Making use of Eqs. 310.02 and 310.04 of Ref. 5, we obtain after simplification

$$\overline{x^2}(H,\mu) = a^2 - \frac{2a(a-c)}{K}(K-E) + \frac{(a-c)^2}{3K}\{(2+k^2)K - 2(1+k^2)E\} \tag{23}$$

where $E = E(k)$ is the complete elliptic integral of the second kind. Note that $k^2 = 0$ implies $a = b = 0$ and $K = E$, from which $\overline{x^2} = 0$, as expected, since this corresponds to the center at $(x, \dot{x}) = (0, 0)$.

Thus, we have reduced the study of Eq. (16) with $\dot{\mu} = \epsilon$ to the study of the averaged slow flow described by

$$\bar{H}' = \overline{x^2}(\bar{H}, \mu) \tag{24}$$

where $\overline{x^2}$ is given at Eq. (23) and $(\ )' = \mathrm{d}(\ )/\mathrm{d}\mu$. This equation is valid to $\mathcal{O}(\epsilon)$ as long as the flow stays in region 2a (Fig. 1), and it can be shown that if $\mu(0) > 0$ and $\epsilon > 0$, then trajectories originating in region 2a remain in that region.[9] Numerical comparisons show that solutions to Eq. (24) agree quite well with the "exact" solution to Eqs. (2) and (3).

In this example there is only one region of phase space where the unperturbed solution is periodic. In case there is more than one such region [e.g., when $V(x;\mu)$ is quartic], then the form of the $\epsilon = 0$ solution and, hence, the form of the right-hand side of Eq. (14) are different in different regions. When the slow flow passes from one region to another, the averaged equation may lose validity. This is because the transition may involve crossing an *instantaneous separatrix* of the unperturbed system. At a separatrix, the period of the $\epsilon = 0$ solution becomes infinite, so that the average computed in Eq. (13) is over an infinite time interval, violating the conditions of the averaging theorem (see Ref. 9 for further discussion of separatrix crossing).

Conclusions

We have presented a general formulation for application of the method of averaging to a specific class of nonlinear equations. The method exploits the existence of an energy integral (the Hamiltonian) for the unperturbed system and leads to a single first-order equation for the slow evolution of the Hamiltonian. By using the canonical coordinate x as the fast variable, the need to identify the rapidly varying phase angle (as in Kruskal's method) is eliminated. As shown in the example, application is relatively straightforward when the form of the potential leads to an explicit solution to the unperturbed problem.

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Controller Design with Regional Pole Constraints: Hyperbolic and Horizontal Strip Regions

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Introduction

IN Ref. 1, fixed-structure synthesis techniques were used to design feedback controllers that place the closed-loop poles within specified regions in the open left half-plane. Specifically, circular, elliptic, parabolic, vertical strip and sector regions were considered with both static and dynamic output feedback controllers. The purpose of the present Note is to extend the results of Ref. 1 by considering two regions that were not considered in Ref. 1, namely, hyperbolic and horizontal strip regions. In practice, the hyperbolic region, which was considered in Refs. 2–9, imposes a lower bound on the damping ratio of the closed-loop poles, whereas the horizontal strip region, briefly discussed in Ref. 10, imposes an upper bound on the damped natural frequencies of the closed-loop poles. The complicating aspect of both of these regions is that each region is reflected into the right half-plane. Hence, it is necessary to exclude from consideration the right-half portion of the constraint region. The proofs of the following theorems are lengthy and hence are omitted in this paper. Details are given in Ref. 11.

