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Final version of IMaGe logo created by Allen K. Philbrick from original artwork from the Founder.

IMaGe Silver Jubilee

SOLSTICE, VOLUME XXI, NUMBER 1; June, 2010

Front matter: June, 2010.
Editorial Board, Advice to Authors, Mission Statement

Awards

MatheMaPics

Sandra L. Arlinghaus and Joseph J. Kerski

The Perimeter Project, Part 3:
Fragile Lands Protection Using Cemetery Zoning
Associated .kmz file
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Zipf's Hyperboloid--Revisited:
Compression and Navigation--Canonical Form
Sandra L. Arlinghaus
with input from an earlier article as part of an ongoing collaboration with Michael Batty

Fractals Take A Non-Euclidean Central Place
Sandra L. Arlinghaus

Solstice Archive

Solstice: An Electronic Journal of Geography and Mathematics,
Volume XXI, Number 1
Institute of Mathematical Geography (IMaGe).

sarhaus@umich.edu
http://www.imagenet.org
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Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2001 (top 80 out of over 1000 entries worldwide)

One article in Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2003 (Spatial Synthesis Sampler).

Solstice is listed in the Directory of Open Access Journals maintained by the University of Lund where it is maintained as a "searchable" journal.

Solstice is listed on the journals section of the website of the American Mathematical Society, http://www.ams.org/

Solstice is listed in *Geoscience e-Journals*

IMaGe is listed on the website of the Numerical Cartography Lab of The Ohio State University: http://ncl.sbs.ohio-state.edu/4_homes.html

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Congratulations to all *Solstice* contributors.

Remembering those who are gone now but who contributed in various ways to *Solstice* or to IMaGe projects, directly or indirectly, during the first 25 years of IMaGe:

Allen K. Philbrick | Donald F. Lach | Frank Harary | H. S. M. Coxeter | Saunders Mac Lane | Chauncy D. Harris | Norton S. Ginsburg | Sylvia L. Thrupp | Arthur L. Loeb | George Kish |

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SOLSTICE: AN ELECTRONIC JOURNAL OF GEOGRAPHY AND MATHEMATICS

http://www.imagenet.org

June, 2010

VOLUME XXI, NUMBER 1

ANN ARBOR, MICHIGAN

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MISSION STATEMENT

The purpose of Solstice is to promote interaction between geography and mathematics. Articles in which elements of one discipline are used to shed light on the other are particularly sought. Also welcome are original contributions that are purely geographical or purely mathematical. These may be prefaced (by editor or author) with commentary suggesting directions that might lead toward the desired interactions. Individuals wishing to submit articles or other material should contact an editor, or send e-mail directly to sarhaus@umich.edu.

SOLSTICE ARCHIVES

Back issues of Solstice are available on the WebSite of the Institute of Mathematical Geography, http://www.imagenet.org and at various sites that can be found by searching under "Solstice" on the World Wide Web. Thanks to Bruce Long (Arizona State University, Department of Mathematics) for taking an early initiative in archiving Solstice using GOPHER.

PUBLICATION INFORMATION

To cite the electronic copy, note the exact time of transmission from Ann Arbor, and cite all the transmission matter as facts of publication. Any copy that does not superimpose precisely upon the original as transmitted from Ann Arbor should be presumed to be an altered, bogus copy of *Solstice*. The oriental rug, with errors, serves as the model for creating this weaving of words and graphics.

Awards and Recognition

- 2010: S. Arlinghaus invited to
  - serve as Editorial Board member for ***Geographical Analysis***
  - serve as Series Editor for CRC Press
- 2009: Best of 3D Warehouse awards (blue ribbons) in addition to those listed below--for work of Archimedes (aka S. Arlinghaus) updating UM models reflecting change in base plate of aerials made by Google in June of 2007. Current status of awards:
  - Archimedes continues as a "Featured Modeler" in the Google 3D Warehouse. She was selected among the first five when this segment was created and has been featured in it ever since.
  - Downtown adjusted models awarded Blue Ribbon status: 1 (Ann Arbor News, partial), 2 (Main and Liberty).
  - University of Michigan campus models awarded Blue Ribbon status: 1 (Angell Hall), 2 (Burton Tower), 3 (Chemistry Building), 4 (Clements Library), 5 (Crisler Arena), 6 (Dennison), 7 (East Hall), 8 (Graduate Library, Hatcher), 9 (Hill Auditorium), 10 (Michigan League), 11 (Modern Languages Building), 12 (Natural Sciences), 13 (President's House), 14 (Rackham), 15 (Randall Labs), 16 (Shapiro Undergraduate Library), 17 (Tappan), 18 (West Hall).
- 2009: Kerry Ard wins Google Earth KML Research Competition. One of two top awards in the student category. The only US student to win.
- 2009: *Solstice* covers displayed in 'Journal Covers" exhibition at the Science Library of The University of Michigan.
- 2008: S. Arlinghaus invited to speak at Google 3D Warehouse Base Camp in Mountain View, CA at the GooglePlex. Had to decline the invitation; nonetheless, was nice to be thought of as representing higher education in regard to work already done.
- 2008: Best of 3D Warehouse awards number over 50.
- 2007: Best of 3D Warehouse awards (blue ribbons); these buildings come up default in all free downloads of Google Earth when the "3d buildings" checkbox is checked. They are designed for planning, rather than for architectural, purposes; file size is kept small. What is important is giving the "impression" of the building rather than giving large amounts of detail. View the associated .kmz files in Google Earth to understand the context; they are attached to the linked pages below. Be sure to turn on the "terrain" switch, otherwise buildings made in older software (older versions of Google SketchUp) will float above the surface.
  - Archimedes's models (S. Arlinghaus is "Archimedes").
    - Campus models of Arlinghaus: 1 (Alumni Center), 2 (Angell Hall), 3 (Angell Hall Complex), 4 (Art Museum, first model), 5 (Art Museum, second model), 6 (Bagnoud Building), 7 (Biomedical Sciences Building), 8 (Bursley Hall), 9 (C. C. Little Building), 10 (Chemistry Building), 11 (Clements Library, first model), 12 (Clements Library, second model), 13 (Crisler Arena), 14 (Dennison Building, first model), 15 (Dennison Building, second model), 16 (East Hall, first model), 17 (East Hall, second model), 18 (Frieze Building), 19 (Hatcher Library North), 20 (Hatcher Library South), 21 (Haven Hall), 22 (Hill Auditorium, first model), 23 (Hill Auditorium, second model), 24 (Kraus Natural Science Building), 25 (Michigan League, first model), 26 (Michigan League, second model), 27 (Literature, Science, and the Arts Building), 28 (Mason Hall), 29 (Michigan Stadium), 30 (Modern Language Building), 31 (Northwood IV), 32 (Pharmacy College), 33 (Power Center), 34 (Rackham Building, first model), 35 (Rackham Building, second model), 36 (Randall Laboratory), 37 (Schembechler Hall), 38 (Shapiro Library), 39 (Tappan Hall, second model), 40 (Tisch Hall), 41 (University Hospitals), 42 (West Hall, first model), 43 (West Hall, second model).

        - DDA models of Arlinghaus: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
    - Build Your Campus competition models--student participants each won at least one blue ribbon, as a Best of 3D Warehouse award
        - Lauren Leigh Hoffman: Dana Building
        - Juan Sergio Ponce de Leon: Yost Arena, South Quad
        - Andrew Walton: Golf Course Clubhouse

- 2007: University of Michigan models of about 300 buildings included in the online folder resulting from the "Build Your Campus" competition.
- 2007: Archimedes selected by Google as a "Featured Modeler."
- 2006: Google 3D Warehouse, "Google Picks" then go to "Cities in Development" http://sketchup.google.com/3dwarehouse/ to see textured models of downtown Ann Arbor buildings.
- 2006: *3D Atlas of Ann Arbor, Version 2*. Google Earth Community, ranked a "Top 20 Rated Post" on Entrance page, December 8, 2006.
- 2006: *3D Atlas of Ann Arbor, Version 2*. Rated a 5 globe production (top score) in Google Earth Community, November 2006.
- 2004: Sandra L. Arlinghaus and William C. Arlinghaus, Spatial Synthesis Sampler, *Solstice*, Summer 2004. Semi-Finalist, Pirelli 2003 INTERNETional Award Competition.
- 2004: Sandra Lach Arlinghaus, recipient, The Presidentâ€™s Volunteer Service Award, March 11, 2004.
- 2003: Jeffrey A. Nystuen, won the 2003 Medwin Prize in Acoustical Oceanography given by the Acoustical Society of America. The citation was "for the innovative use of sound to measure rainfall rate and type at sea". It is awarded to a young/mid-career scientist whose work demonstrates the effective use of sound in the discovery and understanding of physical and biological parameters and processes in the sea.
- 2002: Sandra L. Arlinghaus, William C. Arlinghaus, and Frank Harary. *Graph Theory and Geography: an Interactive View (eBook)*, published by John Wiley and Sons, New York, April 2002. Finished as a Finalist in the 2002 Pirelli INTERNETional Award Competition (in the top 20 of over 1200 entries worldwide).
- 2001: *Solstice*, Semi-Finalist, Pirelli 2001 INTERNETional Award Competition in the Environmental Publishing category.
- 1992: *Solstice*, article about it by Ivars Peterson in *Science News*, 25 January, 1992..
- 1991: *Solstice*, article about it by Joe Palca, *Science* (AAAS), 29 November, 1991.

