NEAR WAKE OF THE RAREFIED PLASMA FLOWS
AT MESOTHERMAL SPEEDS

by

V. C. LIU and H. JEW
The University of Michigan
Ann Arbor, Michigan

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This paper discusses the plasma interaction between a negatively charged body and a tenuous plasma stream. The mean free path of the ions is many orders larger than the size of the body which is, in turn, large compared with the Debye length of the ambient plasma. The speed of the free stream is much larger than the ion thermal speed and yet much smaller than the electron thermal speed. The Vlasov-Poisson system of equations is solved by a self-consistent method.

The free stream particle density is many orders larger than unity. The electromagnetic and the gasdynamic aspects of the charged particles and field potential in the near wake under various ambient plasma and body environments are presented. Discussion of the validity of an often-used hydrodynamic approximation for the ion density and Maxwell-Boltzmann distribution for the electron density is given. The significance of the results pertaining to the ionospheric gasdynamics of an orbiting satellite is discussed.

I. Introduction

The launching of artificial satellites and spacecrafts has brought to bear a new challenge to the aerodynamicists. The motion of a conducting body, e.g., a satellite, in an extremely tenuous and ionized gaseous medium such as the upper ionosphere, permeated by the geomagnetic field introduces many new aspects to the gasdynamic problem. It is obvious that the Knudsen number of the flow ($K_n = \frac{R}{L}$) where $R$ denotes a characteristic length of the moving body; $L$ the mean free path of the ambient ions, is many orders larger than unity. One is thus led to consider the problem in the light of the gasdynamics of free molecules. The fact, however, that the free stream particles of interest are electrically charged adds great complexity to the problem. Consider a negatively charged metallic body moving in an ionized gas at a given temperature. The incident flux of electrons which on the average out-race the ions will deposit a net amount of negative charge to the body assuming both species of particles are singly charged. The surface charges tend to repel the electrons and attract the ions. This effect leads finally to an equilibrium state of the surface potential ($\phi_0$) such that the electron flux balances the ion flux in the charge input to the body. The time constant for a satellite to attain such an equilibrium state is extremely short. The fact that the body is charged and moves in a medium of charged particles considerably complicates the state of the "free molecules," e.g., the motion of the charged particles is now under the influence of the space charge potential as well as the surface charge potential which, in turn, depends on the motion of the charged particles themselves. It is this coupling of the field potential and the particles that gives the new complexity of the ionospheric gasdynamics. The development in a plasma of significant electric and possibly magnetic fields that exert a strong and frequently decisive influence on its motion is a fundamental feature of ionospheric gasdynamics, distinguishing it from the dynamics of neutral gases. This feature becomes manifest to the fullest degree in the motion and spreading of the free stream plasma into wake cavity right behind a rapidly moving body.

In the above discussion, no specific mention is made about the external forces, e.g., externally applied magnetic field which will exert on a moving charge an additional force, the magnetic component of the Lorentz force on a charged particle. It is also noted that the presence of neutral particles in the free stream is not considered. This stems from the fact that in the dynamics of the collisionless flows, the neutral and charged particles can be treated independently.

This generalization of gasdynamics to the plasma flows ramifies the gasdynamic variables of interest. The new quantities include, among others, the distributions of field potential and charged particle densities in the disturbed plasma near the body.

The electromagnetic consequences of the flow contribute to much of the apparent anomalous phenomena pertaining to the electromagnetic scattering from a satellite moving in the upper ionosphere, often called the ghost radar cross section phenomenon and the ionospheric plasma resonances monitored in the satellite ionograms, etc. In other words, the crude approximations in the classical electromagnetic theory which neglects the coupling between the electromagnetic and the gasdynamic aspects of the plasma flows needs refinement when the physical situation demands.

As a modest start on the investigation of the plasma interaction, the present study limits itself to the steady state problem. Consequently the interesting aspects of instabilities, wave excitations, etc., will be excluded. It is noted that the disturbed plasma field near a rapidly moving body can be conveniently divided into two sub-regional fields: the sheath near the frontal stagnation point facing the stream and the wake behind the body. They are characterized as the region of condensation and the region of rarefaction respectively. It is to the near wake that the present study is directed.