Characterization of the Hyperbolic Constraint Region

To begin, consider the two-sided hyperbolic region $\mathcal{H}(a,b)$ defined by

$$\mathcal{H}(a,b) \triangleq \left[\lambda \in \mathbb{C} : \frac{(\mathrm{Re}\lambda)^2}{a^2} - \frac{(\mathrm{Im}\lambda)^2}{b^2} > 1\right]$$

where a and b are positive real numbers. To specify the left-half region that is of interest for stability, we focus on the subset $\mathcal{H}_L(a,b) \triangleq [\lambda \in \mathcal{H}(a,b) : \mathrm{Re}\lambda < 0]$, which corresponds to the left branch of the hyperbola. It is often convenient to write $\lambda = -\zeta\omega_n + j\omega_d$, where $0 \le \zeta \le 1$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. It is also known that the settling time is related to Reλ. In practice, design criteria may involve the damping ratio ζ and the reciprocal of the settling time $\eta = \zeta\omega_n$. The constraint $\zeta \ge \zeta_{\min}$ and $\eta \ge \eta_{\min}$ can be enforced by the hyperbola parameters a and b by choosing $a = \eta_{\min}$ and $b = (\eta_{\min}/\zeta_{\min})\sqrt{1-\zeta_{\min}^2}$. Next it can be shown that the region $\mathcal{H}(a,b)$ can be equivalently characterized by

$$\mathcal{H}(a,b) \triangleq \left[\lambda \in \mathbb{C} : 1 + 2\delta(\mathrm{Re}\lambda^2) + \gamma|\lambda|^2 < 0\right]$$

where

$$\delta \triangleq -\frac{a^2+b^2}{4a^2b^2}, \qquad \gamma \triangleq \frac{a^2-b^2}{2a^2b^2} \tag{1}$$

This leads to the following result. Let "spec" denote spectrum.

Proposition 1: Let $A \in \mathbb{R}^{n\times n}$, let $V_h \in \mathbb{R}^{n\times n}$ be positive definite, and let δ and γ be real numbers such that $\delta < 0$ and $2\delta < \gamma < -2\delta$. Then, if there exists an $n \times n$ positive definite matrix Q_h satisfying

$$0 = Q_h + \delta(A^2Q_h + Q_hA^{2T}) + \gamma AQ_hA^T + V_h \tag{2}$$

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then spec$(A) \subset \mathcal{H}(a,b)$, where

$$a \triangleq (1/-2\delta-\gamma)^{1/2}, \qquad b \triangleq (1/-2\delta+\gamma)^{1/2} \tag{3}$$

Note that $\mathcal{H}(a,b)$ includes regions lying in the open left half-plane $\mathcal{C}^-$ and in the open right half-plane $\mathcal{C}^+$. Proposition 1 applies to all of $\mathcal{H}(a,b)$, not just $\mathcal{H}_L(a,b)$. Considering stability, we now combine the standard Lyapunov equation with Eq. (2). Thus, the characteristic roots will be constrained to lie inside the left hyperbolic constraint region.

Theorem 2: Let A, V, V_h, Q, and $Q_h \in \mathbb{R}^{n\times n}$ and V and V_h be positive definite matrices. Then, if there exist positive-definite matrices Q and Q_h and real numbers δ and γ such that $\delta<0$ and $2\delta<\gamma<-2\delta$ satisfying

$$0 = Q_h + \delta(A^2Q_h + Q_hA^{2T}) + \gamma AQ_hA^T + V_h \tag{4}$$

$$0 = AQ + QA^T + V \tag{5}$$

then spec$(A) \subset \mathcal{H}_L(a,b)$, where a and b are given by Eq. (3).

Let $\mathcal{A}$ and $\tilde{\mathcal{A}} \in \mathbb{R}^{2n\times 2n}$ be defined by

$$\mathcal{A} \triangleq I + \delta(A^2 \oplus A^2) + \gamma A \otimes A$$

$$\tilde{\mathcal{A}} \triangleq -I - \frac{1}{\delta}(A \oplus A)^{-2}\left[I + (\gamma-2\delta)A \otimes A\right]$$

where $\otimes$ and $\oplus$ denote Kronecker product and sum.

Proposition 3: Let δ and γ be real numbers such that $\delta<0$, and $2\delta<\gamma<-2\delta$, and let a and b be given by Eq. (3). Then the following statements hold.

