Analytical Solution for Horizontal Gliding Flight

by

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This paper presents the analytical solutions for atmospheric flight in the horizontal plane. The flight conditions considered include maximum range flight, maximum endurance flight, chattering flight, and constant lift turn flight. They enable us to have a wide understanding of the characteristics of flight in the horizontal plane.

I. Introduction

In general, there are five state variables and three control variables for flight in the horizontal plane. The five state variables are two components of the position vector, two components of the velocity vector, and mass. The three control variables are two components of the thrust vector and the combination of flight and bank angle in order to keep the flight at constant altitude. For gliding flight, the thrust is zero and the problem is reduced to four state variables and one control variable. This problem had been studied extensively in two eminent books 1,2 and many published papers. 3–11 The purpose of this paper is twofold: first of all, to summarize the analytical solutions obtained before, and secondly to elaborate possible extensions.

II. Equations of Motion

For gliding flight in the horizontal plane, the equations of motion are 1,2

\[
\frac{dX}{dt} = V \cos \psi
\]

(1a)

\[
\frac{dY}{dt} = V \sin \psi
\]

(1b)

\[
\frac{dV}{dt} = -\frac{\rho SV^2 C_d}{2m}
\]

(1c)

\[
\frac{d\psi}{dt} = \frac{\rho SV C_l}{2m} \sin \sigma
\]

(1d)

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where $X$ is down range, $Y$ is lateral range, $V$ is speed, $\psi$ is heading angle, $\rho$ is atmospheric density, $S$ is reference area of vehicle, $C_D$ is drag coefficient, $C_L$ is lift coefficient, $m$ is mass of vehicle, $\sigma$ is bank angle, and $t$ is the time. In order to keep the flight in the horizontal plane, a constraining relation between $C_L$ and $\sigma$ is

$$\frac{1}{2} \rho V^2 SC_L \cos \sigma = mg = W \quad (2)$$

where $W$ is weight of vehicle. The drag coefficient is modeled by the parabolic polar

$$C_D = C_{D_0} + KC_L^2 \quad (3)$$

where $C_{D_0}$ is zero lift drag coefficient and $K$ is induced drag factor. By introducing the dimensionless variables

$$x = \frac{gX}{V_0^2}, \quad y = \frac{gY}{V_0^2}, \quad u = \frac{V}{V_0}, \quad \theta = \frac{gt}{V_0}, \quad \omega = \frac{2W}{\rho SV_0^2 C_L'} \quad (4)$$

where the subscript $0$ denotes initial condition, $g$ is gravitational acceleration, and $C_L'$ is lift coefficient for maximum lift-to-drag ratio, we obtain the dimensionless equations of motion

$$x' = u \cos \psi \quad (5a)$$
$$y' = u \sin \psi \quad (5b)$$
$$u' = -\frac{u^2}{2E'}\left(1 + \frac{\omega^2}{u^4 \cos^2 \sigma}\right) \quad (5c)$$
$$\psi' = \frac{\tan \sigma}{u} \quad (5d)$$

where the prime denotes the derivative taken with respect to the dimensionless time $\theta$, and $E'$ is maximum lift-to-drag ratio. The constraining relation in Eq. (2) becomes

$$\cos \sigma = \frac{\omega}{\lambda u^2} \quad (6)$$

where $\lambda = C_L / C_L'$. In Eqs. (5), the four state variables are $x$, $y$, $u$ and $\psi$, and $\sigma$ is the sole control variable. The required lift coefficient can be calculated from Eq. (6) and $C_L'$. In this paper we shall assume that both $C_{D_0}$ and $K$ are constant and therefore $C_L' = (C_{D_0} / K)^{1/2}$ is also constant. The load factor is defined to be

$$n = \frac{L}{W} = \frac{1}{\cos \sigma} \quad (7)$$

It is clear that the dimensionless constraining relation described by Eq. (6) has the limitations
\[-1 \leq \cos \sigma = \frac{\omega}{\lambda u^2} \leq 1\]  

(8)

Therefore, if the maximum and minimum values of \( \lambda \) are assumed to be

\[
\lambda_{\text{max}} = 2 \quad \text{(9a)}
\]

\[
\lambda_{\text{min}} = -2 \quad \text{(9b)}
\]

we have the maximum maneuverable domain as shown in Fig. 1.