The Perimeter Project:
Cemetery Zoning Used in Fragile Lands Protection--Part III

Sandra L. Arlinghaus and William E. Arlinghaus

Presented at the
Second Annual GooglEarth Day Conference
Held in 2024 Dana Building
School of Natural Resources and Environment
The University of Michigan
April 22, 2010

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Conceptual Overview: US Burial Practice

- Conventional US burial practice involving chemical embalming and cement vaults damages the environment. Scattering of ashes following cremation removes this difficulty but offers little constructive to the environment. Indeed, cremation by fire adds to the carbon footprint. Green or natural burial removes these difficulties and may offer even more to the environment when done with care.

- Scattering and Green burial approaches offer few means for memorialization. The Internet does so, when website memorials are trust-funded in the same manner as conventional cemeteries. Such memorialization has the added advantage of integrating far-flung family members in virtual space.

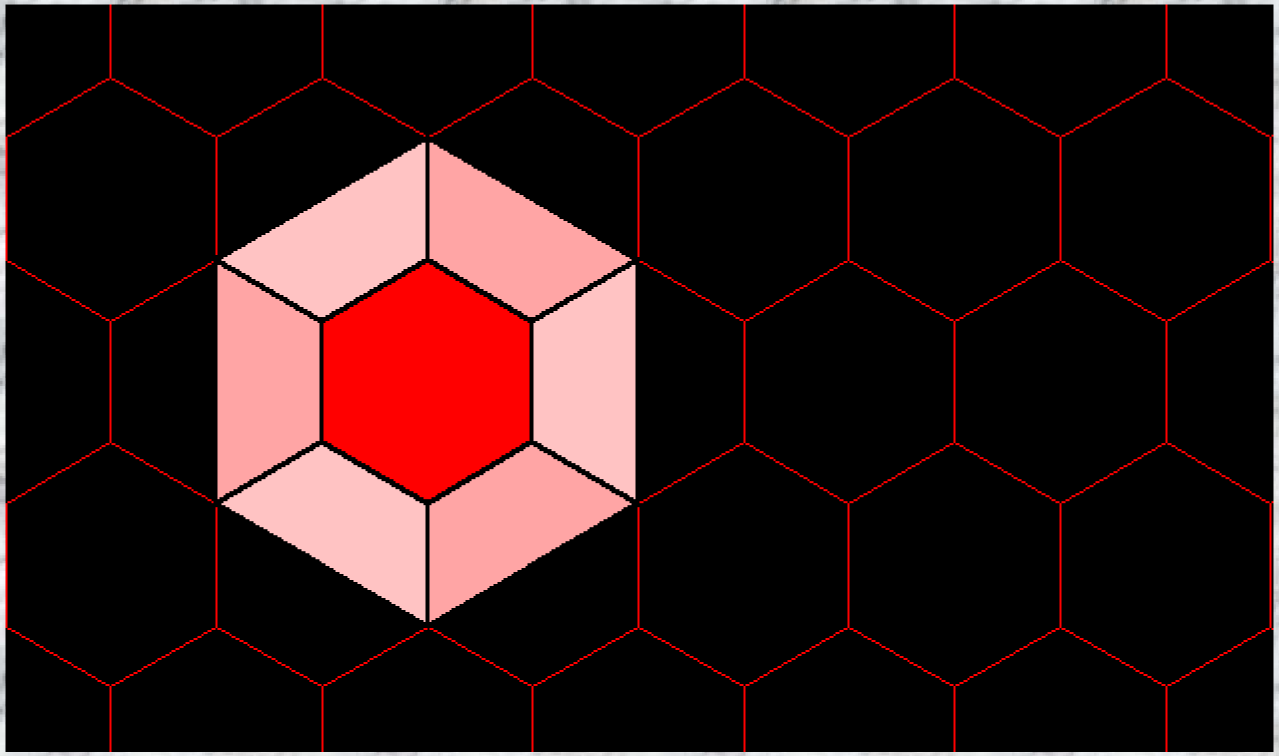
Memorialization: Zoning Implications

- One might employ online memorials as a means to encourage more environmentally-sensitive burial practice.

- Furthermore, with such encouragement in place, one might turn the idea around and attempt to protect fragile lands by acquiring cemetery zoning for them (which is the most difficult to change--zoning can be a moving target that responds to the political whim of varying administrations).

- An initial approach to protecting broad swaths of land might be to endow existing sets with cemetery zoning and dedicate parts of the existing land use to cemetery use—as a “mixed use development” in much the way that condos are clustered on one area of a parcel, all zoned for condos, while a large portion of the parcel is dedicated to passive parkland.

Mixed Use Visualization:
Abstraction of Seamless Integration of Disparate Landuse Types



Environmental Rationale:

Cemeteries and Golf Courses: Creative Mixed Use

Contemporary environmental science views golf courses as difficult uses of large tracts of land.

Use golf course non-playing area, endowed with restrictive cemetery zoning, as a site for green and natural scattering or burial of remains. Integrate the uses now; as the baby-boomer population ages, the need to expand the cemetery land holdings may well increase dramatically. Instead begin now to use portions of existing large tracts (golf courses) for burial; natural burial may enhance the vegetative cover and make these grounds a showplace for variety in environmentally sound gardening principles

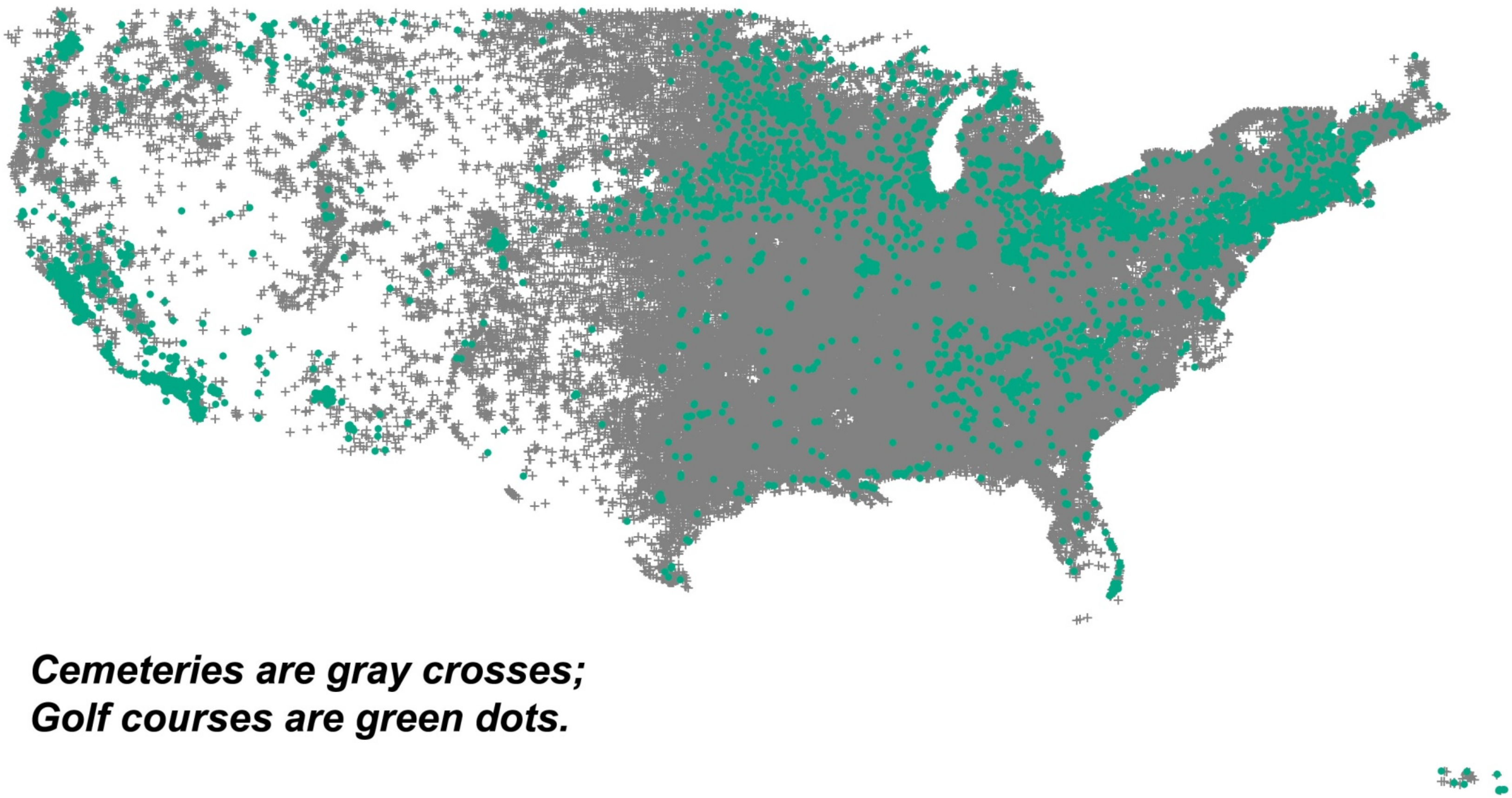
- **Golf courses:** EPA data (http://www.epa.gov/oppefed1/models/water/golf_course_adjustment_factors.htm)—15,827 golf courses (March 2003) range in size from 110-200 acres. Consider 150 as a middle ground (many, but not urban or resort, 18 hole courses range in size from 150-200 acres). That would put total acreage at: 15,827 * 150 = **2,374,050 acres**.

