Efficient Multi-Stage Time Marching for Viscous Flows via Local Preconditioning

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A new method has been developed to accelerate the convergence of explicit time-marching, laminar, Navier-Stokes codes through the combination of local preconditioning and multi-stage time marching optimization. Local preconditioning is a technique to modify the time-dependent equations so that all information moves or decays at nearly the same rate, thus relieving the stiffness for a system of equations. Multi-stage time marching can be optimized by modifying its coefficients to account for the presence of viscous terms, allowing larger time steps. We show it is possible to optimize the time marching scheme for a wide range of cell Reynolds numbers for the scalar advection-diffusion equation, and local preconditioning allows this optimization to be applied to the Navier-Stokes equations. Convergence acceleration of the new method is demonstrated through numerical experiments with circular advection and laminar boundary-layer flow over a flat plate.

Nomenclature

Roman letters

\(A\) Cell area
\(a, b\) Horizontal and vertical components of advection velocity, \(q\)
\(c\) Speed of sound
\(C\) Viscous flux Jacobian in the \(x\)-direction
\(C_n\) Condition number, ratio of largest to smallest eigenvalues
\(d\) Horizontal shift for Tchebyshev polynomial transformation
\(E\) Energy per unit volume
\(E\) Viscous flux Jacobian in the \(y\)-direction
\(F\) Inviscid flux vector, \(x\)-component
\(G\) Inviscid flux vector, \(y\)-component
\(H\) Total enthalpy per unit volume
\(h\) Ratio of cell area to the length of the diagonal
\(H\) Inviscid flux vector
\(J\) Viscous flux vector

\(l\) Length from a cell corner to the diagonal along the advection velocity direction
\(L_0\) Reference length
\(M\) Mach number
\(p\) Pressure
\(P\) Preconditioning matrix
\(P_n\) \(N\)th-order polynomial
\(Pr\) Prandtl number, 0.72
\(q\) Advection velocity
\(Q_0\) Reference velocity
\(q_{x,y}\) Heat flux components
\(R\) Viscous flux vector, \(x\)-component
\(R_S\) Negative Real extent of the spatial operator’s Fourier Footprint
\(R_T\) Negative Real extent of the temporal operator
\(Re\) Reynolds number
\(Re_A\) Cell Reynolds number (generic)
\(Re_{Ax}\) Cell Reynolds number based on \(\Delta x\)
\(Re_h\) Cell Reynolds number based on \(h\)
\(Re_x\) Reynolds number based on \(x\)-coordinate
\(Res\) Residual
\(S\) Conserved scalar quantity
\(s\) Cell length
\(S\) Viscous flux vector, \(y\)-component
\(T\) Temperature
\(T_n\) \(N\)th-order Tchebyshev polynomial
\(U\) Blasius freestream velocity
Introduction

This work is motivated by the poor convergence of explicit time-stepping Computational Fluid Dynamics (CFD) codes for viscous flow problems. The recent development of local preconditioning methods\textsuperscript{1-4} and the flexibility of multistage time-marching schemes provide the tools to relieve this bottleneck.\textsuperscript{a} With the strong shift toward cache-based parallel architectures, implicit schemes are beginning to show limitations for this emerging environment.

Typically an explicit time marching scheme is limited by the minimum of an inviscid time step\textsuperscript{5,6} and a viscous time step.\textsuperscript{7} The inviscid portion relates to the advective part of the equations and the viscous portion represents the dissipation component. In terms of Von Neumann stability analysis\textsuperscript{8} these portions correspond to the extents of the Fourier Footprint\textsuperscript{b} along the imaginary axis and negative real axis, respectively.

In the beginning of the "Euler era" (early 1980s), multistage time-marching schemes were optimized for the maximum imaginary extent of the stability domain, given the number of stages of the scheme.\textsuperscript{9} This allows inviscid problems to be solved efficiently on a single grid. Very soon, multigrid relaxation made its entrance,\textsuperscript{10} and the emphasis in multi-stage design shifted to choosing the coefficients such that maximum high-frequency damping results for a given number of stages, typically at half the maximum allowable time step.\textsuperscript{11-13} This optimization technique originally was based on a scalar analysis, losing most of its validity when applied to a system of equations. The introduction of local preconditioning for the Euler equations,\textsuperscript{4} which tends to equalize the advective time scales and concentrates the Fourier footprint, finally made it possible to optimize high-frequency damping for all types of waves admitted by the inviscid system.\textsuperscript{14-16}

For a typical viscous flow problem, however, a majority of the computational cells are limited in the size of their time step by the viscous criterion, due to the limited extent of the stability boundary along the negative real axis. By making the multistage coefficients a function of the local cell Reynolds number, $Re_A$, the stability boundary can be reshaped to alleviate this restriction.\textsuperscript{15,16}

For systems of equations, local preconditioning is necessary for equalizing various time scales admitted by the system.\textsuperscript{16-20} However, the analytical preconditioning analysis for the Navier-Stokes equations is much more complicated than for the Euler equations, and is still far from complete. As a result, the current paper uses a simple block-Jacobi method to extend the

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\textsuperscript{a} Also, as a fringe benefit of local preconditioning, the solution accuracy at low Mach numbers is enhanced.

\textsuperscript{b} The locus of the Fourier transform in the complex plane.
Euler preconditioner to viscous problems.\textsuperscript{21}

In the work of Lynn,\textsuperscript{15,16} multistage methods are optimized based on the precise Fourier footprint of the spatial operator. The optimization must be performed separately for each choice of cell Reynolds number. The multistage time steps can be tabulated as functions of $Re_A$, or, more practically, given as functions that closely fit the data. Neither Lynn nor D. Lee\textsuperscript{28} actually carry out Navier-Stokes calculations with $Re_A$-dependent multistage coefficients.