III. Variational Formulation

For variational formulation, the Hamiltonian can be formed as\(^2\),

\[H = p_x u \cos \psi + p_y u \sin \psi - p_x \left( \frac{u^2}{2E_0} \frac{1}{\omega^2} \right) \left[ 1 + \frac{\omega^2}{u^4 \cos^2 \sigma} \right] + p_\psi \frac{\tan \sigma}{u}\]  

(10)

where \( p_x, \ p_y, \ p_\psi, \) and \( p_\psi \) are adjoint variables corresponding to \( x, \ y, \ u, \) and \( \psi, \) respectively. The variational problem has the integrals\(^2\)

\[
H = C_0
\]

(11a)

\[
p_x = C_1
\]

(11b)

\[
p_y = C_2
\]

(11c)

\[
p_\psi = -C_2 x + C_1 y + C_3
\]

(11d)

where \( C_0, C_1, C_2, \) and \( C_3 \) are constants of integration. When the optimal bank control is interior, we have \( \partial H / \partial \sigma = 0 \) and

\[
\tan \sigma = \frac{(p_\psi)(E^* u)}{(p_u / \omega)}
\]

(12)

The first integral \( H = C_0 \) comes from the fact that \( \partial H / \partial \theta = 0 \) since \( H \) is not an explicit function of the independent variable \( \theta \). In the cases the final time \( \theta_f \) is free, in other words, it is neither specified nor being extremized, we have \( H_f = C_0 = 0 \). Also, we shall have \( C_1 = 0 \) if \( x_f \) is free, and \( C_2 = 0 \) if \( y_f \) is free. It happens in many cases the final heading \( \psi_f \) is free, and the transversality condition may result that \( p_\psi = -C_2 x_f + C_1 y_f + C_3 = 0 \). Actually, one of the integrals can be eliminated as long as it is not zero. For example when \( C_0 \neq 0 \), by letting \( k_1 = C_1 / C_0, \ k_2 = C_2 / C_0, \) and \( k_3 = C_3 / C_0 \), the following binomial equation for the bank angle can be derived

\[
(-k_2 x + k_1 y + k_3) \tan^2 \sigma + 2u[1 + u(k_1 \cos \psi + k_2 \sin \psi)] \tan \sigma - \frac{(-k_2 x + k_1 y + k_3)}{\omega^2} (\omega^2 + u^4) = 0
\]

(13)
IV. Rectilinear Flight

A. Maximum Range and Maximum Endurance Glide

For rectilinear motion in the horizontal plane, we have $\sigma = 0$, $\psi = 0$ and the equations of motion are simply:

\[ x' = u \]  \hspace{1cm} (14a)
\[ u' = -\frac{u^2}{2E^*\omega}(1 + \omega^2) \]  \hspace{1cm} (14b)

The two equations (14a) and (14b) can be combined to give:

\[ \frac{dx}{du} = \frac{-2E^*\omega u^3}{u^4 + \omega^2} \]  \hspace{1cm} (15)

If we specify the final speed $u_f$ to be the stall speed which occurs at $\lambda_{\text{max}}$, then from Eq. (6) we have:

\[ u_f = \sqrt{\frac{\omega}{\lambda_{\text{max}}}} \]

The integration from $x_0 = 0$ and $u_0 = 1$ to $x_f = x_{\text{max}}$ and $u_f = \sqrt{\omega/\lambda_{\text{max}}}$ gives the maximum range for rectilinear flight:

\[ x_{\text{max}} = \frac{1}{2} E^* \omega \log\left[ \frac{(1 + \omega^2)\lambda_{\text{max}}^2}{\omega^2(1 + \lambda_{\text{max}}^2)} \right] \]  \hspace{1cm} (16)

The inverse of Eq. (14b) is:

\[ \frac{d\theta}{du} = -2E^*\omega(\frac{u^2}{u^4 + \omega^2}) \]  \hspace{1cm} (17)

