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NON-LINEAR MODAL ANALYSIS OF THE FORCED RESPONSE OF STRUCTURAL SYSTEMS

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Abstract

A non-linear modal analysis procedure is presented for the forced response of non-linear structural systems. It utilizes the notion of invariant manifolds in the phase space, which was recently used to define non-linear normal modes and the corresponding non-linear modal analysis for unforced vibratory systems. For harmonic forcing, a similar procedure could be formulated, simply by augmenting the size of the free vibration problem. However, in order to accommodate general, non-harmonic external excitations, the invariant manifolds associated with the unforced system are used herein for the forced response analysis. The procedure allows one to generate reduced-order models for the forced analysis of structural systems. Although strictly speaking the invariance property is violated, good results are obtained for the case study considered. In particular, it is found that fewer non-linear modes than linear modes are needed to perform a forced modal analysis with the same accuracy. For systems with small and/or diagonal damping, approximate invariant manifolds are determined, which are shown to yield good results for both the unforced and forced responses.

1 Introduction

The analysis of the free and forced responses of linear dynamic systems is a well established field, with many analytical and numerical tools available.[1,3] In particular, modal analysis allows one to break a problem into smaller, more easily solved sub-problems, and then to consider the solution of the original problem as, in some sense, a post-processing product, using the theorem of superposition. Typically, these sub-problems involve second-order, forced or unforced, linear oscillators (under some non-degeneracy conditions), called modal oscillators. In practice, for the (forced or unforced) analysis of large-scale structural linear systems, model reduction procedures have been developed, where only a few normal modes are retained while the others are ignored (and so are the components of the external forcing that might excite them).

Such formal procedures have not yet been developed for non-linear dynamic systems, partly because (1) until recently, the concept of non-linear normal modes was not well defined for arbitrary, vibratory, non-linear systems, and (2) the theorem of superposition does not hold, preventing immediate use of the non-linear normal modes. Traditional perturbation methods[4,6] can be used to account for some non-linear modes or harmonically forced motions, but not, in general, for arbitrary, multi-mode, forced motions. Besides, because they typically seek the solutions in time of all the differential equations of motion simultaneously, perturbation techniques are not easily usable for systems with many degrees of freedom, and do not provide a model reduction technique (which is crucial for large-scale structural applications). It might be argued that the normal form theory enables one to by-pass both problems (1) and (2), and provides a procedure similar to the modal analysis of linear systems (see reference 7 for such a treatment). However, again, this method is cumbersome for large systems, does not provide a model reduction technique, and is limited to special kinds of external forcing (the normal form theory, as applied in reference 7 for forced response problems, requires that the forcing functions be considered as the solution of a suitable ordinary differential equation (known *a priori*), which essentially reduces its use to problems with harmonic forcing or no forcing). In references 8 and 9, an attempt at using a linear combination of non-linear modal components is proposed for systems with non-linear stiffness and harmonic forcing, with good results for near-resonance excitations. However, this formulation disregards any possible interactions between the various non-linear modes involved, which may prove important in some instances, in particular when internal resonances exist or when the non-linear modal coupling is strong.

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In references 10-12, a non-linear modal analysis technique was introduced for the free response of non-linear systems, where the interactions between the various non-linear modes of interest are preserved. In the system's phase space, this non-linear modal analysis is defined in terms of a high-dimensional invariant manifold whose dimension is twice the number of modes retained in the analysis. The reduced dynamics of the system takes place on this multi-mode invariant manifold and is governed by coupled, non-linear modal oscillators -as many oscillators as there are modeled modes. When not all modes are modeled (i.e., retained in the analysis), the invariance property ensures that no contamination of (and from) the non-modeled modes can occur, so that only the modeled modes need to be simulated. Interactions between the modeled modes are allowed and automatically accounted for, including internal resonances. The dynamics on the invariant manifold can then, if desired, be simplified by use of the normal form theory on the (reduced) set of modal oscillators, in view of an analysis by perturbation methods (see reference 13 for a case with an internal resonance). This invariant manifold procedure is geometric in nature, and is theoretically applicable to many non-linear structural systems, including gyroscopic and/or non-proportionally damped ones. It was shown in references 10-12 to provide as accurate multi-mode dynamic responses as the traditional linear modal analysis of the non-linear system (i.e., the projection of the equations of motion onto the modes of the linearized system, a commonly employed technique), but with significantly fewer modes.

When external forcing is present, the invariant manifolds can be shown to be time-varying about the manifolds of the unforced system. However, determining such time-varying manifolds can quickly become very cumbersome and computationally demanding. An attempt toward this end has been presented in reference 14, where the effect of the excitation was assumed to be additive in the description of the manifold. In this case, it can be seen that determining the time-varying part of the invariant (single- or multi-mode) manifold itself requires to solve exactly as many ordinary differential equations (ODE's) as there are non-modeled modes, thus bringing the total number of ODE's back to the same number as in the original system. When more general time variations of the manifolds are allowed, the computational burden is even increased further.

This article investigates the possibility of neglecting the time variations of the manifolds and, consequently, of utilizing the invariant single- or multi-mode manifolds of the unforced system to perform a non-linear modal analysis of the forced response of the system. This allows one to develop an efficient and systematic model reduction procedure, where only the modes that are mainly excited by the external forces need to be modeled. The linear modes that are mainly excited by the internal non-linear modal interactions are then effectively recovered by the invariant manifold of the unforced system. This approximation is shown to be valid for small amplitudes of forces, and particularly effective at or near resonance of one of the modeled modes, where significantly fewer non-linear modes than linear modes (i.e., modes of the linearized system) can be used for a given accuracy in the system response. The efficiency of this forced non-linear modal analysis procedure is also demonstrated on a case of non harmonic excitation.

It should be noted that, in practice, the damping of many structural systems is only approximately known. Consequently, it may not be worthwhile (or even physically meaningful) to characterize fully the invariant manifolds of the (unforced) damped system, and approximations of them of low order in the damping may often be sufficient. This is particularly true for the proposed forced non-linear modal analysis, for which the invariant manifolds of the unforced system are used (which should typically prove to be a more restrictive approximation). Consequently, approximate invariant manifolds can be generated for the case of small damping, and can be utilized for unforced and forced non-linear modal analyses. As will be seen, in both cases, good agreement is obtained with this small damping approximation.

The remainder of this paper is organized as follows. Section 2 describes approximate methods to obtain the single- or multi-mode invariant manifolds of a weakly damped (unforced) non-linear structural system given those of its undamped counterpart. This yields an approximate non-linear modal analysis of the free response of the damped system. Section 3 describes the proposed non-linear modal analysis of the forced response of non-linear systems, and Section 4 closes on a few conclusions.

2. Non-Linear Modal Analysis of the Free Response of Damped Systems

2.1. Non-Linear Modes and Invariant Manifolds

Several definitions of non-linear normal modes of vibrations have been proposed in the past for non-linear structural systems (see, in particular, references 15-27). In most cases, the system is assumed to be conservative, so that, essentially, modal motions are periodic and, in the configuration space, all coordinates are parametrized by only one of them. For non-conservative systems, it is obvious that such assumptions are too restrictive. Indeed, even for linear systems, damping generally causes non-periodic motions and complex normal modes, each of which can also be viewed as all coordinates being parametrized by one of them plus by the corresponding velocity (time-derivative). In this case, the appropriate place to study normal mode motions is clearly the phase space (rather than the configuration space), where the normal modes are represented by invariant planes and the (not necessarily periodic) motions on them. Similarly, for non-linear systems, non-linear normal modes can be defined as motions occurring on invariant, curved manifolds in the system's phase space.[28,34] For non-degenerate cases, these invariant manifolds are bi-dimensional, and are tangent at the origin to the eigenplanes of the linearized system. For weakly non-linear systems, asymptotic approximations of the invariant manifolds can be determined up to any order of accuracy, and the non-linear modal dynamics on each manifold are described by second-order, non-linear modal oscillators. Traditional perturbation methods can also be used to determine non-linear normal modes,[24,33] and combinations of invariant manifold techniques and perturbation methods allow one to analyze the dynamics of the system in a given non-linear mode for multi-degree of freedom systems.[32]

Since the theorem of superposition does not hold for non-linear systems, a direct use of the non-linear normal modes defined above for multi-mode motions is not as obvious as it is in the case of linear systems. (Along this line, a linear superposition of non-linear modal coordinates was proposed in references 8 and 9, and another was described in reference 28 and used with some success in reference 35). A fundamentally new non-linear modal analysis was presented in references 10-12 for autonomous systems (i.e., for free responses), where a single high-dimensional invariant manifold encompasses the influence of all the non-linear modes of interest. The dimension of this invariant manifold (in the phase space) is twice the number of modeled modes, and the corresponding dynamics on it are given by coupled, second-order, non-linear, modal oscillators -as many oscillators as modes used in the non-linear modal analysis, i.e., as modes describing the manifold. Similar to the case of the single-mode invariant manifolds, asymptotic approximations can be determined for weakly non-linear systems.

While the invariant manifold procedures are very general in nature and can be applied systematically to gyroscopic, non-conservative systems, the determination of the single- or multi-mode manifolds is typically much easier for undamped systems than for damped ones. Besides, the damping of a structural system is usually small and poorly known, except perhaps in a linear modal sense. Thus, if a (single- or multi-mode) invariant manifold of an undamped system is known, determining the corresponding invariant manifold of the damped system may seem an unnecessary, or even physically irrelevant task.