II. Physical Model

In order to gain physical insight into the interaction between the field potential and the plasma particles associated with a rapidly moving body we introduce an idealized physical model. Consider a body of size $R$ with a constant surface potential $\phi_0$ ($\phi_0 \leq 0 = $ free stream field potential)

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**It is noted that with the presence of other charging sources, e.g., electron emission due to the impact of solar radiation, the equilibrium potential may be displaced.
which moves in a rarefied plasma of singly-charged particles. It is assumed that the free stream plasma is in a bi-thermal equilibrium which has an electron temperature not necessarily equal to the ion temperature \( T_i \). Note that each species of particles is in thermal equilibrium by itself. This quasi-equilibrium state prevails in the upper ionosphere when it is exposed to the direct solar radiation. The free stream velocity \( V \) with respect to the coordinates fixed to the body is steady at a mesothermal speed which implies that \( C_i \ll V \ll C_e \) where \( C_i \) and \( C_e \) denote the ion thermal speed \( (2kT_i/m_i)^{1/2} \) and the electron thermal speed \( (2kT_e/m_e)^{1/2} \), respectively. These conditions comply with typical satellite motions in the upper ionosphere.

The accommodation of the charged particles upon collision with the body must depend on the surface condition as well as the impact energy. For our present purpose where the kinetic energy of the incident particle is not more than a few electron volts, it is reasonable to assume that an electron, which collides with body, becomes absorbed; an ion, neutralized.

In plasma dynamics, the Debye length \( d = (\epsilon N/4\pi e^2 n_0)^{1/2} \) is an important characteristic length which denotes the distance over which a static electric charge is screened by polarization of the ambient plasma with density \( n_0 \). We shall further assume that the present model satisfies the characteristic conditions: \( I \gg R \gg d \) which comply with the typical satellite motions in the upper ionosphere. As an illustration we may cite a satellite\(^2\) moving with a speed \( V = 8 \text{ km/sec} \) at an altitude of 1000 km where \( T_e = T_i = 5000^\circ \text{K} \) (\( C_i \approx 2 \text{ km/sec} \), \( C_e \approx 350 \text{ km/sec} \), \( d = 1 \text{ cm} \), \( I = 8 \text{ km} \) and ion Larmor radius \( R_l = 20 \text{ m} \). If the ion Larmor radius corresponding to the local geomagnetic field is small compared with \( R \), it is expected that the magnetic field effect on the plasma disturbance in the near wake would be negligible. The condition \( I \gg R \) decouples the gasdynamic effects of the neutral and the charged particles and makes it possible to treat the charged particles as collision-free.

In the following analysis we shall use the normalized quantities; e.g., the linear displacement \( x \) in units of \( R \); field potential \( \phi \), in \( kT_i/e \) (\( e = \text{electron charge} \); velocity \( (2kT_i/m_i)^{1/2} \); number density \( n_{\text{in}} = n_0 \), the free stream electron density.

### III. Electron Distribution

The parameter \((V/C_i)^2\) can be considered as an index for the comparison of the rates of generation and annihilation of the electron gasdynamic disturbances as the body moves. Note that the thermal speed indicates the capacity of randomization which tends to restore the particle motions to their equilibrium state for a closed system. The given conditions* \( I \gg R \) and \( C_e \gg V \) make it possible to obtain simple approximations for the electron distribution in the following cases.