Its integration from $\theta_0 = 0$ and $u_0 = 1$ to $\theta_f = \theta_{\text{max}}$ and $u_f = \sqrt{\omega/\lambda_{\text{max}}}$ gives the maximum endurance for rectilinear flight:

\[ \theta_{\text{max}} = \frac{E^*\sqrt{\omega}}{2\sqrt{2}} \left[ \log\left( \frac{1 + \sqrt{2}\lambda_{\text{max}} + \lambda_{\text{max}}}{1 - \sqrt{2}\lambda_{\text{max}} + \lambda_{\text{max}}} \right) + 2 \tan^{-1}\left( \frac{\sqrt{2}\lambda_{\text{max}}}{1 - \lambda_{\text{max}}} \right) - \log\left( \frac{1 + \sqrt{2}\omega + \omega}{1 - \sqrt{2}\omega + \omega} \right) - 2 \tan^{-1}\left( \frac{\sqrt{2}\omega}{1 - \omega} \right) \right] \]  \hspace{1cm} (18)

For numerical computation, we shall assume the following data for the aircraft model:

\[ C_{D_0} = 0.0125, \ K = 0.05, \ C_D^* = 2C_{D_0} = 0.025, \ C_L^* = \sqrt{\frac{C_{D_0}}{K}} = 0.5, \ E^* = \frac{C_L^*}{C_D^*} = 20, \]  \hspace{1cm} (19)

\[ C_{\max} = 1, \ \lambda_{\text{max}} = 2 \]
The maximum range and maximum endurance as functions of dimensional altitude are plotted in Fig. 2. It is seen from Fig. 2 that when the altitude is too high, the rectilinear flight in the horizontal plane does not exist. There is an altitude called ceiling where both $x_{\text{max}}$ and $\theta_{\text{max}}$ are zero. From Eqs. (16) and (18), it is easy to find the ceiling altitude is

$$\omega_{\text{ceiling}} = \lambda_{\text{max}} = 2$$  \hspace{1cm} (20)

There is a global optimal altitude for global maximum range. At that point we have $\frac{\partial x_{\text{max}}}{\partial \omega} = 0$ and it can be derived, so that the $\omega_{\text{maximum}}$ satisfies the relation

$$\frac{2}{1 + \omega^2} = \log\left[\frac{(1 + \omega^2)\lambda_{\text{max}}^2}{\omega^2(1 + \lambda_{\text{max}}^2)}\right]$$  \hspace{1cm} (21)

Eq. (21) provides the solution $\omega_{\text{maximum}} x = 0.411$. On the other hand, there is a $\omega_{\text{maximum}} \theta$ where the endurance is globally optimal. It satisfies $\frac{\partial \theta_{\text{max}}}{\partial \omega} = 0$ and we have the relation

$$\log\left[\frac{1 + \sqrt{2\lambda_{\text{max}}^2 + \lambda_{\text{max}}^2}}{1 - \sqrt{2\lambda_{\text{max}}^2 + \lambda_{\text{max}}^2}}\right] + 2\tan^{-1}\left(\frac{\sqrt{2\lambda_{\text{max}}^2 + \lambda_{\text{max}}^2}}{1 - \lambda_{\text{max}}}ight) - \log\left[\frac{1 + 2\omega + \omega}{1 - \sqrt{2\omega + \omega}}\right] + 2\tan^{-1}\left(\frac{\sqrt{2\omega}}{1 - \omega}\right) - 2\tan^{-1}\left(\frac{\sqrt{2\omega}}{1 + \omega^2}\right) = 0$$  \hspace{1cm} (22)

With some elaborative calculation, we obtain $\omega_{\text{maximum}} \theta = 0.27465$ from Eq. (22).

**B. Chattering**

The relation shown in Eq. (18) is unique. In other words, it takes the time $\theta_{\text{max}}$ to decelerate the aircraft from $u_0 = 1$ to $u_f = \sqrt{\omega / \lambda_{\text{max}}}$. Then, what is the $\theta_{\text{min}}$ value for the same speed reduction? To investigate this problem, we have to go back to Eq. (7). By letting $\sigma = 0$ in Eq. (6), we have

$$\omega = \lambda u^2$$  \hspace{1cm} (23)

Using the relation in Eq. (17) gives

$$\frac{d\theta}{du} = -2E^r\left(\frac{\lambda}{1 + \lambda^2}\right)$$  \hspace{1cm} (24)