Consequently, alternatives to determining the invariant manifolds of a damped system (and, most importantly, the dynamics on them) are considered below. One approach consists of treating the modal manifolds of a weakly damped system as perturbations of those of its undamped counterpart. Small variations from the undamped manifolds can be obtained analytically, which yield modified, non-linear, modal oscillators corresponding to this small-damping approximation. (Note that, strictly speaking, those manifolds are not invariant any longer). For the particular case of diagonal linear damping (i.e., modal damping in the linear modal coordinates), an even simpler approach consists of applying the linear damping directly to the non-linear modal oscillators of the undamped system, while keeping the modal manifolds of the undamped system unchanged. This is the direct analog for non-linear systems of the diagonal damping assumption for linear systems, where the normal modes are unchanged while the modal oscillators are individually and independently damped.[2] For non-linear systems, however, diagonal linear damping does not affect the linear part of the modal manifold, but does affect the higher-order terms. For weak diagonal damping, these variations are small and are neglected in this particular approach.

2.2. Procedure for General Non-Linear Systems

2.2.1. Overview

The non-linear modal analysis procedure presented in references 10-12 is geometric in nature. It defines invariant (single- or multi-mode) manifolds in the system's phase space on which (single-mode or multi-mode) free response motions occur. The dimension of the manifold itself depends solely on the number of non-linear modes considered in the analysis of the motion. For weakly non-linear systems, a constructive technique was developed to construct local approximations of the invariant manifold about the origin of the phase space. This technique follows closely the one developed for the generation of non-linear normal modes in references 28-31 (where single-mode invariant manifolds are determined), which itself was inspired by the generation of center manifolds in the theory of non-linear dynamical systems.[36,37]

Essentially, all motions involving, say, M non-linear modes, necessitate $2M$ independent variables to be fully described in the system's phase space. All the dependent variables are then uniquely determined by these $2M$ non-linear modal coordinates, and the relationship between the dependent and independent variables represents exactly the equation of the desired invariant multi-mode manifold. For small oscillations about the equilibrium position of interest, a Taylor series expansion of the manifold can be performed with respect to the $2M$ non-linear modal coordinates. The coefficients of this asymptotic expansion can then be determined uniquely by solving successive sets of linear algebraic equations, one order of approximation at a time (Section 2.2.2 provides a brief review of the practical steps involved in this process). At the linear order, the traditional span of the M eigenvectors of the linearized system is recovered, while at the higher orders, the influence of the various linear modes on the M modeled non-linear modes is taken into account. In the case of a single non-linear normal mode model, the single-mode invariant manifold thus obtained can be thought of as a generalized, non-linear, amplitude-dependent eigenvector.

The restriction of the equations of motion to the invariant manifold obtained (by enforcing the relationship between the dependent and independent variables) then provides the dynamics of the M modeled non-linear modes in terms of M coupled, second-order, non-linear, modal oscillators.

This methodology is applicable to general gyroscopic, non-conservative structural systems. However, it should be noted that, for systems with no first-order time-derivatives in the equations of motion, the equations of motion are symmetric in time, as replacing the time t by $t' = -t$ yields the same equation, albeit backwards in time. Consequently, initial conditions at $t = 0$ yield the same solution in forward and backward times. In the phase space, this is only possible if all the monomials in the Taylor series expansion of the invariant manifold contain only appropriate powers of the independent velocities (recall that, in the phase space, half of the $2M$ independent variables are the velocities of the other half): even powers of the independent velocities for the expansions of the dependent generalized displacements, and odd powers of them for those of the dependent generalized velocities (see Section 2.2.2 below). It is therefore immediate from these symmetry considerations that, for systems with no first-order time-derivatives in the equations of motions, half of the coefficients in the Taylor series expansions are zero. In particular, this result is applicable to non-gyroscopic, undamped systems having non-linear stiffness (such as arising from large deformations or non-dissipative material non-linearities, for example).

Consequently, the Taylor series expansion of an invariant manifold of a damped non-gyroscopic system with non-linear stiffness contains twice as many terms as that of its undamped counterpart. For small damping (which is typical in structural dynamics), perturbation approximations from the invariant manifolds of the undamped system may therefore be an attractive alternative.

2.2.2. Practical Determination of Multi-Mode Invariant Manifolds

The generic equations of motion of (discretized) non-linear structural systems are assumed to be of the form

$$\dot{x}_i = y_i, \quad \dot{y}_i = f_i(x_1, \cdots, x_N, y_1, \cdots, y_N), \qquad (1)$$

for $i = 1, \cdots, N$ and where, in general, f_i contains all damping, gyroscopic, stiffness, and non-linear forces. For simplicity, it is assumed that these equations have been cast into the modal coordinates of the associated undamped linearized system. When M non-linear modes are modeled, the $2M$ modal variables required to describe the (multi-mode) invariant manifold can be chosen to be those corresponding to the M linear modes to which the manifold has to be tangent, as

$$u_k = x_k, \quad v_k = y_k, \qquad (2)$$

for $k \in S_m$, and where S_m denotes the subset of indices corresponding to the modeled modes. The $2N - 2M$ remaining variables are then functionally related to the modeled modes as

$$x_j = X_j(\mathbf{u}_m, \mathbf{v}_m), \quad y_j = Y_j(\mathbf{u}_m, \mathbf{v}_m), \qquad (3)$$

for $j \notin S_m$, and where $\mathbf{u}_m$ and $\mathbf{v}_m$ represent the vectors of the non-linear modal coordinates and velocities, i.e., they are the collections of the u_k's and v_k's, $k \in S_m$ (bold-face characters denote vector or matrix quantities). For weakly non-linear systems, Taylor series expansions of X_j and Y_j , for $j \notin S_m$, can be expressed as

$$X_j(\mathbf{u}_m, \mathbf{v}_m) = \sum_{k \in S_m} a^k_{1,j} u_k + a^k_{2,j} v_k$$

$$+\sum_{k\in S_m}\sum_{l\in S_m} a_{3,j}^{k,l}u_k u_l + a_{4,j}^{k,l}u_k v_l + a_{5,j}^{k,l}v_k v_l$$
$$+\sum_{k\in S_m}\sum_{l\in S_m}\sum_{q\in S_m} a_{6,j}^{k,l,q}u_k u_l u_q + a_{7,j}^{k,l,q}u_k u_l v_q$$
$$+a_{8,j}^{k,l,q}u_k v_l v_q + a_{9,j}^{k,l,q}v_l v_k v_q + \cdots \quad (4)$$

$$Y_j(\mathbf{u}_m,\mathbf{v}_m) = \sum_{k\in S_m} b_{1,j}^{k}u_k + b_{2,j}^{k}v_k$$
$$+\sum_{k\in S_m}\sum_{l\in S_m} b_{3,j}^{k,l}u_k u_l + b_{4,j}^{k,l}u_k v_l + b_{5,j}^{k,l}v_k v_l$$
$$+\sum_{k\in S_m}\sum_{l\in S_m}\sum_{q\in S_m} b_{6,j}^{k,l,q}u_k u_l u_q + b_{7,j}^{k,l,q}u_k u_l v_q$$
$$+b_{8,j}^{k,l,q}u_k v_l v_q + b_{9,j}^{k,l,q}v_l v_k v_q + \cdots \quad (5)$$

where, for undamped, non-gyroscopic systems with non-linear stiffness, all a_2's, a_4's, a_7's, a_9's, and b_1's, b_3's, b_5's, b_6's, b_8's, are zero by symmetry in time. Determining the Taylor series coefficients of X_j and Y_j can be performed by utilizing the jth pair of equations of motion, Eq. (1), as

$$\left\{\begin{array}{lcl} \sum_{k\in S_m}\left[\frac{\partial X_j}{\partial u_k}v_k + \frac{\partial X_j}{\partial v_k}f_k\right] & = & Y_j \\ \sum_{k\in S_m}\left[\frac{\partial Y_j}{\partial u_k}v_k + \frac{\partial Y_j}{\partial v_k}f_k\right] & = & f_j \end{array}\right. \quad (6)$$

for $j \notin S_m$. At each order of approximation, say p, the equations for the Taylor series coefficients can be put in matrix form as

$$\mathbf{A}_j^{(p)}\mathbf{a}_j^{(p)} = \mathbf{b}_j^{(p)} \quad (7)$$
$$\mathbf{A}_j^{(p)}\mathbf{b}_j^{(p)} = \mathbf{f}_j^{(p)} \quad (8)$$

where $\mathbf{a}_j^{(p)}$ and $\mathbf{b}_j^{(p)}$ represent the collection of the order-p coefficients describing the manifold, and $\mathbf{f}_j^{(p)}$ is problem dependent. Obtaining Eqs. (7) and (8) requires, upon substitution of Eqs. (4) and (5) into Eq. (6), to equate terms of identical powers in the non-linear modal coordinates. The left-hand-sides of Eqs. (7) and (8) are symmetric in $\mathbf{a}_j^{(p)}$ and $\mathbf{b}_j^{(p)}$, because the left-hand-sides in Eq. (6) are symmetric in X_j and Y_j. In general, $\mathbf{f}_j^{(p)}$ can be expressed as

$$\mathbf{f}_j^{(p)} = \mathbf{C}_0^{(p)} + \mathbf{C}_{1,j}^{(p)}\mathbf{a}_j^{(p)} + \mathbf{C}_{2,j}^{(p)}\mathbf{b}_j^{(p)}, \quad (9)$$

where $\mathbf{C}_{1,j}^{(p)}$ and $\mathbf{C}_{2,j}^{(p)}$ typically arise from a linear stiffness term and a linear damping (or gyroscopic) term in f_j, respectively, and $\mathbf{C}_0^{(p)}$ is typically due to the non-linear terms in f_j (and may contains coefficients determined at lower orders). Equation (8) can be re-written as

$$\mathbf{A}_j^{(p)}\mathbf{b}_j^{(p)} = \mathbf{C}_0^{(p)} + \mathbf{C}_{1,j}^{(p)}\mathbf{a}_j^{(p)} + \mathbf{C}_{2,j}^{(p)}\mathbf{b}_j^{(p)}, \quad (10)$$

which can be recombined with Eq. (7) as

$$\left(\mathbf{A}_j^{(p)2} - \mathbf{C}_{1,j}^{(p)} - \mathbf{C}_{2,j}^{(p)}\mathbf{A}_j^{(p)}\right)\mathbf{a}_j^{(p)} = \mathbf{C}_0^{(p)} \quad (11)$$

The coefficients of order p describing the invariant manifold of interest can then be obtained by solving Eq. (11), and then Eq. (7). For the actual determination of $\mathbf{a}_j^{(p)}$ and $\mathbf{b}_j^{(p)}$, it is important to realize that, at each order p, the Taylor series expansion of the invariant manifold (Eqs. (4) and (5)) comprises monomials involving one non-linear mode only, monomials involving two non-linear modes only, etc., and monomials involving at most p modeled non-linear modes. The equations for the corresponding coefficients can be decoupled from one another and, in practice, one does not have to solve at once the (potentially large) problem given in Eqs. (11) and (7). Rather, one can solve a succession of small problems of the same form, first for the coefficients involving the first modeled mode only, then for those involving the second mode only, etc-, then for the coefficients involving the first and second modeled modes only, and so on (see reference 11).