1) Suppose $a \geq b$. Then $\mathcal{A}$ and A are asymptotically stable if and only if spec$(A) \subset \mathcal{H}_L(a,b)$.

2) Suppose $a<b$. Then $\tilde{\mathcal{A}}$ and A are asymptotically stable if and only if spec$(A) \subset \mathcal{H}_L(a,b)$.

Lemma 4: Let spec$(A) \subset \mathcal{H}_L(a,b)$, where a and b are given by Eq. (3), and let V and $V_h \in \mathbb{R}^{n\times n}$ be positive-definite matrices. Let δ and γ be given by Eq. (1). Then there exist unique $n\times n$ positive-definite matrices Q and Q_h satisfying

$$0 = Q_h + \delta(A^2Q_h + Q_hA^{2T}) + \gamma AQ_hA^T + V_h \tag{6}$$

$$0 = AQ + QA^T + V \tag{7}$$

Controller Synthesis

Based on Eqs. (6) and (7), we can now perform controller synthesis. Here we consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t) \tag{8}$$

$$y(t) = Cx(t) \tag{9}$$

where $x(t)$, $u(t)$, $w(t)$, and $y(t)$ are n-, m-, d-, and l-dimensional vectors, and A, B, C, and D_1 are corresponding constant matrices. With static output feedback of the form

$$u(t) = Ky(t) \tag{10}$$

it is our goal to select K such that the closed-loop system has the following properties:

1) The closed-loop poles are constrained to lie in the hyperbolic constraint region $\mathcal{H}_L(a,b)$.

2) The performance index

$$J \triangleq \lim_{t\to\infty} \mathcal{E}\frac{1}{t}\int_0^t \left[x(t)^TR_1x(t) + 2x(t)^TR_{12}u(t) + u(t)^TR_2u(t)\right] \mathrm{d}t \tag{11}$$

is minimized.

The closed-loop system (8–10) is given by

$$\dot{x}(t) = A_sx(t) + D_1w(t) \tag{12}$$

where $A_s = A + BKC$. To determine a feedback gain K satisfying properties 1 and 2, we begin by defining an open set of feedback gains $\mathcal{K}_s \triangleq [K : \text{spec}(A_s) \subset \mathcal{H}_L(a,b)]$, which place the closed-loop poles in $\mathcal{H}_L(a,b)$. We assume that $\mathcal{K}_s$ is not empty. Equation (11) can be written as

$$J(K) = \lim_{t\to\infty} \mathcal{E}\frac{1}{t}\int_0^t (x^TR_sx)\, \mathrm{d}t \tag{13}$$

Furthermore, by defining the nonnegative-definite state covariance

$$Q \triangleq \lim_{t\to\infty} \mathcal{E}\frac{1}{t}\int_0^t (xx^T)\, \mathrm{d}t \tag{14}$$

the system (8–11) combined with criterion 2 will be as follows: Minimize $J(K) = \text{tr}\, QR_s$, where $R_s \triangleq R_1 + R_{12}KC + (R_{12}KC)^T + (KC)^TR_2KC$ subject to

$$0 = A_sQ + QA_s^T + V_s$$

where $V_s \triangleq D_1D_1^T$. However, to impose criterion 2, we may overbound the desired performance index as shown in Lemma 5 so that a minimization procedure can be carried out later.

Lemma 5: Let $K \in \mathcal{K}_s$ and let V_s and $V_h \in \mathbb{R}^{n\times n}$ be positive-definite matrices. Then there exist $n\times n$ positive-definite matrices Q and Q_h satisfying

$$0 = Q_h + \delta(A_s^2Q_h + Q_hA_s^{2T}) + \gamma A_sQ_hA_s^T + V_h \tag{15}$$

$$0 = A_sQ + QA_s^T + V_s \tag{16}$$

Furthermore, $J(K) < \mathcal{J}(K)$, where $\mathcal{J}(K) \triangleq \text{tr}(QR_s + Q_h)$.

