Estimating Parameterized Post-Newtonian Parameters from Spacecraft Radiometric Tracking Data

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The theory of general relativity can be tested by precisely measuring small changes in the trajectory of a spacecraft traveling near the sun. An important aspect of such a measurement is the potential of estimating the parameterized post-Newtonian parameters \( \gamma \) and \( \beta \) independently. We present a detailed covariance analysis of such a trajectory, analyzing uncertainties in the spacecraft state and \( \gamma \) and \( \beta \). The radiometric data types simulated in our analysis are range, very long baseline interferometry, and Doppler measurements. Also included are the effects of Earth and spacecraft relative geometries, station-location errors, stochastic accelerations, and uncertainties in the solar quadrupole moment \( J_2 \). For an X-band tracking system, we show that if the steady-state stochastic accelerations, station-location, and solar quadrupole moment errors are known better than \( 10^{-14} \) km/s\(^2\), 0.1 m, and \( 10^{-8} \), respectively, the experiment can achieve unbiased estimates \( \sigma_{\gamma} = 8.90 \times 10^{-5} \) and \( \sigma_{\beta} = 4.09 \times 10^{-4} \). To achieve this level of precision on accelerations requires a drag-free spacecraft or accurate accelerometers.

### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>linear map of the spacecraft dynamics</td>
</tr>
<tr>
<td>( a )</td>
<td>total acceleration acting on the spacecraft, km/s(^2)</td>
</tr>
<tr>
<td>( a_{GR}, a_{\frac{3}{2}} )</td>
<td>acceleration vectors caused by general relativity and solar quadrupole moment, km/s(^2)</td>
</tr>
<tr>
<td>( a_0 )</td>
<td>initial semimajor axis, km</td>
</tr>
<tr>
<td>( c )</td>
<td>speed of light, km/s</td>
</tr>
<tr>
<td>( E, E_0 )</td>
<td>actual and initial (Keplerian) sets of orbital elements</td>
</tr>
<tr>
<td>( e_0 )</td>
<td>initial eccentricity</td>
</tr>
<tr>
<td>( H_k, H_y, H_D )</td>
<td>partial derivatives of the range, very long baseline interferometry (VLBI), and Doppler measurements with respect to the state</td>
</tr>
<tr>
<td>( I_{n \times n} )</td>
<td>( n \times n ) identity matrix</td>
</tr>
<tr>
<td>( i_0 )</td>
<td>initial inclination, deg</td>
</tr>
<tr>
<td>( J_2 )</td>
<td>solar quadrupole moment</td>
</tr>
<tr>
<td>( k )</td>
<td>discrete-time index for measurement times, s</td>
</tr>
<tr>
<td>( M_0 )</td>
<td>initial mean anomaly, deg</td>
</tr>
<tr>
<td>( P )</td>
<td>covariance matrix</td>
</tr>
<tr>
<td>( p )</td>
<td>parameter vector</td>
</tr>
<tr>
<td>( q )</td>
<td>measurement model parameter vector</td>
</tr>
<tr>
<td>( R )</td>
<td>square-root information filter matrix</td>
</tr>
<tr>
<td>( R_E )</td>
<td>radius of the Earth, km</td>
</tr>
<tr>
<td>( R_H )</td>
<td>upper triangularized square-root information filter matrix</td>
</tr>
<tr>
<td>( R_0 )</td>
<td>radius of the sun, km</td>
</tr>
<tr>
<td>( r )</td>
<td>norm of the spacecraft position vector, km</td>
</tr>
<tr>
<td>( r_0 )</td>
<td>spacecraft position vector, km</td>
</tr>
<tr>
<td>( r_E )</td>
<td>sun–Earth vector, km</td>
</tr>
<tr>
<td>( r_p )</td>
<td>spacecraft periapsis vector, km</td>
</tr>
<tr>
<td>( r_a )</td>
<td>station-location vector, km</td>
</tr>
<tr>
<td>( r_s )</td>
<td>tracking station vector from the Earth center, km</td>
</tr>
<tr>
<td>( T_{ij} )</td>
<td>orthogonal Householder transformation matrix</td>
</tr>
<tr>
<td>( t, t_0 )</td>
<td>current and epoch times, s</td>
</tr>
<tr>
<td>( v_{\infty} )</td>
<td>excess velocity, km/s</td>
</tr>
<tr>
<td>( V )</td>
<td>range, VLBI, and Doppler measurements</td>
</tr>
<tr>
<td>( v )</td>
<td>norm of the spacecraft velocity vector, km/s</td>
</tr>
<tr>
<td>( v_p )</td>
<td>spacecraft velocity vector, km/s</td>
</tr>
<tr>
<td>( v_p )</td>
<td>spacecraft periapsis velocity, km/s</td>
</tr>
<tr>
<td>( v_v )</td>
<td>velocity vector from the sun to Earth tracking station, km/s</td>
</tr>
<tr>
<td>( x )</td>
<td>spacecraft state vector</td>
</tr>
<tr>
<td>( y )</td>
<td>estimated state vector</td>
</tr>
<tr>
<td>( Z_{\alpha}, Z_{\beta} )</td>
<td>range, VLBI, and Doppler measurements</td>
</tr>
<tr>
<td>( Z_{r}, Z_{v}, Z_D )</td>
<td>right ascension of the Earth tracking station, deg</td>
</tr>
<tr>
<td>( \beta )</td>
<td>parameterized post-Newtonian parameter appearing in the ( g_{00} ) component of the metric tensor</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>parameterized post-Newtonian parameter appearing in the spatial component of the metric tensor</td>
</tr>
<tr>
<td>( \Delta V )</td>
<td>change in spacecraft velocity vector, km/s</td>
</tr>
<tr>
<td>( \delta )</td>
<td>declination of the Earth tracking station, deg</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Nordtvedt effect parameter</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>information matrix</td>
</tr>
<tr>
<td>( \mu )</td>
<td>gravitational constant of the sun, ( \mu = GM_\odot = 1.327 \times 10^{11} ) km(^3)/s(^2)</td>
</tr>
<tr>
<td>( \xi )</td>
<td>angle of the sun’s tangent line from the Earth center, deg</td>
</tr>
</tbody>
</table>

