 0.9975 \end{bmatrix}$$

and the modal mass after the simple mass normalization is

$$[\mathfrak{M}] = \begin{bmatrix} 1.0000 & -0.8706 & -0.1914 & -0.1104 & 0.9708 \\ -0.8706 & 1.0000 & 0.1932 & 0.0381 & -0.8998 \\ -0.1914 & 0.1932 & 1.0000 & 0.0404 & -0.2014 \\ -0.1104 & 0.0381 & 0.0404 & 1.0000 & 0.0106 \\ 0.9708 & -0.8998 & -0.2014 & 0.0106 & 1.0000 \end{bmatrix}$$

but after a full quadratic normalization, the modal mass is much better:

$$[\mathfrak{M}] = \begin{bmatrix} 1.0000 & -0.0625 & -0.0118 & -0.0185 & 0.1189 \\ -0.0625 & 1.0000 & 0.0425 & 0.0058 & -0.1543 \\ -0.0118 & 0.0425 & 1.0000 & 0.0440 & -0.1375 \\ -0.0185 & 0.0058 & 0.0440 & 1.0000 & 0.0102 \\ 0.1189 & -0.1543 & -0.1375 & 0.0102 & 1.0000 \end{bmatrix}$$

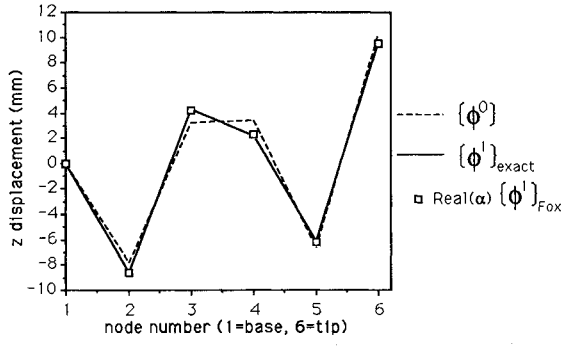

Fig. 8 Fourth bending mode [normalization by Re(α)].

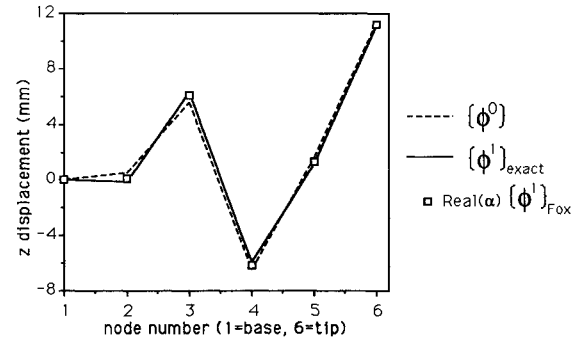

Fig. 9 Seventh bending mode [normalization by Re(α)].

Table 3 Quality indices (design value set 2, full modal set)

	Fourth mode		Seventh mode	
	Norm. q.i.[a]	Acc. q.i.[b]	Norm. q.i.[a]	Acc. q.i.[b]
Re(α) root	1.0010	1.0002	1.0152	1.0061
High's linear	1.0014	0.9997	1.0196	1.0096
Simple mass	1.0000	0.9990	1.0000	0.9999

[a]Normalization quality index. [b]Accuracy quality index.

Table 4: Accuracy quality indices (design variable set 1, truncated modal set)

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
High's linear	0.9988	0.9997	0.9976	1.0048	0.9961
Simple mass	1.0000	1.0000	0.9789	0.4325	0.2058
Full quadratic	1.0000	1.0000	0.9896	0.9804	0.9812

Note that the modal mass matrix would ideally be the identity matrix. Normalization provides for unit values on the diagonal, and orthogonalization provides for null values off the diagonal. The diagonal values are in fact the normalization quality indices discussed previously. It is evident that High's normalization is failing in the actual normalization task, since some diagonal elements (normalization quality indices) have grown large. The simple mass normalization performs the normalization exactly as the diagonal elements are exactly unity, but has lost the desired orthogonality in that some off-diagonal values have grown large. This shows that the simple mass normalization is not controlling error since the baseline information has not been preserved. The full quadratic normalization performs the normalization exactly and retains orthogonality by preserving as much baseline information as possible. Examining the accuracy quality indices in Table 4 clarifies the picture further.