We can now formulate the auxiliary minimization problem: determine $K \in \mathcal{K}_s$ that minimizes $\mathcal{J}(K)$ where the positive-definite matrices Q_h and Q satisfy Eqs. (15) and (16).

Theorem 6: Let $K \in \mathcal{K}_s$ minimize $\mathcal{J}(K)$. Then there exist positive-definite matrices Q_h, Q, P_h, and $P_s \in \mathbb{R}^{n\times n}$ satisfying

$$0 = \hat{A}Q + Q\hat{A}^T + V_s \tag{17}$$

$$0 = Q_h + \delta(\hat{A}^2Q_h + Q_h\hat{A}^{2T}) + \gamma\hat{A}Q_h\hat{A}^T + V_h \tag{18}$$

$$0 = \hat{A}^TP + P\hat{A} + \hat{R}_s \tag{19}$$

$$0 = I + \delta(\hat{A}^{2T}P_h + P_h\hat{A}^2) + \gamma\hat{A}^TQ_h\hat{A} + P_h \tag{20}$$

where, under the assumption that Π defined next is nonsingular,

$$\hat{A} \triangleq A - B(\text{vec}^{-1}\Pi^{-1}\text{vec}\ \Omega)C$$

$$\begin{aligned}\hat{R}_s \triangleq{}& R_1 - R_{12}(\text{vec}^{-1}\Pi^{-1}\text{vec}\ \Omega)C \\ &- C^T(\text{vec}^{-1}\Pi^{-1}\text{vec}\ \Omega)^TR_{12}^T \\ &+ C^T(\text{vec}^{-1}\Pi^{-1}\text{vec}\ \Omega)^TR_2(\text{vec}^{-1}\Pi^{-1}\text{vec}\ \Omega)C\end{aligned}$$

$$\begin{aligned}\Omega \triangleq{}& R_{12}^TQC^T + \delta(B^TA^TP_hQ_hC^T + B^TP_hQ_hA^TC^T) \\ &+ \gamma B^TP_hAQ_hC^T + B^TPQC^T\end{aligned}$$

$$\begin{aligned}\Pi \triangleq{}& CQC^T \otimes R_2 + \delta\left[(CQ_hP_hB \otimes B^TC^T)U_{m\times l}\right. \\ &\left.+ (CB \otimes B^TP_hQ_hC^T)U_{m\times l}\right] + \gamma(CQ_hC^T \otimes B^TP_hB)\end{aligned}$$

such that the feedback gain K is given by

$$K = -\text{vec}^{-1}\Pi^{-1}\text{vec}\ \Omega \tag{21}$$

Let us now design a full-order dynamic compensator satisfying pole constraints with regulator/estimator separation. Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t) \tag{22}$$

$$y(t) = Cx(t) + D_2 w(t) \tag{23}$$

where $x(t)$, $u(t)$, $w(t)$, and $y(t)$ are n-, m-, d-, and l-dimensional vectors, and A, B, C, D_1, and D_2 are corresponding constant matrices. Now the goal is to choose A_c, B_c, C_c such that the dynamic compensator

$$\dot{x}_c(t) = A_c x_c + B_c y(t) \tag{24}$$

$$u(t) = C_c x_c \tag{25}$$

satisfies properties 1 and 2.

The closed-loop system and performance criterion of Eq. (11) can be restated as follows:

Minimize

$$J(A_c, B_c, C_c) = \operatorname{tr} QR_d \tag{26}$$

subject to

$$0 = A_d Q + QA_d^T + V_d$$

where

$$A_d = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad R_d = \begin{bmatrix} R_1 & R_{12}C_c \\ C_c^T R_{12}^T & C_c^T R_2 C_c \end{bmatrix}$$

$$V_d = \begin{bmatrix} V_1 & V_{12} B_c^T \\ B_c V_{12}^T & B_c V_2 B_c^T \end{bmatrix}$$

The set of dynamic compensators that places the closed-loop poles in $\mathcal{H}_L(a,b)$ is defined by

$$\mathcal{K}_d \triangleq [(A_c, B_c, C_c) : \operatorname{spec}(A_d) \subset \mathcal{H}_L(a,b)]$$

The following result is analogous to Lemma 5.