- **Cemeteries:** The attribute table in ArcGIS 9.1.3 shows 126,166 cemeteries. We do not know the total acreage in cemeteries.
  - If the total acreage in golf courses equaled the total acreage in cemeteries, then the average cemetery size would be: 2,374,050 / 126,166 = 18.82 acres. Arlington National Cemetery is 254 acres and has 300,000 interred. (http://en.wikipedia.org/wiki/Arlington_National_Cemetery)
  - How big is the average cemetery? A typical cemetery plot is 4 feet wide by 10 feet long (40 square feet). Suppose a typical cemetery buries 365 people per year. That's 14,600 square feet of land per year, excluding land for landscaping, interior surface routes, maintenance areas, houses for staff, chapels, and so forth. Over the entire country (126,166 cemeteries) that's 1,842,023,600 square feet (42,287 acres) per year devoted to burial.
  - Over 50 years, that's **2,114,352 acres** of land in burial (only). Reuse of existing burial sites may take place, in many states, after 50 years. Cemetery requirements on land use are more restrictive than are others.
  - As the baby-boomer population ages, there may be more need, in the short run, for cemetery land.

- **It appears reasonable to assume that the acreage in cemetery use is similar in size to the acreage for golf courses (there are many more cemeteries than golf courses):**

Entire 18 hole course: 2,374,050 acres – Burial sites only: 2,114,352 acres

There are more cemeteries than golf courses.

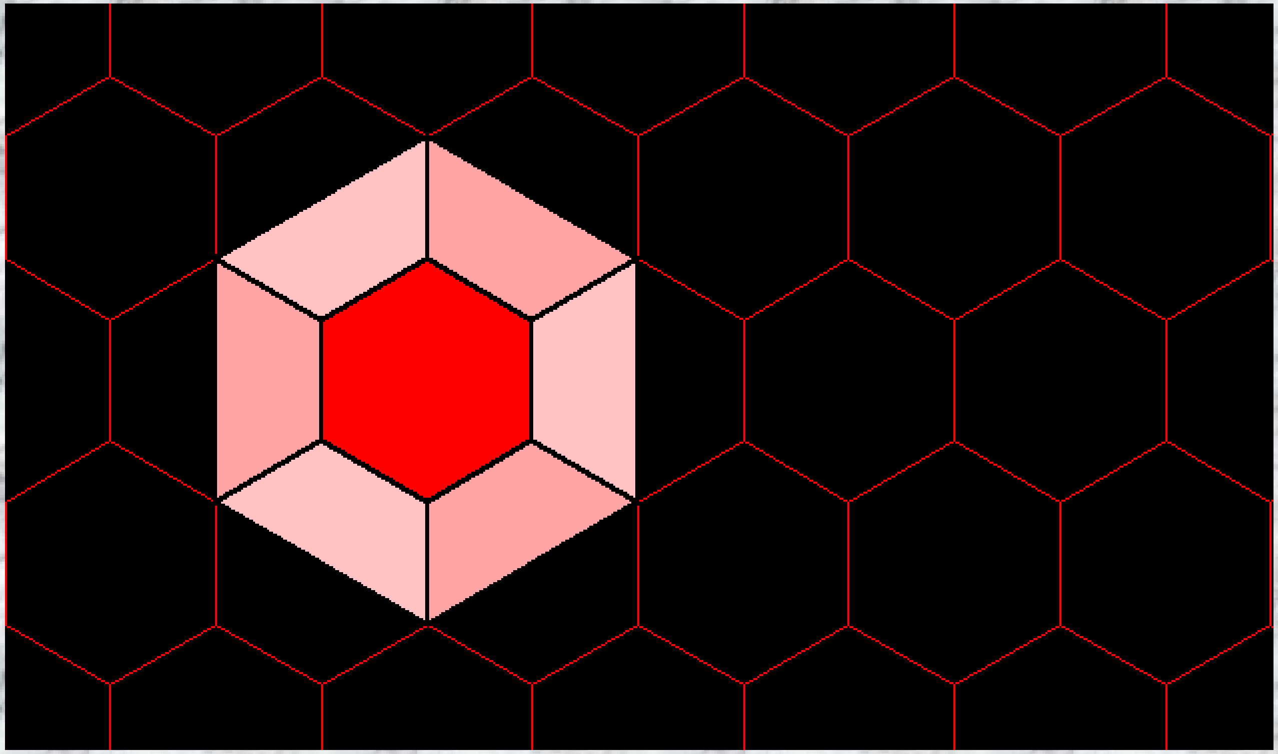


Cemeteries are gray crosses;
Golf courses are green dots.

A closer look: the case of Michigan

- However, the spatial distribution of cemeteries is more widely scattered than is the more clustered distribution of golf courses.
- A tempting thought is to consider these as two sets and attempt to maximize their intersection in order to minimize impact on the land.
  - Currently, these sets are disjoint (have empty intersection), both in terms of
    - Actual use.
    - Zoning.
  - To create a non-empty intersection
    - Change golf course zoning to cemetery zoning (the more restrictive zoning).
    - Zone the entire golf course as "cemetery" use, but imagine the cemetery use clustered away from the fairways and such—much as condo complexes might cluster residential units in one area and parkland in another.
- Use existing golf courses for recreational use as well as for cemetery use that offers constructive environmental input.
  - Scattering of cremation ashes in selected, protected areas.
  - Scattering of resomation (water extraction) waters in selected, protected areas.
  - Natural whole-body burial (no dangerous embalming fluids, steel, or concrete) in selected, protected areas.
- At the outset, existing golf courses rezoned as "cemetery" might serve the needs of some urban and resort populations. Future planning might base locational decisions on some of the data shown in the maps above.

Mixed Use Visualization:
Cemetery use on Golf Course



Burial Alternatives:
Cemeteries and Golf Courses

Comments from William E. Arlinghaus, B. A.
General Manager, Chapel Hill Memorial Gardens, Grand Rapids
President, The Compass Group

Cremation
Resomation
Natural Burial
Legal, Perception, and Related Issues

Memorialization:
Cemeteries and Golf Courses

When remains are integrated into the environment, rather than compactly stored in a vault or similar object, memorial needs may change…no longer are there marble monuments clearly and directly associated with individual remains.