In the present work the optimization procedure is reversed, and thereby greatly simplified. We start with a family of multistage schemes that has stability domains of desirable shape. Then, using the scalar advection-diffusion operator as a foothold, we fit the spatial footprint within the temporal stability boundary to achieve stability and possibly some other desirable property, like prescribed high-frequency damping. Related work which optimizes the time step without consideration for the associated damping is given by Lorber et al.\textsuperscript{22,23} The transition from the advection-diffusion operator to the Navier-Stokes operator is accommodated by the local preconditioning. This procedure provides a numerical relationship between $Re_A$ and the multistage time step coefficients. The resulting $Re_A$-dependent marching schemes are used to solve the advection-diffusion equation and compute laminar boundary-layer flow.

**Runge-Kutta Design**

Optimizing the time-marching scheme for efficiency or high-frequency damping is most easily accomplished by working in the complex space of the Fourier transform via Von Neumann stability analysis.\textsuperscript{8} The overriding goal, of course, is stability, i.e., to contain the Fourier Footprint of the spatial discretization inside the stability boundary of the time-marching scheme. A secondary goal is to provide some level of high-frequency damping. If only single-grid marching is desired, the tradition is to go for the largest time step possible. For multigrid marching it is necessary to maximize high-frequency damping, which means giving up the maximum time step. The latter strategy actually turns out to also be more efficient for single-grid marching, as was shown by Tai.\textsuperscript{15} Maximizing high-frequency damping without setting a target level for the damping, though, is not a good idea for a viscous equation solver, as this tends to reduce the time step to unacceptably low values.\textsuperscript{16,16}

For the sake of simplicity we will start our analysis by examining the scalar model equation for the Navier-Stokes system of equations: the advection-diffusion equation,\textsuperscript{c}

\[ S_t + a S_x + b S_y = \frac{1}{Re} (S_{xx} + S_{yy}), \quad (1) \]

where $S$ is some conserved, scalar quantity; $a$ and $b$ are horizontal and vertical components of the advection velocity, respectively; and $Re$ is the Reynolds number.

Once we choose a discretization of the spatial terms, we are stuck with the resulting Fourier Footprint, so we will study it first. Afterward we will examine the temporal operator, and finally, the procedure used to fit one to the other.

**Spatial Operator**

For this study we will use central differencing for the viscous terms and the $\kappa$-scheme\textsuperscript{24,25} for the advective terms. This yields,

\[
F = - \left( (1 - \kappa) \frac{a \Delta t}{\Delta x} + 2 \frac{\Delta t}{Re \Delta x^2} \right) (1 - \cos \beta_x) \\
- \left( (1 - \kappa) \frac{a \Delta t}{\Delta x} + \frac{1}{4} \left( 1 - \cos 2 \beta_x \right) \right) \\
- \left( (1 - \kappa) \frac{b \Delta t}{\Delta y} + 2 \frac{\Delta t}{Re \Delta y^2} \right) (1 - \cos \beta_y) \\
- \left( (1 - \kappa) \frac{b \Delta t}{\Delta y} \right) \frac{1}{4} \left( 1 - \cos 2 \beta_y \right) \\
- \left( \frac{b \Delta t}{\Delta y} \sin \beta_y + i \frac{(\kappa - 1) b \Delta t}{2 \Delta y} \sin 2 \beta_y \right),
\]

where $\kappa \in [-1,1]$, providing a blending between fully upwind and central differencing, respectively. In a moment we will look at the resulting Fourier Footprint, but first let us define some parameters,

\[ q = \sqrt{a^2 + b^2}, \quad \text{the advection speed,} \]
\[ \phi = \arctan \frac{b}{a}, \quad \text{the flow angle,} \]
\[ \mathcal{R} = \frac{\Delta x}{\Delta y}, \quad \text{cell aspect ratio,} \]

and scale the footprint by fixing the position of the high frequency components ($\beta_{x,y} = \pi$) at $-R_S$. With these definitions, Eq 2 becomes,

\[
- R_S = -2 \Delta t \left[ (1 - \kappa) \frac{q}{\Delta x} (\cos \phi + \mathcal{R} \sin \phi) \right. \\
\left. + 2 \frac{2}{Re \Delta x^2} (1 + \mathcal{R}^2) \right]. \quad (3)
\]

In this form, two length scales have emerged,

\[ h = \frac{\Delta x}{\sqrt{1 + \mathcal{R}^2}}, \quad (4) \]

the ratio of cell area to the length of the diagonal, and

\[ l = \frac{\Delta x}{\cos \phi + \mathcal{R} \sin \phi}, \quad (5) \]

which is the length from a cell corner to the diagonal along the advection velocity direction. Figure 1...
shows the cell geometry and the associated lengths. Moreover, \( h \) and \( l \) are related by \( l = h/\cos \theta \) where \( \theta \) is the angle between \( h \) and the advection velocity.