Lemma 7: Let the triple $(A_c, B_c, C_c) \in \mathcal{K}_d$, and let V_d and $V_h \in \mathbb{R}^{n \times n}$ be positive-definite matrices. Then there exist positive-definite matrices Q and $Q_h \in \mathbb{R}^{n \times n}$ satisfying

$$0 = Q_h + \delta(A_d^2 Q_h + Q_h A_d^{2T}) + \gamma A_d Q_h A_d^T + V_h \tag{27}$$

$$0 = A_d Q + QA_d^T + V_d \tag{28}$$

Furthermore, $J(A_c, B_c, C_c) < \mathcal{J}(A_c, B_c, C_c)$, where $\mathcal{J}(A_c, B_c, C_c) \triangleq \operatorname{tr}(QR_d + Q_h)$.

Here we enforce regulator/estimator separation for determining (A_c, B_c, C_c). Thus, the dynamic compensator is assumed to be of the form

$$\dot{x}_c = Ax_c + Bu + B_c(y - Cx_c) \tag{29}$$

$$u = C_c x_c \tag{30}$$

such that $A_c \triangleq A + BC_c - B_c C$. To exploit this, it is useful to design the estimator by defining the tracking error $e \triangleq x - x_c$ such that

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BC_c & -BC_c \\ 0 & A + B_c C \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} D_1 \\ D_1 - B_c D_2 \end{bmatrix} w \tag{31}$$

Then the goal is to separately place the eigenvalues of the error dynamics and regulator in the hyperbolic constraint region $\mathcal{H}_L(a,b)$. From Eq. (31), it is noticed that there are in fact two separate problems for determining B_c and C_c. The subproblem for the estimator can be formulated such that the weighted estimator cost is given by

$$J_e(B_c) = \lim_{t \to \infty} \mathcal{E} \frac{1}{t} \int_0^t (e^T W e) \, dt \tag{32}$$

where W is a given $n \times n$ positive-definite matrix. However, Eq. (32) can be rewritten as

$$J_e(B_c) = \operatorname{tr} Q_e W \tag{33}$$

Note that Q_e satisfies the Lyapunov equation

$$0 = A_e Q_e + Q_e A_e^T + V_e \tag{34}$$

where $A_e = A + B_c C$ and $V_e = V_1 - B_c V_{12}^T - V_{12} B_c + B_c V_2 B_c^T$. For the regulator, we consider

$$\dot{x} = Ax + Bu + D_1 w \tag{35}$$

$$u = C_c x \tag{36}$$

which implies that

$$\dot{x} = A_r x + D_1 w \tag{37}$$

The corresponding cost is

$$J_r(C_c) = \lim_{t \to \infty} \mathcal{E} \frac{1}{t} \int_0^t (x^T R_1 x + 2x^T R_{12} u + u^T R_2 u) \, dt = \operatorname{tr} Q_r R_r \tag{38}$$

where Q_r satisfies

$$0 = A_r Q_r + Q_r A_r^T + V_1$$

where $A_r = A + BC_c$ and $R_r = R_1 + R_{12} C_c + (R_{12} C_c)^T + C_c^T R_2 C_c$. Now let $\tilde{\mathcal{K}}_s$ be defined as $\tilde{\mathcal{K}}_s \triangleq [K : \operatorname{spec}(A + B_c C) \subset \mathcal{H}_L(a,b)]$ to characterize a dual set of gains for the closed-loop pole assignment. To place the eigenvalues of error dynamics and regulator in $\mathcal{H}_L(a,b)$, it is also required that A_e and A_r be stable as guaranteed by Proposition 1.