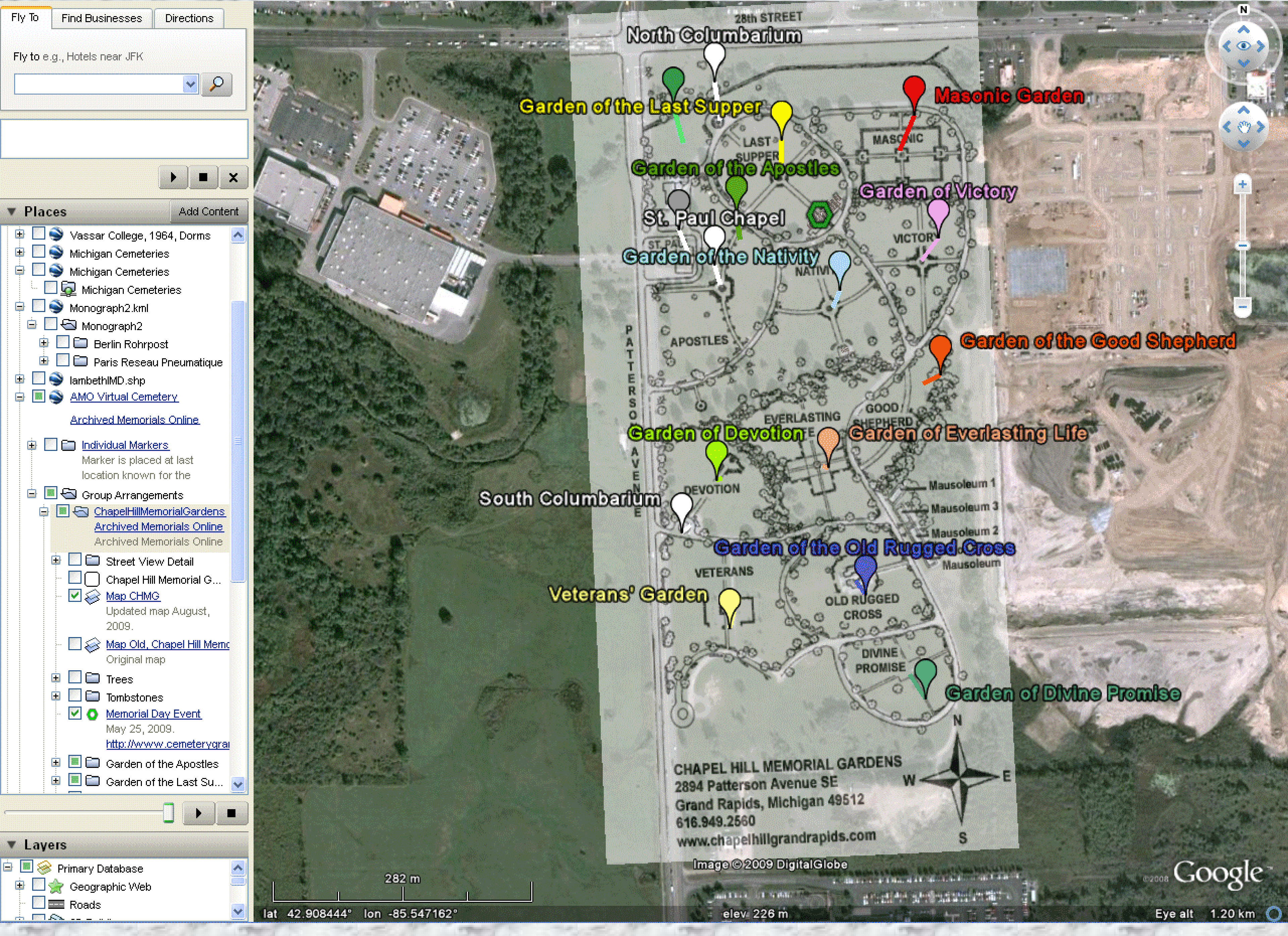
The Internet offers direct, individual memorialization opportunity that becomes permanent when website maintenance is trust-funded as is traditional monument maintenance.

Pilot Project: Chapel Hill Memorial Gardens Grand Rapids, Michigan

- A sample of the more than 300 online memorials appears in the trust-funded Virtual Cemetery of Archived Memorials Online (AMO). A small set of AMOs has been online since 2003 (and trust-funded since 2002)
  - The Pilot Project at Chapel Hill began January 1, 2009 as the first time integration of AMOs with an actual cemetery occurred
  - That project has generated hundreds of new Basic AMOs as part of the standard burial package.
- Cemetery maps and actual lot locations, along with trees and 3D objects and memorial text and images are located in Google Earth.
- http://www.ArchivedMemorialsOnline.com

Virtual Cemetery Visualized Using Google Earth and Google SketchUp

- Associated cemetery maps are embedded on the Google globe
- Balloons mark burial location
- AMOs popup when clicked on within the Google Earth browser interface
- Associated features add extra reality
  - Mausoleum buildings created in SketchUp
  - 3D trees found online
  - Street views interior to the cemetery from field photographs—useful for site-benchmarking, as well.
  - Special events visuals of various kinds



Fly To
Find Businesses
Directions
Fly to e.g., Hotels near JFK
Places
Add Content
Vassar College, 1964, Dorms
Michigan Cemeteries
Michigan Cemeteries
Michigan Cemeteries
Monograph2.kml
Monograph2
Berlin Rohrpost
Paris Reseau Pneumatique
lambethMD.shp
AMO Virtual Cemetery
Archived Memorials Online
Individual Markers
Marker is placed at last location known for the
Group Arrangements
ChapelHillMemorialGardens
Archived Memorials Online
Archived Memorials Online
Street View Detail
Chapel Hill Memorial G...
Map CHMG
Updated map August, 2009.
Map Old, Chapel Hill Memo
Original map
Trees
Tombstones
Memorial Day Event
May 25, 2009.
http://www.cemeterygra
Garden of the Apostles
Garden of the Last Su...
Layers
Primary Database
Geographic Web
Roads
28th STREET
North Columbarium
Garden of the Last Supper
Masonic Garden
Garden of the Apostles
Garden of Victory
St. Paul Chapel
Garden of the Nativity
Garden of the Good Shepherd
Garden of Devotion
Garden of Everlasting Life
South Columbarium
Garden of the Old Rugged Cross
Veterans' Garden
Garden of Divine Promise
LAST SUPPER
MASONIC
VICTORY
APOSTLES
EVERLASTING
GOOD SHEPHERD
DEVOTION
Mausoleum 1
Mausoleum 3
Mausoleum 2
Mausoleum
VETERANS
OLD RUGGED CROSS
DIVINE PROMISE
PATTERSON AVENUE
CHAPEL HILL MEMORIAL GARDENS
2894 Patterson Avenue SE
Grand Rapids, Michigan 49512
616.949.2560
www.chapelhillgrandrapids.com
N
W
E
S
Image © 2009 DigitalGlobe
Google
282 m
lat 42.908444° lon -85.547162°
elev 226 m
Eye alt 1.20 km

Pilot Project: Planning of Memorialization

- The materials already present in the Virtual Cemetery range in complexity from “basic” to “simple.” As there is a wide-range in traditional physical cemetery memorialization so too might there be in virtual memorialization.
- Advance planning enables one to create an AMO while alive (where in this case “AMO”=“Active Memory Online”).
  - Simpler forms of this file might involve links to existing persistent files elsewhere on the internet.
  - Other more complex forms might involve the creation (by the individual) of a complete biography, in GEOMAT or other format, to be entered into the Virtual Cemetery at the appropriate time.
- Sample ‘GEOMAT’ personal biography (in progress),

  derivative of work over a period of a few years with Ann E. Larimore and Rob Haug: http://www.MyLovedOne.com/GEOMAT/Sandy/

Directions…Memorialization

- Beyond the Basic AMO

  Custom AMOs with added visuals or videos—possible associated contacts: PWilliams productions

- Facebook—Personal Memorialization Wall,

  in association with Jen Osburn, CHMG

- Teaching of new staff and consumers
  - Handbook in progress
  - Online materials in progress

Directions…Related Pilot Projects

- Extensions to other existing cemeteries might offer opportunities to learn more about database management and related issues
- Municipal extension might involve
  - Matt Naud, City of Ann Arbor Environmental Coordinator
  - Roger Rayle, CSF Research Associate and Chair of Scio Residents for Safe Water
  - Allen Creek Greenway or other citizen groups
- Regional extension might involve various groups, such as golf courses, in association with land acquisition and zoning issues.
  - The Memorial? Pebble Beach, Neptune Society, as well?
  - Contact with a variety of local and regional golf course and cemetery experts.
  - Work with software companies to integrate TV walls in golf course clubhouses to display internet archived memorials.
  - Land acquisition tied to locally unwanted land uses, such as water tower sites.
- Publication of results in online and conventional media.
- International extension to developing nations and database management issues will involve DevInfo and Kris Oswalt (CSF).

Directions…Connections and Feedback

- In cremation it is necessary (lest the crematorium explode), and in green burial it is desirable, to remove pacemakers and other metal from the body. We have necessary contacts, from funeral homes, to handle such removals.
- Kim Eagle, M.D. and Timir Baman M.D. (Cardiovascular Center, University of Michigan) have a pacemaker recycling project in progress in which pacemakers will be recycled in hospitals in developing nations.
  - When hospitals and online legal forms are ready from UM, we are prepared to supply pacemakers.
  - As both projects evolve, there may be further opportunity for synergistic effort in the international arena.
- Possible discussions with others in the fields of golf course management and architecture, natural burial, and mortuary science—note audience members with such background.

Many thanks to:
The School of Natural Resources and Environment for room use;
Kris Oswalt of Community Systems Foundation (CSF) for software support;
Google Earth for a software donation to CSF.

CSF archive

http://www.csfnet.org

Information related to this topic

http://www.MyLovedOne.com
http://www.ArchivedMemorialsOnline.com
http://www.ChapelHillGrandRapids.com

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Best viewed in Firefox browser (but should work in any current browser). Works best with a high speed internet connection.
Final version of IMaGe logo created by Allen K. Philbrick from original artwork from the Founder.

Zipf's Hyperboloid--Revisited: Compression and Navigation--Canonical Form

Sandra L. Arlinghaus, Ph.D., The University of Michigan
with input from an earlier article as part of an ongoing collaboration with
Michael Batty, Ph.D., University College London

Introduction

A set of cities may be rank-ordered according to population (or any number of other variables). When it is, and the largest city is assigned the rank of 1, the next largest the rank of 2, and so forth, it follows that there is an inverse relationship between rank and population size. Zipf characterized this relationship as the "rank-size rule." The pattern inferred from plotting actual data corresponds to the equation $xy=K$ which represents an hyperbola in the plane. West, Brown, Enquist, and Savage characterize this rule as:

> The empirically observed regularity is that settlements of rank r in the descending (size) array of settlements have a size equal to 1/r of the size of the largest settlement in the system. In other words, when the population size of towns is plotted against their frequency on a logarithmic scale, we see an approximately straight line with an exponent of -1. This relationship is known as the 'rank-size rule'.

