Maximum Entropy Controller Synthesis for Colocated and Noncolocated Systems

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I. Introduction

MAXIMUM entropy controller synthesis was developed specifically for the robust control of flexible structures. The goal of this paper is to provide well-documented numerical examples that illustrate the characteristics of the method. The examples considered in this note were chosen to contrast the properties of maximum entropy controllers in two cases: colocated and noncolocated. Our results confirm previous observations, namely, that maximum entropy controllers employ positive real phase stabilization in the colocated case and wider and deeper notch gain stabilization in the noncolocated case. The computations were performed using a standard quasi-Newton technique in conjunction with the appropriate cost gradient expressions.

II. Maximum Entropy Controller Synthesis

Consider the structural model

\[
\dot{x} = \left( A + \sum_{i=1}^{n} \sigma_i A_i \right) x + Bu + D_i w
\]

\[
y = C x + D_2 w
\]

with feedback controller

\[
x_c = A_c x_c + B_c y
\]

\[
u = C_c x_c
\]

performance variables

\[
z = E_c x + E_c u
\]

and performance measure

\[
J(A_c, B_c, C_c) = \lim_{s \to \infty} \left\{ \frac{1}{s} \int_0^s z(t)z(t)dt \right\}
\]

where 

\[
x, y \in \mathbb{R}^n, u \in \mathbb{R}^m, \quad w \in \mathbb{R}^r, \quad C \in \mathbb{R}^{r \times n}, \quad \sigma_i \text{ is uncertain parameter representing uncertainty in } \omega_m \text{, and } A_i = \text{diag} \left\{ 0, \ldots, 0, \frac{1}{\omega_m}, 0, \ldots, 0 \right\}
\]

so that the \( i \)th \( 2 \times 2 \) diagonal block is the only nonzero entry in \( A_i \). The disturbance \( w \) is a standard white noise signal and \( c \) denotes expectation. The matrix \( A \) is assumed to be in real coordinates, that is,

\[
A = \text{diag} \left\{ -\eta_1, \omega_m, \ldots, -\eta_r, -\omega_m, -\eta_r \right\}
\]

In maximum entropy theory, the performance \( J(A_c, B_c, C_c) \) is given by

\[
J(A_c, B_c, C_c) = tr \tilde{Q} E \tilde{E}
\]

where \( \tilde{Q} \) satisfies the maximum entropy covariance equation

\[
0 = \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^T + \sum_{i=1}^{n} \delta_i \left[ \frac{1}{2} \tilde{A}_i \tilde{Q} + \tilde{Q} \tilde{A}_i^T + \frac{1}{2} \tilde{Q} \tilde{A}_i^T \right] + \tilde{D} \tilde{D}^T
\]

and where \( \tilde{A}, \tilde{A}_i, \tilde{D}, \) and \( \tilde{E} \) are defined by

\[
\tilde{A} \triangleq \begin{bmatrix} A & BC \\ B & A_c \end{bmatrix}, \quad \tilde{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}
\]

\[
\tilde{E} \triangleq [E,EcC_c]
\]

and \( \delta_i \) is a measure of the magnitude of the uncertainty \( \sigma_i \).

To minimize \( J(A_c, B_c, C_c) \) given by Eq. (9) where \( \tilde{Q} \) satisfies Eq. (10), we define a Lagrangian function

\[
L(A_c, B_c, C_c, \tilde{Q}) \triangleq tr \tilde{Q} E \tilde{E} + tr \tilde{P} \left( \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^T \right)
\]

\[
+ \sum_{i=1}^{n} \delta_i \left[ \frac{1}{2} \tilde{A}_i \tilde{Q} + \tilde{Q} \tilde{A}_i^T + \frac{1}{2} \tilde{Q} \tilde{A}_i^T \right] + \tilde{D} \tilde{D}^T
\]

where \( \tilde{P} \) is a nonzero Lagrange multiplier. Now by partitioning \( \tilde{Q} \) and \( \tilde{P} \) as
The first example is a two-mass system with a colocated sensor/actuator pair as shown in Fig. 1, where the measured output $y_c$ is the velocity of mass $M_1$ ($y_c$ and $y_m$ denote the outputs for the colocated and noncolocated cases). The dynamics of the system are given by

