Controller Design with Regional Pole Constraints: Hyperbolic and Horizontal Strip Regions

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Introduction

In Ref. 1, fixed-structure synthesis techniques were used to design feedback controllers that place the closed-loop poles within specified regions in the open left half-plane. Specifically, circular, elliptic, parabolic, vertical strip and sector regions were considered with both static and dynamic output feedback controllers. The purpose of the present Note is to extend the results of Ref. 1 by considering two regions that were not considered in Ref. 1, namely, hyperbolic and horizontal strip regions. In practice, the hyperbolic region, which was considered in Refs. 2-9, imposes a lower bound on the damping ratio of the closed-loop poles, whereas the horizontal strip region, briefly discussed in Ref. 10, imposes an upper bound on the damped natural frequencies of the closed-loop poles. The complicating aspect of both of these regions is that each region is reflected into the right half-plane. Hence, it is necessary to exclude from consideration the right-half portion of the constraint region. The proofs of the following theorems are lengthy and hence are omitted in this paper. Details are given in Ref. 11.

Characterization of the Hyperbolic Constraint Region

To begin, consider the two-sided hyperbolic region \( \mathcal{H}(a, b) \) defined by

\[
\mathcal{H}(a, b) = \{ \lambda \in \mathbb{C} : (\text{Re} \lambda)^2 > \frac{(\text{Im} \lambda)^2}{b^2} \}
\]

where \( a \) and \( b \) are positive real numbers. To specify the left-half region that is of interest for stability, we focus on the subset \( \mathcal{H}_0(a, b) = \{ \lambda \in \mathcal{H}(a, b) : \text{Re} \lambda < 0 \} \), which corresponds to the left branch of the hyperbola. It is often convenient to write \( \lambda = -\omega_n + j\omega_d \), where \( 0 \leq \omega_d^2 \leq 1 \) and \( \omega_n \geq \omega_d \geq 1 - \frac{1}{\sqrt{2}} \). It is also known that the settling time is related to \( \text{Re} \lambda \). In practice, design criteria may involve the damping ratio \( \xi \) and the reciprocal of the settling time \( \eta = \ell_s/\omega_n \). The constraint \( \xi \geq \xi_{\text{min}} \) and \( \eta \geq \eta_{\text{min}} \) can be enforced by the hyperbola parameters \( a \) and \( b \) by choosing \( a = \frac{\xi_{\text{min}}}{\omega_n} \) and \( b = \sqrt{\xi_{\text{min}}^2 + 1 - \xi_{\text{min}}^2} \). Next it can be shown that the region \( \mathcal{H}(a, b) \) can be equivalently characterized by

\[
\mathcal{H}(a, b) = \{ \lambda \in \mathbb{C} : 1 + 2\delta(\text{Re} \lambda^2) + \gamma |\lambda|^2 < 0 \}
\]

where

\[
\delta = \frac{a^2 + b^2}{4a^2b^2}, \quad \gamma = \frac{a^2 - b^2}{2ab^2}
\]

This leads to the following result. Let "spec" denote spectrum.

Proposition 1: Let \( A \in \mathbb{R}^{n \times n} \), let \( V_k \in \mathbb{R}^{n \times k} \) be positive definite, and let \( \delta \) and \( \gamma \) be real numbers such that \( \delta < 0 \) and \( 2k \eta_{\text{spec}} < -2\delta \). Then, if there exists an \( n \times n \) positive definite matrix \( Q_0 \) satisfying

\[
0 = Q_0 + \delta(A^T Q_0 + Q_0 A^T) + \gamma A^T Q_0 A + V_k
\]

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then spec(A) ⊂ \mathcal{L}(a, b)$, where
\[ a \triangleq (1 - 2\delta - \gamma)^{\frac{1}{2}}, \quad b \triangleq (1 - 2\delta + \gamma)^{\frac{1}{2}} \tag{3} \]