By using log paper, they (as do most others) convert hyperbolae to straight lines. A different, and more direct, approach involves viewing Zipf curves as hyperbolae and seeing them as part of a basic coordinate system on an hyperboloid of two sheets

(see *Solstice* (June 2006) Zipf's Hyperboloid).

THE HYPERBOLOID OF TWO SHEETS

Part of the rationale for plotting on log paper and converting to straight lines is that lines are easier than curves to handle. Such conversion, however, implies a functional relationship that may or may not be there. Why convert curves as models, whose equations are known, to what they in fact are not? Log paper masks the true shape of the curves. Instead, Arlinghaus wished to look for a surface from which to derive these curves, as a set, in order to understand, rather than to mask, fundamental geometric structure.

The branches of the hyperbola in the 1st and 3rd quadrants may be rotated about the line $y=x$ to generate a surface of revolution composed of two dish or bowl-shaped objects facing outward from the origin; follow this link and scroll down to find an image containing an hyperboloid of two sheets that the user can manipulate ("sheets" is the 3-dimensional equivalent of the 2-dimensional "branches").

To ease visualization of the pattern of intersecting planes with the hyperboloid sheets, consider the shapes in an upright position. In analogy with the sphere, one sees, looking at the animations in Figure 1, that the parallels are circular sections sliced by the cutting plane in Figure 1a and that the meridians are hyperbola of varying curvature (Figure 1b) reminiscent of the pattern in the Zipf plot in its hyperbolic representation.

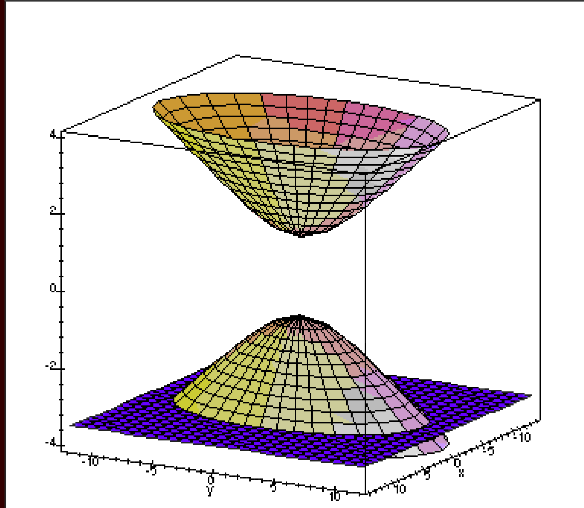


Figure 1a. Reproduced with permission of Putz, John F. Animated Demonstrations for Multivariable Calculus. Found at: http://archives.math.utk.edu/ICTCM/EP-10/C9/html/paper.html

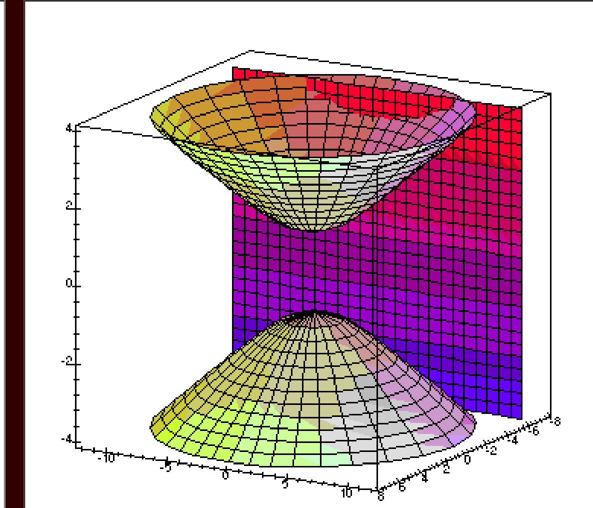


Figure 1b. Reproduced with permission of Putz, John F. Animated Demonstrations for Multivariable Calculus. Found at: http://archives.math.utk.edu/ICTCM/EP-10/C9/html/paper.html

The center of the solid, in each case, is the origin of the three dimensional coordinate system. Each solid has two poles: on the sphere, these are opposite ends of a diameter of the surface. On the hyperboloid, these are opposite ends of the axis of bilateral symmetry of the surface, piercing each bowl at its base. The "great hyperbolae" on the hyperboloid pass through a pole; they serve as geodesics on the surface. The "small hyperbolae" do not pass through a pole.

In terms of Zipf plots, the farther a small hyperbola is from a great hyperbola, as suggested in the animation in Figure 1b, the larger is the biggest city and all subsequent cities by rank in the smaller hyperbola. When the planes are parallel, visualization is easy in either two or three dimensions. When they are not, and curves cross each other in the plane, visualization on the three dimensional surface permits visual disentanglement of form.

MAPPING THE HYPERBOLOID OF TWO SHEETS

When the hyperboloid of two sheets is orthographically mapped to the plane (through the origin) the visualization is as follows. First, imagine that the plane is extended to include points at infinity so that the concept of parallel lines is removed. The Euclidean space is converted to a non-Euclidean one. Now, orthographic projection from the origin sends the upper sheet to the interior of a disk whose boundary is composed of the points at infinity: what had been unbounded become compressed inside a single disk. Points on the boundary of the disk are places where the hyperbolae touch the asymptotes (at infinity). In this model, a great hyperbola, a geodesic passing through a pole (bottom of the bowl), maps to a diameter of the disk; small hyperbolae map to circular arcs with arc endpoints on the disk boundary (images in this link suggest how some of these mappings occur). This systematic mapping of the hyperboloid of two sheets carries the viewer into the non-Euclidean world of hyperbolic geometry. The disk is the "Poincaré" disk. This hyperbolic geometry model served as the base for Lobachevsky's geometry which has often been realized in relation to space-time problems in a variety of disciplines.

ZIPF'S HYPERBOLOID?

If Lobachevsky's hyperboloidal geometry is a natural base from which to drop Zipf curves, as Zipf's Hyperboloid, then how might that geometry be useful in understanding the emergence, timing, and decay of these systems? Figure 2 shows mappings of Zipf curves in a compact form in the Poincaré disk. Because the hyperbolae that pass a pole on the hyperboloid of revolution of two sheets map to diameters of the Poincaré Disk model of the hyperbolic plane, one has the freedom in the Zipf model to choose any Zipf hyperbola as the one about which to perform the revolution. Thus, one might tie such choice to cultural or other considerations. Suppose one wished, for example, to consider pre- and post-World War II patterns. Then, 1950 might be selected as a date about which to create the surface of revolution. Thus, in Figure 2a, imagine that the red diameter represents a U.S.A. rank-size database from 1950, extrapolated according to some procedure to infinity in either direction. Then, the red circles, each with 10 percent fill, on either side, represent the corresponding extrapolated U.S.A. rank-size curves on either side of 1950; those farther from the diameter are farther from 1950. Further, in Figure 2b, imagine that the green diameter represents a European rank-size (extrapolated) database from 1950. The green circles on either side of the diameter, each with 10 percent fill, represent the corresponding European rank-size (extrapolated) databases on either side of 1950. These figures have both been drawn with no crossings of rank-size curves. Were there crossings, the corresponding circles would have perimeters that intersect, affecting fill concentration (which might be used to suggest urban or other concentration). They have also been drawn assuming data on both sides of the date selected for revolution; naturally, if no data were available on one side, then no fill would appear on that side.