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\[\rho \] = range from the Earth tracking station
to the spacecraft, km
\[\rho \] = position vector of the spacecraft from the Earth
tracking station, km
\[\dot{\rho} \] = unit position vector of the spacecraft from the
Earth tracking station
\[\sigma_x \] = steady-state stochastic accelerations, km/s²
\[\sigma_i \] = standard deviation of \(i\)
\[\sigma_{ij} \] = covariance of \(i\) and \(j\)
\[\sigma_{\gamma}, \sigma_{\beta}, \sigma_{\alpha} \] = approximated range, VLBI, and Doppler
measurement uncertainties
\[\tau \] = correlation time for steady-state stochastic
acceleration, days
\[\Phi \] = state transition matrix
\[\phi \] = initial phase angle, deg
\[\chi \] = spacecraft-Earth-sun angle, deg
\[\psi \] = Earth obliquity, deg
\[\Omega_0 \] = initial argument of ascending node, deg
\[\omega \] = argument of perihelion, deg
\[\omega_E \] = Earth rotation rate, deg/day
\[\omega_0 \] = initial argument of perihelion, deg

Introducing

The theory of general relativity (GR) has been explored with
great intensity, both theoretically and experimentally. Previous
experimental tests of GR not only improved our understanding of
the theory, but also introduced new technologies, such as the Mössbauer
effect and atomic clocks in the case of the gravitational red shift.
Several GR mission tests, such as Gravity Probe-B, BepiColombo,
and the Laser Interferometry Space Antenna, are currently planned
to validate and further investigate our modern understanding of gravita-
tional theories.

Tests of GR, such as the precession of Mercury’s perihelion and
the Viking radar time delay, confirmed the theory by measuring the
Eddington–Robertson–Schiff parameters \(\gamma\) and \(\beta\), which appear in
the parameterized post-Newtonian (PPN) formulation. The param-
eter \(\gamma\) arises in the \(g_{ij}\) components of the metric tensor \(g_{ij}\), while \(\beta\) characterizes the \(g_{00}\) component. Measuring these constants will
essentially describe how well GR accounts for the law of gravita-
tion in the weak field limit. In Einstein’s formulation of GR, both \(\gamma\)
and \(\beta\) are unity, although both parameters can differ from unity in
some alternatives to GR. A recent experiment by Bertotti et al., in
which the radio links with the Cassini spacecraft were calibrated,
provided a new estimate \((\gamma - 1) = (2.1 \pm 2.3) \times 10^{-5}\). In contrast,
the parameter \(\beta\) has never been measured independently, but only
estimated indirectly from linear combinations involving \(\gamma\), such as
the perihelion advance effect \(2\beta + 3\gamma\), or the Nordvedt effect, \(\mu\), which
measures \(\mu = (4\beta - 3\gamma)\). Let us consider the precession
of Mercury’s perihelion, where the secular effect of GR on Mer-
cury’s trajectory is measured. In this observation, the gravitational
shift is highly correlated with the solar quadrupole moment \(J_2\), a
measure of the solar oblateness. If we assume that GR is correct,
then our observations can be used to measure the solar quadrupole
moment. Alternatively, if we can assume a value for \(J_2\), an esti-
mate for \(\gamma\) and \(\beta\) can be found. A recent study by Pireaux and
Rozelot provides an estimated value of the solar quadrupole
moment \(J_2 = (2.0 \pm 0.4) \times 10^{-7}\). The purpose of the present paper is
to verify the feasibility of the test of GR proposed by Longuski et al.,
by carrying out a covariance analysis aimed at computing the level
of accuracy to which \(\gamma\), \(\beta\), and \(J_2\) can be estimated simultaneously.

A test of GR with use of a heliocentric spacecraft trajectory with
small perihelion was previously discussed by Mease et al., who
studied the possibility of estimating the PPN parameters and
solar gravitational harmonics by using the proposed Starprobe space-
craft. Their baseline covariance analysis showed potential estimates
of the PPN parameter \(\gamma\) and solar \(J_2\) to accuracies of \(10^{-2}\) and
\(10^{-3}\), respectively; however, the parameter \(\beta\) was unobservable
at the level of \(10^{-3}\) because of large station-location uncertainties.
In a subsequent study, Mease et al. carried out a more detailed co-

variance analysis including additional quantities of interest, such as
the preferred-frame effect parameter \(\alpha\) and the Moffat parameter \(\zeta\).
However, neither \(\gamma\) nor \(\beta\) is observable to a significant level using
the assumed tracking capability.

More recently, Longuski et al. have reanalyzed a variation of
this test and shown that GR can be tested by precisely measuring
small changes in the position of a spacecraft as it escapes from the
sun on a hyperbolic trajectory. Their analysis used a heuristic argu-
ment, supported by solving the deflection equation for a hyperbolic
trajectory, to establish that \(\gamma\) and \(\beta\) could be measured to an
accuracy of \(10^{-3}\) assuming foreseeable improvements in spacecraft
tracking technology. This theoretical discussion motivated a new
consideration of this test of GR. Our follow-on studies based on an
analytic approach verified that the PPN parameters are measurable
to the level \(\sim 1 \times 10^{-3}\) in principle. Another important result from
that study was that simultaneous estimates of the PPN parameters
could be obtained separately (i.e., with very little correlation be-
tween the estimates of \(\gamma\) and \(\beta\)). This result makes the proposed GR
experiment unique because other GR tests generally cannot clearly
disentangle \(\gamma\) and \(\beta\) from each other.

