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COLLEGE OF ENGINEERING
Department of Electrical Engineering
Space Physics Research Laboratory

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MEASUREMENTS OF ATMOSPHERIC PRESSURE, TEMPERATURE, DENSITY, AND COMPOSITION AT VERY HIGH ALTITUDES

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I. INTRODUCTION

This is one of a series of reports on a research effort whose objective is the determination of the ambient pressure, temperature, density, and composition of the earth's atmosphere at altitudes above the level where the mean free path of the various particles is appreciably greater than the dimensions of the measuring object.

The research effort is devoted to several tasks:

(a) a theoretical study of the general measurement problem, and several associated problems;

(b) development of suitable sensors;

(c) development of associated instrumentation to permit fruitful employment of the sensors; and

(d) the development of an ultra-high vacuum system capable of achieving pressures as low as the state of the art permits, with the final objective of sensor calibration and testing.

II. DISCUSSION

During the quarter, the effort was concerned with items (a), (b), and (d) above, as follows.

1. THEORETICAL STUDY

There are naturally several areas in which theoretical studies are essential or helpful in realization of the objectives of the program. These include:

(a) General development of the measurement technique employing sensors in spherical or possibly other geometries, in both the orbiting and near vertical trajectory case.[12] In this treatment, a sphere with an appropriate chamber and opening is assumed to be moving at a specified velocity in a specified manner. A general solution is obtained for the relationships between "internal" and "external" pressure and density.

(b) Consideration of the orifice problem, that is; determination of the optimum configuration for the port through which the sensor must sample the external gas. That is, should it be "knife-edge," a cylindrical port, or otherwise?

(c) Consideration of the general problem of the motion of an ejected sphere or other geometry in free flight with the objective of arriving at an arrangement which will maximize the probability of achieving the desired motion after ejection. The considerations of (a) above suggest the optimum orientation pattern.

(d) Study of various aspects of particle motion in the sensor, including the energy situation, effect and importance of initial energy, optimum configuration to maximize sensitivity, and other considerations appropriate to use of an ionization-gage-type device.

Of the four listed topics, (a) has received considerable attention. As a result of this work, it is now possible to include a general mathematical development in this report as an appendix. Work on topics (b) and (d) has been initiated and will be discussed in a later report, when some significant conclusions have been reached.

2. DEVELOPMENT OF SENSORS

As discussed previously, a device which is considered to offer considerable promise for use in measurements of the type constituting the objectives of this program is the omegatron (Fig. 1).* The chief reasons for this choice are, first, that it is small compared to other possible devices such as the r-f spectrometer and time-of-flight spectrometer, and accordingly appears greatly to simplify problems arising from the necessity of maintaining good diffusion and flow equilibrium between "external" and "internal" gas; and second, that it is a simple device physically, easily constructed and thus less subject to influence by externally applied forces and other disturbing effects.

An inevitable result of the small size and simplicity is less sensitivity, and a need for a strong, d-c, magnetic field. Recent advances in instrumentation techniques, however, offer considerable promise of greatly increasing the useful range with respect to altitude, and the investigation is proceeding along these lines.

On the basis of a study of the literature and the advice and counsel of personnel of the Tube Shop of the Department of Electrical Engineering a first omegatron (Fig. 2) has been constructed.

This first model is very crude, and is expected to be useful primarily as a means of gaining familiarity with constructional procedures to show how a more

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*Several references appear at the end of the report.

useful model should be built. It is planned, however, to operate the device to become acquainted with its operational aspects, and to provide some experimental results for substantiation of published and computed data.

Two general schemes of detecting resonant particles are being considered and will be explored. The first technique involves direct collection of the ions and thus provides, as an output signal, a current resulting from recombination at the collector. Most tubes reported in the references have employed this technique as it afforded, in laboratory use, adequate sensitivity. The second method reported was based upon the concept of measuring the energy absorbed from the r-f field by the resonant ions.[6] Although this technique appears to hold greater immediate promise insofar as sensitivity and relative complexity are concerned, amplification techniques applied to the direct-collection case offer attractive possibilities. Both schemes will be employed until the better becomes apparent.

It is expected that, by the end of the next reporting period, newer models of the omegatron will have been operated using both detection techniques.

3. ULTRA-HIGH VACUUM SYSTEM

The ultra-high vacuum system discussed in the previous report has been completed and is operating continuously as of the end of the reporting period. Figure 3 is a photograph of the system; and Fig. 4 is a diagram detailing the various major components of which the system is composed.

The system has been developed on the basis of years of experience in this laboratory with "standard" vacuum systems, and on the basis of information relative to the achievement of "ultra-high vacuum" recorded in the literature.[5,7]

The system has been brought into operation gradually to insure proper operation of the various elements. At the close of the period covered by this report, all elements had been checked out and the system was operating in the interval 10^{-6} mm Hg to 10^{-7} mm Hg. The ion-gage control employed at that time was inadequate for proper gage degassing and functioning and thus was limiting the attainment of lower pressures.

BIBLIOGRAPHY

The following listing includes papers and reports relevant to the subject of the research effort, and which may have been mentioned in this or the previous quarterly report.

1. Sommer, H., Thomas, H. A., and Hipple, J. A., "Measurement of e/m by Cyclotron Resonance," Phys. Rev., 82, 697-702 (June, 1951).

2. Edwards, A. G., "Some Properties of a Simple Omegatron Type Mass Spectrometer," Brit. J. Appl. Phys., 6, 44-48 (February, 1955).

3. Bell, R. L., "Omegatron as a Leak Detector," J. Sci. Instr., 33, 269 (July, 1956).

4. Wagener, J. S., and Marth, P. T., "Analysis of Gases at Very Low Pressures by Using the Omegatron Spectrometer," J. Appl. Phys., 28, 1027-1030 (September, 1957).

5. Alpert, D., "New Developments in the Production and Measurement of Ultra-High Vacuum," J. Appl. Phys., 24, 860 (July, 1953).

6. Sommer, H., and Thomas, H. A., "Detection of Magnetic Resonance by Ion Resonance Absorption," Phys. Rev., 78, 806 (June, 1950).

7. Alpert, D., and Buritz, R. S., "Ultra-High Vacuum II. Limiting Factors on the Attainment of Very Low Pressures," J. Appl. Phys., 25, 202 (February, 1954).

8. Berry, C. E., "Ion Trajectories in the Omegatron," J. Appl. Phys., 25, 28 (January, 1954).

9. Brubaker, W. M., "Influence of Space Charge on the Potential Distribution in Mass Spectrometer Ion Sources," J. Appl. Phys., 26, 1007 (August, 1955).

10. Brubaker, W. M., and Perkins, G. D., "Influence of Magnetic and Electric Field Distribution on the Operation of the Omegatron," Rev. Scien. Inst., 27, 720 (September, 1956).