$$M_1 \ddot{q}_1 + C_1 \dot{q}_1 + K_1 q_1 = u + C_2(q_2 - q_1) + K_2(q_2 - q_1)$$

$$M_2 \ddot{q}_2 + C_2(q_2 - q_1) + K_2(q_2 - q_1) = 0$$

with the parameter values given in Fig. 1. After transforming to real normal coordinates the following $A$, $B$, and $C$ are obtained:

$$A = \begin{bmatrix} -0.0002 & 0.2208 & 0 & 0 \\ -0.2208 & -0.0002 & 0 & 0 \\ 0 & 0 & -0.0103 & 1.4322 \\ 0 & 0 & -1.4322 & -0.0103 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.1439 \\ 0.2168 \\ -0.0426 \\ 1.1892 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.0545 & 0.0819 & -0.0352 & 0.8181 \end{bmatrix}$$

The performance criterion was chosen so that LQG synthesis would place a notch at the second mode. This is accomplished when

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix $E_1$ weights the amplitude of the first mode but does not penalize the amplitude or velocity of the second mode. In practice this performance criterion reflects the situation in which the closed-loop performance depends primarily upon the lower frequency modes, while the higher frequency modes are uncertain.

Note that in Eq. (21) the nominal damped natural frequencies and damping ratios are $\omega_{d1} = 0.2208$, $\omega_{d2} = 1.4322$, $\zeta_1 = 0.0011$, and $\zeta_2 = 0.0072$, and that the plant is positive real. In Table 1 and Fig. 2 we compare the standard LQG design to three maximum entropy designs, where the only parameter that is varied is $\delta_2$, which is a measure of the magnitude of the uncertainty $\sigma_2$ in the second damped natural frequency. When compared to the LQG compensator, the maximum entropy method with a small measure of uncertainty $\delta_2$ first adjusts the phase so that the controller is stable. Then the method continues to alter the phase as $\delta_2$ increases, yielding positive real controllers as $\delta_2$ becomes large. The increase in robustness obtained by increasing $\delta_2$ was also assessed by determining the range of values of $\sigma_2$ for which the closed-loop system remains stable. This range of values, which increases with $\delta_2$, is a measure of the magnitude of the uncertainty $\sigma_2$ in the second damped natural frequency.

It is important to stress that, although the use of positive real controllers in the colocated case is standard practice to achieve robustness, the maximum entropy method is the only technique we know of that yields such controllers as a direct consequence of uncertainty.

IV. Illustrative Example: Noncolocated Case

In the second example we examine the same two-mass system as in Sec. III. However, in this example the sensor/actuator pair is noncolocated with measured output $y_m = q_2 = Cx$, where

![Fig. 1 Two-mass system.](image)

![Fig. 2 Compensator transfer functions.](image)
Also the matrix $E$ in Eq. (22) is increased by a factor of 10, so as to enhance the notch characteristics of the LQG compensator and to better demonstrate the properties of the maximum entropy controller synthesis. Based on these examples, we can conclude that maximum entropy controllers achieve robustness by tending toward phase stabilization in the colocated case and employing robustified notch filters in the noncolocated case. These are, of course, alternative methods for robustifying LQG designs, such as loop shaping, frequency weighting, and $\mathcal{H}_\infty$ theory. A comparison of these techniques with maximum entropy controllers remains a topic for future investigation.