The present paper presents a detailed numerical analysis of
this test, which includes realistic error sources, such as uncertain
nongravitational accelerations, station-location uncertainties, and
uncertainty in the solar quadrupole moment, using current and pro-
jected measurement accuracies. We first analyze individual contribu-
tions from these error sources to the \(\gamma\) and \(\beta\) estimation accu-
racies. We then show how these errors degrade the performance of
radiometric measurements and discuss the level of precision needed
to carry out this experiment at a meaningful level. Also included is
the effect of solar occultations, which is a critical problem in the
proposed test. This problem is a consequence of the fact that the
gravitational shift is at a maximum in proximity of the sun, and
a loss of measurements in the early stage of the trajectory signif-
ically degrades the quality of the \(\gamma\) and \(\beta\) estimates. In addition to
the hyperbolic trajectory considered in the previous work, we
also analyze different tracking geometries between the spacecraft
and Earth, as well as elliptic orbits with periapsis close to the sun.
In practice, an elliptic orbit would be more useful as it can provide
multiple passages close to the sun. An elliptic orbit allows for re-
petition of the experiment, and for longer data arcs near periapsis
carried by the slower periapsis speed compared to a hyperbolic orbit.
We also present the uncertainty distributions of the PPN parameters
as functions of periapsis distance and eccentricity.

Our analysis differs in the following ways from the previous work
of Mease et al.: 1) very long baseline interferometry (VLBI) mea-
surements is included as a data type in addition to range and Doppler,
and 2) more accurate measurement precisions are considered. Fur-
thermore, we ignore PPN parameters other than \(\gamma\) and \(\beta\) and con-
sider a wider range of spacecraft orbit geometries for the sensitivity
study. Over the past few decades, there have been significant im-
provements in the tracking accuracies of radiometric measurements
(e.g., S- to X-band tracking systems). The current implementation
of K-band tracking will further enhance our ability to accurately track
spacecraft. Future improvement can be anticipated if laser
communication is implemented; however, we do not consider this
possibility in the present paper. Moreover, VLBI provides high-
precision angular measurements of the spacecraft trajectory, a key
capability for obtaining better estimates of \(\gamma\) and \(\beta\) (Ref. 4). In this
analysis, we ignore higher-order PPN terms because their effects
are negligible. Also, because each radiometric data type measures
a fundamentally different aspect of the GR perturbation, and is
sensitive to the relative orbital geometries between the spacecraft
and the Earth, we explore a wide range of orbit orientations. Our re-
results exhibit the influence of geometry on the GR estimates as a
function of initial phase angle (i.e., the angle \(\phi\) shown in Fig. 1).
We also include the \(J_2\) perturbation in the model and study how
estimates of this parameter, along with \(\gamma\) and \(\beta\), vary as a function
of inclination, argument of periapsis, and longitude of ascending
node. Finally, we show that a mission similar to the proposed Solar
Probe mission can satisfy the trajectory requirements for this test of GR.
Transient Effects of $\gamma$ and $\beta$

One can consider the test we are investigating as an extension of previous GR experiments, such as measurements of Mercury’s perihelion shift or the deflection of light as a result of the sun’s gravity. A novel aspect of this new experiment comes from the potential of separately estimating $\gamma$ and $\beta$. Precisely knowing these predicted parameters allows us to determine how well GR accounts for gravity data. The classic test of GR, observation of the precession of Mercury’s perihelion, is based on the secular effect of GR over long time spans. In actuality, the transient oscillations of Mercury’s orbit as a result of GR are much larger (over one orbit) than the secular effect; however, their amplitudes are small in absolute terms, and their effect averages out over time. Hence, only the secular effects can be measured over very long time baselines, and these provide estimates of linear combinations of $\gamma$ and $\beta$, such as the perihelion advance effect ($2 + 2\gamma - \beta$) or the Nordtvedt effect ($\gamma = 4\beta - \gamma - 3$). Our approach measures transient deviations in the trajectory caused by GR over a short period of time, where the effect is large enough to measure, and this allows us to disentangle $\gamma$ and $\beta$.

If we let $E_0$ be a set of Keplerian orbital elements, the change in $E_0$ as a result of GR can be defined as $\Delta E = E - E_0$, where $E$ is computed by solving the two-body equations of motion incorporating the GR perturbation. The total acceleration acting on the spacecraft is given by

$$a(t) = -(\mu/r^3)r + a_{GR}$$

where $a_{GR}$ is the GR perturbing acceleration. Its dimensional representation is

$$a_{GR} = (\mu/c^2 r^3)(2(\gamma + \beta)\mu/r - \gamma v^2 r + 2(\gamma + 1)(r \cdot v)v)$$

The changes in orbital elements as a result of this perturbation are shown in Fig. 2, where we assume that the spacecraft is initially located at perihelion (four solar radii) with hyperbolic excess velocity 39 km/s. We note that the GR perturbation acts only in the orbital plane, and therefore the inclination and argument of ascending node are unchanged. The largest change in orbital elements caused by $\gamma$ and $\beta$ occurs very early in the trajectory and essentially disappears after a few days. This is expected because the GR perturbation is at its maximum in close proximity to the sun. One outcome of this observation is a drive to extend radiometric tracking as close to the sun as possible. Furthermore, Longuski et al.\textsuperscript{7} discuss the importance of the partial derivatives of the orbital elements with respect to the GR parameters as a spacecraft travels along its trajectory. An important conclusion from their study is that the partials of the argument of periapsis $\omega$, with respect to $\beta$ and $\gamma$, are unique. The ratios of these partials, which represent the correlation between $\beta$ and $\gamma$, converge to a constant value slower than the ratios of the partials of the other orbital elements, indicating that there might be sufficient information to disentangle $\gamma$ and $\beta$ by tracking the spacecraft close to perihelion.

In addition to independent $\gamma$ and $\beta$ measurements, our proposed heliocentric spacecraft trajectory offers an opportunity to check GR under a fundamentally different flight regime (i.e., a sub-light-speed experiment). As discussed earlier by Longuski et al.,\textsuperscript{7} deflection of a spacecraft trajectory as a result of GR is greater than that of light at the same periapsis distance $r_p$. Therefore, proper tracking of a spacecraft can, at least in principle, provide PPN estimates that are comparable to the light-speed experiments.