11. Hopkins, N. J., "A Magnetic Field Strength Meter Using the Proton Magnetic Moment," Rev. Scien. Inst., 20, 401 (June, 1949).

12. Spencer, N. W., Boggess, R. L., Lagow, H. E., and Horwitz, R., "On the Use of Ionization Gage Devices at Very High Altitudes," Jour. Amer. Rocket Soc., to be published.

13. Spencer, N. W., and Boggess, R. L., "Radioactive Ionization Gage Pressure Measurement System," Jour. Amer. Rocket Soc., 29, 68 (January, 1959).

14. Spencer, N. W., Bi-polar Probe Instrumentation No. 1, UMRI Report 2521, 2816, 1-1-S, Ann Arbor, October, 1958.

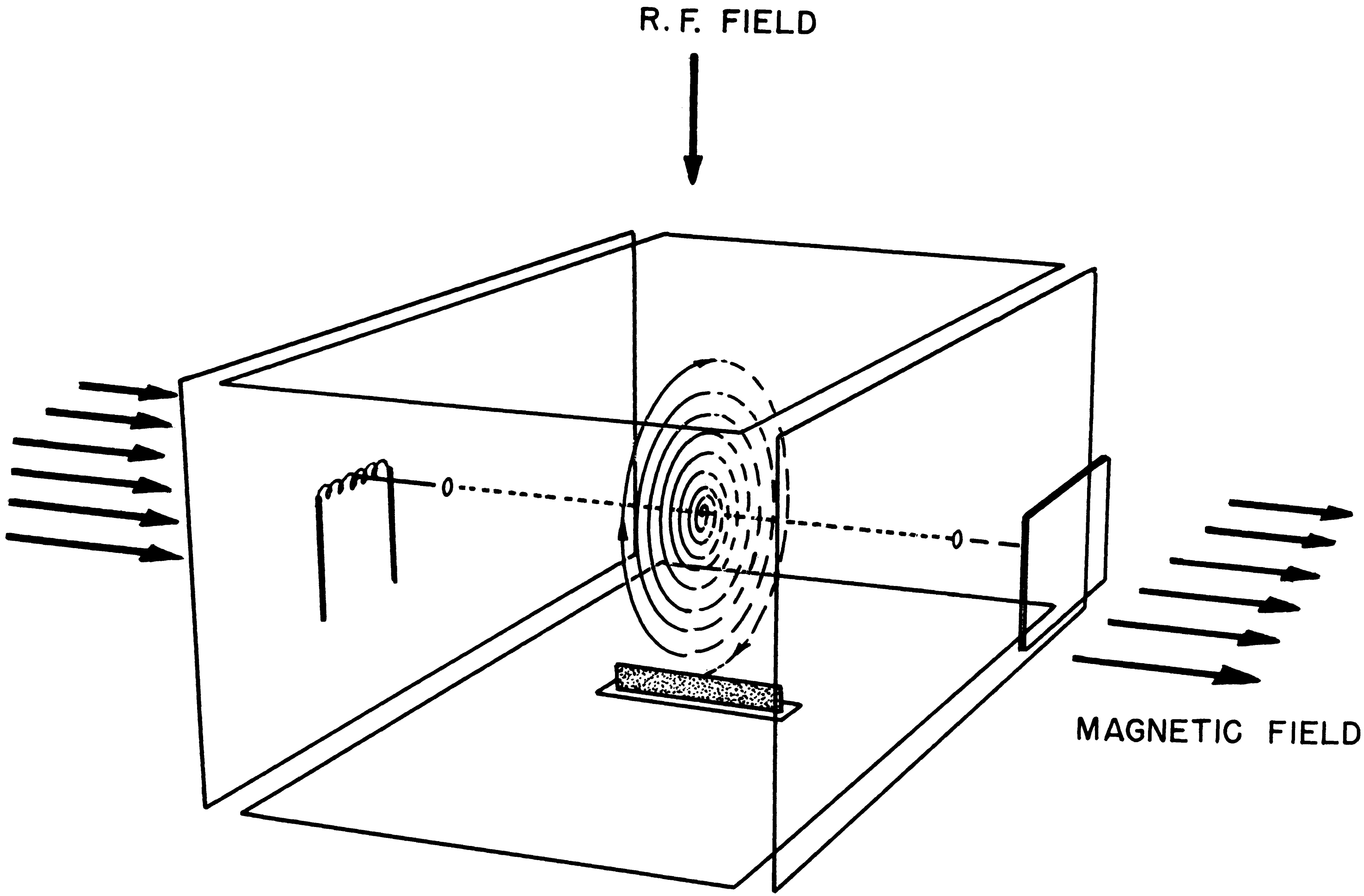


Fig. 1. Functional diagram of omegatron.