High's normalization and the full quadratic normalization retain accuracy since they scale $[\Delta\Phi]$ prior to addition to $[\Phi^0]$, whereas the simple mass normalization retains little accuracy for the large mode changes since it scales the sum.

For design variable scaling set 2, complex α_i scaling factors were evident and the following data were extracted at iteration 2:

$$[\Delta\Phi]_{\text{unscaled}} =$$

$$\begin{bmatrix} -40.993 & -7.1956 & -19.315 & -1.2772 & 0.1380 \\ 0.4204 & 0.0427 & 0.1943 & 0.0094 & -0.0013 \\ -146.22 & -21.686 & -71.960 & -3.0535 & 0.2394 \\ 0.6244 & 0.1009 & 0.3184 & 0.0088 & 0.0002 \\ -292.34 & -43.913 & -141.83 & -5.5908 & 0.0484 \\ 0.7999 & 0.1174 & 0.3768 & 0.0169 & 0.0016 \\ -456.94 & -69.657 & -220.89 & -9.4925 & -0.3233 \\ 0.8373 & 0.1401 & 0.4100 & 0.0216 & 0.0021 \\ -628.09 & -99.830 & -304.37 & -14.011 & -0.7463 \\ 0.8648 & 0.1553 & 0.4201 & 0.0230 & 0.0021 \end{bmatrix}$$

The modal mass after High's normalization is

$$[\mathfrak{M}] = \begin{bmatrix} 5840.6 & 90243 & 2237.3 & 108.21 & -76.249 \\ 90,243 & 1,395,481 & 34,563 & 1654.3 & -1177.7 \\ 2237.3 & 34563 & 858.17 & 41.467 & -29.209 \\ 108.21 & 1654.3 & 41.467 & 2.9908 & -1.4540 \\ -76.249 & -1177.7 & -29.209 & -1.4540 & 0.9973 \end{bmatrix}$$

and the modal mass after the simple mass normalization is

$$[\mathfrak{M}] = \begin{bmatrix} 1.0000 & 0.9946 & 0.9994 & 0.8068 & -0.9991 \\ 0.9946 & 1.0000 & 0.9937 & 0.7933 & -0.9932 \\ 0.9994 & 0.9937 & 1.0000 & 0.8068 & -0.9986 \\ 0.8068 & 0.7933 & 0.8068 & 1.0000 & -0.8298 \\ -0.9991 & -0.9932 & -0.9986 & -0.8298 & 1.0000 \end{bmatrix}$$

Table 5: Accuracy quality indices (design variable set 2, truncated modal set)

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
High's linear	0.9987	0.9355	1.1334	1.8647	0.9248
Simple mass	1.0000	0.5580	0.0364	0.1004	0.0144
Full quadratic	1.0000	1.0277	1.0104	0.9968	0.9912

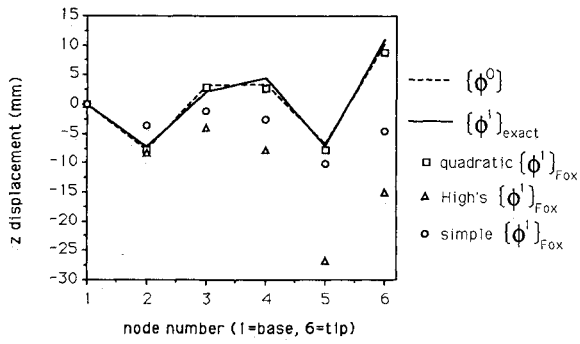

Fig. 10 Large eigenvector change due to iteration wandering (fourth mode).

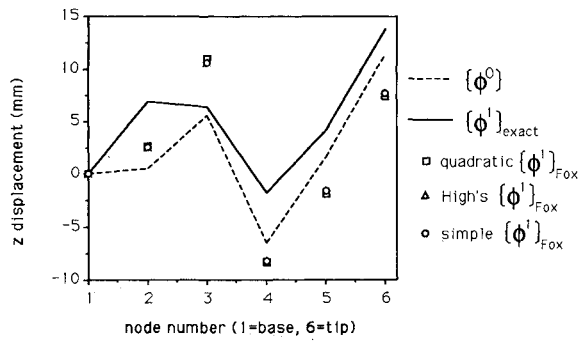

Fig. 11 Large eigenvector change due to large design parameter change (seventh mode).