Using the new length scale, \( h \), we solve for \( \Delta t \) to fix the high frequency portion of the footprint at \(-R_\ast\),

\[
\Delta t = \frac{\frac{R_S}{2}}{(1 - \kappa) \cos \theta + \frac{2}{Re h^2}} .
\]

Factoring out the \( q/h \) and converting the Reynolds number, \( Re \), to the cell Reynolds number, \( Re_h \), we find,

\[
\Delta t = \frac{R_S h}{2 q Re_h (1 - \kappa) \cos \theta + 2}.
\]

Substituting this time step into Eq. 2 gives our scaled Fourier Footprint of the form,

\[
\mathcal{F} = -\frac{R_S h}{2 \Delta x Re_h \cos \theta + 2} \left[ \left( \frac{R_{n+1} \cos \phi + 2}{Re_h} \right) \left( 1 - \cos \beta_z \right) + i \cos \phi \sin \beta_z \\
+ \beta \left( \frac{R_{n+1} \sin \phi + 2}{Re_h} \right) \left( 1 - \cos \beta_y \right) \right].
\]

Finally, we reveal the Fourier Footprint in Fig. 2 which shows the effect of various \( \kappa \)'s for \( R_S = 1, \beta = 45 \text{deg}, Re_h = 1, \beta_x, \beta_y \in [0, 2\pi], \Delta \beta = 2\pi/40 \).

The resulting coefficients of \( P_n(z) \) are chosen such that the \(|P_n(z)| = 1\) stability boundary remains simply connected and the blended polynomial is scaled properly, i.e.,

\[
P_n(0) = 1,
\]

and

\[
\left. \frac{dP_n}{dz} \right|_{z=0} = 1.
\]

For example, a four stage scheme would take the form,

\[
P_4(z) = 1 - \frac{32d^3 - 16d^2 + \varepsilon^2}{D} + \frac{48d^2 - 8d^2 + \varepsilon^2}{D} \frac{32d^3 + 8}{D^3} z^4,
\]

where \( D = 8d^4 - 8d^2 + \varepsilon^4 \). To satisfy the conditions given by Eqs. 11 and 12 we find,

\[
\varepsilon = \sqrt{4d(2+d) + 16d^2(2+d)^2 - 8d^2(d+4)}.
\]

Thus, we now have a one-parameter family of stability boundaries governed by \( d \).

Next, we recall that a 4-stage Runge-Kutta scheme has the form of

\[
P_4(z) = 1 + \alpha_4 z + \alpha_4 \alpha_2 z^2 + \alpha_4 \alpha_3 \alpha_2 z^3 + \alpha_4 \alpha_3 \alpha_2 \alpha_1 z^4.
\]
So by equating the coefficients in Eqs. 13 and 15 we can determine the coefficients, \( \alpha_k \), for the corresponding Runge-Kutta scheme.

**Fitting**

The procedure employed is analogous to blowing up a balloon into a bear trap: a given stability boundary (the bear trap) is chosen; next, a given high frequency damping level is located on the negative real axis; and then the cell Reynolds number is increased until some part of the spatial operator's Fourier Footprint (the balloon) reaches a specified damping contour of the temporal operator. Figure 3 depicts the parameters, the stability boundary, and the spatial operator Fourier Footprint.

The fitting algorithm is as follows:

1. Pick a value for \( d \). This sets the shape and extent of the Tchebychev stability boundary.

2. Compute \(-RT\), the negative Real extent of the Temporal operator. (Hint: \( RT = 2d \).)

3. Given the negative Real extent of the stability domain, \(-RT\), as a starting point, find the negative Real extent of the spatial operator, \(-RS\), which satisfies the prescribed high frequency damping, \( \sigma_n \), by moving toward the origin along the Real axis, computing \( |P_n(z)| \).

4. Using the viscous limit \( (Re_h \to 0) \) as an initial guess increase the cell Reynolds number until the prescribed damping conditions for the entire footprint or its high-frequency part are met.

Figure 4 shows a sequence of stability plots for different values of \( d \). Superimposed are the spatial operator Fourier Footprints which yield the maximum time step given a high-high frequency damping of 0.5. The contours in Fig. 4 represent levels of the temporal operator's amplification (or damping) factor, \( |P| \), from 0.1 to 1.0 in increments of 0.1.

Notice the large negative extent of the stability domain along the real axis, made possible by the Tchebychev polynomial. The domain maximally extends to \(-2n^2\) for an \( n \)-stage scheme; however, for this choice the scheme is only stable in the limit of \( Re_h = 0 \) (refer to Fig. 4(d)).

Figure 5 shows the reference time step and Runge-Kutta coefficients as a function of the cell Reynolds number for the resulting 4-stage Runge-Kutta scheme. This provides the link between the cell Reynolds number, the time step, and the multistage coefficients for a given cell. Note that the time step maintains an appreciable value all the way down to \( Re_h = 0.1 \), which is smaller than many people use to resolve the boundary layer in a Navier-Stokes calculation.

**Extension to Systems (Local Preconditioning)**

Local preconditioning is a technique to remove stiffness from a system of equations. In the aeronautical CFD community, the set of time-dependent, Reynolds-averaged, compressible, Navier-Stokes equations—sometimes simplified to the (inviscid) Euler equations—are typically solved in an iterative fashion to achieve a steady-state solution. Although in most cases only the steady-state solution is desired, the time-dependent equations are still employed so that the marching problem is well posed for all Mach numbers. The time-dependent Euler equations are hyperbolic and admit real wave-like fundamental solutions; the Navier-Stokes equations are mixed parabolic-hyperbolic and admit damped traveling waves as well as non-propagating damped modes.