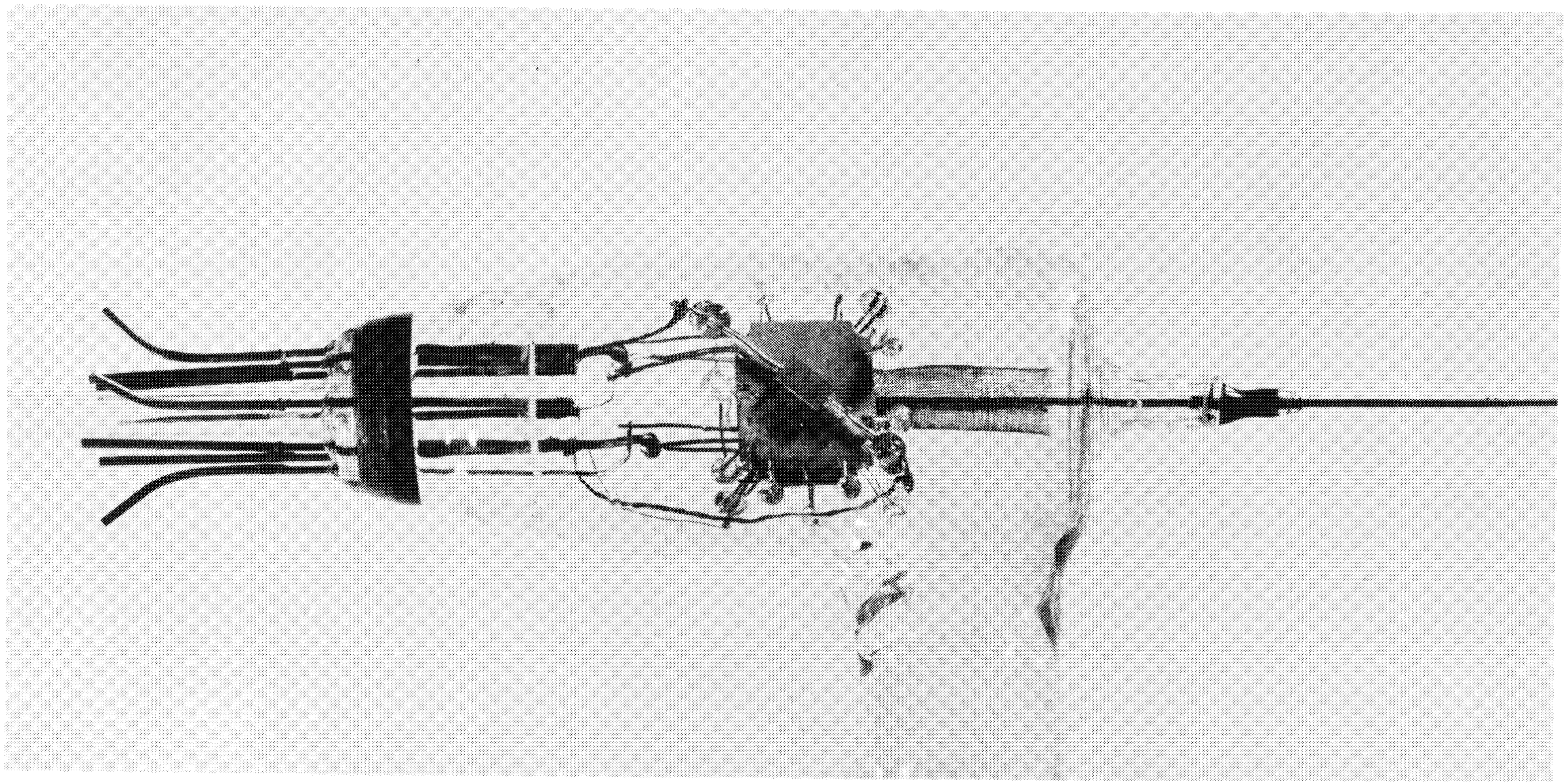


Fig. 2. Experimental omegatron.

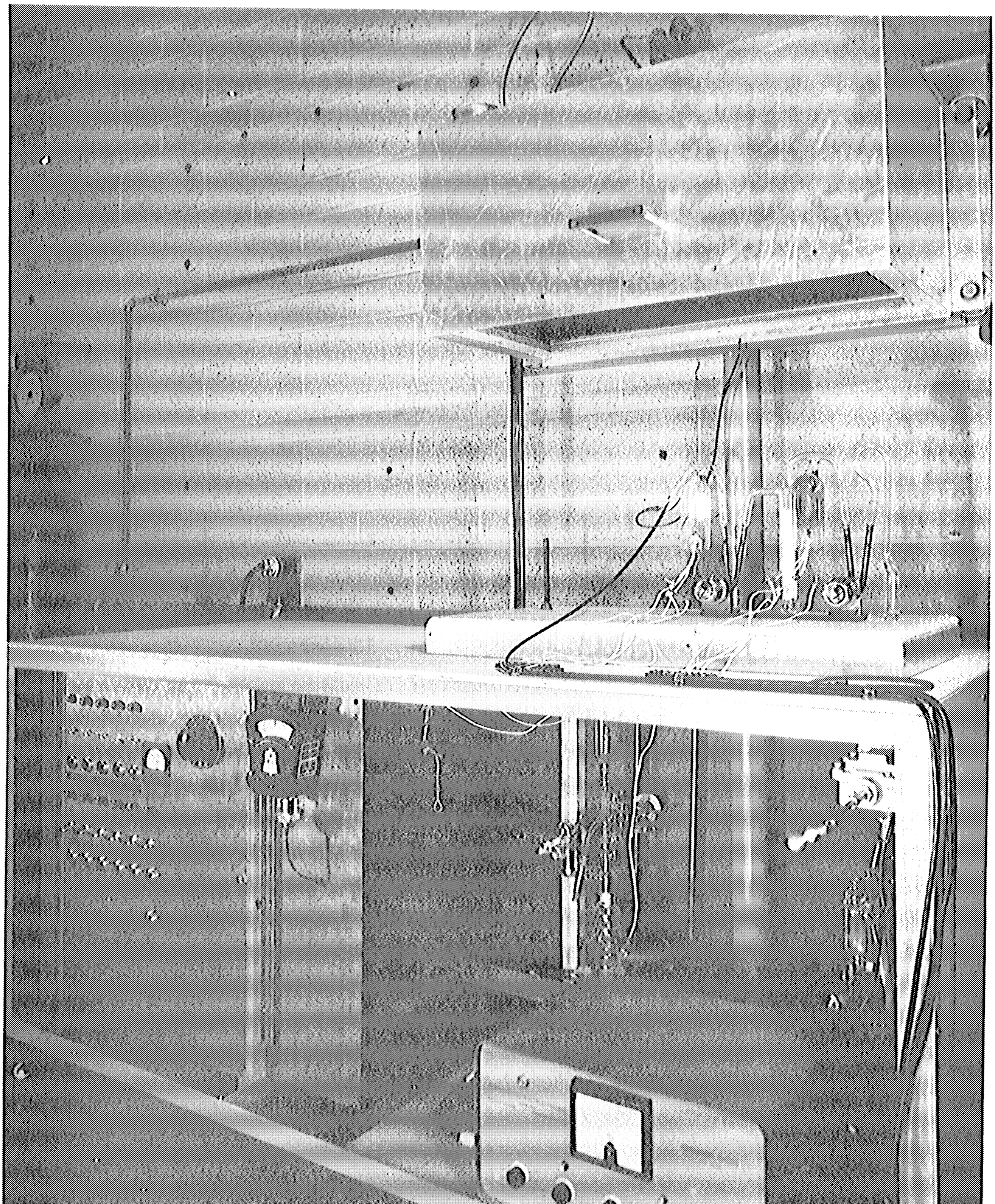


Fig. 3. Ultra-high vacuum system.