#### A. The Maxwell-Boltzmann Distribution

In the statistical mechanics of a thermal equilibrium state, the distribution of the particles is governed by the Maxwell-Boltzmann law. Under the condition \( V < C_e \), the electron arrangement can be approximated by the Maxwell-Boltzmann distribution provided the detailed balancings of the particle motions can be maintained. It is noted that an electron of sufficiently high energy can reach the body against the repelling surface potential and is lost to the system of the equilibrium free stream without replenishment. Consequently, the higher the value of the repelling surface potential, the smaller the loss of the high energy particles to the equilibrium system which is governed by the Maxwell-Boltzmann distribution

\[
\rho_e = \exp(\beta T_i/T_e)
\]  

The electron loss to the body has been estimated.\(^3\)

#### B. Field-Free Approximation

When the field potential is uniformly small, it is valid to approximate the electron distribution as if they were neutral particles. The neutral particle distribution in the wake can be obtained from integration of the Maxwellian distribution over the velocity space considering the interception of the particles by the body which, for a sphere,

\[
\rho_{\text{ph}} = \frac{1}{2} \left[ 1 + \sqrt{1 - (\rho^2 \sigma^2)^{-1}} \right]
\]

and for a long cylinder,

\[
\rho_{\text{cyl}} = \frac{1}{2} \left[ 1 + \sqrt{1 - (\rho^2 \sigma^2)^{-1}} \right]
\]

when the free stream speed is negligibly small compared with the thermal speed of the neutral particles. Adopting the Boltzmann factor, we can approximate the electron distribution in the wake of a sphere,

\[
\rho_{\text{cyl}} = \rho_{\text{ph}} \exp(\beta T_i/T_e)
\]

of a long cylinder,

\[
\rho_{\text{cyl}} = \rho_{\text{cyl}} \exp(\beta T_i/T_e)
\]

provided that \( V \ll C_e \) and \( |\phi| \ll 1 \).

Although the above approximation appears reasonable for bodies with zero surface potential, it turns out from the self-consistent analysis that the field potential \( |\phi| \) can have high values in the wake even when \( C_e = 0 \) as will be shown later.

### IV. Equations Governing the Ion Distribution

In view of the condition \( V \gg C_i \), the ion distribution in the wake is expected to deviate significantly from its equilibrium state. Under the disturbance of the moving body, the ions will stream in space with variable accelerations which
depend upon the local field potential. An adequate coordinate system to represent the ion distribution must be, at least, a six-dimensional phase space \( (r, \Omega, \phi, t) \). With a rarefied plasma where the binary collisions are negligible, the ion distribution \( f(\vec{r}, \Omega, \phi, t) \) which represents the phase density in the phase space is governed by the collisionfree Boltzmann equation together with the Poisson equation for the field potential \( \phi(r) \) as follows:

\[
\frac{\partial f}{\partial \Sigma} - \frac{1}{2} (\frac{\partial \phi}{\partial \Sigma})^2 = \frac{\partial f}{\partial \tau} = 0 \quad (6)
\]

where \( n_i = \int f d\Sigma \). This system of equations, often called the Vlasov-Poisson system, when solved simultaneously with appropriate boundary conditions gives the self-consistent solution of the particle-field distributions for a given electron distribution \( n_e(\phi) \).

The inherent mathematical difficulty of treating such a system of coupled, nonlinear equations has induced the use of various ad hoc approximations to decouple the system. One of the often-used scheme is to approximate the ion distribution as if they were neutral particles hence ignoring the field effect on their motion. Equation (7) is then used to solve for the field potential \( \phi(r) \).

\[
(d/R)^2 d\phi = n_e - n_i \quad (7)
\]

The invalidity of such pseudo-neutral approximation to the near wake has been discussed and illustrated. Note that with the sheath, which is sandwiched between a mesotherm stream and a blunt body facing the stream, the field component most influential to the ion motion aligns with the free stream. Inasmuch as the field effect on the ion motion which is assumed the same order as the ion thermal speed is small compared with the free stream ion speed and is justifiably neglected. On the other hand, with the wake where the cavity-filling action is primarily attributed to the thermal velocity normal to the free stream velocity there is no justification for ignoring the field effect in comparison with the thermal velocity effect.

An approach which gives self-consistent solutions for the ion density \( n_i(r) \) and the field potential \( \phi(r) \) has been introduced and will be used here.