Convergence to steady-state for inviscid calculations is impaired in some flow regimes due to the spread in the characteristic wave speeds. The spread is largest for waves moving in the streamwise direction, in which case their speeds are \( q, q+c, \) and \( q-c \), where \( q \) is the total flow speed and \( c \) is the speed of sound. The ratio of the largest to smallest wave speed is termed the condition number, \( C_n \), and serves as a measure of "stiffness". Figure 6 shows the condition number as a function of Mach number for the Euler equations. The solid line represents the original Euler equations and it is apparent that infinite stiffness occurs in both the subsonic and transonic regimes; as the Mach number further increases, the condition number asymptotes to the ideal value of one.

The dashed line in Fig. 6 represents the Euler equations preconditioned with the Van Leer-Lee-Roe preconditioner. This preconditioner comes closest to achieving equalization of wave speeds, without affecting their direction of propagation or their hyperbolicity. The dashed line shows it is possible to eliminate completely the stiffness as the Mach number goes to zero, greatly reduce the transonic stiffness, and, in general, substantially lower the condition number for the system of equations. This makes the system behave as

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\[^{4}\text{Or, if unable to find the requested damping, use the maximum damping available.}\]
Fig. 4 Optimized stability plots as a function of the temporal operator parameter, $d$, for Fromm's Scheme ($\kappa = 0$) with a prescribed high frequency damping of 0.5 with contours of damping every 0.1 and the size of the frequency symbols scaled by their respective damping.
Fig. 5 The reference time step and Runge-Kutta coefficients as a function of the cell Reynolds number for the fitting presented in Fig. 4.

Fig. 6 Condition number as a function of Mach number for the Euler equations.

A scalar equation which has only a single wave speed, so we can apply the Runge-Kutta scheme developed in the previous section after accounting for some scaling factors.

The inviscid portion of local preconditioning used in this paper is that of Van Leer-Lee-Roe with a modification according to D. Lee and Lynn. In stream-aligned, hyperbolic symmetrizing variables, 
\[
\left[ \frac{dp}{\rho \sigma}, \frac{dn_x}{\sigma}, \frac{dn_y}{\sigma}, \frac{dn_z - \xi^2 dp}{\sigma} \right]^T,
\]
its form,
\[
P_E = \begin{bmatrix}
\frac{\xi M_x}{\sigma} & -\frac{\xi M_x}{\sigma} & 0 & 0 \\
-\frac{\xi M_x}{\sigma} & 1 + \frac{\xi}{\sigma} & 0 & 0 \\
0 & 0 & \xi & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]
with \( \xi = \sqrt{1 - M^2} \) and \( \xi = \sqrt{1 + \frac{\alpha_x^2}{\sigma}} \), where \( \alpha_x \) is an effective streamwise aspect ratio given by summing the appropriate velocity projections for each cell face, \( i \),
\[
\alpha_x = \frac{\sum_i (u - u_x^i) \Delta x_i + v \Delta y_i}{\sum_i (u - u_x^i) \Delta x_i - v \Delta y_i}.
\]

This form is due to Darmofal.

Preconditioning of the Navier-Stokes equations is accomplished according to the method proposed in Ref. 21, namely, the addition of the viscous Jacobians:
\[
P_{NS}^{-1} = P_E^{-1} + \frac{2}{\Delta x(1 - \kappa)} (C + A E),
\]
where \( C \) and \( E \) are the Jacobians of the viscous fluxes in the \( x \) and \( y \) directions, respectively.

Thus, instead of marching to the steady state with
\[
U_t = \text{Res}(U),
\]
march with a preconditioned residual
\[
U_t = P(U) \text{Res}(U).
\]

This can be viewed as marching with a matrix time step as opposed to a scalar time step. However, the following three elements of an existing code need to be modified to account for the new, preconditioned system:

- Time-step definition
- Artificial dissipation
- Boundary conditions

First, since preconditioning serves to collapse all wave speeds to the total flow speed, \( q \), the time step should be based on the flow speed, \( q \), and not the largest acoustic wave speed, \( q + c \), which is typically used in the unconditioned case.

Next, for an upwind scheme the artificial dissipation matrices \( |A| \) and \( |B| \) need to be evaluated in terms of preconditioned quantities, i.e., they become \( P^{-1} |PA| \) and \( P^{-1} |PB| \) for the preconditioned system.

And lastly, if one is employing "weak" (image cell) boundary conditions and using the modified upwind scheme to solve for the boundary fluxes, no change is necessary for the boundary conditions. However, if one is using explicit characteristic boundary conditions and/or employs "strong", specified boundary fluxes, these procedures need to be modified to account for the preconditioned equations that are now being solved.

**Numerical Experiments**

Two numerical experiments were performed, the first on the scalar model equation for the Navier-Stokes equations: advection-diffusion; and the second experiment solved the Navier-Stokes system of equations for boundary layer flow over a flat plate.
Circular Advection

To remove the added complication of preconditioning a system of equations, the scalar advection-diffusion model problem was first considered. The case considered here is that of circular advection in the upper half plane, i.e., the domain consisted of 40 equally-spaced cells along the z direction from -1 to 1 and 20 equally-spaced cells along the y direction from 0 to 1, with a top-hat profile entering at the lower left rotating clockwise to exit at the lower right. Figure 7 shows a representative solution for a Reynolds number of 500.

The number of iterations to converge the $L_2$ error norm to $10^{-10}$ are recorded in Table 1 for a host of Reynolds numbers for both the variable-coefficient Runge-Kutta scheme and the constant-coefficient scheme. The fixed-coefficient scheme uses advection-optimized Runge-Kutta coefficients due to Tai for maximizing damping over frequencies from $\pi/4$ to $\pi$ while the variable-coefficient scheme uses the cell Reynolds number-dependent coefficients that were developed in the preceding section. Note that for this problem the average cell Reynolds numbers are a factor of 20 less than the Reynolds number and actually tend toward zero near the center of the advection velocity field since $Re_h = Re \, h \, q$. However, for very low values of $q$ we found that the local time-step given by Eq. 7 must be capped to avoid numerical instability regardless of the marching method used.