Note that \( \mathcal{L}(a, b) \) includes regions lying in the open left half-plane \( \mathbb{C}^- \) and in the open right half-plane \( \mathbb{C}^+ \). Proposition 1 applies to all of \( \mathcal{L}(a, b) \), not just \( \mathcal{L}_c(a, b) \). Considering stability, we now combine the standard Lyapunov equation with Eq. (2). Thus, the characteristic roots will be finite matrices \( Q \) and \( Q_0 \) and real numbers \( \delta \) and \( \gamma \) such that \( \delta < 0 \) and \( 2\delta < \gamma < -2\delta \) satisfying

\[
0 = Q_h + \delta(A^2Q_h + Q_hA^T) + \gamma A Q_h A^T + V_h \tag{4}
\]

\[
0 = A Q + QA^T + V \tag{5}
\]

then spec(A) ⊂ \( \mathcal{L}_c(a, b) \), where \( a \) and \( b \) are given by Eq. (3). Let \( \mathcal{G} \) and \( \mathcal{G} \) be defined by

\[
\mathcal{G} \triangleq I + \delta(A^2 \otimes A^2) + \gamma A \otimes A \]

\[
\mathcal{G} \triangleq I - \frac{1}{\delta}(A \otimes A)^{-1}[I + (\gamma - 2\delta)A \otimes A] \]

where \( \otimes \) and \( \oplus \) denote Kronecker product and sum.

**Proposition 3:** Let \( \delta \) and \( \gamma \) be real numbers such that \( \delta < 0 \), and \( 2\delta < \gamma < -2\delta \), and let \( a \) and \( b \) be given by Eq. (3). Then the following statements hold.

1) Suppose \( a \geq b \). Then \( \mathcal{G} \) and \( \mathcal{A} \) are asymptotically stable if and only if spec(\( \mathcal{A}(a, b) \)) ⊂ \( \mathcal{L}(a, b) \).

2) Suppose \( a < b \). Then \( \mathcal{G} \) and \( \mathcal{A} \) are asymptotically stable if and only if spec(\( \mathcal{A}(a, b) \)) ⊂ \( \mathcal{L}(a, b) \).

**Lemma 4:** Let spec(A) ⊂ \( \mathcal{L}_c(a, b) \), where \( a \) and \( b \) are given by Eq. (3), and let \( V \) and \( V_h \) ∈ \( \mathbb{R}^{n \times n} \) be positive-definite matrices. Let \( \delta \) and \( \gamma \) be given by Eq. (1). Then there exist unique \( n \times n \) positive-definite matrices \( Q \) and \( Q_0 \) satisfying

\[
0 = Q_h + \delta(A^2Q_h + Q_hA^T) + \gamma A Q_h A^T + V_h \tag{6}
\]

\[
0 = A Q + QA^T + V_h \tag{7}
\]

**Controller Synthesis**

Based on Eqs. (6) and (7), we can now perform controller synthesis. Here we consider the linear time-invariant system

\[
x(t) = Ax(t) + Bu(t) + D_1w(t) \tag{8}
\]

\[
y(t) = Cx(t) \tag{9}
\]

where \( x(t) \), \( u(t) \), \( w(t) \), and \( y(t) \) are \( n \), \( m \), \( d \), and \( l \)-dimensional vectors, and \( A \), \( B \), \( C \), and \( D_1 \) are corresponding constant matrices. With static output feedback of the form

\[
u(t) = Ky(t) \tag{10}
\]

it is our goal to select \( K \) such that the closed-loop system has the following properties:

1) The closed-loop poles are constrained to lie in the hyperbolic constraint region \( \mathcal{L}(a, b) \).

2) The performance index

\[
J = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ x(t)^T R_x x(t) + 2x(t)^T R_u u(t) + u(t)^T R_u u(t) \right] dt \tag{11}
\]

is minimized.