Lemma 8: Let $B_c \in \tilde{\mathcal{K}}_s$ and $C_c \in \mathcal{K}_s$, and let V_e, V_{he}, V_1, and $V_{hr} \in \mathbb{R}^{n \times n}$ be positive-definite matrices. Then there exist positive-definite matrices Q_{he} and $Q_e \in \mathbb{R}^{n \times n}$ satisfying

$$0 = Q_{he} + \delta(A_e^2 Q_{he} + Q_{he} A_e^{2T}) + \gamma A_e Q_{he} A_e^T + V_{he} \tag{39}$$

$$0 = A_e Q_e + Q_e A_e^T + V_e \tag{40}$$

such that $J_e(B_c) < \mathcal{J}_e(B_c) = \operatorname{tr}(Q_e W + Q_{he})$. Furthermore, there exist positive-definite matrices Q_{hr} and $Q_r \in \mathbb{R}^{n \times n}$ satisfying

$$0 = Q_{hr} + \delta(A_r^2 Q_{hr} + Q_{hr} A_r^{2T}) + \gamma A_r Q_{hr} A_r^T + V_{hr} \tag{41}$$

$$0 = A_r Q_r + Q_r A_r^T + V_1 \tag{42}$$

and such that $J_r(C_c) < \mathcal{J}_r(C_c) = \operatorname{tr}(Q_r R_r + Q_{hr})$.

Theorem 9: Let $B_c \in \tilde{\mathcal{K}}_s$ and $C_c \in \mathcal{K}_s$ where $\mathcal{J}_e(B_c)$ and $\mathcal{J}_r(C_c)$ are minimized, and let V_e, V_{he}, V_1, and $V_{hr} \in \mathbb{R}^{n \times n}$ be positive-definite matrices. Then there exist positive-definite matrices P_{he}, P_e, Q_e, and $Q_{he} \in \mathbb{R}^{n \times n}$ satisfying

$$0 = \hat{A}_e Q_e + Q_e \hat{A}_e^T + V_e \tag{43}$$

$$0 = Q_{he} + \delta(\hat{A}_e^2 Q_{he} + Q_{he}\hat{A}_e^{2T}) + \gamma \hat{A}_e Q_{he} \hat{A}_e^T + V_{he} \tag{44}$$

$$0 = \hat{A}_e^T P_e + P_e \hat{A}_e + W \tag{45}$$

$$0 = I + P_{he} + \delta(\hat{A}_e^{2T} P_{he} + P_{he}\hat{A}_e^2) + \gamma \hat{A}_e^T P_{he} \hat{A}_e \tag{46}$$

and positive-definite matrices P_{hr}, P_r, Q_r, and $Q_{hr} \in \mathbb{R}^{n\times n}$ satisfying

$$0 = \hat{A}_r Q_r + Q_r \hat{A}_r^T + V_1 \tag{47}$$

$$0 = Q_{hr} + \delta(\hat{A}_r^2 Q_{hr} + Q_{hr}\hat{A}_r^{2T}) + \gamma \hat{A}_r Q_{hr} \hat{A}_r^T + V_{hr} \tag{48}$$

$$0 = \hat{A}_r^T P_r + P_r \hat{A}_r + \hat{R}_r \tag{49}$$

$$0 = I + P_{hr} + \delta(\hat{A}_r^{2T} P_{hr} + P_{hr}\hat{A}_r^2) + \gamma \hat{A}_r^T P_{hr} \hat{A}_r \tag{50}$$

where, under the assumption that Π_e and Π_r defined next are nonsingular matrices,

$$\hat{A}_e \triangleq A - (\mathrm{vec}^{-1}\Pi_e^{-1}\,\mathrm{vec}\Omega_e)C$$

$$\hat{A}_r \triangleq A - B(\mathrm{vec}^{-1}\Pi_r^{-1}\,\mathrm{vec}\Omega_r)$$