\begin{table}  
\caption{Compensator comparison—colocated case}  
\begin{tabular}{cccccccc}
\hline
$\delta$ & Stable? & Minimum phase? & Positive real? & $\mathcal{J}_2$ Nominal state cost & $\mathcal{J}_2$ Nominal control cost & Stability boundary \\
\hline
0 (LQG) & No & Yes & No & 13.8522 & 1.9189 & 1.4318 & 1.4655 \\
0.3 & Yes & Yes & No & 14.2884 & 1.7881 & 1.5933 & 10^{-15} \\
10 & Yes & Yes & Yes & 15.2425 & 1.7933 & $-10^{-15}$ & 10^{-14} \\
1000 & Yes & Yes & Yes & 15.1942 & 1.8463 & $-10^{-15}$ & 10^{-14} \\
\hline
\end{tabular}  
\end{table}

\begin{table}  
\caption{Compensator comparison—noncolocated case}  
\begin{tabular}{cccccccc}
\hline
$\delta$ & Stable? & Minimum phase? & Positive real? & $\mathcal{J}_2$ Nominal state cost & $\mathcal{J}_2$ Nominal control cost & Stability boundary \\
\hline
0 (LQG) & Yes & Yes & Yes & 772.9009 & 11.0468 & 1.4245 & 1.4341 \\
0.2 & Yes & Yes & No & 776.1827 & 10.4267 & 1.3242 & 1.4887 \\
0.5 & Yes & No & No & 786.9195 & 8.5317 & 1.1482 & 1.7400 \\
1.0 & Yes & No & No & 816.7371 & 5.4372 & 1.0300 & 10^{-16} \\
\hline
\end{tabular}  
\end{table}

\begin{figure}  
\centering
\includegraphics[width=\textwidth]{colocated_case}
\caption{Colocated Case}
\end{figure}

\begin{figure}  
\centering
\includegraphics[width=\textwidth]{noncolocated_case}
\caption{Noncolocated Case}
\end{figure}

\begin{equation}  
C = [-0.1063 \quad 0.1597 \quad 0.0018 \quad -0.0419]  \quad (23)
\end{equation}

\section{Conclusions}

The purpose of this note was to contrast the robustness of maximum entropy controllers in the colocated and noncolocated cases, and to demonstrate a new computational technique for maximum entropy controller synthesis. Based on these examples, we can conclude that maximum entropy controllers achieve robustness by tending toward phase stabilization in the colocated case and employing robustified notch filters in the noncolocated case. The starting point for these designs was LQG theory, which, in this case, yielded rather sensitive controllers. There exist, of course, alternative methods for robustifying LQG designs, such as loop shaping, frequency weighting, and $\mathcal{H}_\infty$ theory. A comparison of these techniques with maximum entropy controllers remains a topic for future investigation.

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\section{References}

Multiobjective Controller Design Using Eigenstructure Assignment and the Method of Inequalities

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Introduction

In recent years, eigenstructure assignment has been an active topic of research in multivariable control theory. Since the degrees of freedom are available over and above pole assignment using state or output feedback,1 respectively, numerous researchers have exercised those degrees of freedom to make the systems have good insensitivity to perturbations in the system parameter matrices via eigenstructure assignment.1-5

Most eigenstructure assignment techniques in the last decade only pay attention to the optimal solutions for some special performance indices, e.g., \( \|V_B\|_2 \|V_R\|_2 \) where \( V_B \) is the right eigenvector matrix. However, many practical control systems are required to have the ability to satisfy simultaneously different and often conflicting performance criteria, for instance, closed-loop stability, low feedback gains, and insensitivity to model parameter variations.

In this Note, we provide a new approach to make the closed-loop system satisfy a set of required performance criteria with less conservatism, using eigenstructure assignment and the method of inequalities.6

Multiobjective Controller Design

Consider a linear multivariable time-invariant, completely controllable, state feedback system:

\[
\dot{x} = Ax + Bu, \quad u = Kx
\]

where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the control input vector, \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( K \in \mathbb{R}^{m \times n} \). Then the closed-loop system representation is given by \( \dot{x} = (A + BK)x \). Introducing an \( n \times n \)-dimensional eigenvector matrix \( V_B = [V_{B1}, V_{B2}, \ldots, V_{Bn}] \), where \( V_{Bi} \) (\( i = 1, 2, \ldots, n \)) is the right eigenvector corresponding to the eigenvalue \( \lambda_i \), a general solution for this problem can be given in the form of a parametric expression, \( \mathbf{K}(A, V_B) \), for all feedback gain matrices \( K \) which assign the self-conjugate set of eigenvalues \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \) to the closed-loop system. If both the vector \( A \) and the right eigenvector matrix \( V_B \) of \( (A + BK) \) are specified, the controller \( K \) is determined.