Covariance Analysis

Baseline State To Be Estimated

In our numerical approach we follow the baseline trajectory analyzed in Refs. 4 and 7, which places the spacecraft on an escape hyperbolic trajectory at perihelion equal to four solar radii, $r_p = 4R_\odot$ ($R_\odot = 6.96 \times 10^8$ km). The periapsis velocity is $v_p = 31.1$ km/s with corresponding excess velocity $V_\infty = 39$ km/s. The nominal trajectory has initial orbital elements of semimajor axis $a_0 = 0.58$ astronomical unit ($\text{AU} = 6.725 \times 10^8$ km, eccentricity $e_0 = 1.03$, and $i_0 = 0\text{deg} = \Omega_0 = M_0 = 0\text{deg}$). The spacecraft and Earth orbits are assumed to be coplanar about a spherical sun. The Earth’s orbit is circular with radius of 1 AU. The spacecraft trajectory is obtained by numerically integrating two-body equations of motion that incorporate the GR perturbation. While the initial sun–Earth-spacecraft angle (phase angle $\phi$) remains as a free parameter, we assume a phase angle of zero for our baseline case.

Our hypothetical trajectory approaches perihelion on an elliptic orbit (with aphelion at Jupiter ~5.2 AU), then boosts into a hyperbolic escape trajectory (using a perihelion maneuver of $\Delta v \sim 3$ km/s at $4R_\odot$). Hence, to constrain the initial velocity error we assume the presence of accelerometers to measure the large perihelion burn. At epoch the conservative initial uncertainties (variances) for the initial covariance matrix are $\sigma_\gamma = \sigma_\beta = 1$ km, $\sigma_{\omega_0} = \sigma_{\Omega_0} = \sigma_{M_0} = 1$ m$^2$/s$^2$, and $\sigma_{v_0} = \sigma_{e_0} = 1$ with zero correlation. The boost into this hypothetical trajectory introduces new errors. This maneuver is a part of the baseline trajectory only because it was used as a nominal case in earlier studies. We show later that the perihelion boost is not necessarily needed to obtain the same level of $\gamma$ and $\beta$ estimates by analyzing a wider range of trajectories that do not include this maneuver.

In addition to the state and PPN parameters $\gamma$ and $\beta$, we also include other parameters of interest, such as the uncertainties in the station-location vector and in solar $J_2$, and study their impact on the $\gamma$ and $\beta$ uncertainties. Our goal is to analyze their individual contributions to the PPN estimates, as this allows us to constrain

![Fig. 1 Hyperbolic flyby of a spacecraft about the sun.](image1.png)

![Fig. 2 Change in the semimajor axis $a$, eccentricity $e$, argument of periapsis $\omega$, and mean anomaly $M$ caused by GR.](image2.png)
the uncertainties in these parameters and to determine the level of accuracy required in carrying out this experiment. Additionally, we briefly discuss the potential for improving estimated uncertainties of these parameters. Lastly, we model the effect of stochastic acceleration perturbations on the trajectory, which can have a critical impact on $\gamma$ and $\beta$ estimation uncertainties.

### Definition of the Information and Covariance Matrix

Our filter model is based on ordinary first-order least-squares approximation theory and is carried out by using modern orbit determination procedures. The state vector we estimate in our analysis can be represented as $y(t) = [x^T \ p^T \ q^T]^T$ for an arbitrary system. For the baseline analysis we define the state vector as $y = [r^T \ v^T \ \gamma \ \beta]^T$, which assumes no measurement parameters when additional parameters are included, we simply add them to the state and carry out the same process.

Numerical precision is often lost in the computation of covariance as a result of numerical integration and matrix inversion. A method for retaining precision is to use the square-root information filter (SRIF):

$$\mathbf{L}(t_0; t, t_0) = \mathbf{R}^T(t_0; t, t_0) \mathbf{R}(t_0; t, t_0)$$

Here $\mathbf{L}(t_0; t, t_0)$ is the initial state information matrix, which represents the information content of the radiometric data added in the time interval from $t_0$ to $t$, and $\mathbf{R}(t_0; t, t_0)$ is the initial state SRIF matrix, which we propagate, instead of the information matrix, to update data in each time increment. The SRIF matrix is related to the adjoint of the state transition matrix (STM) and ideally maps as

$$\Phi(t_0; t, t_0) = \mathbf{R}(t_0; t, t_0) \Phi(t_0; t, t_0)$$

where $\Phi(t_0; t_0)$ describes the linear mapping of the state perturbation about the nominal trajectory. Let $T_H$ be an orthogonal Householder transformation matrix such that the updated SRIF matrix is defined as

$$\mathbf{L}(t_0; t, t_0) = T_H \begin{bmatrix} \mathbf{R}(t_0; t_k, t_0) \\ \frac{1}{\sigma_R} \mathbf{H}_R(t_{k+1}) \Phi(t_{k+1}, t_0) \\ \frac{1}{\sigma_\gamma} \mathbf{H}_\gamma(t_{k+1}) \Phi(t_{k+1}, t_0) \\ \frac{1}{\sigma_D} \mathbf{H}_D(t_{k+1}) \Phi(t_{k+1}, t_0) \end{bmatrix}$$

where $\Phi(t_{k+1}, t_0)$ describes the linear mapping of the state perturbation about the nominal trajectory. Let $T_H$ be an orthogonal Householder transformation matrix such that the updated SRIF matrix is defined as

$$\mathbf{L}(t_0; t, t_0) = T_H \begin{bmatrix} \mathbf{R}(t_0; t_k, t_0) \\ \frac{1}{\sigma_R} \mathbf{H}_R(t_{k+1}) \Phi(t_{k+1}, t_0) \\ \frac{1}{\sigma_\gamma} \mathbf{H}_\gamma(t_{k+1}) \Phi(t_{k+1}, t_0) \\ \frac{1}{\sigma_D} \mathbf{H}_D(t_{k+1}) \Phi(t_{k+1}, t_0) \end{bmatrix}$$