The closed-loop system (8–10) is given by

\[
x(t) = A_x x(t) + D_1w(t) \tag{12}
\]

where \( A_x = A + BKC \). To determine a feedback gain \( K \) satisfying properties 1 and 2, we begin by defining an open set of feedback gains \( \mathcal{K} \triangleq \{K : \text{spec}(A_x) \subset \mathcal{L}_c(a, b)\} \), which place the closed-loop poles in \( \mathcal{L}_c(a, b) \). We assume that \( \mathcal{K} \) is not empty. Equation (11) can be written as

\[
J(K) = \lim_{T \to \infty} \frac{1}{T} \int_0^T (x^T R_x x) dt \tag{13}
\]

Furthermore, by defining the nonnegative-definite state covariance

\[
Q = \lim_{T \to \infty} \frac{1}{T} \int_0^T (xx^T) dt \tag{14}
\]

the system (8–11) combined with criterion 2 will be as follows:

Minimize \( J(K) = \text{tr} QR_x \), where \( R_x \triangleq R_1 + R_2 + (R_2KC)^T + (KC)TR_2KC \) subject to

\[
0 = A_x Q + QA_x^T + V_h \tag{15}
\]

\[
0 = A_x Q + QA_x^T + V \tag{16}
\]

Furthermore, \( J(K) \) ≤ \( J(\mathcal{K}) \), where \( J(\mathcal{K}) = \text{tr} (QR_x + Q_h) \).

We can now formulate the auxiliary minimization problem: determine \( K \in \mathcal{K} \), that minimizes \( J(\mathcal{K}) \) where the positive-definite matrices \( Q \) and \( Q_0 \) satisfy Eqs. (15) and (16).

**Theorem 6:** Let \( K \in \mathcal{K} \) minimize \( J(\mathcal{K}) \). Then there exist positive-definite matrices \( Q_h \), \( P \), and \( P_0 \) that satisfy

\[
0 = \dot{A}P + P\dot{A}^T + \bar{R}_s \tag{17}
\]

\[
0 = Q_h + \delta(A^2Q_h + Q_hA^T) + \gamma A Q_h A^T + V_h \tag{18}
\]

\[
0 = \dot{A}^T P + PA + \bar{R}_s \tag{19}
\]

\[
0 = I + \delta(A^2P_h + P_hA^T) + \gamma A^TQ_hA + P_h \tag{20}
\]

where, under the assumption that \( \mathcal{I} \) defined next is nonsingular,

\[
\bar{A} \triangleq A - B(\text{vec}^{-1} \Pi^{-1} \text{vec} \Omega)C \]

\[
\bar{R}_s \triangleq R_1 - R_{12}(\text{vec}^{-1} \Pi^{-1} \text{vec} \Omega)C \]

\[
- C^T(\text{vec}^{-1} \Pi^{-1} \text{vec} \Omega)D_1R_1^T \]

\[
+ C^T(\text{vec}^{-1} \Pi^{-1} \text{vec} \Omega)D_1R_2(\text{vec}^{-1} \Pi^{-1} \text{vec} \Omega)C \]

\[
\Omega \triangleq R_1^TQC + \delta(B^T A^T P_h Q_h A^T + B^T P_h Q_h A^T C^T) \]

\[
\gamma B^T P_h A Q_h C^T + B^T P C^T \]

\[
\Pi \triangleq C Q C^T \oplus R_2 + \delta([C^T \text{vec} P_h B \otimes B^T C^T]U_{m \times l}) \]

\[
+ (C B \otimes B^T P_h Q_h C^T)U_{m \times l} \]

\[
+ (C B \otimes B^T P_h Q_h C^T)U_{m \times l} \]

\[
+ (C B \otimes B^T P_h Q_h C^T)U_{m \times l} \]

such that the feedback gain \( K \) is given by

\[
K = - \text{vec}^{-1} \Pi^{-1} \text{vec} \Omega \tag{21}
\]
Let us now design a full-order dynamic compensator satisfying pole constraints with regulator/estimator separation. Consider the linear time-invariant system

\[ \dot{x}(t) = Ax(t) + Bu(t) + D_1w(t) \]  
\[ y(t) = Cx(t) + D_2w(t) \]

where \( x(t), u(t), w(t), \) and \( y(t) \) are \( n, m, d, l \)-dimensional vectors, and \( A, B, C, D_1, \) and \( D_2 \) are corresponding constant matrices. Now the goal is to choose \( A_c, B_c, C_c \) such that the dynamic compensator

\[ \dot{x}_c(t) = A_cx_c + BCy(t) \]
\[ u(t) = Ccx_c \]

satisfies properties 1 and 2.