Approximations of increasing order can be computed sequentially in this manner[§]. Once the multi-mode manifold of interest has been approximated to the desired order, the dynamics of the system on it are obtained by solving the reduced set of equations of motion corresponding to the modeled modes, that is,

$$\left\{\begin{array}{lcl} \dot{u}_k & = & v_k \\ \dot{v}_k & = & f_k(\mathbf{u}_m,\mathbf{v}_m) \end{array}\right. \quad (12)$$

for $k \in S_m$, where Eqs. (4) and (5) have been utilized where necessary. It is important to note that the dynamics on the invariant manifold can be obtained to a higher order of approximation than the invariant manifold itself.[11,31,34] Specifically, for a system where the lowest non-linearity is of order Q, the order of approximation of the dynamics is

[§]The sensitivity of the approximation of an invariant manifold to low-order model uncertainties may be investigated using Eqs. (12) and (13), in particular, by analyzing the manner in which low-order coefficients affect the determination of the higher-order coefficients.

$N' + Q - 1$, where N' is the order of approximation of the manifold. Stated differently, the N'th-order shape correction is needed in the invariant manifolds in order to obtain the $(N'+Q-1)$th-order dynamics legitimately.

2.3. Alternative Procedure for Small Damping

For undamped and damped systems, the equations for the order-p coefficients describing the invariant manifold of interest (Eqs. (7) and (10)) can be expressed as, respectively,

$$\mathbf{A}\mathbf{a}^{(p)}_{j,und} = \mathbf{b}^{(p)}_{j,und} \quad (13)$$

$$\mathbf{A}\mathbf{b}^{(p)}_{j,und} = \mathbf{C}_0 + \mathbf{C}_1\mathbf{a}^{(p)}_{j,und} + \mathbf{C}_2\mathbf{b}^{(p)}_{j,und} \quad (14)$$

and

$$\mathbf{A}_d\mathbf{a}^{(p)}_{j,d} = \mathbf{b}^{(p)}_{j,d} \quad (15)$$

$$\mathbf{A}_d\mathbf{b}^{(p)}_{j,d} = \mathbf{C}_0 + \mathbf{C}_1\mathbf{a}^{(p)}_{j,d} + \mathbf{C}_2\mathbf{b}^{(p)}_{j,d} + \mathbf{D}_0 + \mathbf{D}\mathbf{b}^{(p)}_{j,d} \quad (16)$$

where the subscript and superscript on the matrices have been dropped for clarity. The subscripts *und* and d refer to the undamped and damped systems, respectively. The vectors $\mathbf{a}^{(p)}_{j,und}$ and $\mathbf{b}^{(p)}_{j,und}$ (resp., $\mathbf{a}^{(p)}_{j,d}$ and $\mathbf{b}^{(p)}_{j,d}$) are the order-p coefficients of the invariant manifold of the undamped (resp., damped) system. It should be noted that the similarities between the two cases are direct consequences of Eq. (6), where only the f_j's and f_k's differ by the damping terms. In particular,

$$\mathbf{A}_d = \mathbf{A} + \mathbf{D}_1 \quad (17)$$

is issued from the left-hand-side of Eq. (6) ($\mathbf{D}_1$ is due to the damping in the f_k's), while $\mathbf{D}_0$ and $\mathbf{D}$ are due to the right-hand-side of this same equation (from the damping in f_j). Note that $\mathbf{A}, \mathbf{C}_1, \mathbf{C}_2$, and $\mathbf{C}_0$ are identical in both damped and undamped cases, and that the influence of the damping on these equations is additive ($\mathbf{D}_0, \mathbf{D}_1$ and $\mathbf{D}$) for the equations in $\mathbf{a}^{(p)}_{j,d}$ and $\mathbf{b}^{(p)}_{j,d}$.

Equations (15) and (16) can be combined as

$$\boldsymbol{\Delta}\mathbf{a}^{(p)}_{j,d} = \mathbf{C}_0 + \mathbf{D}_0 \quad (18)$$

$$\mathbf{b}^{(p)}_{j,d} = \mathbf{A}_d\mathbf{a}^{(p)}_{j,d} \quad (19)$$

where

$$\boldsymbol{\Delta} = \mathbf{A}_d^2 - \mathbf{C}_1 - \mathbf{C}_2\mathbf{A}_d - \mathbf{D}\mathbf{A}_d. \quad (20)$$

The solution of Eqs. (18) and (19) for all j provides the order-p coefficients of the Taylor series expansion of the desired multi-mode invariant manifold for the damped system.

Small damping approximation

For small damping, Eqs. (15) and (16) can be considered as perturbations of Eqs. (13) and (14) and, accordingly, the solution of Eqs. (18) and (19) can be expressed in terms of the solution for the undamped system (recall that, for an undamped, non-gyroscopic system with non-linear stiffness, half of the solution of Eqs. (13) and (14) is *a priori* known to be zero). To that end, $\boldsymbol{\Delta}$ can be re-written as

$$\boldsymbol{\Delta} = \boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_1 + \boldsymbol{\Delta}_2, \quad (21)$$

where

$$\boldsymbol{\Delta}_0 = \mathbf{A} - \mathbf{C}_1 - \mathbf{C}_2\mathbf{A} \quad (22)$$

$$\boldsymbol{\Delta}_1 = \mathbf{A}\mathbf{D}_1 + \mathbf{D}_1\mathbf{A} - \mathbf{C}_2\mathbf{D}_1 - \mathbf{D}\mathbf{A} \quad (23)$$

$$\boldsymbol{\Delta}_2 = \mathbf{D}_1^2 - \mathbf{D}\mathbf{D}_1 \quad (24)$$

where $\boldsymbol{\Delta}_0$ corresponds to the undamped system (see Eq. (11)), and $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ are of order one and two in the damping, respectively. For small damping, the inverse of $\boldsymbol{\Delta}$ can then be expressed, to first-order in the damping, as

$$\boldsymbol{\Delta}^{-1} = \boldsymbol{\Delta}_0^{-1} - \boldsymbol{\Delta}_0^{-1}\boldsymbol{\Delta}_1\boldsymbol{\Delta}_0^{-1} + \cdots \quad (25)$$

Expressing the solution of Eqs. (18) and (19) as a perturbation of that of Eqs. (13) and (14) as

$$\mathbf{a}^{(p)}_{j,d} = \mathbf{a}^{(p)}_{j,und} + \delta\mathbf{a}^{(p)}_j \quad (26)$$

$$\mathbf{b}^{(p)}_{j,d} = \mathbf{b}^{(p)}_{j,und} + \delta\mathbf{b}^{(p)}_j \quad (27)$$

and combining with Eqs. (18) and (19), one obtains

$$\delta\mathbf{a}^{(p)}_j = -\boldsymbol{\Delta}_0^{-1}\boldsymbol{\Delta}_1\mathbf{a}^{(p)}_{j,und} + \boldsymbol{\Delta}_0^{-1}\mathbf{D}_0 \quad (28)$$

$$\delta\mathbf{b}^{(p)}_j = (\mathbf{D}_1 - \mathbf{A}\boldsymbol{\Delta}_0^{-1}\boldsymbol{\Delta}_1)\mathbf{a}^{(p)}_{j,und} + \mathbf{A}\boldsymbol{\Delta}_0^{-1}\mathbf{D}_0 \quad (29)$$

It can be seen from Eqs. (28) and (29) that the small perturbations of the multi-mode manifold due to the damping can be expressed analytically from the manifolds of the undamped system and from the characteristics of the damping. To these small damping approximations correspond approximate non-linear modal oscillators of the type of Eq. (12).

Case of diagonal damping

In the case of diagonal damping (i.e., when the damping does not couple the equations of motion at linear order), one finds that $\mathbf{D}_0 = 0$ (and $\mathbf{D}_1$ and $\mathbf{D}_2$ are diagonal). In particular, since $\mathbf{C}^{(1)}_{0,j} = 0$ for all $j \notin S_m$ (using the subscript and superscript notations of Section 2.2.2) at the linear order for equations cast into the linear modal coordinates, then

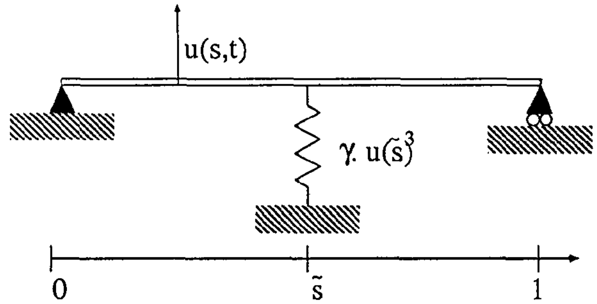

Figure 1: Simply-supported Euler-Bernoulli (linear) beam constrained by a purely cubic spring and subject to ground motion.