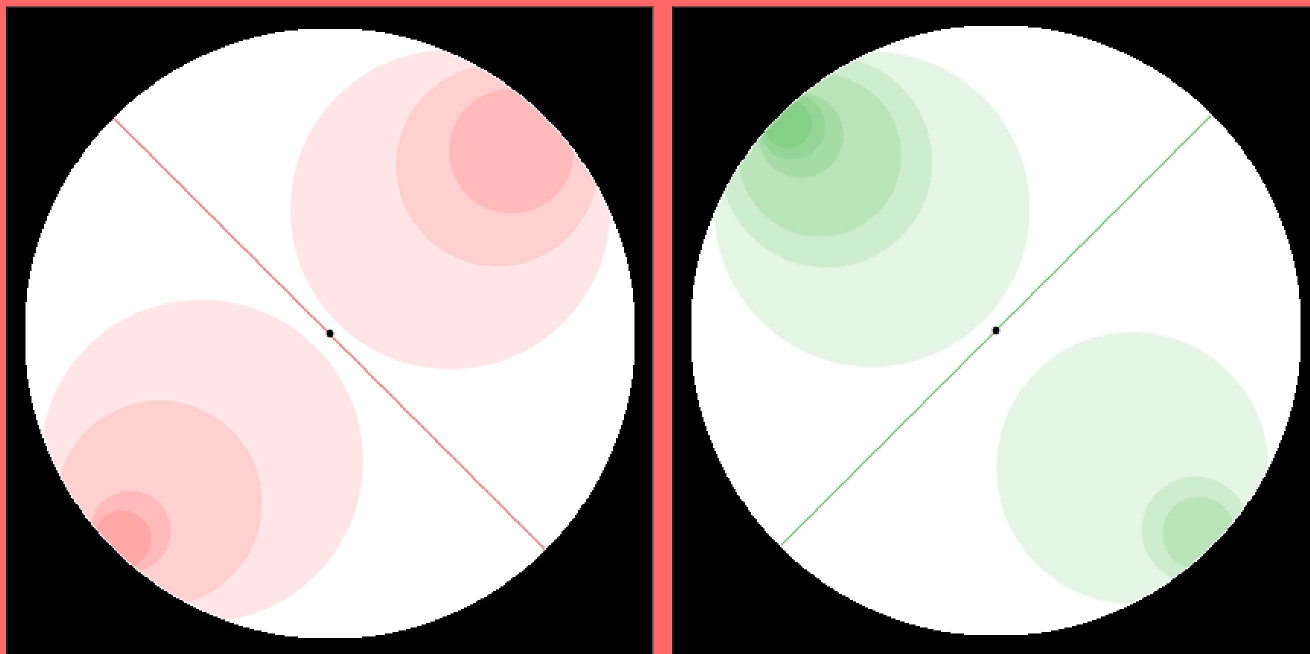


Figure 2a. Poincaré Disk with one set of rank-size curves.

Figure 2b. Poincaré Disk with another set of rank-size curves.

When the two Disk layers are superimposed, as in Figure 3, one advantage to the compact Poincaré Disk model becomes clear: Sets of rank-size curves from different parts of the world can be clearly visualized, simultaneously. It is perhaps an interesting question to consider what overlapping regions might represent: perhaps an opportunity for a Thiessen-style of transformation leading to rank-size art? Indeed, the simple pattern in Figure 3 suggests the style of deeper pattern that can arise in association with complex issues: it is the method of pattern creation so beautifully depicted in the art of Escher's Circle Limit series (inside Poincaré Disks).

Naturally, extrapolated rank-size curves need not be used; if instead, actual data only is used, the circles do not extend to the edge of the limit circle (points at infinity) of the Poincaré Disk. We show the extrapolated curves here to indicate the power of the method. Not only may it be used for the simultaneous display of complex datasets from disparate locations but also it handles finite as well as infinite processes.

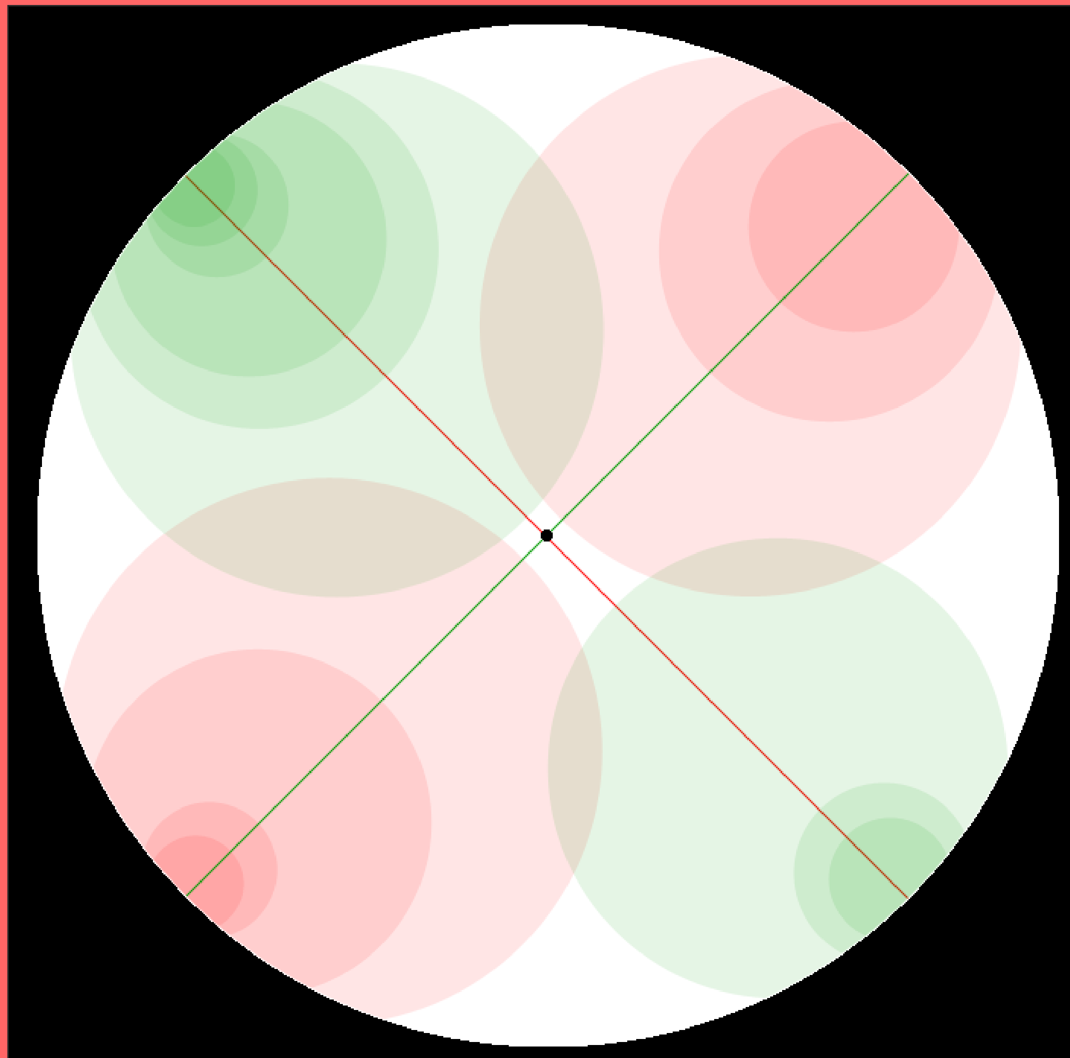


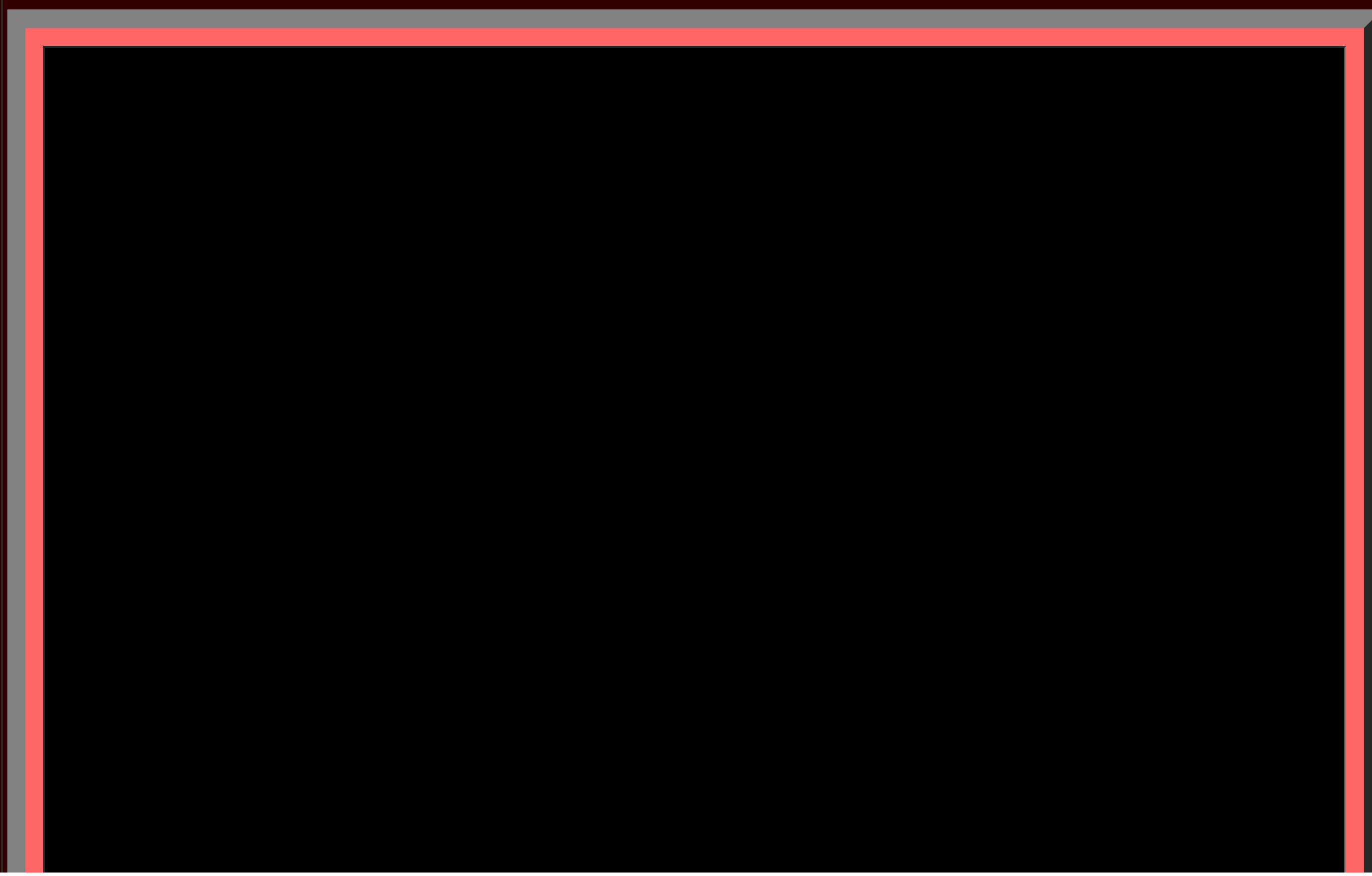
Figure 3. Poincaré Disk model of the hyperbolic plane with two sets of rank-size curves, from disparate locations, embedded.