Table 1 clearly demonstrates the devastating effects that low Reynolds numbers (and hence low cell Reynolds numbers) have on the standard advection-optimized time marching scheme. As anticipated by the Fourier analysis, the variable-coefficient scheme retains a healthy convergence rate even with local cell Reynolds numbers well below 0.01.

### Laminar Boundary Layer

The flow solver for this portion of the study was purposely kept as simple as possible to isolate the effect of the variable Runge-Kutta coefficients. The solver consists of a cellcentered, finite-volume-based scheme which uses Roe's Flux Difference Splitting for the inviscid fluxes and central differencing for the viscous

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**Table 1** Comparison of constant-coefficient Runge-Kutta scheme to variable-coefficient Runge-Kutta scheme for Circular Advection on a 40 $\times$ 20 grid.

<table>
<thead>
<tr>
<th>Re</th>
<th>Variable</th>
<th>Fixed</th>
</tr>
</thead>
<tbody>
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<td>4,271</td>
</tr>
<tr>
<td>10</td>
<td>422</td>
<td>2,736</td>
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<tr>
<td>25</td>
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<td>1,374</td>
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<td>50</td>
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<td>500</td>
<td>85</td>
<td>136</td>
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<tr>
<td>1,000</td>
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<td>5,000</td>
<td>81</td>
<td>98</td>
</tr>
<tr>
<td>10,000</td>
<td>82</td>
<td>99</td>
</tr>
</tbody>
</table>

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$^a$Coefficients: $\alpha_1=0.2131$, $\alpha_2=0.4364$, $\alpha_3=0.7641$ with $\nu=1.1727$. 

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Fig. 7 Circular advection for a Reynolds number of 500 including velocity vectors and solution contours every 0.1 from 0.05 to 0.95.
terms on a structured grid of quadrilaterals. In integral form,

\[ \int_A \mathbf{U} \cdot dA + \oint_S \mathbf{H} \cdot ds = \oint_S \mathbf{J} \cdot ds, \]

where \( \mathbf{U} \) is the conserved state vector defined as \((\rho, \rho u, \rho v, \rho E)^T\), \( \mathbf{H} \) is the inviscid flux vector, \((\mathbf{F} \delta + \mathbf{G} \delta) \cdot d\delta\), and \( \mathbf{J} \) is the viscous flux vector, \((\mathbf{R} \delta + \mathbf{S} \delta) \cdot d\delta\).

The Cartesian components of the inviscid flux are

\[ \mathbf{F} = (\rho u, \rho u^2 + p, \rho uv, \rho uH)^T, \]

and

\[ \mathbf{G} = (\rho v, \rho v^2 + p, \rho vw, \rho vH)^T, \]

while the viscous components are

\[ \mathbf{R} = (0, \tau_{xx}, \tau_{xy}, \nu \tau_{xx} + \nu \tau_{xy} - q_x)^T, \]

and

\[ \mathbf{S} = (0, \tau_{xy}, \tau_{yy}, \nu \tau_{xy} + \nu \tau_{yy} - q_y)^T. \]

The governing equations are non-dimensionalized by freestream speed of sound, \(c\), and density, \(\rho\), so that the viscous stresses and heat flux terms are given by

\[ \tau_{xx} = \frac{M}{Re} \mu \frac{2}{3} (2u_x - v_y), \]

\[ \tau_{yy} = \frac{M}{Re} \mu \frac{2}{3} (2v_y - u_x), \]

\[ \tau_{xy} = \frac{M}{Re} \mu (u_x + v_y), \]

\[ q_x = \frac{M}{(\gamma - 1) Re Pr \mu} \nu \tau_{xx}, \]

\[ q_y = \frac{M}{(\gamma - 1) Re Pr \mu} \nu \tau_{yy}. \]

The system is closed with an equation of state,

\[ \rho = (\gamma - 1) \left[ E - \frac{1}{2} (u^2 + v^2) \right], \]

where \(E\) is the total energy per unit volume, \(H\) is the total enthalpy per unit volume, \(T\) is the temperature, \(\gamma\) is the ratio of specific heats (1.4), \(\mu\) is the viscosity, \(Pr\) is the Prandtl number (0.72), and \(Re\) is the Reynolds number.

The numerical test problem chosen is that of a laminar boundary layer flow; specifically, two-dimensional subsonic flow over a flat plate. The unit-length Reynolds number is 10,000 and the freestream Mach number is varied between 0.05 and 0.3. The computational domain and mesh are shown in Fig. 8. The plate is 4 units long, with the upstream boundary 2 units away from the leading edge and the upper boundary is placed at 1.2 units. There are 36 cells evenly distributed along the x-direction (24 cells on the plate and 12 upstream) and 40 cells exponentially stretched in the y-direction according to

\[ y_j = y_n \frac{e^{\frac{j-1}{n-1}} - 1}{e^\frac{1}{n} - 1} \quad \text{for } j = 1, \ldots, n_j \]

where the wall spacing is set to have an acoustic cell Reynolds number of 10, yielding a stretching parameter, \(c\), of 1.1360. By employing characteristic boundary conditions, neither second-order boundary conditions nor solution-assumed grids are necessary to capture the boundary layer gradients (see Fig. 9 where the computed results are compared to the Blasius results).