In the computation, we adopt QR factorization as the Householder transformation. Here, $\mathbf{H}$ are the partial derivatives of the measurement with respect to the state computed at $t = t_{k+1}$; the subscripts $R, V$, and $D$ denote range, VLBI, or Doppler measurements, respectively. The noise factors (i.e., $\sigma_R, \sigma_\gamma,$ and $\sigma_D$) are given in Table 1, where they represent the approximate measurement uncertainties provided by S-, X-, and K-band tracking systems. After the Householder transformation is applied, the updated information matrix becomes

$$\mathbf{L}(t_0; t_{k+1}, t_0) = \mathbf{R}^T_H(t_0; t_{k+1}, t_0) \mathbf{R}_H(t_0; t_{k+1}, t_0)$$

and the updated covariance matrix becomes

$$\mathbf{P}(t_0; t_{k+1}, t_0) = \mathbf{L}^{-1}(t_0; t_{k+1}, t_0)$$

$$\mathbf{R}^T H(t_0; t_{k+1}, t_0) \mathbf{R}_H(t_0; t_{k+1}, t_0)$$

### Table 1 Summary of measurement accuracies

<table>
<thead>
<tr>
<th>Tracking system</th>
<th>$\sigma_R$, m</th>
<th>$\sigma_\gamma$, rad</th>
<th>$\sigma_D$, mm/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-band</td>
<td>10</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>X-band</td>
<td>1</td>
<td>1</td>
<td>0.1</td>
</tr>
<tr>
<td>K-band</td>
<td>0.1</td>
<td>0.1</td>
<td>0.01</td>
</tr>
</tbody>
</table>

*For 1-min count time.

which represents the evolution of the a priori uncertainties in the state vector.

### Computation of the State Transition Matrix

The STM represents the linear mapping of the state perturbation as a function of time and is defined by the initial spacecraft state vector (i.e., $\Phi(t_0; t_0; \rho_0, v_0, \gamma, \beta)$). For the given state vector, the time propagated STM is represented as

$$\Phi(t_k, t_0) = \frac{\partial y(t)}{\partial y(t_0)}$$

with $\Phi(t_k, t_0)$ an $8 \times 8$ identity matrix. The time derivative of the STM is a simple linear relation, $\Phi(t_k, t_0) = \mathbf{A} \Phi(t_k, t_0)$, where the linear mapping matrix $\mathbf{A}$ is given by

$$\mathbf{A}(t) = \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 1} \\ \frac{\partial a}{\partial r} & \frac{\partial a}{\partial v} & \frac{\partial a}{\partial \beta} \\ 0_{2 \times 3} & 0_{2 \times 3} & 0_{2 \times 1} \end{bmatrix}$$

The partial derivatives of the acceleration vector with respect to the state variables yield

$$\frac{\partial a}{\partial r} = -\frac{\mu}{r^3} [I_{3 \times 3} - 3 \mathbf{r} \mathbf{r}^T] + \frac{\partial (a_{GR})}{\partial y} \sim -\frac{\mu}{r^3} [I_{3 \times 3} - 3 \mathbf{r} \mathbf{r}^T]$$

$$\frac{\partial a}{\partial v} = \frac{\partial (a_{GR})}{\partial y} \sim 0_{3 \times 3}$$

$$\frac{\partial a}{\partial \beta} = \frac{2 \mu \mathbf{r}^T \mathbf{r}}{c^2 r^2}$$

where the unit vector $\mathbf{r}$ is the normalized spacecraft-position vector. The partials of $a_{GR}$ with respect to spacecraft states (i.e., $r$ and $v$) are included in the simulation; however, their contribution is negligible when compared to Newtonian gravity.

### Implementing Stochastic Acceleration Effects

Time-correlated random accelerations can be included in the SRIF matrix as discussed in Scheeres et al. The perturbations caused by process noise directly feed into the SRIF matrix at time $t_k$, and therefore, we solve instead for the SRIF matrix directly rather than solving for the STM. We define the current-state information matrix and related quantities as

$$\mathbf{A}(t_k; t_0) = \mathbf{R}^T(t_k; t_0) \mathbf{R}(t_k; t_0)$$

$$\mathbf{P}(t_k; t_0) = \mathbf{A}^{-1}(t_k; t_0)$$

$$\mathbf{R}(t_k; t_0) = -\mathbf{R}(t_k; t_0) \mathbf{A}(t_k)$$

The current-state-covariance matrix [Eq. (15)] represents the uncertainties in the state variables at time $t_k$, whereas the preceding definition is for the a priori estimates. The resulting $\gamma$ and $\beta$ uncertainties do not change because they are not dynamical variables. In the absence of stochastic perturbations, the two definitions of covariance matrices can be mapped into each other by

$$\mathbf{P}(t_k; t_0) = \Phi(t_k, t_0) \mathbf{P}(t_k, t_0) \Phi^T(t_k, t_0)$$

However, this ideal relation is no longer preserved in the presence of stochastic perturbations. The time derivative of the perturbed SRIF matrix is

$$\mathbf{R}(t_k; t_0) = -\mathbf{R}(t_k; t_0) \mathbf{A}(t_k)$$

$$= -\frac{1}{2} \mathbf{R}(t_k; t_0) \mathbf{B}_p \mathbf{R}^T(t_k; t_0) \mathbf{R}(t_k; t_0)$$

$$= \frac{1}{2} \mathbf{R}(t_k; t_0) \mathbf{B}_p \mathbf{R}^T(t_k; t_0) \mathbf{R}(t_k; t_0)$$

where $\mathbf{B}_p$ represent the perturbations in the SRIF matrix.
where

\[ B = \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 2} \end{bmatrix}^T \]  

(19)

\[ P_w = 2\tau \sigma_n^2 I_{3 \times 3} \]  

(20)

We assume that the stochastic accelerations are not being estimated during the orbit-determination process and continuously affect the state. For this reason, we map the covariance caused by the process noise \( P_w \) only into the velocity space by using the matrix \( B \). In the actual computation the correlation time \( \tau \) is set to 0.5 days, and we vary the steady-state acceleration noise to study its impact on the actual computation the correlation time \( \tau \).