The closed-loop system and performance criterion of Eq. (11) can be restated as follows:

Minimize

\[ J(A_c, B_c, C_c) = \text{tr} QR_d \]

subject to

\[ 0 = A_dQ + QA_d^T + V_d \]

where

\[ A_d = \begin{bmatrix} A & BC_c \\ B_c & A_c \end{bmatrix}, \quad R_d = \begin{bmatrix} R_1 & R_{12}C_c \\ C_c^TR_{12}^T & C_c^TR_cC_c \end{bmatrix}, \quad V_d = \begin{bmatrix} V_1 & V_{12} \\ V_{12}^T & B_cV_{12} + B_cB_c^T \end{bmatrix} \]

The set of dynamic compensators that places the closed-loop poles in \( \mathcal{K}_d(a,b) \) is defined by

\[ \mathcal{K}_d \triangleq [(A_c, B_c, C_c) : \text{spec}(A_d) \subset \mathcal{K}_d(a,b)] \]

The following result is analogous to Lemma 5.

Lemma 7: Let the triple \((A_c, B_c, C_c) \in \mathcal{K}_d \) and let \( V_d \) and \( V_b \) be positive-definite matrices. Then there exist positive-definite matrices \( Q \) and \( P_b \) satisfying

\[ 0 = Q + \delta(A_2^2Q + Q^TA_2^T) + \gamma A_dQ + QA_d^T + V_b \]
\[ 0 = A_dQ + QA_d^T + V_d \]

Furthermore, \( J(A_c, B_c, C_c) < \beta(A_c, B_c, C_c) \), where \( \beta(A_c, B_c, C_c) \)

\[ \Delta \text{tr}(QR_d + Q_b) \]

Here we enforce regulator/estimator separation for determining \((A_c, B_c, C_c)\). Thus, the dynamic compensator is assumed to be of the form

\[ \dot{x}_c = Ax_c + Bu + B_c(y - Cx_c) \]
\[ u = Ccx_c \]

such that \( A_c \triangleq A + BC_c - B_cC_c \). To exploit this, it is useful to design the estimator by defining the tracking error \( e \triangleq x - x_c \) such that

\[ \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BC_c & -BC_c \\ 0 & A + B_cC_c \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} D_1 \\ D_1 - B_cD_2 \end{bmatrix}w \]

Then the goal is to separately place the eigenvalues of the error dynamics and regulator in the hyperbolic constraint region \( \mathcal{K}_e(a,b) \). From Eq. (31), it is noticed that there are in fact two separate problems for determining \( B_c \) and \( C_c \). The subproblem for the estimator can be formulated such that the weighted estimator cost is given by

\[ J_e(B_c) = \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} (e^TWe) dt \]

where \( W \) is a given \( n \times n \) positive-definite matrix. However, Eq. (32) can be rewritten as

\[ J_e(B_c) = \text{tr}Q_1W \]

Note that \( Q_b \) satisfies the Lyapunov equation

\[ 0 = A_eQ_b + Q_bA_e^T + V \]

where \( A_e = A + B_cC_c \) and \( V_b = V_1 - B_cV_{12} - V_{12}B_c + B_cB_c^T \). For the regulator, we consider

\[ \dot{x} = Ax + Bu + D_1w \]
\[ u = C_cx \]

which implies that

\[ \dot{x_c} = Ax_c + D_1w \]

The corresponding cost is

\[ J_r(C_c) = \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} (x^TR_1x + 2xTR_2u + u^TR_2u) dt = \text{tr}Q_2R_r \]

where \( Q_r \) satisfies

\[ 0 = A_eQ_r + Q_eA_e^T + V_r \]

where \( A_e = A + BC_c - B_cC_c \) and \( R_r = R_1 + R_{12}C_c + (R_{12}C_c)^T + C_c^TR_cC_c \). Now let \( \mathcal{K}_e \) be defined as \( \mathcal{K}_e \triangleq \{K : \text{spec}(A + B_cC_c) \subset \mathcal{K}_e(a,b)\} \) to characterize a dual set of gains for the closed-loop pole assignment. To place the eigenvalues of error dynamics and regulator in \( \mathcal{K}_e(a,b) \), it is also required that \( A e \) and \( A_e \) be stable as guaranteed by Proposition 1.