Table 2 shows the number of iterations to converge the L₂ error norm to 10⁻⁶ for a range of Mach numbers for both the fixed-coefficient Runge-Kutta scheme and the variable-coefficient Runge-Kutta scheme. The fixed-coefficient scheme was run with and without preconditioning. The results clearly show the benefits of preconditioning, with local preconditioning providing nearly Mach number independent convergence. However, we note some degradation as the cell Reynolds numbers become lower as the Mach number is lowered. This effect was predicted by D. Lee and shows the limitation of simply using the viscous Jacobians to augment the Euler preconditioning (see Eq. 18).

Most important, the results for the variable-coefficient scheme show even further improvement in convergence efficiency: the variable-coefficient scheme reaches convergence with nearly a factor of five fewer iterations than the fixed-coefficient scheme.

<table>
<thead>
<tr>
<th>Mach Number</th>
<th>Fixed Coefficient</th>
<th>Variable Coefficient</th>
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<tbody>
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<td>3,102</td>
</tr>
<tr>
<td>0.1</td>
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</tbody>
</table>

Parabolic stretching, as used in Refs. 30 and 31, was originally tried until, upon further examination, it became apparent that it had a canonical first-point stretching factor of three! The unconditioned and preconditioned solutions are indistinguishable. Strictly speaking, Blasius is only valid for incompressible flow, but up to Mach 0.3 compressibility effects are negligible. CPU timing data are not presented for this proof-of-concept study since no optimization of the new implementation has been attempted.
Conclusions

A four-stage time-marching scheme was optimized for a viscous model problem. The Runge-Kutta coefficients as a function of cell Reynolds number were designed to yield a specified damping level while maintaining large time steps. These variable coefficients were applied to a two-dimensional, cell-centered, Navier-Stokes solver which incorporated local preconditioning. The results for a subsonic flat plate boundary layer flow indicate that by employing local Runge-Kutta coefficients as a function of cell Reynolds number, convergence to steady state is greatly enhanced.

Since this is a proof-of-concept study, many other variables were purposely held constant so that the effects of the cell Reynolds number-dependent Runge-Kutta coefficients could be ascertained. Given the qualitative improvement over the standard method, the new approach appears to justify further development.

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Colophon

This paper was typeset in Donald Knuth’s 10pt Computer Modern Font using the free, multi-platform\LaTeX\ typesetting system and Kleb’s aiaa bundle.\textsuperscript{35} The \textit{psfrag} package was used to augment figures generated with \textit{TECPLOT} while the \textit{fix2col} package was used to correct single and double-column float placement. The nomenclature section was generated automatically with the \textit{nomencl} package and variable, automatic page referencing was done with the \textit{varioref} package.

References

\textsuperscript{1}Chorin, A. J., “A Numerical Method for Solving Incompressible Viscous Flow Problems,” \textit{Journal of Computational}
Appendix

A Non-dimensionalization of the Advection-Diffusion Equation

We begin with

\[ S_a + a S_x + b S_y = \mu (S_{xx} + S_{yy}), \]  

(A.1)

then define non-dimensional quantities,

\[ \tilde{S} = \frac{S}{S_0}, \quad \tilde{a} = \frac{a}{Q_0}, \quad \tilde{b} = \frac{b}{Q_0}, \quad \tilde{\mu} = \frac{\mu}{\mu_0}, \quad \tilde{\alpha} = \frac{\alpha}{L_0}, \quad \tilde{y} = \frac{y}{L_0}, \quad \text{and} \quad \tilde{t} = \frac{t}{Q_0/L_0}. \]  

(A.2)
Substituting into Eq. A.1 yields,
\[
\frac{\partial (S_0 \bar{S})}{\partial (L_0/Q_0 \bar{t})} + Q_0 \frac{\partial (S_0 \bar{S})}{\partial (L_0 \bar{x})} + Q_0 \bar{b} \frac{\partial (S_0 \bar{S})}{\partial (L_0 \bar{y})} = \mu_0 \bar{\mu} \left( \frac{\partial^2 (S_0 \bar{S})}{\partial L_0 \bar{z} \partial L_0 \bar{x}} + \frac{\partial^2 (S_0 \bar{S})}{\partial L_0 \bar{y} \partial L_0 \bar{y}} \right). \tag{A.3}
\]
Collecting constant terms gives,
\[
\frac{S_0 Q_0}{L_0} \bar{S}_z + \frac{S_0 Q_0}{L_0} \bar{S}_y + \frac{S_0 Q_0}{L_0} \bar{S}_y = \frac{\mu_0 Q_0}{L_0^2} \bar{\mu} \left( \bar{S}_{zz} + \bar{S}_{yy} \right). \tag{A.4}
\]
while multiplying through by \(L_0/(S_0 Q_0)\) and defining \(Re = S_0 L_0/\mu_0\) yields,
\[
S_t + a S_x + b S_y = \frac{1}{Re} (S_{xx} + S_{yy}). \tag{A.5}
\]
If we chose \(\mu_0 = \mu\) and drop the bar notation, we have simply,
\[
S_t + a S_x + b S_y = \frac{1}{Re} (S_{xx} + S_{yy}). \tag{A.6}
\]

\section*{B Time-Step Normalization for the Advection-Diffusion Equation}

Let us try to find a non-dimensional time step that includes both advective and diffusive effects.