$a^{(1)}_{j,d} = b^{(1)}_{j,d} = 0$ from Eqs. (18) and (19), so that at linear order, the manifolds of the damped and undamped systems are identical. At higher order, however, Eqs. (28) and (29) indicate a perturbation of the manifolds due to the damping.

Nevertheless, for small diagonal damping, it is particularly interesting to neglect these perturbations, since this approximation allows the use of the invariant manifolds of the undamped system for performing the non-linear modal analysis of the damped system. The non-linear modal oscillators governing the dynamics are then simply those of the undamped system, with the addition of the diagonal damping terms (the perturbations at higher orders in the dynamics are automatically neglected since they would have occurred from those of the perturbed manifold).

2.4. Example: A Simply Supported Euler-Bernoulli Beam Constrained by a Non-Linear Spring and Subject to Ground Excitation

The above procedures have been applied to a homogeneous, simply supported Euler-Bernoulli beam constrained by a non-linear cubic spring -see Fig. 1. If the beam is of length $l = 1$, the equation of transverse motion of the system can be shown to be, in non-dimensional form:

$$\ddot{u} + \alpha u_{,ssss} + \beta u^3 \delta(s - \tilde{s}) = 0, \tag{30}$$

$s \in (0, 1)$, where $\alpha = EI/m$, $\beta = \gamma/m$, E is the Young's modulus of the beam, I is its second moment of area, m is its mass per unit length, γ is the non-linear stiffness of the spring, s represents the abscissa along the beam, $\tilde{s}$ denotes the spring location along the beam, $u(s,t)$ is the transverse deflection of the beam, $\cdot_{,s}$ denotes a derivative with respect to s, an overdot represents a derivative with respect to time, and δ is the Dirac function. The associated boundary conditions are $u(0) = u(1) = 0$ and $u_{,ss}(0) = u_{,ss}(1) = 0$. The beam deflection, $u(s,t)$, is discretized using the natural modes of the linearized system, $\phi_j(s) = \sin(j\pi s)$ (see references 30-31), as

$$u(s,t) \sim \sum_{j=1}^{N} \eta_j(t)\phi_j(s), \tag{31}$$

where N is the number of terms in the expansion, i.e., the number of terms that would be retained for a linear modal analysis of the non-linear system. Projection of the equation of motion onto the ith linear mode yields

$$\ddot{\eta}_i + \alpha(i\pi)^4\eta_i + 2\beta\left[\sum_{j=1}^{N} \eta_j \sin(j\pi\tilde{s})\right]^3 \sin(i\pi\tilde{s}) = 0, \tag{32}$$

$i = 1, \cdots, N$. If diagonal damping is included in the model, it is typically added at this stage, so that Eq. (32) becomes, for $i = 1, \cdots, N$,

$$\ddot{\eta}_i + 2\xi_i\sqrt{\alpha}(i\pi)^2\dot{\eta}_i + \alpha(i\pi)^4\eta_i + 2\beta\left[\sum_{j=1}^{N} \eta_j \sin(j\pi\tilde{s})\right]^3 \sin(i\pi\tilde{s}) = 0, \tag{33}$$

(where ξ_i is the damping ratio of the ith linear mode), which can be written in first-order form as

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = f_i(x_1, \cdots, x_N, y_1, \cdots, y_N) \end{cases} \tag{34}$$

$i = 1, \cdots, N$, with $x_i = \eta_i$, $y_i = \dot{\eta}_i$, and

$$f_i = -2\xi_i\sqrt{\alpha}(i\pi)^2 y_i - \alpha(i\pi)^4 x_i - 2\beta\left[\sum_{j=1}^{N} x_j \sin(j\pi\tilde{s})\right]^3 \sin(i\pi\tilde{s}). \tag{35}$$

The set of differential equations, Eq. (34), is what is simulated for a typical linear modal analysis. Alternatively, the procedures described previously can be applied to Eq. (34).

Following the developments of Sections 2.2 and 2.3, three sets of (multi-mode) manifolds have been determined for comparison purposes: the invariant manifolds of the damped system (denoted IMDS hereafter) obtained from Eqs. (18)-(19), the small-damping approximation of them (denoted SDAIM) obtained from Eqs. (26)-(29), and the invariant manifolds of the undamped system (denoted IMUS) obtained from Eqs. (13)-(14). The results for the latter case (IMUS) are available in reference 11, while

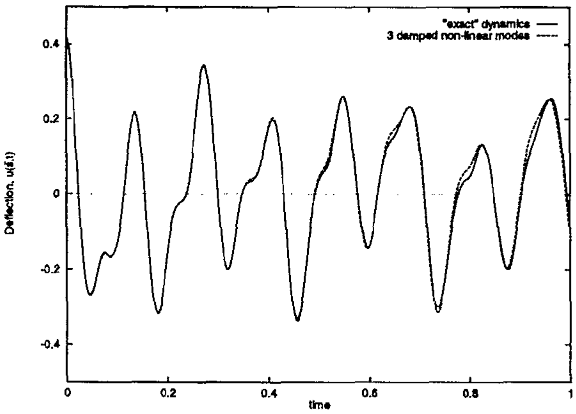

Figure 2: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode damped invariant manifold (free response motion). $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0.2, v_1(0) = v_2(0) = v_3(0) = 0$.

those for the first and second cases (IMDS and SDAIM) are too lengthy to give here but can be obtained analytically using Mathematica^{TM}. (Those calculations have also been performed by numerically solving the equations for the Taylor series coefficients. This was found to be a very rapid computation compared to the numerical time-integration of the modal dynamics, and is a worthy alternative.) In all cases, the (multi-mode) manifolds have been determined up to cubic order.

Correspondingly, one obtains three sets of coupled, non-linear, modal oscillators for the free responses, in the form of Eq. (12). For the case of the IMUS, diagonal damping is directly added to the modal oscillators, while the modal oscillators resulting from the IMDS and from the SDAIM contain the appropriate damping terms. Using cubic-order determination of the manifolds (IMDS, SDAIM or IMUS), these dynamics can be obtained up to fifth-order.

Results using the first three non-linear modes (which are internally resonant in the undamped case[11]) are presented in Figs. 2-7. In those plots, the "exact" solution has been determined by using Eq. (34) with many (25 or more) linear modes. Simulations using increasing numbers of linear modes are compared with simulations using three non-linear modes (using either the damped non-linear modes (IMDS), the small damping approximation of them (SDAIM), or the undamped non-linear modes (IMUS)). In this case, significantly fewer non-linear modes than linear modes are needed to achieve a given level of

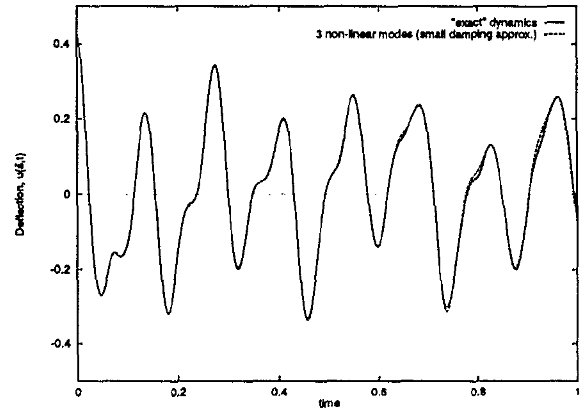

Figure 3: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold determined with a small damping approximation (free response motion). $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0.2, v_1(0) = v_2(0) = v_3(0) = 0$.

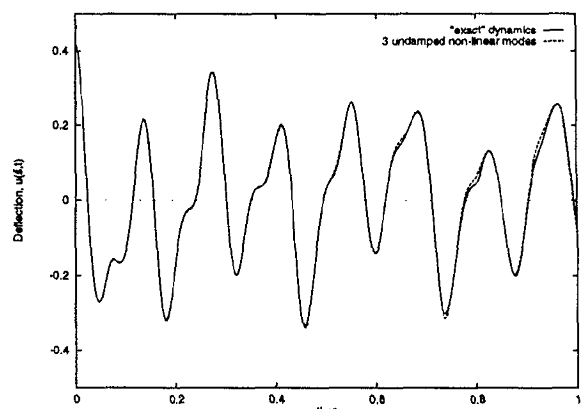

Figure 4: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold of the undamped system (free response motion). $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0.2, v_1(0) = v_2(0) = v_3(0) = 0$.

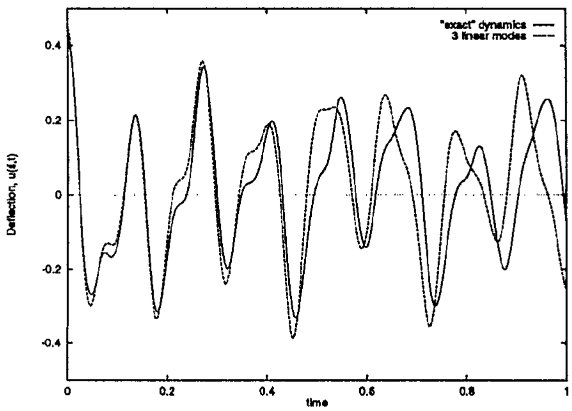

Figure 5: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a three-mode linear modal analysis of the non-linear system (free response motion). $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0.2, v_1(0) = v_2(0) = v_3(0) = 0$.