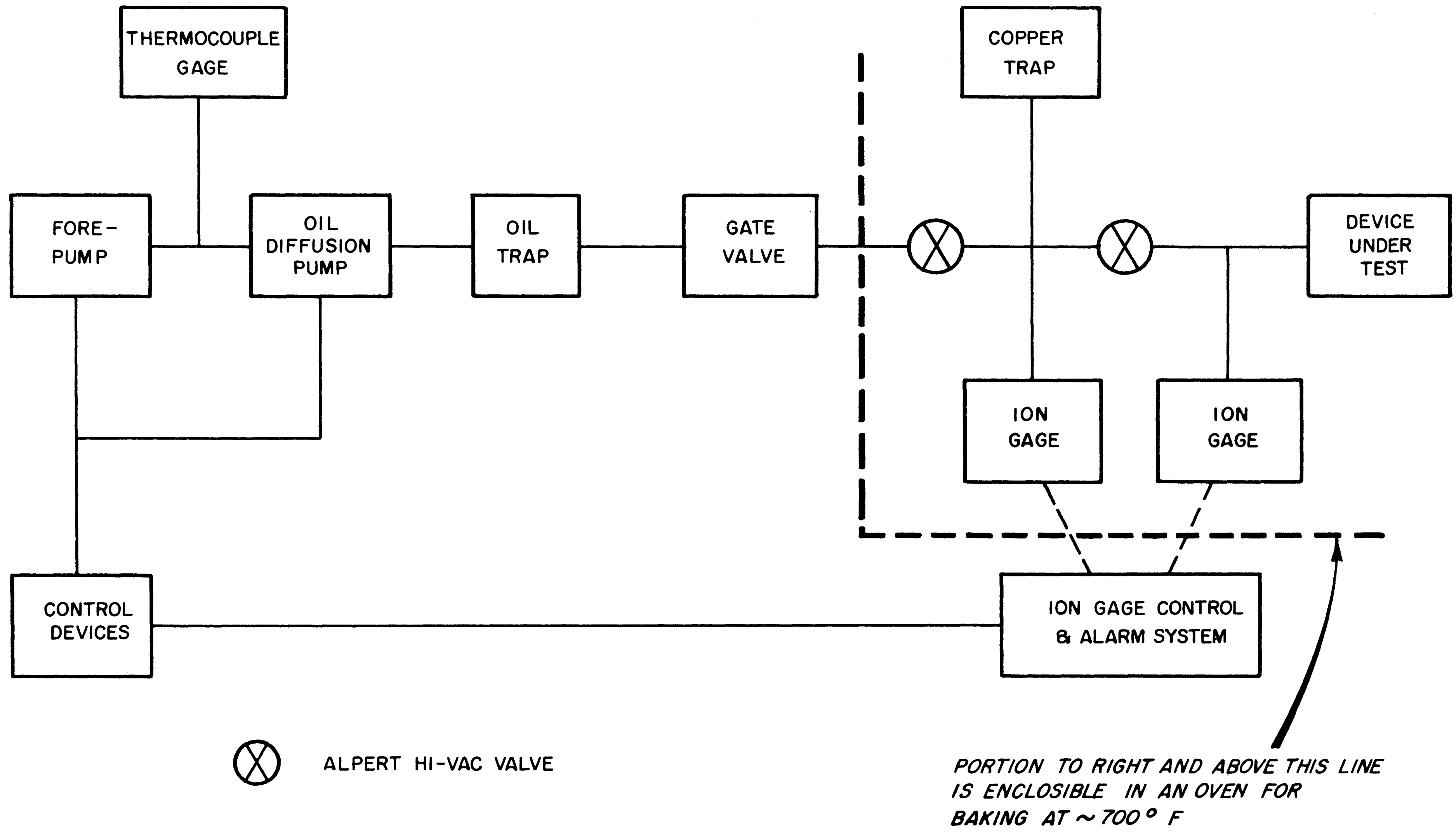


Fig. 4. Block diagram of ultra-high vacuum system.

APPENDIX

ON THE DETERMINATION OF PRESSURE, TEMPERATURE, AND DENSITY OF UPPER ATMOSPHERE

The following development was prepared by Mr. Madhoo Kanal for inclusion in this report.

TABLE OF SYMBOLS

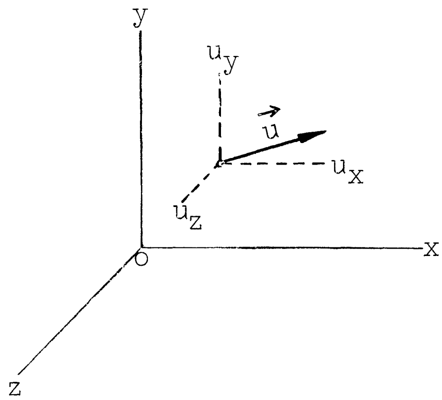
P_i = Pressure inside the chamber.
T_i = Temperature inside the chamber.
P_o = Pressure outside.
T_o = Temperature outside.
θ_x = Angle between the direction of sphere motion and x-axis.
θ_y = Angle between the direction of sphere motion and y-axis.
θ_z = Angle between the direction of sphere motion and z-axis.
u_x = x-component of the particle velocity.
u_y = y-component of the particle velocity.
u_z = z-component of the particle velocity.
θ = Angle between x-axis and the normal component of u_x.
ϕ = Angle between y-axis and the normal component of u_y.
ψ = Angle between z-axis and the normal component of u_z.
α = Angle between the direction of sphere motion and the orientation of the chamber hole with respect to the origin.
c_{m_i} = The most probable velocity of the particle inside the chamber.
c_{m_o} = The most probable velocity of the particle outside.
N_i = The number of particles per unit volume inside the chamber.
N_o = The number of particles per unit volume outside the chamber.
G_i = The number of particles leaving the chamber in time Δt.
G_o = The number of particles entering the chamber in time Δt.
W = The velocity of the sphere.
$\vec{V}$ = The imaginary cylindrical volume created by the normal components of W_x, W_y, W_z, u_x, u_y, u_z outside.
$\vec{V}'$ = The imaginary cylindrical volume created by the normal component of u_x inside the chamber.
m = The mass of a particle.

INTRODUCTION

Sometimes a few assumptions make a conception more comprehensible. These assumptions are justifiable and necessary. Throughout the solution of this problem I will not define the most probable particle velocity (c_m). Nevertheless, there are some distributions, and Maxwellian distribution is one of them, that define 'c_m' to some degree of approximation. I therefore sincerely hope to leave the readers' intuition unoffended at the end of the paper.

I do not mean to restrict myself to that limited situation which leaves the results reached in doubt; on the contrary, some reflection on the statistical behavior of the systems will throw light on the fact that random behavior of the subsystems is combined in one system as a whole, which in turn is controlled by these subsystems. In the theory of Boral sets these subsystems (as I prefer to call them) are termed subsets. The distribution of these subsets is very important as regards their relevancy to the behavior of the system as a whole. The most important factors in their distribution are (1) that the subsets should abut each other, and (2) that there are no empty subsets present in the system.

Consider a system of randomly distributed particles moving about in a random manner. Let a particle selected at random move in a certain arbitrary direction with velocity u. Let u_x, u_y and u_z be the components of the velocity along x, y and z directions. The particle will continue to move along its path with velocity u as long as it does not collide with another particle, resulting in its change of direction of path and velocity. But as we know that the collision does occur and that the change in its path does take place, resulting in the change of its energy and hence velocity, we are thus bound to restrict our situation to the statistical behavior of the system as a whole and neglect the behavior of the particle as an individual. Nevertheless, we are not yet at the end of our journey because if we cannot study the actual behavior of an individual particle, we still can study its probable behavior. Then the question arises whether there does exist a distribution function which can foretell the probable behavior of an individual particle at any instant. This is the point, mentioned earlier, regarding the most probable particle velocity, and at present the question is best answered by the Maxwellian distribution. Now the probability that the velocity of a particle selected at random shall have components lying between u_x and $u_x + du_x$, u_y and $u_y + du_y$, and u_z and $u_z + du_z$ is given by the velocity distribution function. Mathematically



$$P = f(u_x, u_y, u_z)du_x\, du_y\, du_z \tag{1}$$

where P is the probability and f the function of distribution. If N is the number of particles per unit volume with the above velocity distribution and c_m is the most probable particle velocity, then

$$N = \iiint k\, e^{-\frac{(u_x^2+u_y^2+u_z^2)}{c_m^2}} \frac{du_x}{c_m}\frac{du_y}{c_m}\frac{du_z}{c_m} \qquad (2)$$

where k is a constant.

$$\therefore \quad N = k \cdot \pi^{3/2}, \text{ or } k = N/\pi^{3/2} \qquad (3)$$

N is the number density; therefore, k has the units of number density and differs from N by the factor of $\pi^{-3/2}$ in magnitude.