### Extracting \( \gamma \) and \( \beta \) Estimate Uncertainties

Each component in a covariance matrix \( \sigma_{ij} \) represents the variance of 

\[ \gamma \] and \( \beta \) by computing \( \sigma_{ij} = \sqrt{\sigma_{ij}^2} \) and \( \sigma_{ij} = \sqrt{\sigma_{ij}^2} \), respectively. The unique opportunity of the proposed test is the potential of the spacecraft trajectory in the plane of sky relative to the tracking station.12

Combined with range measurements, the three-dimensional position of the spacecraft can be obtained. We represent this measurement as a set of angles

\[ Z_V = [Z_m \ Z_d]^T \]  

(26)

Taking partials with respect to \( y \) yields

\[ H_V = \frac{\partial Z_V}{\partial y} = \begin{bmatrix} \hat{m}_0 & \rho & 0_1 \times 5 \end{bmatrix} \]  

(27)

where we define

\[ \hat{m}_0 = \hat{l}_0 \]  

(28)

\[ \hat{n}_0 = \frac{\hat{z} - (\hat{z} \cdot \hat{l}_0)\hat{l}_0}{|\hat{z} - (\hat{z} \cdot \hat{l}_0)\hat{l}_0|} \]  

(30)

and where \( \hat{z} = [0 \ 0 \ 1]^T \).

The final data type we consider is Doppler measurements \( Z_D \),

\[ Z_D = \frac{d}{dt} |r - r_E - r_n| = \hat{\rho} \cdot \hat{\rho} \]  

(31)

which are widely used for interplanetary missions. This data type gives range rate via Doppler frequency shifts in the transmitted signals and, because of the Hamilton–Melbourne effect,13 provide angular information on the trajectory as well. The partial derivative of \( Z_D \) results in

\[ H_D = \frac{\partial Z_D}{\partial y} = \begin{bmatrix} \hat{\rho}^T \left( \frac{\partial \hat{\rho}}{\partial r} \right) & \hat{\rho}^T & 0_{1 \times 2} \end{bmatrix} \]  

(32)

where

\[ \frac{\partial \hat{\rho}}{\partial r} = \frac{1}{\rho} \hat{I}_{3 \times 3} - \hat{\rho} \hat{\rho}^T \]  

(33)

### Solar Occultation Effects

When the spacecraft passes in front of (or behind) the sun (Fig. 3), we cannot obtain radiometric measurements. Because the trajectory originates close to the sun, this can be an important effect in the early stages of the experiment.

Let us define

\[ \chi = \cos^{-1} \left[ \frac{\rho - (-r_E)}{\rho r} \right] \]  

(34)

Based on the geometry of the Earth and sun, the angle between \( r_E \) and the tangent vector from center of the Earth to the outer radius of the sun \( \hat{\xi} \) is computed to be 0.267 deg. We assume that no measurements are taken (i.e., \( \chi = 0 \)) if \( \chi \leq \xi + 0.5 \) deg for Doppler and VLBI measurements and \( \chi \leq \xi + 5 \) deg for range measurements.14
where $v$

To analyze the effect of uncertainties in the perturbing acceleration caused by solar oblateness [i.e.,

sis, we modify the spacecraft acceleration vector by incorporat- and estimate it along with $\gamma$

ance analysis. The only information needed are the partials of the order 1 space ($\partial r_i / \partial J_2$) in the state vector

between the PPN parameters and $\beta$. Hence, we include the solar

Gravitational shift.3 Hence, we include the solar

Effect of Solar Quadrupole Moment $J_2$

As discussed by Mease et al.,5,6 this test also offers the potential of estimating the solar quadrupole moment $J_2$ to a precision of order $1 \times 10^{-8}$ for an X-band mission and an order of magnitude better if K-band tracking is employed. Because of large correlations between the PPN parameters and $J_2$, we must search the parameter space ($\gamma$, $\beta$, $J_2$) to study their simultaneous contributions to the gravitational shift.1 Hence, we include the solar $J_2$ as a state variable and estimate it along with $\gamma$ and $\beta$.

To incorporate the solar quadrupole moment in the analysis, we modify the spacecraft acceleration vector by incorporating the perturbing acceleration caused by solar oblateness [i.e.,

$\mathbf{a} = -(\mu/r^3)\mathbf{r} + a_{\text{GR}} + \mathbf{a}_{J_2}$, where

\[
\mathbf{a}_{J_2} = \frac{\partial}{\partial r} \left[ \frac{\mu}{r} \left( 1 + \frac{R^2 J_2}{2r^4} (r^2 - 3z^2) \right) \right]
\] (39)

Inclusion of $J_2$ changes the spacecraft orbit, in particular the orientation of the orbital plane. We analyze orbit orientation ($i$, $\omega$, $\Omega$) to determine the effect of $J_2$ on the accuracy of our $\gamma$ and $\beta$ estimates. In this geometric analysis, we only consider a 90-deg inclination case to take maximum advantage of measuring $J_2$, whose nominal value is assumed to be $2 \times 10^{-7}$ as discussed in Ref. 3. The partial derivatives of the acceleration vector with respect to the spacecraft state are almost negligible, which is expected because the $J_2$ acceleration $\mathbf{a}_{J_2}$ is even smaller than the GR contribution. The partials with respect to $J_2$ are given as

\[
\frac{\partial \mathbf{a}}{\partial J_2} = -\frac{3\mu R^2}{2r^3} \left[ \begin{array}{c} (1 - 5z^2/r^2) x \\ (1 - 5z^2/r^2) y \\ (3 - 5z^2/r^2) z \end{array} \right]
\] (40)

Results

Unbiased Results

For our analysis, the trajectory conditions given in Refs. 4 and 7 are first verified as the unbiased case, where we assume no error sources are included. The spacecraft is initially located at periapsis of a heliocentric hyperbolic trajectory with $r_p = 4R_\odot$ and $V_\infty = 39$ km/s. Figure 4 shows the evolution of PPN uncertainties as a function of time considering S-, X-, and K-band tracking capabilities. An important result is that measurements in the early stage of the trajectory significantly improve the estimates of $\gamma$ and $\beta$. However, as shown in the correlation plot (Fig. 5), early stage measurements have higher correlation factors, and we need to further investigate ways to calibrate radiometric measurements and disentangle the information content for $\gamma$ and $\beta$ estimates. The covariances are linearly proportional to the measurement noise in the absence of stochastic accelerations. Hence, changing measurement accuracies from X- to K-band tracking can improve $\gamma$ and $\beta$ estimates by an order of magnitude.