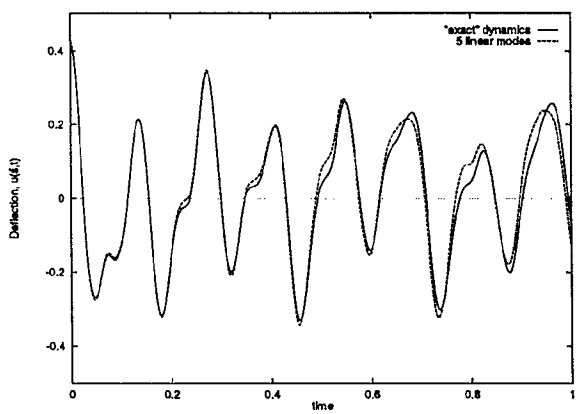

Figure 6: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a five-mode linear modal analysis of the non-linear system (free response motion). $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0.2, v_1(0) = v_2(0) = v_3(0) = 0$.

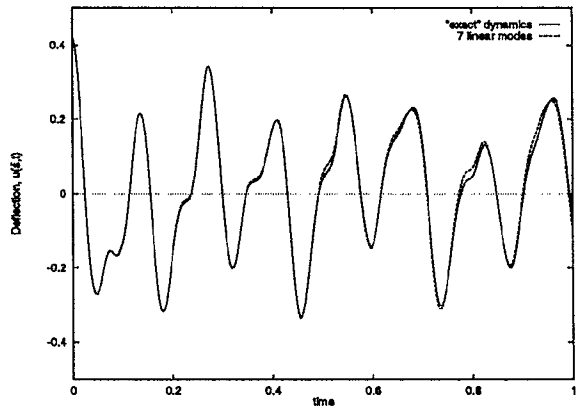

Figure 7: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a seven-mode linear modal analysis of the non-linear system (free response motion). $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0.2, v_1(0) = v_2(0) = v_3(0) = 0$.

accuracy (three non-linear modes instead of seven linear modes). It should be noted that the results obtained by the three non-linear modal analysis procedures are very similar. Thus, for this particular system, the use of the undamped non-linear modes (and of the undamped multi-mode manifolds, the IMUS) is found to be a good approximation when used in conjunction with small diagonal damping.

3. Non-Linear Modal Analysis of Forced Response

3.1. Preliminary Comments

The non-linear modal analysis presented in references 10-12 is explicitly designed to be "exact" (within the accuracy of the asymptotic expansion and for the selected set of modeled modes) for the free response of a wide class of gyroscopic, non-conservative, non-linear, structural systems. This is true even when non-removable interactions exist between non-linear modes, as long as the set of modeled modes is properly selected. Unfortunately, when external excitation is present, a similarly "exact" forced non-linear modal analysis formulation is not presently known.

An exception, however, is the special case when the external excitation is harmonic. In this case the excitation can be considered as the solution of an ordinary differential equation (namely, $\ddot{u} + \omega^2 u = 0$), and the forcing can then be treated as an additional (known) pair of variables, say u and $v = \dot{u}$, with the original system of equations being augmented by the harmonic equation¶. The non-linear modal analysis introduced in references 10-12 can be applied to the resulting (augmented) autonomous system. After the desired invariant manifolds have been determined for the augmented system, one can then replace, both in the equation of the manifold and in the modal oscillator equations, the known variables u and v by the (original) harmonic functions. At this point, all terms that were linear in u and v become harmonic terms of order zero, and likewise for all quadratic (cubic, etc-) terms in u and v. All quadratic terms involving u or v only once become harmonically-varying linear terms (in the original variables), and so on.

It can therefore be observed that, in the presence of external forcing, the invariant manifolds (in terms of the non-augmented set of variables only) become time-varying, with time-dependencies at all orders, including orders zero and one. In general, if the invariant manifold is expressed as

$$x_j = X_j(\mathbf{u}_m, \mathbf{v}_m; t),\ y_j = Y_j(\mathbf{u}_m, \mathbf{v}_m; t), \quad (36)$$

for $j \notin S_m$, the Taylor series expansions for small oscillations becomes

$$X_j = X_j(0,0;t) + \frac{\partial X_j}{\partial \mathbf{u}_m}(0,0;t)\cdot \mathbf{u}_m + \frac{\partial X_j}{\partial \mathbf{v}_m}(0,0;t)\cdot \mathbf{v}_m + N_{1,j}(\mathbf{u}_m, \mathbf{v}_m; t) \quad (37)$$

$$Y_j = Y_j(0,0;t) + \frac{\partial Y_j}{\partial \mathbf{u}_m}(0,0;t)\cdot \mathbf{u}_m + \frac{\partial Y_j}{\partial \mathbf{v}_m}(0,0;t)\cdot \mathbf{v}_m + N_{2,j}(\mathbf{u}_m, \mathbf{v}_m; t) \quad (38)$$

where time-dependences appear at all orders ($N_{1,j}$ and $N_{2,j}$ represent the non-linear terms in $\mathbf{u}_m$ and $\mathbf{v}_m$). Notice that, as apparent in the case of harmonic forcing, the order zero time-dependent terms cannot be determined from the linearized system alone (except in the case of forcing with small amplitude).

Reference 14 presents an attempt at determining such time-dependent invariant manifolds in the special case when only order zero time variations are present. However, even in this simple case, the procedure fails to reduce the number of ODE's to be simulated from that of the original system of equations. (Reference 7 also uses order zero time variations for harmonic forcing, in the context of normal forms).

¶This formulation was suggested to us by Richard Rand of Cornell University, and was also used in reference 7.

3.2. Proposed Forced Non-Linear Modal Analysis

An approximate non-linear modal analysis for forced response can be proposed based on the following remark: at all orders of approximation, the time-variation of the multi-mode manifolds might be neglected if the amplitude of the forcing is small compared to the amplitude of the response. When this is verified (as is typically the case at or near the resonance of one of the modeled non-linear modes), the time-independent multi-mode manifolds of the unforced system should provide good approximations of the time-varying manifolds.

With this approximation, the (coupled) non-linear modal oscillators corresponding to the unforced system become

$$\begin{cases} \dot{u}_k &= v_k \\ \dot{v}_k &= f_k(\mathbf{u}_m, \mathbf{v}_m) + g_k(t) \end{cases} \quad (39)$$

for $k \in S_m$, and where $f_k(\mathbf{u}_m, \mathbf{v}_m)$ is unchanged from the unforced case, and $g_k(t)$ is the external forcing on the kth pair of equations of motion, Eq. (1).

It should be noted that, strictly speaking, the invariance property of the (unforced) modal manifold is violated by its use for the forced response. Consequently, this forced non-linear modal analysis is a case where the approximations for small damping or diagonal damping introduced in Section 2 should be well suited. Indeed, the approximation of time-independent manifolds should typically prove to be more restrictive than that of small damping.

Also, in general, external forcing excites all modes simultaneously. Typically, for a given excitation, the non-linear modes can be expected to be classified in three categories: those with a negligible response, those with a significant response primarily due to the external forcing, and those with a significant response primarily due to the internal non-linear modal coupling. Clearly, the first category of modes does not need to be modeled. The second category would typically need to be modeled, while the third category is the one for which this forced non-linear modal analysis should prove to be most useful, as the effect of these modes is expected to be captured by an adequate time-independent multi-mode manifold.

It should be made clear that the primary objective of this (forced) non-linear modal analysis is to generate reduced-order models of potentially large (forced) structural systems, *i.e.*, to reduce large systems of coupled ordinary differential equations such as Eq. (1) to (much) smaller systems of coupled

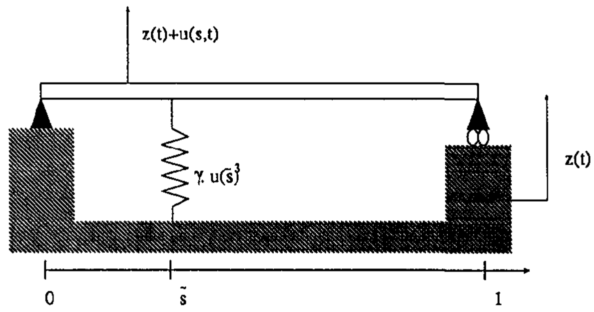

Figure 8: Simply-supported Euler-Bernoulli (linear) beam constrained by a purely cubic spring.

(modal) equations such as Eq. (39). Obtaining the solution of Eq. (39), *per se*, is a separate issue which is beyond the scope of this study. In particular, if a single-mode is modeled and the external forcing is harmonic, a number of perturbation methods can be applied for the analysis of Eq. (39) (see references 4-6). Depending on the method utilized to solve Eq. (39), both the transient and long-term dynamics of the system can be recovered, or only the steady-state behavior. Hereafter, numerical time-integration will be used in all cases. In this respect, this non-linear modal analysis somewhat differs from traditional perturbation methods, which typically aim at obtaining solutions of certain (systems of) differential equations by solving all of them simultaneously. This is usually cumbersome for systems with many degrees of freedom and, even though single- or multi-mode motions can be determined using perturbation methods[24,33] or the normal form theory,[7] typically these techniques cannot generate multi-mode reduced-order models. When analytical work is to be performed on a reduced set of modes of a system, the present non-linear modal analysis procedure is an appropriate means of generating the desired reduced set of coupled modal equations first.