DERIVATION OF DIFFUSION EQUATION

A spherical chamber with an opening of cross-sectioned area Σ is filled with a monotypic gas. It is moving in outer space, where the mean free path of particles is large compared to the dimensions of the vessel, with the drift velocity W. When the sphere is in flight, the chamber hole is opened. The particles inside the chamber will begin to diffuse out into space, and the particles in outer space will begin to diffuse into the chamber, until after some time equilibrium is established between the number of particles that get out (G_i) and the number of particles that get into the chamber (G_o) from outside in time Δt. Mathematically, the equilibrium will exist if

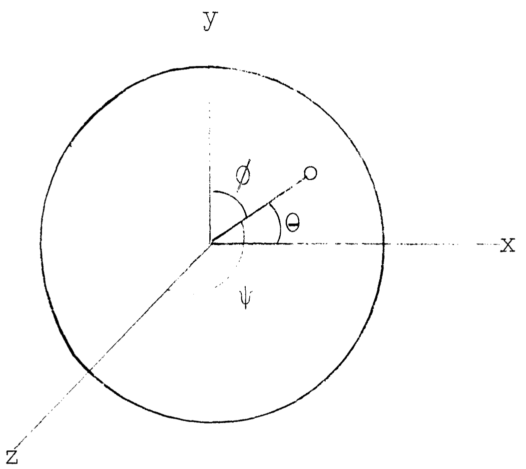
$$G_i = G_o \qquad (4)$$

if P_i is the pressure exerted by the gas particles inside the chamber at the instant when equilibrium is established and T_i is the temperature, and if P_o and T_o are the pressure and temperature outside at the same instant, respectively, then a certain relation is expected to correspond to these parameters. This paper contains the derivation of that relation.

Consider any arbitrary orientation of the chamber hole with respect to the arbitrary axes of reference as shown in the figure. Let W_x, W_y, and W_z be the components of the drift velocity along x, y, and z axes. Then the sum of the normal components of the velocities on the chamber hole is

$$\bar{V} = (W_x+u_x)\hat{i} + (W_y+u_y)\hat{j} + (W_z+u_z)\hat{k} \qquad (5)$$

$$= (W_x+u_x)\cos\theta + (W_y+u_y)\cos\phi + (W_z+u_z)\cos\psi \quad . \qquad (6)$$



The imaginary volume created in time Δt normal to the chamber hole is

$$u = \Big[(W_x+u_x)\cos\theta + (W_y+u_y)\cos\phi + (W_z+u_z)\cos\psi\Big]\Delta t\Sigma \tag{7}$$

The number of particles that enter the chamber hole in time Δt, therefore, is given by

$$G_o = \Sigma\Delta t k \iiint \Big[(W_x+u_x)\cos\theta + (W_y+u_y)\cos\phi + (W_z+u_z)\cos\psi\Big] e^{-\frac{(u_x^2+u_y^2+u_z^2)}{c_{m_o}^2}} \frac{du_x}{c_{m_o}}\frac{du_y}{c_{m_o}}\frac{du_z}{c_{m_o}} \tag{8}$$

The limits of integration for u_x, u_y, and u_z are

$$G_o = \Sigma\Delta t k \int_{u_x=-\infty}^{\infty} \int_{u_y = -\frac{1}{\cos\phi}\left\{(w_x+u_x)\cos\theta + (W_z+u_z)\cos\phi\right\} - W_y} \int_{\left(u_z = -\frac{1}{\cos\psi}\left\{(W_x+u_x)\cos\theta + (W_y+u_y)\cos\phi\right\} - W_z\right)} \Big[(W_x+u_x)\cos\theta + (W_y+u_y)\cos\phi + (W_z+u_z)\cos\psi\Big] e^{-\frac{(u_x^2+u_y^2+u_z^2)}{c_{m_o}^2}} \frac{du_x}{c_{m_o}}\frac{du_y}{c_{m_o}}\frac{du_z}{c_{m_o}} \tag{9}$$

From Eq. (3) we have

$$k = \frac{N_o}{\pi^{3/2}} \tag{10}$$

where N_o is the number of particles per unit volume in the outer atmosphere. Therefore, substituting Eq. (10) in Eq. (9), we get

$$G_o = \frac{\sum \Delta t N_o}{\pi^{3/2}} \iiint \left[(W_x+u_x)\cos\Theta + (W_y+u_y)\cos\phi + (W_z+u_z)\cos\psi \right] e^{-\frac{(u_x^2+u_y^2+u_z^2)}{c_{m_o}^2}} \frac{du_x}{c_{m_o}} \frac{du_y}{c_{m_o}} \frac{du_z}{c_{m_o}} \quad . \tag{11}$$

It will be easier if we change the limits to $-\infty$ to α. Hence Eq. (11) gives

$$G_o = \frac{\sum \Delta t N_o}{\pi^{3/2}} \int \int_{-\infty}^{\infty} \int \left[(W_x+u_x)\cos\Theta + (W_y+u_y)\cos\phi + (W_z+u_z)\cos\psi \right]$$
$$U \left[(W_x+u_x)\cos\Theta + (W_y+u_y)\cos\phi + (W_z+u_z)\cos\psi \right]$$
$$\exp\left[-\frac{(u_x^2+u_y^2+u_z^2)}{c_{m_o}^2} \right] \frac{du_x}{c_{m_o}} \frac{du_y}{c_{m_o}} \frac{du_z}{c_{m_o}} \tag{12}$$

Multiplying and dividing Eq. (12) by c_{m_o} and letting

$$t = \left(\frac{W_x}{c_{m_o}} \cos\Theta + \frac{W_y}{c_{m_o}} \cos\phi + \frac{W_z \cos\psi}{c_{m_o}} \right)$$

$$x = \frac{u_x}{c_{m_o}}$$

$$y = \frac{u_y}{c_{m_o}}$$

and

$$z = \frac{u_z}{c_{m_o}} \quad ,$$

we get from Eq. (12)

$$G_o = \iiint \frac{\sum \Delta t N_o c_{m_o}}{\pi^{3/2}} [t + x\cos\Theta + y\cos\phi + z\cos\psi] \, U [t + x\cos\Theta \; y\cos\phi + z\cos\psi]$$
$$\exp[-x^2 + y^2 + z^2] \, d_x d_y d_z \quad . \tag{13}$$