The results of the $\gamma$ and $\beta$ estimates, with different time spans, are presented in Table 2. Taking measurements before periapsis passage significantly reduces the uncertainties in $\gamma$ and $\beta$. This is expected because taking measurements near periapsis provides more useful information about the trajectory. However, for our hyperbolic trajectory, the orbital maneuver at periapsis can introduce problems. Hence we also consider the possibility of obtaining comparable precision with elliptic orbits. All of the values $\sigma_\gamma$ and $\sigma_\beta$ shown in

<table>
<thead>
<tr>
<th>Initial time, days</th>
<th>Final time, days</th>
<th>A posteriori $\sigma_\gamma \times 10^3$</th>
<th>A posteriori $\sigma_\beta \times 10^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>0.890$^a$</td>
<td>4.09$^a$</td>
</tr>
<tr>
<td>0</td>
<td>30</td>
<td>0.782</td>
<td>3.69</td>
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<tr>
<td>0</td>
<td>100</td>
<td>0.495</td>
<td>3.19</td>
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<tr>
<td>0</td>
<td>200</td>
<td>0.317</td>
<td>0.86</td>
</tr>
<tr>
<td>−5</td>
<td>5</td>
<td>0.607</td>
<td>1.12</td>
</tr>
<tr>
<td>−10</td>
<td>10</td>
<td>0.581</td>
<td>1.08</td>
</tr>
<tr>
<td>−15</td>
<td>15</td>
<td>0.526</td>
<td>0.95</td>
</tr>
<tr>
<td>−100</td>
<td>100</td>
<td>0.123</td>
<td>0.24</td>
</tr>
</tbody>
</table>

$^a$No error sources are included.

$^b$With respect to periapsis passage.

$^c$Unbiased case.

![Spacecraft diagram](image-url)
Fig. 4 Uncertainties of $\gamma$ and $\beta$ as functions of time, excluding error sources.

Fig. 5 Correlation between $\gamma$ and $\beta$ estimates as a function of time with X-band tracking accuracies, excluding error sources.

Table 2 are the final ones (a posteriori) taken at the end of the timespan. These uncertainties will be decreased by an order of magnitude when K-band tracking systems are implemented, implying that both $\gamma$ and $\beta$ can be estimated with accuracies better than $10^{-5}$, at least in principle. The parameter $\gamma$ can be estimated more accurately than $\beta$, which is the result obtained by Mease et al.\textsuperscript{5,6} as well.

Figure 6 shows the uncertainties and correlations between $\gamma$ and $\beta$ with different initial phase angles, with and without the solar occultation effect. The estimates are taken at the end of a 10-day timespan, where the spacecraft starts from its perihelion. The uncertainties are relatively sensitive to the initial phase angle $\phi$. This plot shows that there is an order-of-magnitude difference in the estimates when different spacecraft-Earth geometries are considered, thus emphasizing the importance of trajectory design for this experiment. The inclusion of the VLBI measurements as an additional data type did not contribute a significant improvement in $\gamma$ and $\beta$ estimates when compared to Mease et al.; hence, the accurate range and Doppler measurements and finding the optimal spacecraft orientation are the keys in obtaining improved estimates of the PPN parameters.

For the remainder of this paper, by “unbiased case” we mean tracking the spacecraft for 10 days from perihelion by using X-band measurement noise (i.e., the first line of Table 2). We used this result as a standard as it represents the true information content our measurements can extract from the trajectory.

Fig. 6 Uncertainty and correlation of $\gamma$ and $\beta$ estimates as functions of initial sun-Earth-spacecraft angle, excluding error sources.

Fig. 7 Accuracies of $\gamma$ and $\beta$ estimates as functions of eccentricity $e$, excluding the error sources ($s = r_p/R_\odot$).

Elliptic Orbits

As mentioned earlier, the longer data arcs and repeated measurements possible with an elliptic orbit might improve our estimation results. Figure 7 shows the sensitivity of uncertainties in $\gamma$ and $\beta$ as a function of eccentricity for various periapses, where $e < 1$ represents an elliptic orbit. This figure shows that the overall change in the uncertainty distribution for a fixed $r_p$ is not significant, which means that there is a possibility of conducting this test using elliptic orbits. Also restricting the periapsis radius to $r_p = 4R_\odot$ is not necessary; $r_p$ can be increased to lower the heat shield requirement and thus mission cost. However, when the solar $J_2$ is included, it might still be desirable to minimize periapsis distance because the $J_2$ uncertainty is highly sensitive to $r_p$ (Ref. 5). For orbits with $r_p = 4R_\odot$, the estimate of $\beta$ becomes more accurate as eccentricity decreases, whereas the estimates of $\gamma$ are better with hyperbolic orbits. Considering the repetition in tracking measurements, increased data arcs in the vicinity of the sun, and absence of a large maneuver at periapsis, elliptic orbits provide an opportunity for more accurate estimation of $\gamma$ and $\beta$. This assumes that the spacecraft is capable of enduring multiple close passes of the sun.

Effect of Stochastic Accelerations

The effect of stochastic accelerations on the state variables (i.e., position and velocity components) is usually negligible over a short period of time; however, their effect on $\gamma$ and $\beta$ uncertainties is
Table 3 Effect of stochastic acceleration

<table>
<thead>
<tr>
<th>Steady-state acceleration $\sigma_a$, km/s²</th>
<th>A posteriori $\sigma_a \times 10^4$</th>
<th>A posteriori $\sigma_\gamma \times 10^4$</th>
<th>A posteriori $\sigma_\beta \times 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-10}$</td>
<td>465</td>
<td>231</td>
<td></td>
</tr>
<tr>
<td>$10^{-11}$</td>
<td>63.3</td>
<td>42.8</td>
<td></td>
</tr>
<tr>
<td>$10^{-12}$</td>
<td>7.53</td>
<td>8.62</td>
<td></td>
</tr>
<tr>
<td>$10^{-13}$</td>
<td>1.57</td>
<td>5.18</td>
<td></td>
</tr>
<tr>
<td>$10^{-14}$</td>
<td>0.971</td>
<td>4.41</td>
<td></td>
</tr>
<tr>
<td>Unbiased</td>
<td>0.890</td>
<td>4.09</td>
<td></td>
</tr>
</tbody>
</table>

*aCovariances are evaluated at the end of a 10-day time span.