In practice, it is usually intended to locate the eigenvalue vector \( A \) in a well-defined set to meet the requirements of the practical control system (e.g., stability, speed of response, etc.). This leads to eigenvalue constraints, for example of the form \( \lambda_i \leq \lambda_i \leq \lambda_i \), where \( \lambda_i \in \mathbb{R} \) and \( \lambda_i \in \mathbb{R} \) are the lower bound vector and the upper bound vector, respectively. These constraints may be removed by considering the change of variables given by

\[
\lambda_i(v_i) = \lambda_i + (\lambda_i - \lambda_i)\sin^2(v_i),
\]

with \( v_i \in \mathbb{R} \). Since the system is assumed to be completely controllable, the \( i \)th closed right eigenvector \( V_{Bi} \) is given by

\[
V_{Bi} = (\lambda_iI - A)^{-1}BW_i, \quad i = 1, 2, \ldots, n
\]

where \( W_i \in \mathbb{R}^{m \times 1} \) and the matrix \( W = [W_1, W_2, \ldots, W_n] \).

For the case when the matrix \( (\lambda_iI - A) \) is not invertible, for example when one or more closed-loop eigenvalues are required to be identical to open-loop values, then the following alternative to Eq. (3) by Roppenecker2 and Liu and Patton3 can be used without loss of generality. Clearly, the right eigenvector matrix \( V_B \) is a function of \( Y = [v_1, v_2, \ldots, v_n] \) and \( W \), i.e., \( V_B(Y, W) \). Thus, the parametric formula of the controller matrix \( K \) can be described by \( K(Y, W) \). A parametric representation of the control matrix \( K \) is given by

\[
K(Y, W) = WV_B^*(Y, W)
\]

In most parameter insensitive design methods using eigenstructure assignment, the performance indices are given on the basis of the right eigenvector matrix. For example, a very common performance index is given by

\[
\phi(Y, W) = \|V_B\|_2 \|V_R\|_2
\]

where \( \|V_B\|_2 = \text{(maximum eigenvalue of } V_B^*V_B)_{1/2} \).

Though the performance index \( \phi(Y, W) \) can be used to represent an upper bound of the eigenvalue sensitivities, it is often conservative because of the following relations:

\[
\phi(Y, W) \geq \max\{\phi_i(Y, W) : i = 1, 2, \ldots, n\}
\]

where \( \phi_i(Y, W) \) is the individual sensitivity of the eigenvalue \( \lambda_i \) to perturbations in any of the elements of the matrices \( A \) and \( B \), defined by

\[
\phi_i^2(Y, W) = \frac{(V_B^*V_B)(V_B^*V_B)}{(V_L^*V_B)(V_L^*V_B)}, \quad i = 1, 2, \ldots, n
\]

where the superscript * denotes "conjugate-transposed." \( V_B \) is the \( i \)th closed-loop left eigenvector given by the relation \( V_B = V_{Bi} \) with the left eigenvector matrix \( V_L = [V_{L1}, V_{L2}, \ldots, V_{Ln}] \). Hence, to reduce the conservatism the problem becomes to find a pair \( (Y, W) \) such that

\[
\min_{Y,W}\phi_i(Y, W) \quad \text{for } i = 1, 2, \ldots, n
\]

To give a feel for the usefulness of the multiobjective approach as opposed to single-objective design techniques, let us consider the minimization of the cost functions \( \phi_i(Y, W) \) (\( i = 1, 2, \ldots, n \)). Let the minimum value of \( \phi_i \) be given by \( \phi_i^* \), for \( i = 1, 2, \ldots, n \), respectively. For these optimal values \( \phi_i^* \), there exist corresponding values given by \( \phi_j(\phi_j^*) \) (\( j \neq i \),...