Since the limits are from $-\infty$ to ∞, we will use the bilateral transforms which will give us

$$G_o \doteq \frac{\Sigma \Delta t N_o c_{m_o}}{\pi^{3/2}} \iiint e^{\frac{(x \cos\Theta + y \cos\phi + z \cos\psi) P - x^2 - y^2 - z^2}{P}} d_x d_y d_z \quad . \tag{14}$$

$$\therefore \quad G_o = \frac{\Sigma \Delta t N_o c_{m_o}}{\pi^{3/2}} \int \int_{-\infty}^{\infty} \int \exp\left[-\left(x - \frac{P\cos\Theta}{2}\right)^2 - \left(y - \frac{P\cos\phi}{2}\right)^2 - \left(z - \frac{P\cos\psi}{2}\right)^2 + \frac{P^2\cos^2\Theta}{4} + \frac{P^2\cos^2\phi}{4} + \frac{P^2\cos^2\psi}{4}\right] \frac{du_x}{c_{m_o}} \frac{du_y}{c_{m_o}} \frac{du_z}{c_{m_o}} \quad . \tag{15}$$

$$= \frac{\Sigma \Delta t N_o c_{m_o}}{\pi^{3/2}} e^{\frac{P^2}{4}\{\cos^2\Theta + \cos^2\phi + \cos^2\psi\}} \int \int_{-\infty}^{\infty} \int \exp\left[-\left(x - \frac{P\cos\Theta}{2}\right)^2 - \left(y - \frac{P\cos\phi}{2}\right)^2 - \left(z - \frac{P\cos\psi}{2}\right)^2\right] d_x d_y d_z \quad . \tag{16}$$

$$= \frac{\Sigma \Delta t N_o c_{m_o}}{\pi^{3/2}} e^{\frac{P^2}{4}} \times (\pi^{3/2}) \tag{17}$$

$$\therefore \quad \cos^2\Theta + \cos^2\phi + \cos^2\psi \equiv 1$$

and

$$\int \int_{-\infty}^{\infty} \int \exp\left[-\left(x - \frac{P\cos\Theta}{2}\right)^2 - \left(y - \frac{P\cos\phi}{2}\right)^2 - \left(z - \frac{P\cos\psi}{2}\right)^2\right] d_x d_y d_z \equiv \pi^{3/2} \quad .$$

$$\therefore \quad G_o = \frac{\Sigma \Delta t N_o c_{m_o}}{1} e^{\frac{P^2}{4}} \quad . \tag{18}$$

The inverse transform of $e^{\frac{P^2}{4}}$ is given by

$$e^{\frac{P^2}{4}} \doteq \frac{1}{2} \operatorname{erfc}(-t) \quad , \tag{19}$$

$$\therefore \quad G_o = \frac{\Sigma \Delta t N_o c_{m_o}}{2} \int_{-\infty}^{t} \operatorname{erfc}(-t)\, dt \quad , \tag{20}$$

where t has already been defined.

Let $(-t) = S$

$$\therefore \quad G_o = \frac{\sum \Delta t N_o c_{m_o}}{2} \int_s^{\infty} \operatorname{erfc}(s) ds \quad . \tag{21}$$

Let R be any variable such that; when $R \to \alpha$, Eq. (21) becomes

$$G_o = \frac{\sum \Delta t N_o c_{m_o}}{2} \operatorname{Lim}_{R \to \alpha} \{R + t - \int_0^R \operatorname{erf} s ds + \int_0^s \operatorname{erf} s ds\} \tag{22}$$

$$= \frac{\sum \Delta t N_o c_{m_o}}{2} \operatorname{Lim}_{R \to \alpha} \left\{ R + t - R \operatorname{erf} R - \frac{1}{\sqrt{\pi}} (e^{-R^2} - 1) + (s) \operatorname{erf}(s) + \frac{1}{\sqrt{\pi}} (e^{-t^2} - 1) \right\} \tag{23}$$

which obviously gives us

$$G_o = \frac{\sum \Delta t N_o c_{m_o}}{2} \left\{ t \operatorname{erfc}(-t) + \frac{1}{\sqrt{\pi}} e^{-t^2} \right\} . \tag{24}$$

But t has been defined as

$$t = \frac{W_x}{c_{m_o}} \cos \theta + \frac{W_y}{c_{m_o}} \cos \phi + \frac{W_z}{c_{m_o}} \cos \psi \quad .$$

Substituting for t in Eq. (24), we get

$$G_o = \frac{\sum \Delta t N_o c_{m_o}}{2} \left[\left(\frac{W_x}{c_{m_o}} \cos \theta + \frac{W_y}{c_{m_o}} \cos \phi + \frac{W_z}{c_{m_o}} \right) \operatorname{erfc} \left[- \left(\frac{W_x}{c_{m_o}} \cos \theta + \frac{W_y}{c_{m_o}} \cos \phi + \frac{W_z}{c_{m_o}} \cos \psi \right) \right] + \frac{1}{\sqrt{\pi}} \exp \left[- \left(\frac{W_x}{c_{m_o}} \cos \theta + \frac{W_y}{c_{m_o}} \cos \phi + \frac{W_z}{c_{m_o}} \cos \psi \right)^2 \right] \right] \tag{25}$$

Equation (25) gives the number of particles entering the chamber in time Δt when equilibrium is established. For the particles inside the chamber, $W = 0$. Hence the imaginary volume created by the x-component of particle velocity is

$$\bar{V}' = \sum \Delta t u_x \quad . \tag{26}$$

The fraction of number of particles per unit volume with the velocity components between u_x and $u_x + du_x$ is given by the Maxwellian equation, i.e.,

$$\frac{dN_i}{N_i} = \frac{1}{\sqrt{\pi}} e^{-\frac{u_x}{c_{m_i}}} \frac{du_x}{c_{m_i}} \tag{27}$$

where N_i is the number of particles per unit volume inside the chamber and c_{m_i} the most probable particle velocity of those particles. The number of particles, G_i, leaving the chamber in time Δt when the equilibrium is established is, therefore,

$$G_i = \frac{\Sigma \Delta t N_i c_{m_i}}{\sqrt{\pi}} \int_0^{\alpha} \frac{u_x}{c_{m_i}} e^{-\frac{u_x^2}{c_{m_i}^2}} \frac{du_x}{c_{m_i}} \tag{28}$$

$$\therefore \quad G_i = \frac{\Sigma \Delta t N_i c_{m_i}}{\sqrt{\pi}} \left(\frac{1}{2}\right) \tag{29}$$