3.3. Example: A Beam Constrained by a Non-Linear Spring and Subject to Ground Excitation

The above methodology has been applied to a homogeneous, simply supported Euler-Bernoulli beam with a non-linear cubic spring attached at its middle and excited by ground motion -see Fig. 8. This system is identical to the one studied in Section 2.4, with the exception that the support is now moving. The equation of transverse motion becomes

$$\ddot{u} + \alpha u_{,ssss} + \beta u^3 \delta(s - \tilde{s}) = -\ddot{z},\ s \in (0,1), \quad (40)$$

where $z(t)$ is the displacement of the support and $u(s,t)$ is now the transverse deflection of the beam relative to the support (all other quantities are as defined in Section 2.4). The boundary conditions are identical to those of Section 2.4. With diagonal damping and the motion of the support included, Eq. (35) becomes

$$\begin{aligned} f_i = & -2\xi_i\sqrt{\alpha}(i\pi)^2 y_i - \alpha(i\pi)^4 x_i \\ & -2\beta\left[\sum_{j=1}^{N} x_j \sin(j\pi\tilde{s})\right]^3 \sin(i\pi\tilde{s}) \\ & +\frac{2}{i\pi}[(-1)^i + 1]\ddot{z}, \end{aligned} \quad (41)$$

where ξ_i is the damping ratio of the ith linear mode.

The procedure described in Section 3.2 can be applied to Eq. (34). The multi-mode invariant manifolds of the unforced damped system are first determined, and utilized for the non-linear modal analysis of the forced response. As in Section 2.4, for the unforced damped system, three sets of (multi-mode) manifolds have been determined for comparison purposes: the invariant manifolds of the damped system (IMDS), the small-damping approximation of them (SDAIM), and the invariant manifolds of the undamped system (IMUS). Correspondingly, one obtains three sets of coupled, non-linear, modal oscillators for the forced responses, in the form of Eq. (39). As in Section 2.4, the (multi-mode) manifolds have been determined up to third order in the non-linear modal coordinates, which yields a fifth-order approximation of the dynamics.

A case of harmonic excitation near resonance is presented in Figs. 9-14 for the transient behavior, and in Figs. 15-21 for the steady-state response after a long transient regime. All results shown were obtained by numerical time-integration. Again, the "exact" solution was determined by using Eq. (34) with at least 25 linear modes. As can be observed, for these parameters of excitation, significantly fewer non-linear modes than linear modes are needed to achieve a given level of accuracy, for both the transient and long term dynamics: namely, three non-linear modes as compared to 17 linear modes. Again, the results obtained by the three non-linear modal analysis procedures are very similar, and again, the use of the manifolds of the undamped system is found to be a good approximation when small diagonal damping is utilized. In this case, the seventh, 13th and 17th linear modes are internally resonant in the undamped case (for the undamped system, internal resonances occur for linear modes related as $\omega_j = 2\omega_k - \omega_l$ and $\omega_j = 2\omega_k + \omega_l$, where ω_n is the natural frequency of the nth linear mode[11]). The

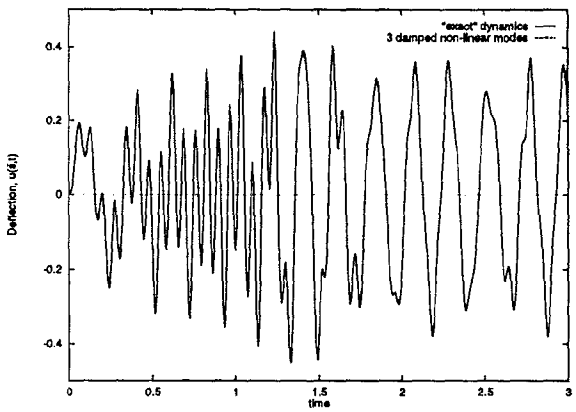

Figure 9: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode damped invariant manifold. Initial transient regime of the forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

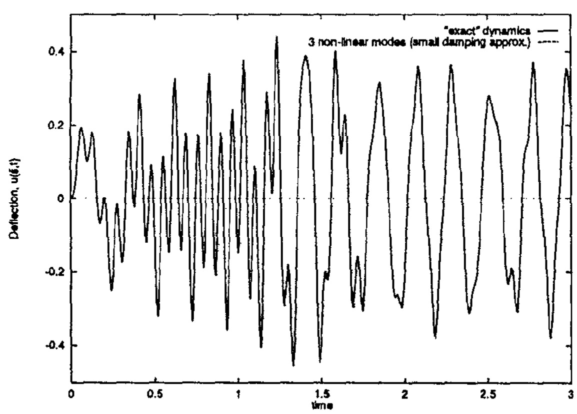

Figure 10: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold determined with a small damping approximation. Initial transient regime of the forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

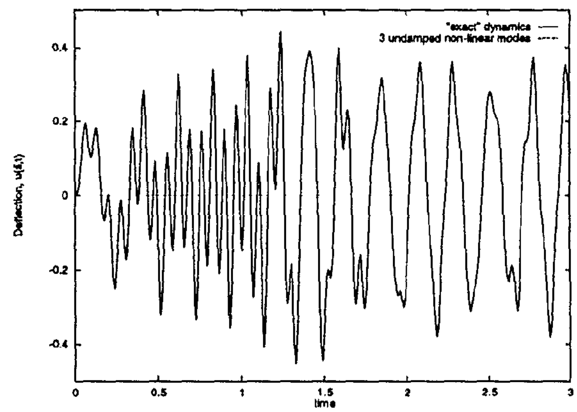

Figure 11: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold of the undamped system. Initial transient regime of the forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

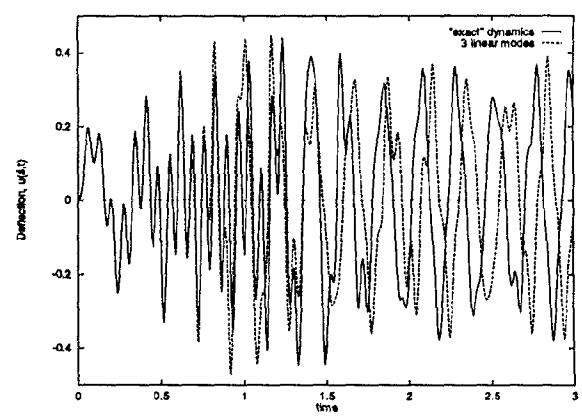

Figure 12: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a three-mode linear modal analysis of the non-linear system. Initial transient regime of the forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

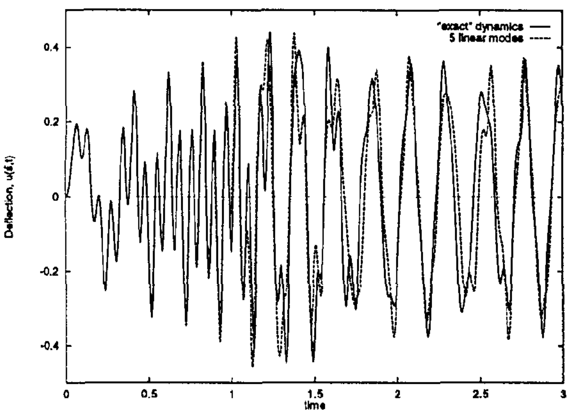

Figure 13: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a five-mode linear modal analysis of the non-linear system. Initial transient regime of the forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

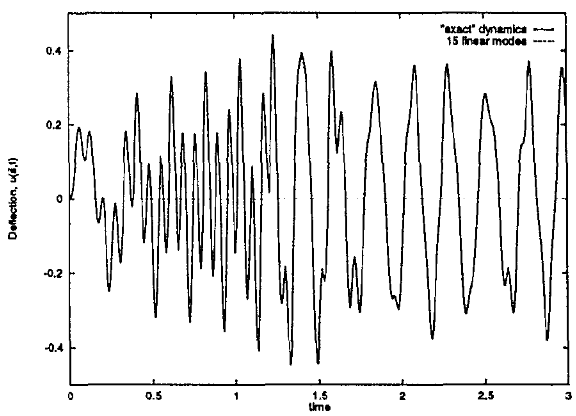

Figure 14: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a fifteen-mode linear modal analysis of the non-linear system. Initial transient regime of the forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

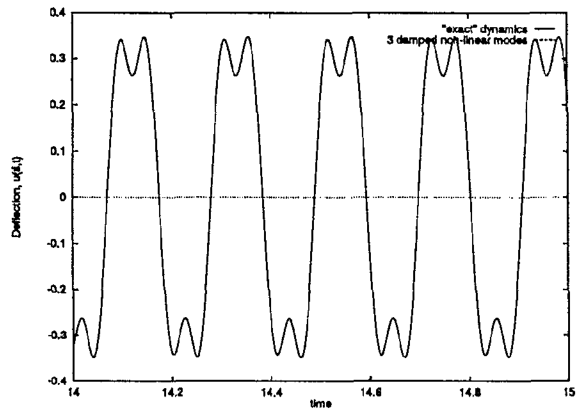

Figure 15: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode damped invariant manifold. Forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

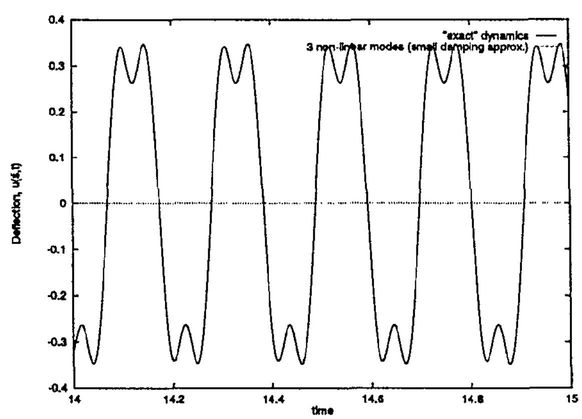

Figure 16: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold determined with a small damping approximation. Forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

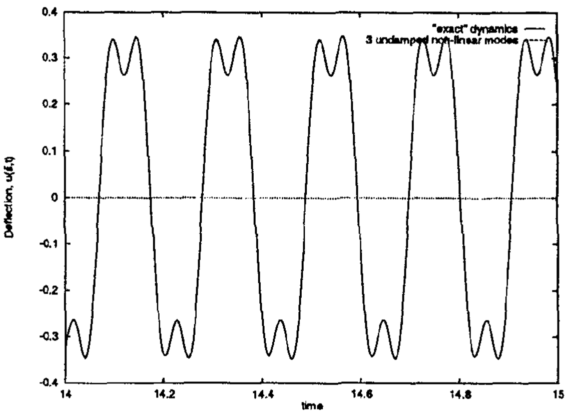

Figure 17: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold of the undamped system. Forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