Lemma 8: Let \( B_c \in \mathcal{K}_e \) and \( C_c \in \mathcal{K}_e \), and let \( V_r, V_{he}, V_1, \) and \( V_{he} \) be positive-definite matrices. Then there exist positive-definite matrices \( Q_{he} \) and \( Q_r \in \mathcal{R}^{n \times n} \) satisfying

\[ 0 = Q_{he} + \delta(A_e^2Q_{he} + Q_{he}A_e^T) + \gamma A_eQ_{he} + QA_e^T + V_{he} \]
\[ 0 = A_eQ_r + QA_e^T + V_r \]

such that \( J_r(B_c) < \beta_r(B_c) = \text{tr}(Q_rW + Q_{he}) \). Furthermore, there exist positive-definite matrices \( Q_{he} \) and \( Q_r \in \mathcal{R}^{n \times n} \) satisfying

\[ 0 = Q_{he} + \delta(A_e^2Q_{he} + Q_{he}A_e^T) + \gamma A_eQ_{he} + QA_e^T + V_{he} \]
\[ 0 = A_eQ_r + QA_e^T + V_r \]

and such that \( J_r(C_c) < \beta_r(C_c) = \text{tr}(Q_rW + Q_{he}) \).

Theorem 9: Let \( B_c \in \mathcal{K}_e \) and \( C_c \in \mathcal{K}_e \) where \( \beta_r(B_c) \) and \( \beta_r(C_c) \) are minimized, and let \( V_r, V_{he}, V_1, \) and \( V_{he} \) be positive-definite matrices. Then there exist positive-definite matrices \( P_{he}, P_r, Q_r, \) and \( Q_{he} \in \mathcal{R}^{n \times n} \) satisfying

\[ 0 = A_eQ_{he} + QA_e^T + V_r \]
0 = Q_{hr} + \delta(\tilde{A}_{r}^{T}Q_{hr} + Q_{hr}\tilde{A}_{r}^{T}) + \gamma A_{hr}A_{hr}^{T} + V_{hr} \tag{44}

0 = \tilde{A}_{r}^{T}P_{r} + P_{r}\tilde{A}_{r} + W \tag{45}

0 = I + P_{hr} + \delta(\tilde{A}_{r}^{T}P_{hr} + P_{hr}\tilde{A}_{r}^{T}) + \gamma \tilde{A}_{hr}P_{hr}\tilde{A}_{r} \tag{46}

and positive-definite matrices $P_{hr}$, $P_{r}$, $Q_{r}$, and $Q_{hr} \in \mathbb{R}^{n \times n}$ satisfying

\begin{align*}
0 &= \tilde{A}_{r}Q_{r} + Q_{r}\tilde{A}_{r}^{T} + V_{1} \tag{47} \\
0 &= Q_{hr} + \delta(\tilde{A}_{r}^{T}Q_{hr} + Q_{hr}\tilde{A}_{r}^{T}) + \gamma A_{hr}A_{hr}^{T} + V_{hr} \tag{48} \\
0 &= \tilde{A}_{r}^{T}P_{r} + P_{r}\tilde{A}_{r} + R_{r} \tag{49} \\
0 &= I + P_{hr} + \delta(\tilde{A}_{r}^{T}P_{hr} + P_{hr}\tilde{A}_{r}^{T}) + \gamma \tilde{A}_{hr}P_{hr}\tilde{A}_{r} \tag{50}
\end{align*}

where, under the assumption that $P_{hr}$ and $P_{r}$ defined next are nonsingular matrices,