Thus for condition (4), we must have

$$\frac{\Sigma \Delta t N_i c_{m_i}}{2\sqrt{\pi}} = \frac{\Sigma \Delta t N_i c_{m_o}}{2} \Bigg[\left(\frac{W_x}{c_{m_o}} \cos\theta + \frac{W_y}{c_{m_o}} \cos\phi + \frac{W_z}{c_{m_o}} \cos\psi\right) \operatorname{erfc}\left\{-\left(\frac{W_x}{c_{m_o}} \cos\theta + \frac{W_y}{c_{m_o}} \cos\phi + \frac{W_z}{c_{m_o}} \cos\psi\right)\right\} + \exp\left\{-\left(\frac{W_x}{c_{m_o}} \cos\theta + \frac{W_y}{c_{m_o}} \cos\phi + \frac{W_z}{c_{m_o}} \cos\psi\right)^2\right\} \Bigg] \tag{30}$$

Cancelling the common terms and rearranging the rest, we get

$$\frac{N_i c_{m_i}}{N_o c_{m_o}} = \Bigg[\sqrt{\pi} \left(\frac{W_x}{c_{m_o}} \cos\theta + \frac{W_y}{c_{m_o}} \cos\phi + \frac{W_z}{c_{m_o}} \cos\psi\right) \operatorname{erfc}\left\{-\left(\frac{W_x}{c_{m_o}} \cos\theta + \frac{W_y}{c_{m_o}} \cos\phi + \frac{W_z}{c_{m_o}} \cos\psi\right)\right\} + \exp\left\{-\left(\frac{W_x}{c_{m_o}} \cos\theta + \frac{W_y}{c_{m_o}} \cos\phi + \frac{W_z}{c_{m_o}} \cos\psi\right)\right\} \Bigg] \tag{31}$$

Now

$$P = \frac{1}{2} m N c_m^2 \tag{32a}$$

and

$$TK = \frac{1}{2} m c_m^2 \tag{32b}$$

If we assume that the particles that enter the chamber are of the same mass as the monotypic gas in the chamber, we get

$$\frac{N_i c_{m_i}}{N_o c_{m_o}} = \frac{P_i}{P_o} \times \sqrt{\frac{T_o}{T_i}} \quad . \tag{33}$$

Substituting Eq. (33) in Eq. (31), we get

$$\frac{P_i}{P_o} = \sqrt{\frac{T_i}{T_o}}\left[\sqrt{\pi}\left(\frac{W_x}{c_{m_o}}\cos\theta + \frac{W_y}{c_{m_o}}\cos\phi + \frac{W_z}{c_{m_o}}\cos\psi\right)\operatorname{erfc}\left\{-\left(\frac{W_x}{c_{m_o}}\cos\theta + \frac{W_y}{c_{m_o}}\cos\phi + \frac{W_z}{c_{m_o}}\cos\psi\right)\right\} + \exp\left\{-\left(\frac{W_x}{c_{m_o}}\cos\theta + \frac{W_y}{c_{m_o}}\cos\phi + \frac{W_z}{c_{m_o}}\cos\psi\right)^2\right\}\right] . \tag{34}$$

Now W_x, W_y and W_z are the components of W such that

and

$$\left.\begin{aligned} W_x &= W\cos\theta_x \\ W_y &= W\cos\theta_y \\ W_z &= W\cos\theta_z \end{aligned}\right\} \tag{35}$$

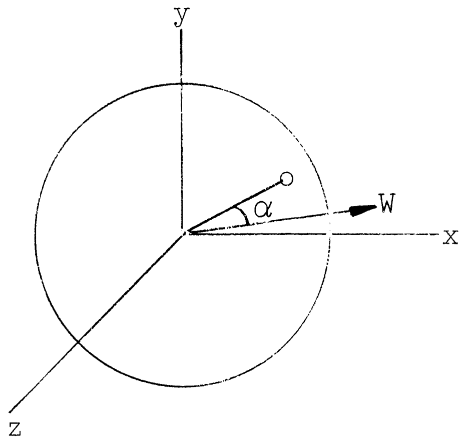
where θ_x, θ_y, and θ_z are the angles that W makes with the x, y, and z axes of reference. Substituting for W_x, W_y, and W_z in Eq. (34), we get

$$\frac{P_i}{P_o} = \sqrt{\frac{T_i}{T_o}}\left[\sqrt{\pi}\,\frac{W}{c_{m_o}}\left(\cos\theta\cos\theta_x + \cos\phi\cos\theta_y + \cos\psi\cos\theta_z\right)\right.$$
$$\operatorname{erfc}\left\{-\frac{W}{c_{m_o}}\left(\cos\theta\cos\theta_x + \cos\phi\cos\theta_y + \cos\psi\cos\theta_z\right)\right\}$$
$$\left. + \exp\left[-\frac{W^2}{c_{m_o}^2}\left(\cos\theta\cos\theta_x + \cos\phi\cos\theta_y + \cos\psi\cos\theta_z\right)^2\right]\right] . \tag{36}$$

From analytical and vector algebra we know that

$$\cos\theta\cos\theta_x + \cos\phi\cos\theta_y + \cos\psi\cos\theta_z = \cos\alpha \tag{37}$$

where α is the angle between the velocity vector W and the line joining the chamber hole and origin as shown in the figure.



Therefore Eq. (36) becomes

$$\frac{P_i}{P_o} = \sqrt{\frac{T_i}{T_o}} \left[\sqrt{\pi} \frac{W}{c_{m_o}} \cos\alpha \; \mathrm{erfc}\left(- \frac{W}{c_{m_o}} \cos\alpha \right) + \exp\left[- \frac{W^2}{c_{m_o}^2} \cos^2\alpha \right] \right] \tag{38}$$

Equation (38) is the required equation.