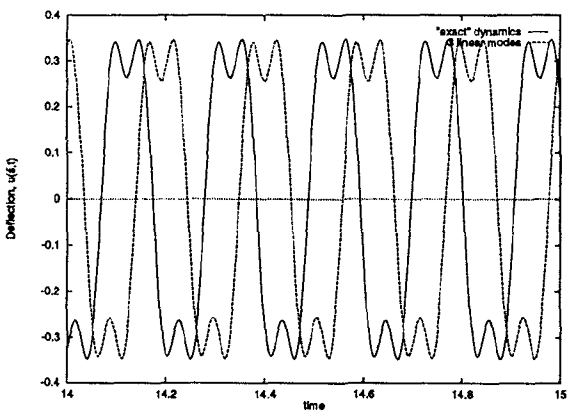

Figure 18: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a three-mode linear modal analysis of the non-linear system. Forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

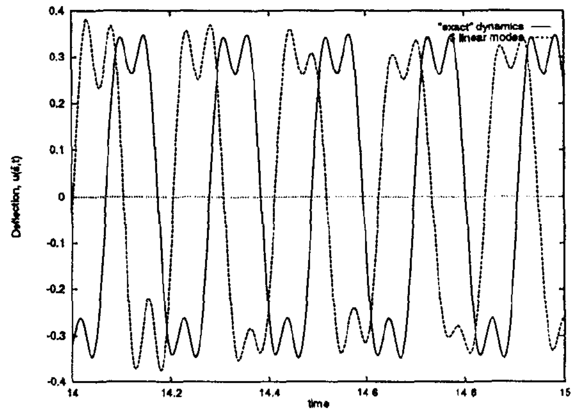

Figure 19: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a five-mode linear modal analysis of the non-linear system. Forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

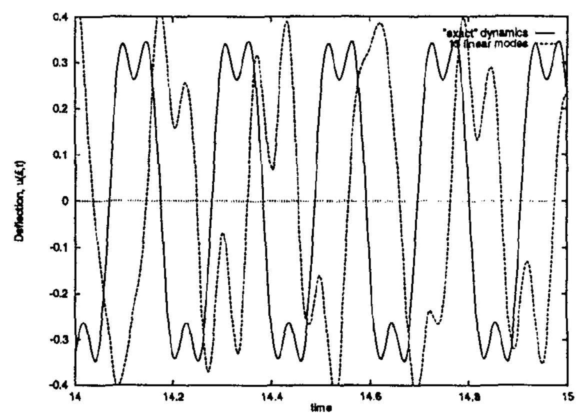

Figure 20: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a fifteen-mode linear modal analysis of the non-linear system. Forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

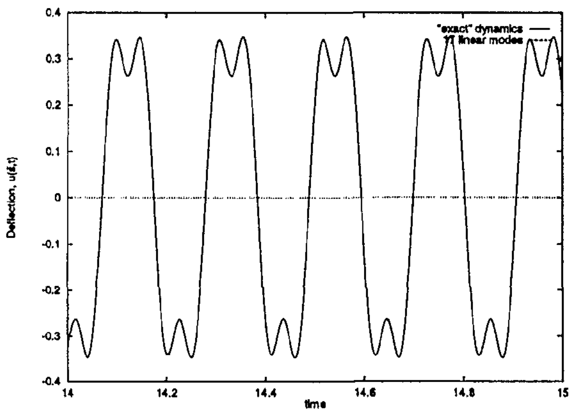

Figure 21: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a seventeen-mode linear modal analysis of the non-linear system. Forced response near the resonance of the third mode, $z(t) = 0.05\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

seventh and 13th linear modes are subject to significant excitation via the non-linear coupling terms and, in turn, the 17th linear mode possesses a significant response and is required in a linear modal analysis of the system. However, because all these non-linear modal coupling terms originate from excitations in the first and third non-linear modes only (due to the forcing exciting mostly theses modes), a reduced-order model comprised of only the first and third non-linear modes includes, qualitatively and quantitatively, all the desired contamination effects.

Figures 22-33 present results for a case of non-harmonic forcing, where two non-linear modes are simultaneously excited near resonance. Again, good agreement is found between the various non-linear modal analysis procedures, and in all three cases, fewer non-linear modes than linear modes are needed to achieve a predetermined accuracy for both the transient and long-term dynamics (namely, three non-linear modes instead of seven linear modes).

As a word of caution, it should be noted that for too large amplitudes, the forced non-linear modal analysis procedure may fail to produce good results, not only because the assumption of time-independent modal manifolds is no longer valid, but also because the amplitude of the response can become larger than the radius of convergence of the series involved. In the latter case, numerical divergence can be observed. This phenomenon is due to the Taylor series expansion (and its truncation) involved in the generation of the single- or multi-mode manifolds, which

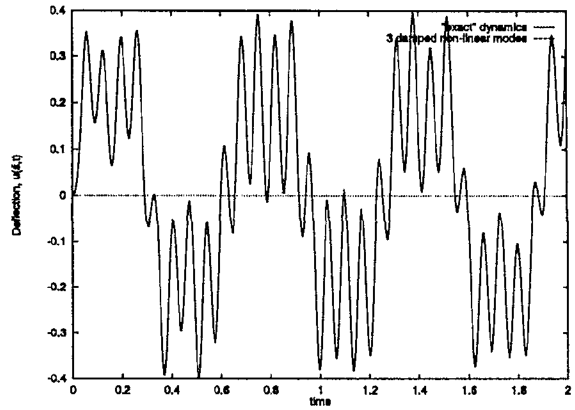

Figure 22: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode damped invariant manifold. Initial transient regime of the forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

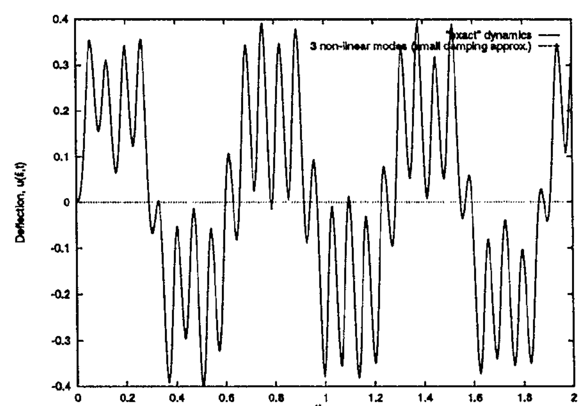

Figure 23: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold determined with a small damping approximation. Initial transient regime of the forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

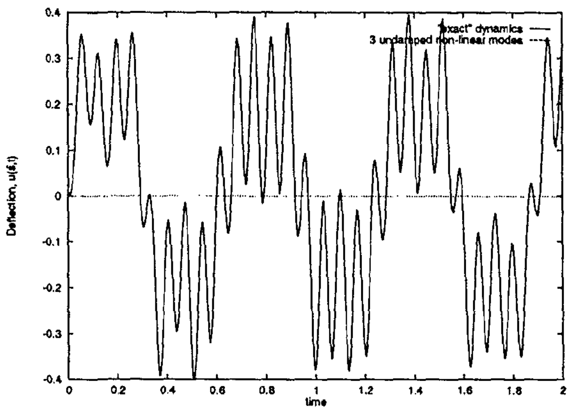

Figure 24: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold of the undamped system. Initial transient regime of the forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

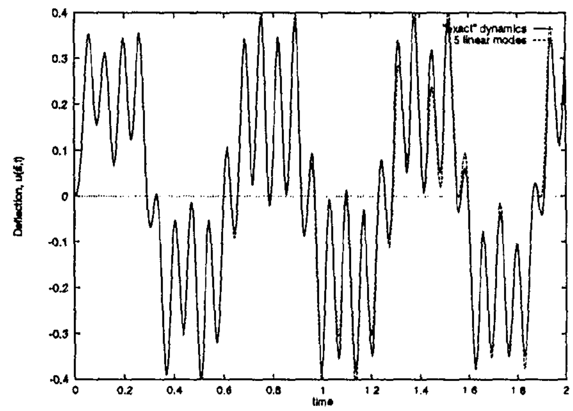

Figure 26: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a five-mode linear modal analysis of the non-linear system. Initial transient regime of the forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

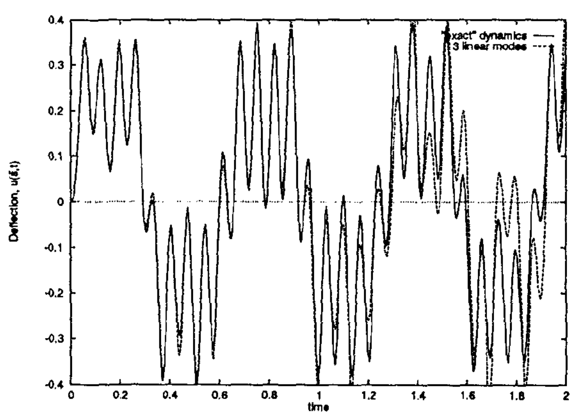

Figure 25: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a three-mode linear modal analysis of the non-linear system. Initial transient regime of the forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