V. Ion Distributions

In the self-consistent approach, a formal solution to Equation (6) for the ion distribution \( f \) is obtained. It is then integrated over the velocity space with appropriate limits to determine the ion density functional \( n_i(\phi) \). Equation (7), after substitution of \( n_0(\phi) \) and \( n_i(\phi) \), becomes a nonlinear, integro-differential equation for the field potential \( \phi(r) \). The solution for \( \phi(r) \) must be constructed which satisfies the boundary conditions at the body and the free stream.

To construct the formal solution for \( f \) from the collisionfree Boltzmann equation or Vlasov equation (6), we suppose to have an ion system which is in steady state. As the ions in the system move, the corresponding phase points will describe phase paths in the six-dimensional phase space \( (r, \Omega, \phi, t) \). Equation (6) represents the fact that the phase density is constant along the phase paths.* Within a specific \( f = F(r, \Omega) \) solving Equation (6) we can consider the field of directions defined by the tangent vectors \( (\partial F/\partial \phi) \). This field of directions is composed of the tangents of a one-parameter family of phase paths in that surface, called characteristic, which are determined by the system of ordinary differential equations

\[
-\frac{d\phi}{d\tau} = \frac{1}{2} (\frac{\partial \phi}{\partial \Sigma})^2 \quad (8)
\]

Any phase path solving the system of Equation (8) is called the a characteristic of the first order quasi-linear partial differential equation (6). It is indeed obvious that Equation (8) is equivalent to the equations of motion of an ion. The integrals of Equation (8) will give us the integrals of motion of an ion. We shall proceed to evaluate these integrals for the following potential fields of specific symmetry:

1. Axisymmetric field which is appropriate for the wake behind a sphere or a circular disk placed normal to the free stream. In this case, Equation (8) in vectorial form is expanded in terms of the components of potential \( \phi \) and \( \theta \) (see the insert in Fig. 1). Two of the integrals of motion which are easily obtained are:

\[
c^2 + c_\theta^2 + c_\phi^2 + \phi = E \quad (9)
\]

where \( E \) denotes the energy constant of an ion; (b) the conservation of angular momentum with respect to the axis of symmetry \( (\Omega) \),

\[
\rho c_\phi = I_\phi \quad (10)
\]

where \( I_\phi \) denotes the angular momentum of an ion. A third integral is needed to determine the ion orbit with given initial conditions for the axisymmetric problem in hand. Equation (8) does not, however, yield a third integral with a universal constant of motion as in (9) or (10). To search for a quasi-integral or locally constant of motion, we simplify the integration by carrying it out for small phase intervals during which the local values of \( \phi \) and \( \theta \) are assumed quasi-constant, thus obtaining

\[
\rho - \rho_0 = c_\phi c_\phi^2 + (2c_\theta c_\phi^2)^2 f'(\partial \phi/\partial \Sigma) \Delta z
\]

\[
- (2c_\theta c_\phi^2)^2 f_{\rho}^2 \Delta z = I_\phi \quad (11)
\]

Note that \( I_\phi \) is only locally constant.

2. Two-dimensional symmetric field (see insert in Fig. 2) which is appropriate for the wake behind a long cylinder with axis along \( x \)-direction and the free stream along \( z \)-direction; or behind a long strip placed in the \( x, y \)-plane. Here the
equation (8) is expanded in terms of the component phase coordinates \( c_y, c_z, y, z \) (see Fig. 2). Again the relation of energy conservation is easily obtained

\[
c^2_y + c^2_z + \phi = E. \tag{12}
\]

Inasmuch as the conservation of angular momentum does not lead to a nontrivial relation, we must develop a second integral to determine the ion orbit with given initial conditions for the problem in question. Again, Equation (8) does not yield another integral with a universal constant of motion. A search for a quasi-integral of motion with similar procedure as in the axisymmetric problem, i.e., by assuming \( c_y \) and \( c_z \) quasi-constants, we obtain

\[
y = c_ya/c_z + \left(2c_y/c_z\right)^{-1} \int \frac{\partial \phi}{\partial y} \, dy = I_3 \tag{13}
\]

where \( I_3 \) is locally constant.

It is noted that with the assumption \( V \gg c_1 \), we expect that \( c^2_y, c^2_z, c^2_\phi \ll c^2_\phi \) on the average.