\begin{align*}
\tilde{A}_{r} &= A - (\text{vec}^{-1}\Pi_{r}^{T}1\text{vec} \Omega_{r})C \\
\tilde{A}_{hr} &= A - B(\text{vec}^{-1}\Pi_{r}^{T}1\text{vec} \Omega_{r}) \\
R_{r} &= R_{1} - R_{12}\text{vec}^{-1}\Pi_{r}^{T}1\text{vec} \Omega_{r} - (\text{vec}^{-1}\Pi_{r}^{T}1\text{vec} \Omega_{r})R_{12}^{T} \\
P_{r} &= (\text{vec}^{-1}\Pi_{r}^{T}1\text{vec} \Omega_{r})R_{2}(\text{vec}^{-1}\Pi_{r}^{T}1\text{vec} \Omega_{r}) \\
\Omega_{r} &= \delta(4P_{hr}Q_{hr}C^{T} + P_{hr}Q_{hr}A^{T}C^{T}) + \gamma P_{hr}A_{hr}C^{T} + P_{r}Q_{r}C^{T} - P_{r}D_{r}D_{r}^{T} \\
\Pi_{r} &= \delta\left[\left(C \otimes (P_{hr}Q_{hr}C^{T})\right)U_{ax} + \left((CQ_{hr}P_{hr}) \otimes C^{T} \right)U_{ax}\right] + \gamma(CQ_{hr}C^{T}) \otimes P_{hr} + (D_{r}D_{r}^{T} \otimes P_{r} \\
\Omega_{r} &= R_{12}^{T}Q_{r} + \delta\left[(B^{T}A^{T}P_{hr}Q_{hr}) + (B^{T}P_{hr}Q_{hr})\right]U_{ax}^{T} \\
\Omega_{r} &= \delta\left[(B \otimes (P_{hr}Q_{hr})U_{ax} + (Q_{hr}B \otimes B^{T})U_{ax}^{T}\right] + \gamma Q_{hr} \otimes B^{T}P_{hr}B \\
\Pi_{r} &= Q_{r} \otimes R_{r} + \delta\left[(B \otimes (P_{hr}Q_{hr})U_{ax} + (Q_{hr}B \otimes B^{T})P_{hr}B\right]
\end{align*}

such that the compensator is given by

\begin{align*}
A_{c} &= A - B_{c}C + BC_{c} \tag{51} \\
B_{c} &= -\text{vec}^{-1}\Pi_{c}^{T}1\text{vec} \Omega_{c} \tag{52} \\
C_{c} &= -\text{vec}^{-1}\Pi_{c}^{T}1\text{vec} \Omega_{c} \tag{53}
\end{align*}

Finally, we briefly discuss regional pole placement within the horizontal strip region. To guarantee stability, we are only interested in the region that is in the open left half-plane. The left portion of the horizontal strip region can be characterized as

\[ 3C_{c}(\omega) \triangleq \left[ \lambda \in \mathbb{C} : \text{Re}\lambda < 0, \ (\text{Im}\lambda)^{2} < \omega^{2} \right] \]

where $\omega$ is the upper bound on the damped natural frequency.

Lemma 10: The set $3C_{c}(\omega)$ is equivalent to

\[ 3C_{c}(\omega) = (\lambda \in \mathbb{C} : -1 + \delta \text{Re}\lambda + \gamma |\lambda|^2 < 0) \]

where

\[ \delta = \frac{1}{4\omega^{2}}, \quad \gamma = \frac{1}{2\omega^{2}} \]

By comparing $3C(\alpha, \beta)$ with $3C_{c}(\omega)$, we immediately notice that the constraint inequalities are similar. The differences only arise at the coefficients of the inequalities. Thus, the major results derived so far for the hyperbolic constraint region can be carried over to be the results for the horizontal strip region with only slight modifications of the coefficients.

**Conclusion**

In this Note we established an upper bound for the cost that can be minimized subject to a pair of matrix root-clustering equations. These equations were used to constrain the poles of the closed-loop system to lie in a hyperbolic or horizontal strip region contained in the left half-plane. The left hyperbolic region was chosen because of its ability to set desired bounds on the damping ratio and settling time. Because of the similarity between root-clustering equations of hyperbolic and horizontal strip regions, the results obtained for the left hyperbolic region can be applied to the left horizontal strip region with minor coefficient changes. Future research will focus on numerical techniques for solving the matrix algebraic equations.

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**References**