It is apparent that the flight time will be smaller if the right hand side of Eq. (24) is less negative. Therefore, when the maximum lift is used, the aircraft will have maximum drag and need minimum time for aerobraking maneuver. However, the aircraft has to bank between $+\sigma_c$ and $-\sigma_c$ (where the subscript c denotes chattering) rapidly to keep the flight rectilinear. The value of $\sigma_c$ can be calculated from the relation
\[ \cos \sigma_c = -\omega / \lambda_{\text{max}} u^2 \]  

(25)

With \( \lambda = \lambda_{\text{max}} \) in Eq. (24), the minimum time required in simply

\[ \theta_{\text{min}} = 2E' \lambda_{\text{max}} \left( u_0 - u_f \right) \]  

(26)

The range of chattering are \( x_c \) can be obtained by integrating Eq. (15) with \( \omega = \lambda_{\text{max}} u^2 \). It results that

\[ x_c = \frac{E' \lambda_{\text{max}}}{1 + \lambda_{\text{max}}^2} (u_0^2 - u_f^2) \]  

(27)

Theoretically, the chattering between +\( \sigma_c \) and -\( \sigma_c \) must be at the rate of infinity. However, it has been found that if the number of control switching is greater than 5, the flight path will be a quasi-straight line and the penalty on the flight time is only several tenth percent.

In Fig. 3, the maximum range \( x_{\text{max}} \) and the chattering range \( x_c \) are plotted. Also shown are the flight times. When the final range is not specified, the solutions obtained are unique. It means that at a given altitude \( \omega \), \( \theta_{\text{max}} \) corresponds to \( x_{\text{max}} \) and \( \theta_{\text{min}} \) corresponds to \( x_c \). In the case where the final range \( x_f \) is specified, further consideration is required. When \( 0 \leq x_f < x_c \), the optimal trajectory for minimum time flight is a two-dimensional turning in the horizontal plane. When \( x_c < x_f < x_{\text{max}} \), the optimal trajectory is simply a combination of chattering and gliding arcs. Let \( x_1 \) be the point where the two arcs join together. For minimum time flight, the aircraft glides from \( x_0 \) to \( x_1 \) in shortest time. Then from \( x_1 \) to \( x_f \) it chatters to reduce speed to \( u_f \) in shortest time. The time for glide from \( x_0 = 0 \) and \( u_0 = 1 \) to \( x_1 \) and \( u_1 \), denoted by \( \theta_1 \), is

\[ \theta_1 = \frac{E' \sqrt{\omega}}{2\sqrt{2}} \left[ \log \left( \frac{u_i^2 + \sqrt{2} \omega u_i + \omega}{u_i^2 - \sqrt{2} \omega u_i + \omega} \right) + 2 \tan^{-1} \left( \frac{\sqrt{2} \omega u_i}{u_i^2 - \omega} \right) - \log \left( \frac{1 + \sqrt{2} \omega + \omega}{1 - \sqrt{2} \omega + \omega} \right) - 2 \tan^{-1} \left( \frac{\sqrt{2} \omega}{1 - \omega} \right) \right] \]

(28)

The time for chattering from \( x_1 \) and \( u_1 \) to \( x_f \) and \( u_f = \sqrt{\omega / \lambda_{\text{max}} \omega} \), denoted by \( \theta_2 \), is

\[ \theta_2 = \frac{2E' \lambda_{\text{max}}}{1 + \lambda_{\text{max}}^2} (u_1 - u_f) \]  

(29)

The total time is equal to the sum of \( \theta_1 \) and \( \theta_2 \), and is the minimum time for flight. We can call it a kind of interior-point boundary value problem (IPBVP). The total range of
flight is the specified $x_f$. It is equal to the sum of the gliding part and the chattering part of
the flight. The integration of Eq. (15) form $u_0$ to $u_1$ and then from $u_1$ to $u_f$ gives

$$
x_f = \frac{1}{2} E^* \omega \log\left(\frac{1+\omega^2}{u_1^2 + \omega^2}\right) + \frac{E^* \lambda_{\text{max}}}{1+\lambda_{\text{max}}^2} (u_1^2 - u_0^2)\tag{30}
$$

The only unknown in Eq. (30) is $u_1$. Therefore, this IPBVP has complete analytic solution.