Effect of Station-Location Errors

Figure 8 presents the time responses of $\sigma_\gamma$ and $\sigma_\beta$ with X-band tracking noises. As seen, station-location uncertainties $\sim 0.1$ m will ensure that our desired accuracy will be achieved. Fortunately, current technology provides this level of precision, and hence X-band radiometric measurements can provide useful information about the GR contribution to the trajectory. However, reducing the noise level by use of K-band tracking does not necessarily improve the $\gamma$ and $\beta$ uncertainties because simultaneous increases in station-location precision are required. Therefore, if the K-band tracking system is implemented, it might be more practical to estimate the station-location vector along with $\gamma$ and $\beta$. Also shown in the plot are oscillations in $\sigma_\gamma$ and $\sigma_\beta$. If the station location is estimated, $\sigma_\gamma$ and $\sigma_\beta$ will always decrease without oscillation.

Effects of Solar Quadrupole Moment

Table 4 shows $\sigma_\gamma$ and $\sigma_\beta$ when $J_2$ is included for a spacecraft orbit with zero inclination. Considering the $J_2$ uncertainty as an unestimated perturbation causes a dramatic degradation in the $\gamma$ and $\beta$ estimates as a result of the high correlations between $\beta$ and $J_2$. Hence, we include $J_2$ in our state estimates. Table 4 shows that the estimate of $\beta$ is highly sensitive to initial uncertainties in $J_2$, which explains why $\beta$ has not been estimated independently from the precession of Mercury’s perihelion (i.e., either $J_2$ or $\beta$ must be known accurately to measure the other).

To understand the effect of orbit orientation, we show the uncertainty distributions of $\gamma$, $\beta$, and $J_2$ as functions of $\omega$ and $\Omega$ in Figs. 9–11. We consider a 90-deg inclination with a priori $J_2 = 10^{-6}$. One important result to note is that $\sigma_\gamma$, $\sigma_\beta$, and $\sigma_{J_2}$ are highly sensitive to spacecraft orbit orientations, analogous to the sensitivity of $\sigma_\gamma$ and $\sigma_\beta$ to the initial phase angle. When solar occultation effects are included, the range of uncertainties changes, but the basic trend of the uncertainty distributions remains the same. The estimates become highly degraded near $\Omega = \pm 90$ deg. One clear explanation comes from the fact that the spacecraft orbit is perpendicular to the Earth line of sight, decreasing the information content of both range and Doppler.
The precession results have exhibited the effects of individual error sources on estimates of $\gamma$ and $\beta$. We now discuss how accurately $\gamma$ and $\beta$ can be estimated when all of the error sources are accumulated in the estimation filter.

Table 5 shows how the estimates of $\gamma$, $\beta$, and $J_2$ vary when different error uncertainties are included in the filter model. The unbiased accuracies are not obtainable with these assumed error bounds; however, uncertainties with accumulated errors can still be compared to current estimates of $\gamma$ and $\beta$. We also see from this table that a decrease in the measurement noise to the K-band tracking capability requires simultaneous reduction in the relevant error sources (e.g., station-location errors), which is expected. We can conclude from this table that the most challenging task of this experiment is placing a spacecraft on a heliocentric elliptic orbit yields results that are essentially equivalent to those obtained from a hyperbolic orbit. Additionally, an elliptic orbit provides multiple perihelion passages, which will enhance the estimation accuracy of $J_2$. The hyperbolic trajectory (with $e = 1.03$) provides a slightly better result for $\gamma$ (but not for $\beta$); however, Solar Probe’s multiple perihelion passages will allow for an overall increase in the level of accuracy. To carry out this experiment, considering the large nongravitational disturbances present near the sun, the spacecraft might need to be equipped with precision accelerometers or a drag-free system. We believe that our results indicate that there is a realistic possibility of carrying out this new GR experiment as an add-on component to the Solar Probe mission objectives.

Conclusions

In this paper, we have analyzed how well the parameterized post-Newtonian (PPN) parameters $\gamma$ and $\beta$ can be estimated from spacecraft radiometric tracking data, such as range, very long baseline interferometry, and Doppler measurements. In the baseline case, the spacecraft starts from a perihelion at four solar radii with $v_p = 311$ km/s, and we obtain radiometric measurements as it propagates along the heliocentric trajectory. Various error sources are included in estimates of $\gamma$ and $\beta$, including stochastic acceleration, station-location errors, and solar $J_2$. Although this is a preliminary analysis, it captures most of the fundamentals required for evaluation of this experiment. Several crucial characteristics of the uncertainties in $\gamma$ and $\beta$ are obtained from this analysis. The first important conclusion is that placing a spacecraft on a heliocentric elliptic orbit yields results that are essentially equivalent to those obtained from a hyperbolic orbit. Additionally, an elliptic orbit provides multiple perihelion passages, which will enhance the estimation accuracy of $\gamma$, $\beta$, and $J_2$. Analysis shows that it might be feasible to carry out this test as part of the Solar Probe mission, which is currently under development. To do so, some improvements in spacecraft-tracking technology will be required. The effect of stochastic accelerations significantly degrades the accuracy of the experiment, and hence it is necessary to decrease the steady-state uncertainty in nongravitational accelerations by either using accurate accelerometers or by implementing a drag-free spacecraft. Station-location errors degrade the accuracies of the PPN parameters, but current estimates appear to be adequate for X-band tracking. Simultaneous reduction in station-location error is required for a K-band tracking system. Finally, the solar $J_2$ can be measured accurately from this experiment.

We have demonstrated that under ideal conditions general relativity can be tested to a significant accuracy by tracking spacecraft trajectories near the sun and have shown that such an experiment is feasible.

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Associate Editor