Compression and Navigation: the Appolonian Gasket and Fractal as Canonical

Form in the Hyperbolic Plane

The layered image in Figure 3 is formed very simply from only a few Zipf-like curves. While it is true that all Zipf curves might be represented in the hyperbolic plane wholly contained within the bounded, compact Poincaré disk, the pattern might become obscured by the complexity of introducing large sets of these curves. If, however, one thinks of the sets of curves in Figure 4 as iterative pattern filling, or not filling, space, then it becomes possible to navigate the compressed forms.

Figure 4 shows a screen capture of a final frame in a movie (created by David Wright, see Indra's Pearls reference below). Play the movie to see how sets of nested circles generate a fractal, called the Appolonian Gasket, shown as a yellow line in the movie. One might suppose that different sequences of nested pattern (thought of as different rank-size curves in the hyperbolic plane, Poincaré disk model) would give rise to different Gaskets. Indeed, from a Euclidean viewpoint, that is the case. Here the advantage of using non-Euclidean geometry is of vital importance. In the hyperbolic plane, all Appolonian Gaskets are in some sense "equivalent" (**Wikipedia. Appolonian Gasket. http://en.wikipedia.org/wiki/Apollonian_gasket).** Hence, the Appolonian Gasket is a canonical form for limiting position of sequences of Zipf curves in the hyperbolic plane (Poincaré Disk model)!



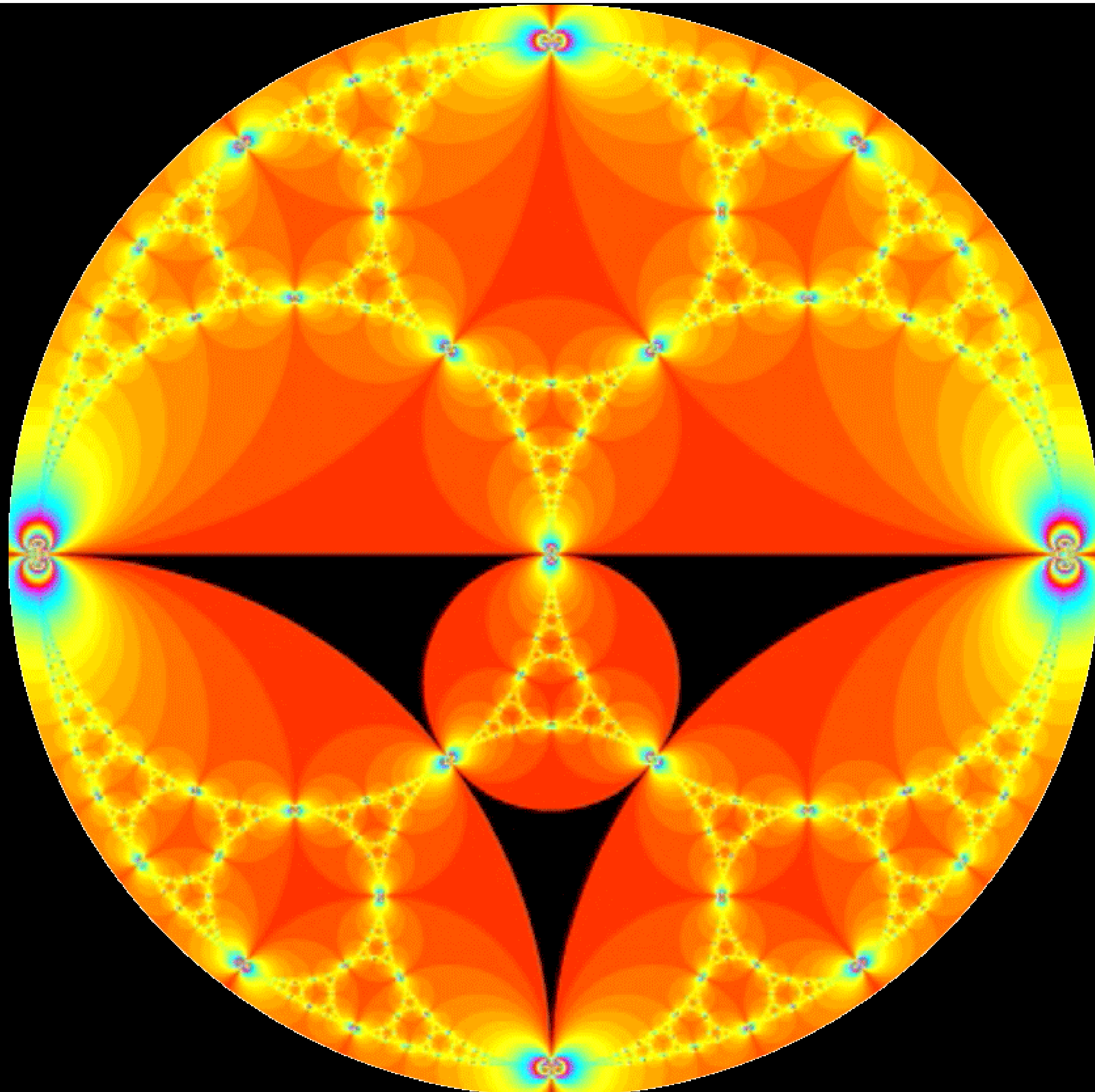


Figure 4. Screen capture from an image created by David Wright (see Indra's Pearls reference below).

What remains is to develop a systematic procedure for converting large data sets into sets of nested curves in the hyperbolic plane and then characterizing them as a Gasket. Both of those processes are substantial endeavors. However, the reason for using the hyperbolic plane is clear and the broad picture is set; what remains is to fill in systematic process.

Related Materials

Links to show materials associated with constructions of Appolonian Gaskets:

- Bourke, Paul. Appolony Fractal. http://local.wasp.uwa.edu.au/~pbourke/fractals/apollony/
- Series, Caroline and Wright, David. Non-Euclidean Geometry and Indra's Pearls. http://plus.maths.org/issue43/features/serieswright/index.html
- Wikipedia. Appolonian Gasket. http://en.wikipedia.org/wiki/Apollonian_gasket

General links and references:

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Congratulations to all *Solstice* contributors.

Remembering those who are gone now but who contributed in

various ways to *Solstice* or to IMaGe projects, directly or indirectly, during the first 25 years of IMaGe:

Allen K. Philbrick | Donald F. Lach | Frank Harary | H. S. M. Coxeter | Saunders Mac Lane | Chauncy D. Harris | Norton S. Ginsburg | Sylvia L. Thrupp | Arthur L. Loeb | George Kish |

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Fractals Take A *Non-Euclidean* Central Place

Sandra L. Arlinghaus, Ph.D., The University of Michigan

Introduction

Central place geometry looks at groupings of hexagons as they tile the Euclidean plane. Pattern within these groupings may be interpreted as representing various elements of how cities share space. The geometry plays out over the unbounded surface of the Euclidean plane; the interpretation surface, however, involves the bounded surface of the Earth (roughly a sphere). One might ask if tilings of regular polygons could suggest other surfaces as a basis for similar interpretation.

Euclid's Fifth ("Parallel") Postulate states that given a line, and a point not on that line, there is exactly one line through that point that does not intersect the given line. That is, the newly drawn line is "parallel" to the given line.

In "Non-Euclidean" geometries the Parallel postulate does not hold (Coxeter, 1961, 1965). One of these non-Euclidean geometries, that admits multiple rather than unique parallels, is called "hyperbolic" geometry. For inspiration for that name, consider the hyperbola: multiple branches through a single point not on a given line might all be asymptotic to the given line--they do not intersect it. Of course, as the process becomes infinite, one might imagine the asymptote getting to the line--of the "parallels" meeting at a point at infinity.

Most of us live in an Euclidean universe. We may function in other ways, but when we

attempt to capture the geometry of our lives we do so, for the most part, in the Euclidean plane. To do otherwise, is jarring to most.