$$\hat{R}_s \triangleq R_1 - R_{12}\mathrm{vec}^{-1}\Pi_r^{-1}\,\mathrm{vec}\,\Omega_r - (\mathrm{vec}^{-1}\Pi_r^{-1}\,\mathrm{vec}\Omega_r)^T R_{12}^T$$
$$+ (\mathrm{vec}^{-1}\Pi_r^{-1}\,\mathrm{vec}\Omega_r)^T R_2 (\mathrm{vec}^{-1}\Pi_r^{-1}\,\mathrm{vec}\Omega_r)$$

$$\Omega_e \triangleq \delta(A^T P_{he} Q_{he} C^T + P_{he} Q_{he} A^T C^T) + \gamma P_{he} A Q_{he} C^T$$
$$+ P_e Q_e C^T - P_e D_1 D_2^T$$

$$\Pi_e \triangleq \delta\left\{\left[C\otimes(P_{he}Q_{he}C^T)\right]U_{n\times l} + \left[(CQ_{he}P_{he})\otimes C^T\right]U_{n\times l}\right\}$$
$$+ \gamma(CQ_{he}C^T)\otimes P_{he} + (D_2 D_2^T)\otimes P_e$$

$$\Omega_r \triangleq R_{12}^T Q_r + \delta(B^T A^T P_{hr} Q_{hr} + B^T P_{hr} Q_{hr} A^T)$$
$$+ \gamma B^T P_{hr} A Q_{hr} + B^T P_r Q_r$$

$$\Pi_r \triangleq Q_r \otimes R_2 + \delta\left[(B\otimes B^T P_{hr} Q_{hr})U_{m\times n}\right.$$
$$\left. + (Q_{hr}P_{hr}B\otimes B^T)U_{m\times n}\right] + \gamma Q_{hr}\otimes B^T P_{hr} B$$

such that the compensator is given by

$$A_c = A - B_c C + B C_c \tag{51}$$

$$B_c = -\mathrm{vec}^{-1}\Pi_e^{-1}\,\mathrm{vec}\,\Omega_e \tag{52}$$

$$C_c = -\mathrm{vec}^{-1}\Pi_r^{-1}\,\mathrm{vec}\,\Omega_r \tag{53}$$

Finally, we briefly discuss regional pole placement within the horizontal strip region. To guarantee stability, we are only interested in the region that is in the open left half-plane. The left portion of the horizontal strip region can be characterized as

$$\mathcal{H}_s(\omega) \triangleq \left[\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0, \quad (\mathrm{Im}\lambda)^2 < \omega^2\right]$$

where ω is the upper bound on the damped natural frequency.

Lemma 10: The set $\mathcal{H}_s(\omega)$ is equivalent to

$$\mathcal{H}_s(\omega) = (\lambda \in \mathbb{C} : -1 + \delta\mathrm{Re}\lambda^2 + \gamma|\lambda|^2 < 0)$$

where

$$\delta \triangleq -\frac{1}{4\omega^2}, \qquad \gamma \triangleq \frac{1}{2\omega^2}$$

By comparing $\mathcal{H}(a,b)$ with $\mathcal{H}_s(\omega)$, we immediately notice that the constraint inequalities are similar. The differences only arise at the coefficients of the inequalities. Thus, the major results derived so far for the hyperbolic constraint region can be carried over to be the results for the horizontal strip region with only slight modifications of the coefficients.

Conclusion

In this Note we established an upper bound for the cost that can be minimized subject to a pair of matrix root-clustering equations. These equations were used to constrain the poles of the closed-loop system to lie in a hyperbolic or horizontal strip region contained in the left half-plane. The left hyperbolic region was chosen because of its ability to set desired bounds on the damping ratio and settling time. Because of the similarity between root-clustering equations of hyperbolic and horizontal strip regions, the results obtained for the left hyperbolic region can be applied to the left horizontal strip region with minor coefficient changes. Future research will focus on numerical techniques for solving the matrix algebraic equations.

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