The general solution of the partial differential equation (6) is an arbitrary function of the integrals of the system of ordinary differential equation (8). We can thus construct the formal solution \( f(x, x) \) as a function of the constants of motion (9), (10), and (11) for the wake with an axisymmetric field potential namely \( f(x, x, I_3) \) behind a sphere; (12) and (13), with a two-dimensional symmetric field namely \( f(x, x, I_3) \), behind a long cylinder. Such an arbitrary function \( f(x, x, I_3) \) or \( f(x, x, I_4) \) would be of no interest to us unless it satisfies the boundary conditions prescribed according to the physical conditions of the problem in question. We shall determine the specific functional forms of the ion distribution \( f \) for the following physical problems:

A. Wake Behind a Sphere

Two physical requirements of the wake which must be satisfied are the undisturbed plasma state at infinity and the ion-neutralizing sphere surface. Relative to a coordinate system fixed to the sphere, the undisturbed plasma state at infinity implies that the ion distribution function at infinity is a Maxwellian displaced by the free stream velocity \( V \), namely

\[
f_{\infty} = \frac{1}{\pi^{3/2}} \exp\left[-\frac{(c_x-V)^2}{2}\right] \tag{14}
\]

The ion distribution at the plane \( z = 0 \) outside of the sphere of radius \( R \), which is large compared with the thickness of the local sheath, is also represented by (14). On the body surface,

\[
f(x, x)\big|_{x=1, c_y > 0} = 0 \tag{15}
\]

where \( c_n \) denotes the ion velocity normal to the wall.

A solution to Equation (6) which satisfies the boundary conditions (14) and (15) is the following

\[
f = \frac{1}{2} (1 + \text{sign } f) f_{\infty} \tag{16}
\]

where \( \text{sign } f = 1 \) for \( f > 0 \) and \( \text{sign } f = -1 \) for \( f < 0 \). From the condition (15) and the assumption \( |\phi| \ll V^2 \), it can be shown that

\[
F = |p - c^2_p/c_2| - 1 = I^2_3 + \frac{(z^2/p^2)(I^2_3/E)} - 1 \tag{17}
\]

Eq(16) degenerates to the displaced Maxwellian at the free stream where \( \phi = 0 \). Note with the condition \( |\phi| \ll V^2 \) where \( |\phi| \) is of the order as the ion thermal energy, the \( f \) given in (16) is a function of the three integrals of motion (9), (10), and (11).

B. Wake of a Long Cylinder Placed Transversely to a Plasma Stream

Following a procedure and approximations similar to those for the wake of a sphere, we obtain the ion distribution in the wake behind a long cylinder,

\[
f = \frac{1}{2} (1 + \text{sign } G) f_{\infty} \tag{18}
\]

where

\[
G = |y - c_x/c_z| - 1 = I^2_5 - 1 \tag{19}
\]

and

\[
f_{\infty} = \frac{1}{\pi^{1/2}} \exp\left[-\frac{(c_x-V)^2}{2}\right] \tag{20}
\]

VI. Self-Consistent Solutions

A. Equations of Field Potentials

The equation governing the field potential in a wake is given by the Poisson equation (7) in which the ion density \( n_i(r) = f(x, x, I_3) \).

1. Wake Behind a Sphere

The number density of ions in the wake behind a sphere is obtained by integrating the distribution function (16) over velocity space, where the velocity range is delineated by the condition \( F = 0 \) or

\[
I^2_3 + \frac{(z^2/p^2)(I^2_3/E)} - 1 = 0 \tag{21}
\]

Equation (21), after retaining only linear terms in \( \phi \) in the integrals \( \int f(x, x, I_3) \, dz \) and \( \int f_{\infty} \, dz \) which appear in \( I_3 \), is solved for \( c_\phi \). In performing the integration for \( n_i(r) \), it is convenient to consider the cases \( p < 1 \) and \( p > 1 \) separately. For \( p < 1 \)...
The substitution of the ion density functional in the Poisson equation will result in a partial integro-differential equation in the field \( \phi \) which must be iterated numerically to obtain a solution.