whereas, after the full quadratic normalization, the modal mass is

$$[\mathfrak{M}] = \begin{bmatrix} 1.0142 & 0.0542 & -0.0305 & 0.0127 & -0.0014 \\ 0.0542 & 1.0243 & 0.0404 & -0.0233 & 0.0003 \\ -0.0305 & 0.0404 & 1.0351 & 0.0210 & 0.0046 \\ 0.0127 & -0.0233 & 0.0210 & 1.0660 & -0.0715 \\ -0.0014 & 0.0003 & 0.0046 & -0.0715 & 1.0000 \end{bmatrix}$$

Once again, High's linear normalization failed in normalization and the simple mass normalization failed in preserving orthogonality. Looking at the accuracy quality indices in Table 5, it is evident that High's normalization retains some accuracy, whereas the simple mass normalization shows extreme inaccuracy for the large mode changes. The full quadratic normalization did not work quite as well for complex α_i roots as it did for real α_i roots, but still greatly outperforms the other methods when large $[\Delta\Phi]$ occur.

Range of Applicability of Quadratic Normalization

The goal of this paper has been to present a fix to small-change eigenvector derivative algorithms (e.g., High, Fox, Nelson). These methods develop trouble when either iterations begin to wander or when the specified design change grows to more than $\pm 10\%$. The two sources of trouble are different but the result is the same: eigenvector changes become too large to handle for conventional normalization schemes. It is important to know the range over which quadratic normalization can control these two problems.

Quadratic normalization is very good at taming eigenvector wandering during iteration. Examples already discussed (and others not documented) have shown eigenvector perturbations that are hundreds of times the normalized values (see the unscaled vectors $[\Delta\Phi]_{\text{unscaled}}$ given earlier). These large eigenvector changes can be triggered by moderate ($\pm 10\%$) design variable changes. Figure 10 shows that the quadratic normalization can scale back the eigenvector perturbation in a typical case, whereas the simple and High normalization methods fail to control this perturbation. The unscaled eigenvector lies far out of the figure. This case is for design variable set 1, for a truncated modal set. The quadratic normalization appears to handle such large eigenvector changes without limitation.

If large design changes are required (e.g., $\pm 60\%$), however, the underlying mechanics of the small-change algorithms are lost. All of the normalization schemes studied (including quadratic) perform poorly. Consider design variable scaling set 3 where the perturbations for the cantilever beam elements are (-61, $+52$, -58, $+64$, and -54%). For such large design changes, Fox's method gives poor estimates of the eigenvector perturbations due to its first-order nature. Figure 11 shows a large difference between baseline and exact and the scaling attempts of the three candidate normalizations (simple, High, and quadratic). There is no particular advantage to the quadratic normalization.

To study the problem of large design parameter changes, one must instead use both a nonlinear perturbation scheme for the equation of motion and a nonlinear normalization. The topic is involved, and the authors have written a separate paper on it.[7] The proper normalization for large change involves a cubic relation for the scalar factor α.

Conclusions

A new mass normalization procedure has been proposed to make certain small change eigenvector derivative algorithms more robust. The normalization controls large oscillations of the eigenvector in iteration methods, and helps extend their stable range to moderate design parameter changes ($\pm 10\%$).

Solution of the quadratic normalization equation for an appropriate α_i root has been successful. Criteria for choosing between α_i roots have been developed for each of the five possible solution circumstances: positive real roots, mixed real roots, negative real roots, repeated real roots, and complex conjugate roots. With these criteria established, the procedure was successfully tested for the case of large mode changes in a benchmark cantilever beam problem. In comparison to the full quadratic normalization, High's linear normalization fails in normalizing large changes since it has dropped terms from the normalization equation. Furthermore, the simple mass scaling is inferior in preserving accuracy and orthogonality, since it does not control error. Thus, although little difference is apparent between the methods when the eigenvector perturbations are small, the full quadratic normalization is strongly recommended when the eigenvector perturbations can become large.

The relative advantage of quadratic mass normalization appears to diminish as design changes become larger, until at $\pm 60\%$ design parameter change, the various normalization methods are equally poor. Well before this occurs, however, one should reformulate the problem with a nonlinear perturbation equation of motion as well as a nonlinear normalization scheme. This topic will be considered in another paper.

Acknowledgments

This research was funded by a Ford Motor Company Grant.

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