As a numerical example, let $(\omega, x_f) = (1.0, 4.25)$. With the numerical values of
$E^* = 20$, $\lambda_{\text{max}} = 2$, $\omega = 1$, $u_f = 0.7071$, and $x_f = 4.25$ insert in Eq. (30), it becomes

$$
\log\left(\frac{2}{1+u_1^2}\right) + 0.8u_1^2 - 0.825 = 0\tag{31}
$$

The solution of Eq. (31) is $u_1 = 0.9344$. The minimum flight time can be calculated from
Eqs. (28) and (29) and is $\theta_f = \theta_1 + \theta_2 = 1.3081 + 3.6368 = 4.9449$, which is 97% smaller
than the pure glide flight time of 5.4729 calculated from Eq. (18). The first term of the
right-hand side of Eq. (30) is $x_1$ and is calculated to be $x_1 = 1.2652$.

V. Constant Lift Turn

By using the relation of Eq. (6) in Eqs. (5c) and (5d), we have

$$
\frac{du}{d\theta} = -\frac{u^2}{2E^* \omega} (1 + \lambda^2)\tag{32}
$$

$$
\frac{d\psi}{d\theta} = \frac{\lambda^2 u^2}{\omega^2} - \frac{1}{u^2} \chi^{1/2}\tag{33}
$$

where $\lambda$ is the control variable. It requires from Eq. (33) that $\omega^2 \leq \lambda^2 u^4$ for the turn to
be possible, as shown in Fig. 4 for positive $\lambda$. For constant lift turn, Eq. (32) can be
integrated to give

$$
\theta_f = \frac{2E^* \omega}{1+\lambda^2} \left(\frac{1}{u_f} - 1\right)\tag{34}
$$

where $\theta_0 = 0$ and $u_0 = 1$ have been used. The combination of Eq. (32) and (33) gives the relation

$$
\frac{d\psi}{du} = -\frac{2E^* \sqrt{\lambda^2 u^4 - \omega^2}}{1+\lambda^2} u^3
$$

or

$$
d\psi = -\frac{E^* \lambda}{\gamma^2} \frac{\sqrt{u^2 - (\omega/\lambda)^2}}{u^2} du\tag{35}
$$
where \( \bar{u} = u^2 \). We can integrate Eq. (35) from the initial condition \( u_0 = 1 \) and \( \psi_0 = 0 \) to obtain

\[
\psi = \frac{E^*}{1 + \lambda^2} \left[ \sqrt{\lambda^2 - \omega^2} + \lambda \log\left(1 + \frac{\sqrt{\lambda^2 - \omega^2}}{\lambda} \right) + \frac{\sqrt{\lambda^2 u^4 - \omega^2}}{u^2} - \lambda \log\left(\frac{\sqrt{\lambda^2 u^4 - \omega^2}}{\lambda} \right) \right] (36)
\]

When \( \lambda = \lambda_{\text{max}} \), we have \( u_f = \sqrt{\omega / \lambda_{\text{max}}} \) and

\[
\psi_f \mid_{\lambda = \lambda_{\text{max}}} = \frac{2 \lambda_{\text{max}}}{1 + \lambda_{\text{max}}^2} \left[ -1 \right] \sqrt{\lambda_{\text{max}}^2 - \omega^2} + \lambda \log\left(1 + \frac{\sqrt{\lambda_{\text{max}}^2 - \omega^2}}{\lambda_{\text{max}}} \right) - \log\left(\frac{\omega}{\lambda_{\text{max}}} \right) \] (37)

The variation of heading angle as a function of \( u \) for constant \( \lambda \) turning is shown in Fig. 5.

VI. Conclusions

It is very difficult to solve flight mechanics problems analytically. The analytic solutions are presented in this paper are focused on the cases of rectilinear flight cases. The maximum range and maximum endurance glide problems are solved completely and discussed extensively. Also solved are the chattering range and chattering endurance. This is a kind of flight that is not widely investigated. For the turning flight, the only problem solved is the constant lift turn. Nevertheless, this is still a research area that attracts many scientists and mathematicians. We believe that some more analytic solutions can be found somewhere at existing literature and also sometime in the future.

References