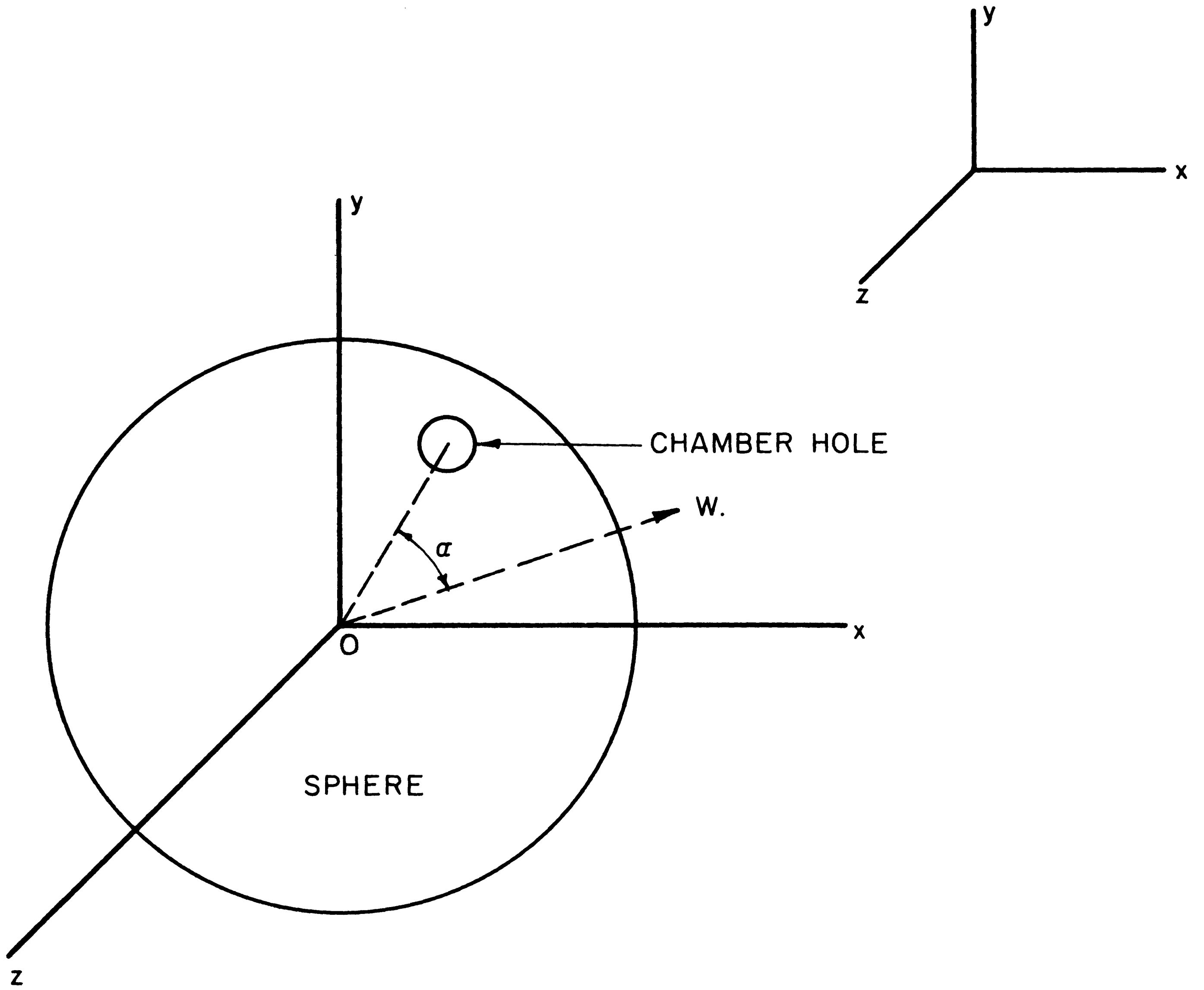


Fig. A-1. Functional diagram illustrating the relationship between the chamber orifice and sphere.

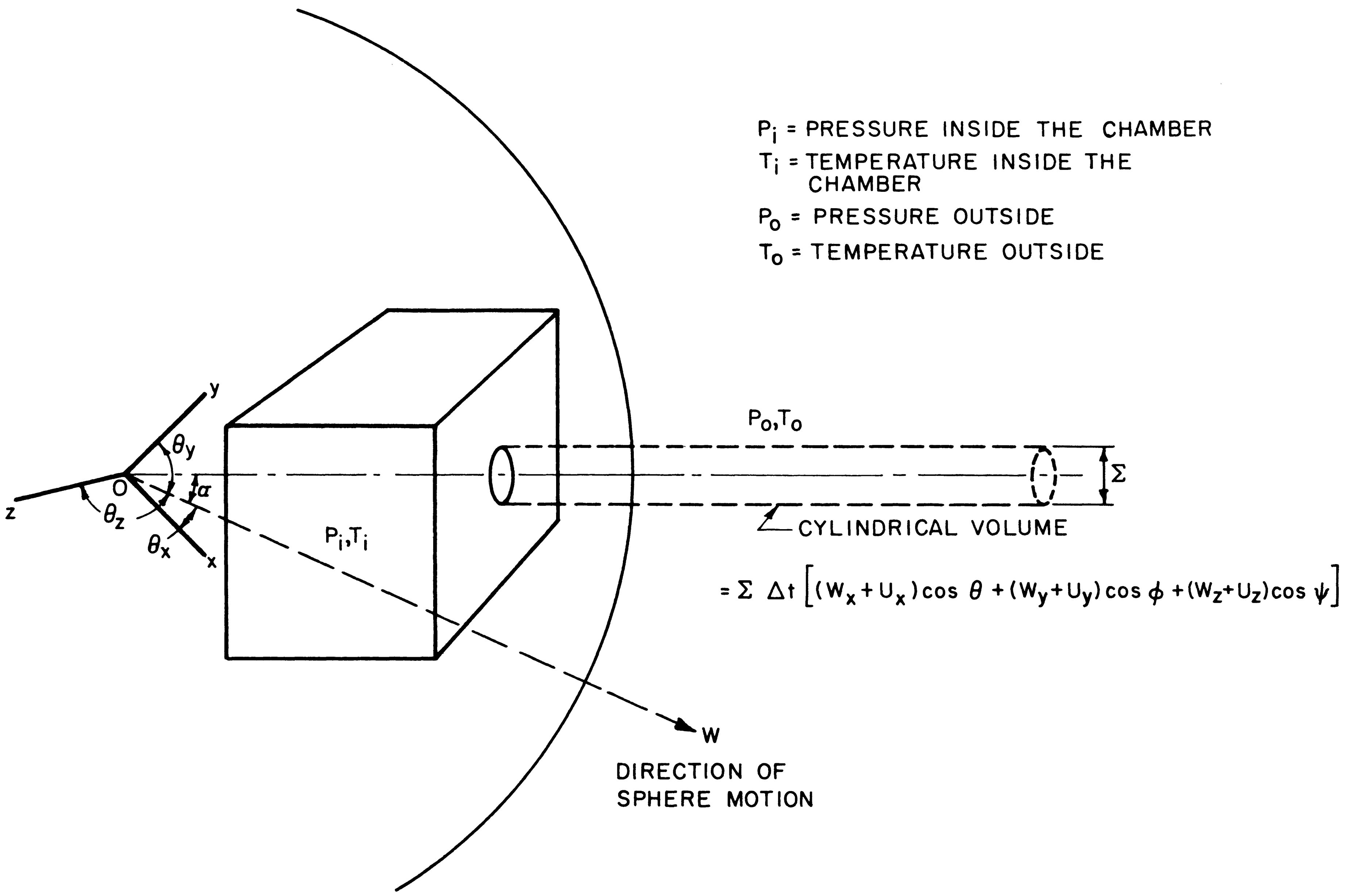


Fig. A-2. Functional diagram of chamber and orifice, illustrating parameters appropriate to the problem.