\subsection*{One-Dimension}

To provide a simple foundation, we begin with the one-dimensional case,
\[
S_t + a S_x = \frac{1}{Re} S_{xx}, \tag{B.1}
\]
where \(S\) is some conserved scalar quantity, \(a\) is the advection speed, and \(Re\) is Reynolds number. Central differencing for the diffusive term and first-order upwind discretization of the advective term yields,
\[
F = - \frac{a \Delta t}{\Delta x} \left( 1 - \cos \beta \right) + i \frac{a \Delta t}{\Delta x} \sin \beta, \tag{B.2}
\]
where \(\beta\) represents the Fourier frequency. Alternatively, employing a higher-order \(\kappa\)-scheme\textsuperscript{24, 25} for the advective term yields,
\[
F = - \left( \frac{a \Delta t}{\Delta x} + \frac{2 \Delta t}{Re \Delta x^2} \right) \left( 1 - \cos \beta \right) - i \frac{a \Delta t}{\Delta x} \sin \beta, \tag{B.3}
\]
where \(\kappa \in [-1, 1]\). Typically one defines parameters according to the advective and diffusive terms, e.g.,
\[
\nu = \frac{a \Delta t}{\Delta x} \quad \text{and} \quad \sigma = \frac{\Delta t}{Re \Delta x^2},
\]
where \(\nu\) is the Courant number\textsuperscript{5} and \(\sigma\) is the Von Neumann number.\textsuperscript{36} This gives a footprint in Fourier space as shown in Fig. B.1 which is an ellipse defined by the Courant number and the Von Neumann number. (The higher-order scheme yields an ellipse only for \(\kappa = 1\); in general it is egg-shaped.)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{footprint.png}
\caption{Fourier footprint for the one-dimensional Advection-Diffusion equation using first-order upwind spatial discretization for the advective term and central differencing for the diffusion term (\(\beta \in [0, 2\pi]\)).}
\end{figure}

Now, suppose we want to fix the location of the negative real extent of this footprint. We can do this by examining the high frequency limit (\(\beta = \pi\)) of the Fourier footprint given in Eq. B.2,
\[
-R_S = -2 \left( \frac{\Delta t}{\Delta x} + \frac{2 \Delta t}{Re \Delta x^2} \right), \tag{B.4}
\]
where \(-R_S\) is the specified negative Real axis extent. Now we simply solve for the time step, \(\Delta t\),
\[
\Delta t = \frac{R_S}{\frac{\alpha}{\Lambda_\pi} + \frac{2}{Re \Lambda_\pi^2}}. \tag{B.5}
\]
For the \(\kappa\)-scheme the result is similar,
\[
\Delta t = \frac{R_S}{\left( \frac{\alpha}{\Lambda_\pi} \right)^2 + \frac{2}{Re \Delta x^2}}. \tag{B.6}
\]
Thus, by specifying such a time step, the high frequency components of the Fourier footprint remained fixed to \(-R_S\). We could also write this in terms of a cell Reynolds number,\textsuperscript{7} \(Re_{\Delta x} = (a \Delta x)/\mu\) where \(\mu = 1/Re\) as shown in Appendix A. For the first-order discretization of the advective term this yields,
\[
\Delta t = \frac{R_S}{\frac{\alpha}{\Lambda_\pi} \left( 1 + \frac{2}{Re \Delta x^2} \right)}, \tag{B.7}
\]
or
\[
\Delta t = \frac{R_S \Delta x}{2 a \left( Re_{\Delta x} + 2 \right)}, \tag{B.8}
\]
while the \(\kappa\)-scheme gives,
\[
\Delta t = \frac{R_S \Delta x}{2 a \left( Re_{\Delta x} (1 - \kappa) + 2 \right)}. \tag{B.9}
\]
Alternatively, we could express these as conditions on the Courant and von Neumann numbers,
\[ R_S = \begin{cases} 
2(\nu + 2\sigma) & \text{first-order} \\
2[(1 - \kappa)\nu + 2\sigma] & \text{higher-order} 
\end{cases} \] (B.10)

Now, we insert our definition of \( \Delta t \) from Eq. B.7 into Eq. B.2 to verify that the footprint is now only dependent on \( R_S \) and the cell Reynolds number,
\[ F = -\frac{R_S}{2} \left[ (1 - \cos \beta) + i \frac{R_e \Delta x}{2 \sin \beta} \right]. \] (B.11)
So \( R_S \) controls the negative real extent and the cell Reynolds number dictates the ellipticity.

Two-Dimensions

While the one-dimensional case is relatively straightforward in terms of parameter choices, etc., adding another dimension creates some “pseudo” ambiguities which are not typically resolved in the CFD community.

The governing equation is now,
\[ S_t + a S_x + b S_y = \frac{1}{R_e} (S_{xx} + S_{yy}). \] (B.12)

Again, using a first-order, upwind approximation for the advective terms and central differencing for the diffusion terms yields,
\[ F = -\left( a \frac{\Delta t}{\Delta x} + 2 \frac{\Delta t}{R_e \Delta x^2} \right) (1 - \cos \beta_x) - i \frac{\Delta t}{\Delta x} \sin \beta_x \\
- \left( b \frac{\Delta t}{\Delta y} + 2 \frac{\Delta t}{R_e \Delta y^2} \right) (1 - \cos \beta_y) - i \frac{\Delta t}{\Delta y} \sin \beta_y. \] (B.13)

Now there are two Fourier frequency components, \( \beta_x \) and \( \beta_y \). The corresponding higher-order result is,
\[ F = -\left( (1 - \kappa)\frac{a \Delta t}{\Delta x} + 2 \frac{\Delta t}{R_e \Delta x^2} \right) (1 - \cos \beta_x) \\
- \left( (1 - \kappa)\frac{b \Delta t}{\Delta y} + 2 \frac{\Delta t}{R_e \Delta y^2} \right) (1 - \cos \beta_y) \\
+ i \left( \frac{\Delta t}{\Delta x} \sin 2\beta_x \right) \\
- i \left( \frac{\Delta t}{\Delta y} \sin 2\beta_y \right). \] (B.14)