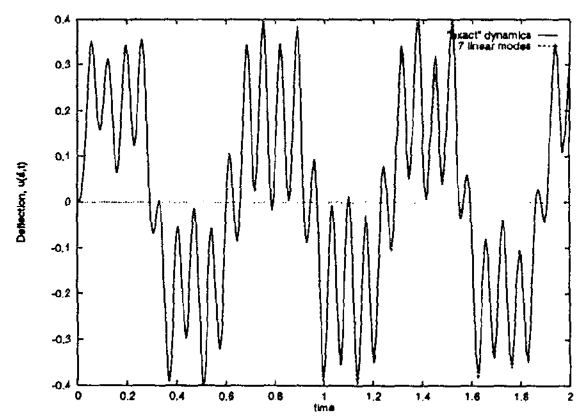

Figure 27: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a seven-mode linear modal analysis of the non-linear system. Initial transient regime of the forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

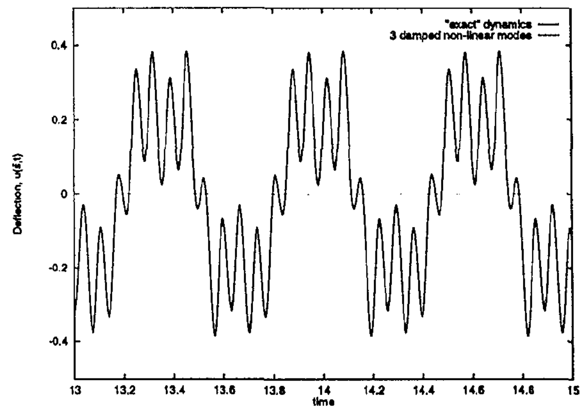

Figure 28: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode damped invariant manifold. Forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

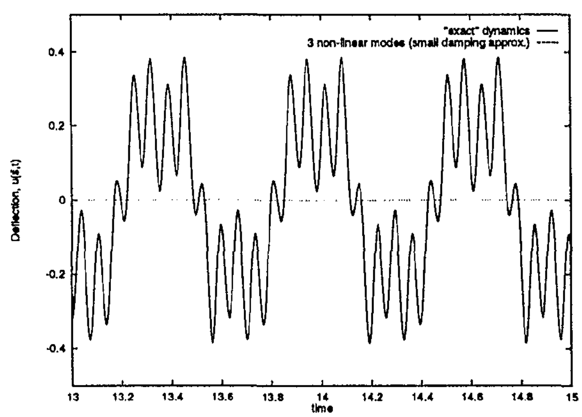

Figure 29: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold determined with a small damping approximation. Forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t){+}0.1\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

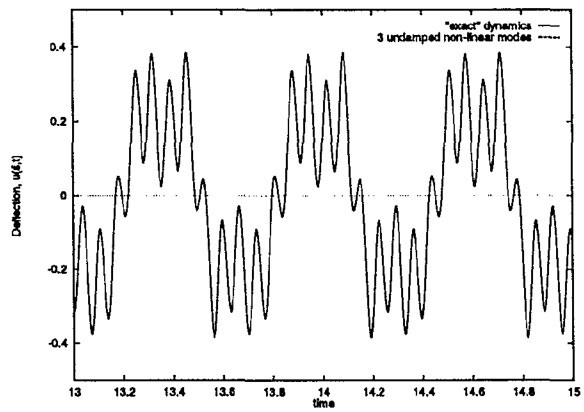

Figure 30: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a third-order accurate three-mode invariant manifold of the undamped system. Forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

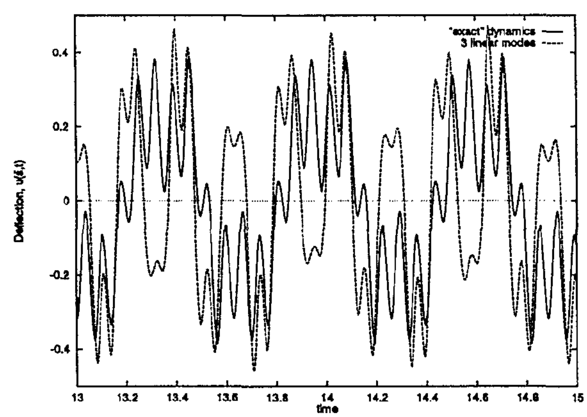

Figure 31: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a three-mode linear analysis of the non-linear system. Forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t){+}0.1\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

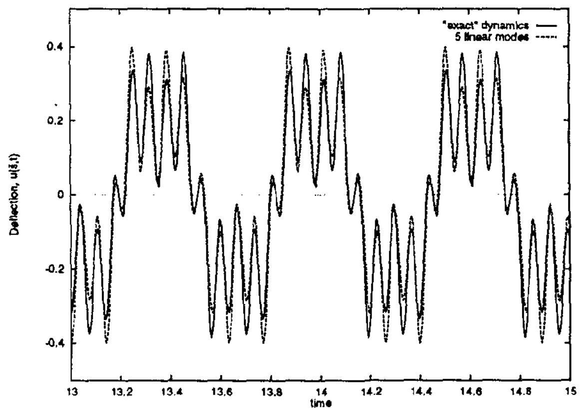

Figure 32: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a five-mode linear modal analysis of the non-linear system. Forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

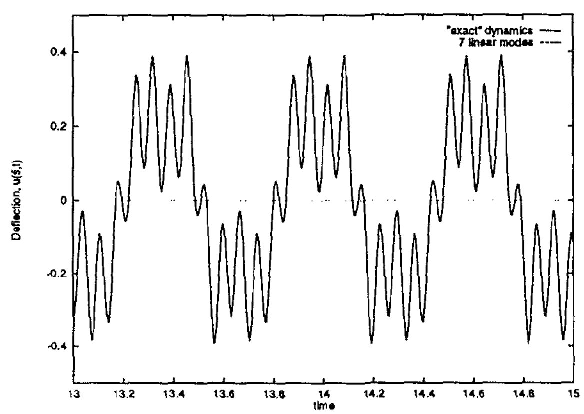

Figure 33: Deflection of the point of abscissa $\tilde{s}$ on the beam as obtained by a seven-mode linear modal analysis of the non-linear system. Forced response near the resonances of the first and third mode, $z(t) = 3\sin(10t) + 0.1\sin(90t)$. , after a long transient regime. $\alpha = 1, \beta = 5000, \tilde{s} = 1/2, \xi_1 = \xi_2 = \xi_3 = 0.01, u_1(0) = u_2(0) = u_3(0) = 0, v_1(0) = v_2(0) = v_3(0) = 0$.

assumes terms of decreasing importance, and implicitly limits the amplitudes of allowed motions. This restriction is also known to occur with the normal form theory (see reference 7). Note that this phenomenon also exists for the free non-linear modal analysis presented in references 10-12 and Section 2.

4. Conclusions

The investigation presented herein reveals some of the advantages and weaknesses of the concept of invariant manifolds when it comes to performing a non-linear modal analysis of the forced response of a non-linear structural system, and to generating reduced-order models for efficient dynamic analyses.

First, the possible use of the modal manifolds associated with systems with no damping or small damping has been investigated for use in the free and forced non-linear modal analyses of damped systems. Excellent results are found for the case study considered, in particular when small diagonal damping is assumed for the system, which is a case of practical significance.

Second, and most important, it is found that, although the invariance of the manifolds obtained in the free response case is clearly violated when they are used in a forced response context, there are ranges of excitation parameters where it is legitimate to do so. In particular, this is the case when the excitation amplitude is sufficiently small compared to the amplitude of the response, in which case the time-dependent invariant manifolds of the forced system can be reasonably approximated by the time-independent manifolds of the unforced system. At or near resonance, the proposed forced non-linear modal analysis can provide significant improvements, in terms of model reduction, over the traditional linear modal analysis of the non-linear system, in particular when non-negligible non-linear modal coupling terms are present. Because it violates the invariance of the modal manifolds by ignoring their time-dependence, this forced non-linear modal analysis is fairly independent of the type of external excitation considered, and it can be applied in cases of small non-harmonic external forces. However, when forcing amplitudes are increased, the proposed method fails, due not so much to the fundamental underlying procedure as to the asymptotic determination of the modal manifolds used in practice. It should be emphasized that, if the determination of the single- or multi-mode manifolds were performed by other means than a Taylor series expansion, these limitations in the amplitude of allowable motions might be attenuated or eliminated.

It should also be noted that, among the modes with a significant response due to the external excitation, it may be possible, in some instances, to not model the "fastest" modes (i.e., those with the smallest relaxation time), by using some form of non-linear static condensation. Examples of such non-linear static condensation techniques are given by the Nonlinear Galerkin methods, recently introduced in the context of Inertial Manifolds in the analysis of turbulent flow in fluid mechanics (see, for instance, 39-45). In such cases, the Nonlinear Galerkin method may be used on these fast modes with significant external forcing, while the time-independent multi-mode manifold procedure would be used for the modes without significant external forces (but with significant coupling). In essence, this would correspond to applying the non-linear static reduction method to the reduced set of non-linear modes obtained using the proposed forced non-linear modal analysis. This combined non-linear modal analysis / non-linear static reduction procedure has not been carried out here.

Finally, notice that perturbation methods could be used to obtain the response of multi-degree of freedom systems under harmonic excitation. However, these methods would typically not be tractable for large-scale problems, because they do not provide a reduced set of equations to be analyzed. Moreover, for general non-harmonic forcing, these methods would typically fail to produce a result. On the other hand, the proposed forced non-linear modal analysis allows one to obtain a reduced set of modal equations capturing the most important features of the dynamics of the system, even for non-harmonic forcing. For harmonic forcing, this reduced-order model can be analyzed, if desired, using perturbation methods, or, alternatively, numerical methods can be utilized (*e.g.*, multi-harmonic balance, time-integration).

The forced non-linear modal analysis presented here is thought to have potentially important applications in the area of structural dynamic analysis, in particular for large-scale non-linear structural systems, where the possibility of generating accurate reduced-order models can be critical.

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