### B. Iteration for Self-Consistent Solution

The self-consistent solution to the field potential and ion density is obtained by a specially devised numerical process of successive iteration (see Appendix). Briefly, the boundary value problem of the field potential \( \phi \) for the semi-infinite wake region is mapped into a semi-circular region by coordinate inversion. A special numerical analysis which combines the techniques of over-relaxation and alternating-direction-implicit iteration is used to iterate the field potential equation successively until a self-consistent state is reached.

### VII. Results and Discussions

The computed results of ion density, electron density and field potential distributions from the self-consistent solutions reveal some insights of the field-particle interactions in the near wake. We shall itemize them and discuss their significance.

#### A. The Potential Valleys in the Wake

When a large conducting body \( (R \gg d) \) with a negative surface potential moves at a mesothermal speed in a tenuous plasma \( (R \ll 1) \), it is expected that both ions and electrons from the free stream, will rush into the wake-cavity created by the rapidly moving body. The electrons, on the average, move faster; consequently the electron and ion components of the plasma always tend to separate.

As a result, the inhomogeneous front moves and spreads with essentially a common mean speed, immediate between those of the ions and the electrons; the internal electric field exerts in this case a strong influence on the particle motion as a whole in the near wake. The potential valleys in the wake of a sphere (see Fig. 1) and of a long cylinder (see Fig. 2) are associated with the above-mentioned fronts of electron-rich mixture. The potential minimum forms a conically shaped surface based on the spherical moving body in question; a double wedge, on a two-dimensional cylindrical moving body. It is seen from the figures that the depth of the potential valley decreases as the body size \((R/d)\) decreases.

The presence of such potential behind a satellite orbiting in the upper ionosphere suggests the possible existence of trapped ions in the valley. The specific modes of oscillations of the trapped ions and electrons, plus the electron cyclotron modes, if the geomagnetic field is taken into account, appear to have shed light on the anomalous oscillations observed in the Ariel I and Alouette I and II satellites.
E. Comparison of the Approximations for Electron Distributions

It is recalled from the discussions in Section III that either the approximation (1) or (2) for the electron distribution is valid under restricting conditions. The results of self-consistent solutions with electron distributions prescribed by the two different approximations are compared (see Fig. 3) and show nontrivial discrepancies. It is worth noting that in view of the presence of an intense potential minimum, when $R \gg d$, even with vanishing surface potential, it appears more justifiable to use the Maxwell-Boltzmann approximation (1).

C. Gurevich's Approximation for Ion Density

In view of the tedious effort in obtaining a self-consistent solution of a plasma wake, it would be desirable to have a simpler approximation with reasonable accuracy. An idea of approximating the field effect on ion motion, first proposed by Gurevich,\(^2\) is worthy of further pursuit. Comparing the free expansions of the charged and the neutral particles into the wake-cavity, the difference stems primarily from the lack of field action on the neutral particles. To estimate the influence of the field potential on the ions, one notes that the field potential which is partly caused by the electrons tends to increase the ion pressure. A total pressure of $kT_i$ can be used with advantage to estimate the ion density in the far wake using free molecular flow theory of the charged and the neutral gas. It is of interest to compare the results from such an approximation and self-consistent solutions of a neutral gas. It is of interest to compare the results from such an approximation and self-consistent solutions of a neutral gas.

References


Appendix. Self-Consistent Solution by Iteration

The two basic steps include coordinate inversion in the unit circle and the iterative solution of Poisson's equation.