From here there are many avenues to follow, one of the most prevalent is to define component-wise Courant numbers and Von Neumann numbers, e.g.,
\[ v_x = \frac{a \Delta t}{\Delta x}, \quad \sigma_x = \frac{\Delta t}{R_e \Delta x^2}, \\\nv_y = \frac{b \Delta t}{\Delta y}, \quad \sigma_y = \frac{\Delta t}{R_e \Delta y^2}. \]
Instead, we propose to express the flow quantities in cylindrical coordinates and the geometric quantities in terms of \( \Delta x \) and the cell aspect ratio,
\[ q = \sqrt{a^2 + b^2}, \quad \phi = \arctan \frac{b}{a}, \]
and \[ \AR = \frac{\Delta x}{\Delta y}. \]
(Cf. Fig. 1 in the main text.) Applying these definitions to Eq. B.13 gives,
\[ F = -\left( \frac{q \cos \phi \Delta t}{\Delta x} + 2 \frac{\Delta t}{R_e \Delta x^2} \right) (1 - \cos \beta_x) \\
- i \frac{q \cos \phi \Delta t}{\Delta x} \sin \beta_x \\
- \left( \AR \frac{q \sin \phi \Delta t}{\Delta x} + 2 \AR^2 \frac{\Delta t}{R_e \Delta x^2} \right) (1 - \cos \beta_y) \\
- i \AR^2 \frac{q \sin \phi \Delta t}{\Delta x} \sin \beta_y. \] (B.15)

As was done for the one-dimensional case, we examine the high frequency limit (\( \beta_x = \beta_y = \pi \)) of the Fourier footprint given in Eq. B.15,
\[ -R_S = -2 \Delta t \left( q \frac{\cos \phi + \AR \sin \phi}{\Delta x} + 2 \frac{1 + \AR^2}{R_e \Delta x^2} \right); \] (B.16)
and, solving for the time step such that the negative Real extent of the footprint remains fixed at \(-R_S\) yields,
\[ \Delta t = \frac{R_S}{q \cos \phi + \AR \sin \phi + 2 \frac{1 + \AR^2}{R_e \Delta x^2}}. \] (B.17)

where \( h \) is a new length scale that corresponds to the ratio of cell area to the cell’s diagonal length and the angle \( \theta \) is the angle between \( h \) and the flow direction. (Again, cf. Fig. 1.) The \( \kappa \)-scheme yields a similar form,
\[ \Delta t = \frac{R_S}{(1 - \kappa)q \cos \phi + 2 \frac{1 + \AR^2}{R_e \Delta x^2}}. \] (B.18)

Again, as was done for the one-dimensional case, we can re-write this in terms of a cell Reynolds number, \( R_{ch} = (qh)R_e \), so Eq. B.17 becomes,
\[ \Delta t = \frac{R_S h}{(1 - \kappa)q R_{ch} \cos \theta + 2}; \] (B.19)
and the \( \kappa \)-scheme yields a similar result,
\[ \Delta t = \frac{R_S h}{2 \frac{1 + \AR^2}{R_{ch}(1 - \kappa) \cos \theta + 2}}. \] (B.20)

Note the differences with respect to Eq. B.8 and Eq. B.9. In the two-dimensional case we have a new
length scale, \( h \), which amounts to something akin to the harmonic average of the two length scales,

\[
    h = \frac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}}}.
\]  

(B.21)

This favors the smaller of the two components. \( h \) can also be expressed as the ratio of the cell area to the length of the diagonal,

\[
    h = \frac{\Delta x \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}.
\]  

(B.22)

Another twist is that the flow speed component in the "\( h \)" direction is what governs the advective portion and not the full flow velocity as one might anticipate.

Now we will substitute our value of \( \Delta t \) into the original Fourier footprint of Eq. B.15,

\[
    \mathcal{F} = \frac{R_S}{2} \frac{h}{\Delta x} \frac{Re_h}{Re_h \cos \theta + 2} \left[ \left( \frac{Re_{\Delta x} \cos \phi + 2}{Re_h} \right) (1 - \cos \beta_x) + i \cos \phi \sin \beta_x 
    \right.
    
    + \mathcal{R} \left( \frac{Re_{\Delta y} \sin \phi + 2}{Re_h} \right) (1 - \cos \beta_y)
    
    \left. + i \mathcal{R} \sin \phi \sin \beta_y \right].
\]  

(B.23)

This form is not as straightforward as the one-dimensional results, but examining the case of \( \mathcal{R} = 1 \), \( \phi = 45 \text{ deg} \), we do recover a similar form;

\[
    \mathcal{F} = -\frac{R_S}{2} \left[ 1 - \frac{(\cos \beta_x + \cos \beta_y)}{2} 
    \right.
    
    \left. + i \frac{Re_h}{Re_h + 2} \frac{(\sin \beta_x + \sin \beta_y)}{2} \right].
\]  

(B.24)

which for \( \beta = \beta_x = \beta_y \), gives the identical result to Eq. B.11. Figure 2 in the main text shows Fourier footprints for four values of \( \kappa \) for the higher-order scheme with \( \mathcal{R} = 1 \), \( \phi = 45 \text{ deg} \), and \( Re_h = 1 \).