A. Coordinate Inversion

We transform Poisson's equation (7) from the physical plane into the image plane by inversion defined by Equations (A.1), (A.2), or (A.3), and (A.4), thereby converting the semi-infinite wake domain into a semi-circle image domain. For the sphere wake, coordinates transform as (primed quantities denote physical coordinates)

$$ p = \frac{p'}{p'+z'^2} \quad \text{and} \quad z = \frac{z'}{z'^2+p'^2} \quad (A.1) $$

and potential transforms as

$$ \psi(p,z) = (p^2+z^2)^{1/2} \psi(p',z') \quad (A.2) $$

where $\psi$ is the image potential and $\psi$ is the electric field potential with coordinates related by Equation (A.1). For the cylinder wake, coordinates transform as

$$ y = \frac{y'}{y'^2+z'^2} \quad \text{and} \quad z = \frac{z'}{z'^2+y'^2} \quad (A.3) $$

and potential transforms as

$$ \psi(y,z) = \psi(y',z') \quad (A.4) $$

where $\psi$ is the image potential and $\psi$ is the electric field potential with coordinates related by Equation (A.3).

B. Iterative Solution of Poisson's Equation

The nonlinear potential equations solved iteratively are, for the sphere

$$ (d/R)^2 (p^2+z^2)^{3/2} \psi = e^{-p^2+z^2} \psi_n(\eta) \quad (A.5) $$

where $n_1(\eta)$ comes from Equations (29) and (23); and, for the cylinder

$$ (d/R)^2 (y^2+z^2)^{3/2} \psi = e^{-y^2+z^2} \psi_n(\eta) \quad (A.6) $$

where $n_2(\eta)$ comes from Equations (24) and (25).

The calculation was carried out numerically in finite differences as a discretized Dirichlet boundary value problem for the domain bounded by $p^2+z^2 = 1$, $0 \leq p \leq 1$, with boundary conditions $\psi = \psi_b = \text{constant}$ on $p^2+z^2 = 1$, $\psi = 0$ along $0 \leq p \leq 1$, and $\partial \psi / \partial n = 0$ along $0 < p < 1$, for the sphere; and the domain bounded by $y^2+z^2 = 1$, $0 \leq x \leq 1$, and $0 < y < 1$, with boundary conditions $\psi = \psi_b = \text{constant}$ on $y^2+z^2 = 1$, $\psi = 0$
along \( y < 1 \), and \( \partial \phi / \partial y = 0 \) along \( 0 < y < 1 \), for the cylinder. In each case, sphere or cylinder, to begin the iteration, an initial input \( \psi(0) \) from solving the Dirichlet problem for Laplace's equation \( \Delta \phi = 0 \), was computed. Using these \( \psi(0) \) values, initial electron and ion densities were calculated. After this, iteration was carried out on Poisson's equation by a combined successive-overrelaxation and alternating-direction iterative technique. The solution \( \phi \) for the electric potential was computed. Using these initial electron and ion densities were calculated. After this, iteration was carried out on Poisson's equation by a combined successive-overrelaxation and alternating-direction iterative technique. The solution \( \phi \) for the electric potential was computed. Using these initial electron and ion densities were calculated.

We shall now briefly describe the mathematics of the solution method itself. Let us record the five-point finite difference equations for uniform mesh \( h \). For sphere wake, we have

\[
(\alpha R)^2(z^2+y^2)^{3/2} - (2\psi_0 - 2p^{-1} \psi_2 - \frac{2p-1}{2p} \psi_4) = 2^{1/2} [n_1(\psi_0) - e^{1/2} (2p^2 \psi_0)]
\]

For cylinder wake, we have

\[
(\alpha R)^2(z^2+y^2)^{3/2} - (2\psi_0 - \psi_2 - \psi_4) = 2^{1/2} [n_1(\psi_0) - e^{1/2} (2p^2 \psi_0)]
\]

In Equations (A.7) and (A.8) subscripts denote neighboring points.

In discretizing the Dirichlet boundary value problem for Poisson's equation, we replace the partial differential equation by a finite difference equation and we replace the region of interest by a set of mesh points in the region. In other words, we solve a finite set of simultaneous difference equations instead of the partial differential equation.

The totality of difference equations describing the discrete problem can be written as a vector equation

\[
\begin{pmatrix}
\dot{\psi}_1 \\
\vdots \\
\dot{\psi}_N
\end{pmatrix} = \mathbf{A} \begin{pmatrix}
\psi_1 \\
\vdots \\
\psi_N
\end{pmatrix}
\]

In Equation (A.9) \( \mathbf{A} \) is a square matrix whose entries are coefficients in the finite difference equation, the column vector \( \dot{\psi} \) denotes the unknown vector whose components consist of values of \( \psi \) on the set of mesh points, and \( \psi \) denotes the column vector whose components are boundary values of \( \psi \) and net charge densities computed from a previous iterate.

An effective iterative method of solving the wake potential problem in the above form is the following. We split the nonsingular matrix \( \mathbf{A} \) into a sum of nonsingular matrices

\[
\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{F}
\]

This can always be done. In Equation (A.10), \( \mathbf{D} \) denotes a diagonal matrix, \( \mathbf{E} \) is a lower triangular matrix, and \( \mathbf{F} \) is an upper triangular matrix. Let

\[
L = D^{-1} E \quad \text{and} \quad U = D^{-1} F.
\]

Then it can be shown that successive overrelaxation iteration is defined by

\[
L (m+1) = (I - \omega L) - \omega (I - \omega U) \mathbf{Y}^m \quad \text{for} \quad m \geq 0
\]

where, for the sphere from Equation (A.7)

\[
(\alpha R)^2(z^2+y^2)^{3/2} - (2\psi_0 - 2p^{-1} \psi_2 - \psi_4)
\]

and

\[
V \psi_0 = (\alpha R)^2(z^2+y^2)^{3/2} - 2p^{-1} \psi_2 - \psi_4
\]

and, for the cylinder from Equation (A.8)

\[
(\alpha R)^2(z^2+y^2)^{3/2} - (2\psi_0 - \psi_2 - \psi_4)
\]

and

\[
V \psi_0 = (\alpha R)^2(z^2+y^2)^{3/2} - \psi_2 - \psi_4
\]

Then it can be shown that an alternating-direction iteration is defined by

\[
\psi^{(m+1)} = P \psi^{(m)} + g(\gamma), \quad m \geq 0
\]

where matrix \( P \) and vector \( g(\gamma) \) have the definitions

\[
P = (V + \alpha I)^{-1}(\alpha I - H)(V + \alpha I)^{-1}(\alpha I - \gamma)
\]

and

\[
g(\gamma) = (V + \alpha I)^{-1}(\alpha I - H)(V + \alpha I)^{-1} I \gamma
\]

The nonsingular matrix \( P \) is called the Peaceman-Rachford matrix, where \( \alpha \) denotes an acceleration parameter.

The coupling of successive overrelaxation to alternating direction was done through introducing the residuals from overrelaxation into the vector \( g(\gamma) \) of alternating direction. This coupled procedure has been proposed as a method of attack on the Dirichlet boundary value problem in rectangular regions for mildly nonlinear elliptic difference equations. In each calculation in this
paper, the region was the interior of a unit quarter circle and the equation was Poisson's equation governing the wake electric field potential in the image space. The uniform mesh constant was \( h = 0.04 \).

\[
F(\frac{1}{2}) = 1, T = \frac{1}{5} = 1, V/C = 8, \phi = -1 (\text{NORM})
\]

\[
r = (x^2 + y^2)^{\frac{1}{2}}
\]

\[
R/d = 1
\]

\[
\frac{n_e}{n_0}
\]

\[
\phi
\]

\[
1A \text{ FIELD POTENTIALS}
\]

\[
\frac{n_i}{n_0}
\]

\[
1B \text{ ION DENSITIES}
\]

\[
\text{IC ELECTRON DENSITIES}
\]

\[
R/d = 1
\]

\[
\frac{n_e}{n_0}
\]

\[
\phi
\]

\[
2A \text{ FIELD POTENTIALS}
\]

\[
\text{FIGURE 1. WAKE OF A SPHERE}
\]

\[
\text{FIGURE 2. WAKE OF A CYLINDER}
\]

\[
\text{[TAN}^{-1}\frac{y}{x}) = 0, T_1/T_0 = 1, V/C = 8, \phi = -1 (\text{NORM}]
\]

\[
(1 + y^2)^{\frac{1}{2}}
\]
2B. ION DENSITIES

2C ELECTRON DENSITIES