Conventional Central Place Geometry: Fractal Characterization

The patterns that emerge from grouping sets of hexagons according to *K*=3, *K*=4, and *K*=7 principles are familiar to even beginning geography students. What is perhaps not as familiar is the use of fractal geometry to produce these patterns from number theoretic properties (Arlinghaus, 1985; Arlinghaus and Arlinghaus, 1989). See the linked materials for a full explanation; Figures 1.a, 1.b, and 1.c show (respectively) how to create a generator to form appropriate pattern for the *K*=7, *K*=4, and *K*=3 hierarchies. Iteration of the generator at different geographic scales produces the entire landscape as suggested in Figure 1.d which shows the next level of generator scaling and iteration (Arlinghaus and Arlinghaus, 2005).

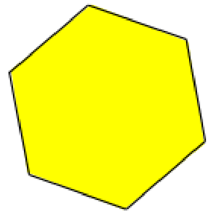

Figure 1.a. *K=7* central place hierarchy: fractal creation

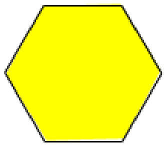

Figure 1.b. *K=4* central place hierarchy: fractal creation

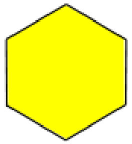

Figure 1.c. *K=3* central place hierarchy: fractal creation

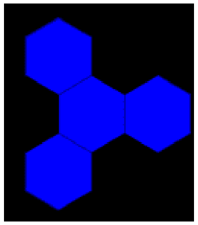

Figure 1.d. Demonstration of iteration of fractal generator for *K=4* hierarchy.

An advantage to the fractal approach is that it permits complete systematic extension of process to higher *K*-values and, at the same time, allows complete solution to existing open questions. One broad concern, however, that is not addressed in this approach or in any other within the Euclidean plane is the issue of bounding the process (rather than allowing it to continue to indefinite extent). In the world of fine art, Maurits Escher addressed this issue elegantly with his Circle Limit Series in which pattern that replicates indefinitely is confined within the boundary of a "limiting" circle.

The Hyperbolic Plane: Poincaré Disk Model

In fact, what Escher did was to embed his interesting carved polygons in the Poincaré Disk model of the non-Euclidean geometry called hyperbolic geometry. The disk contains the entire hyperbolic plane--the tiles appear, from our Euclidean vantage point in the plane of the paper, to disappear as one moves farther toward the limit of the circle edge. Sides of the tiles appear curved, again from our Euclidean standpoint. It is easy to understand this idea (on the sphere) simply by placing a polygon on the Google Earth globe and sliding it around---the animation in Figure 2 illustrates this idea. The reader may download the associated .kmz file here to try it for himself.

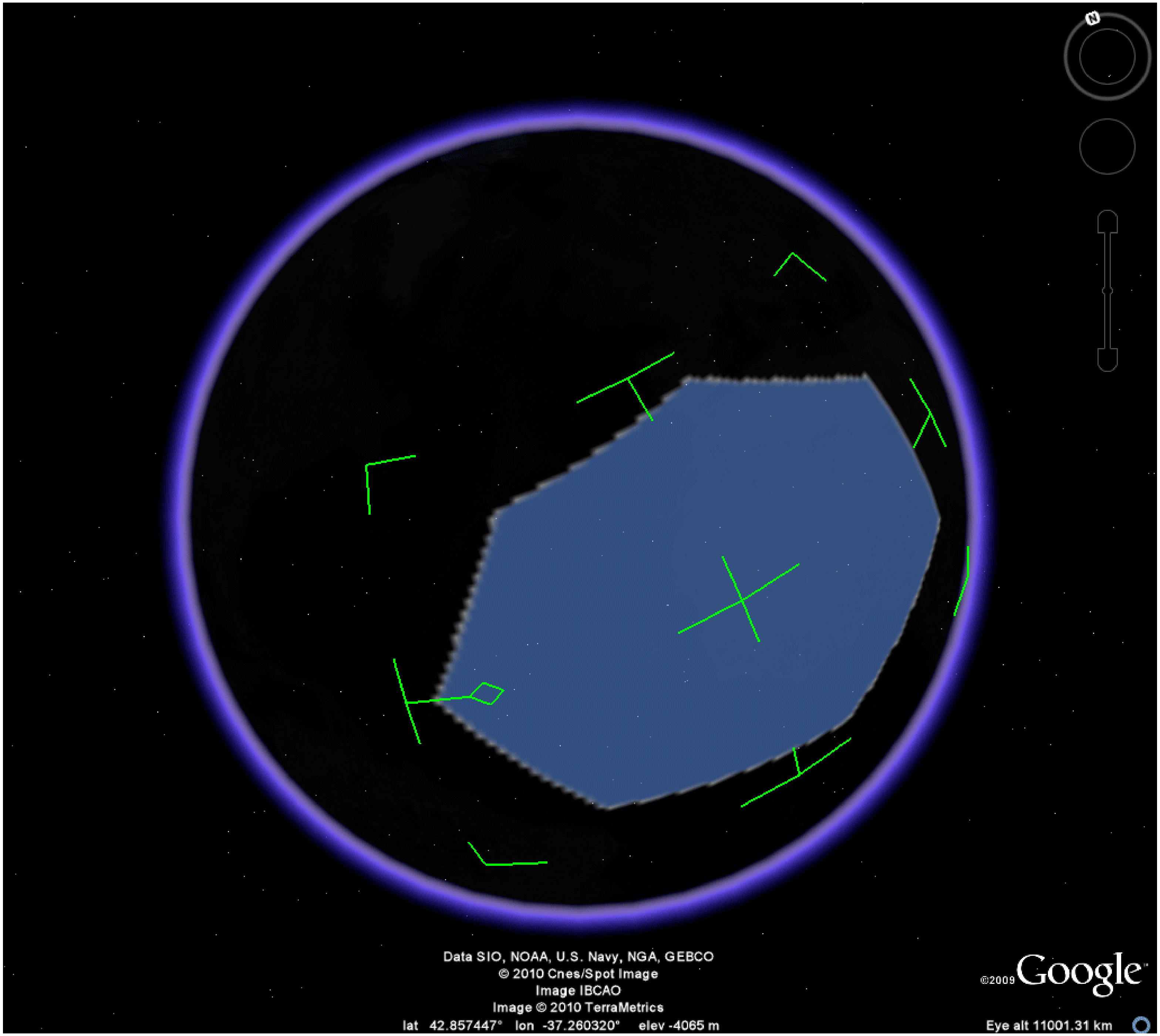
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Image IBCAO
Image © 2010 TerraMetrics
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Eye alt 11001.31 km

Figure 2.

The difference between the polygon on the sphere and a polygon in the hyperbolic plane is that the sphere has constant positive curvature whereas the hyperbolic plane has constant negative curvature, so one must imagine sides that are convex (bulge out) on the sphere would be sides that are concave (sink inward) in the hyperbolic plane (and vice-versa). With the curvature issue, and the shrinking of polygons with distance from the center, in mind, a natural question is to ask about the nature of tilings of this new space.

The Hyperbolic Plane: Tilings

Data compression and the Fibonacci sequence

The Fibonacci sequence is a number pattern that has seen, and continues to see, application in a number of different arenas. The sequence itself is generated as follows:
Start with 1; add the preceding number to it (there is none) generating the next term which is also 1. Then add this new 1 to the previous term generating 2; then add 2 and 1, and so forth, to create the following number pattern:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,...

Some characterizations of this sequence simply have it begin as 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,... Whichever is used, its applications are deep and well known. Indeed, an entire journal, *The Fibonacci Quarterly*, is devoted to its theory and applications. Using it to represent phyllotaxis and the arrangement of spiral leaf patterns on branches or the spiral arrangement of cells on the skin of a pineapple have been with us for a long time. So too has an interpretation of the sequence in terms of the way that rabbits reproduce.

More recent applications employ the sequence in creating a universal computer code that will stem error propogation in computing systems. This property, and others, classify Fibonacci coding as a "Lossless" Entropy Encoding Data Compression Method.

Pattern compression in the hyperbolic plane and the Fibonacci sequence

Series and Wright note that tilings, as well, can create interesting methods of data compression through pattern compression. They note the following example involving similar tilings of both the Euclidean plane, Figure 3.a, and the Hyperbolic plane (realized in the Poincaré Disk Model), Figure 3.b.

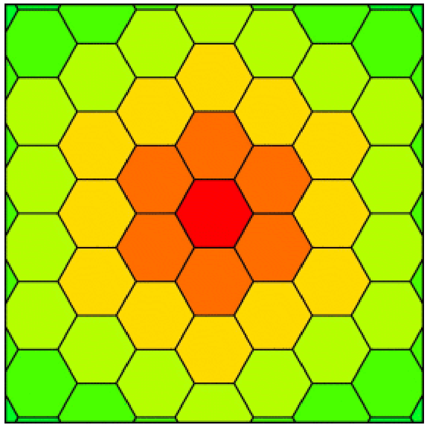

Figure 3.a. Tiling of the Euclidean plane by hexagons. Figure appeared originally in Series and Wright,
http://plus.maths.org/issue43/features/serieswright/index.html

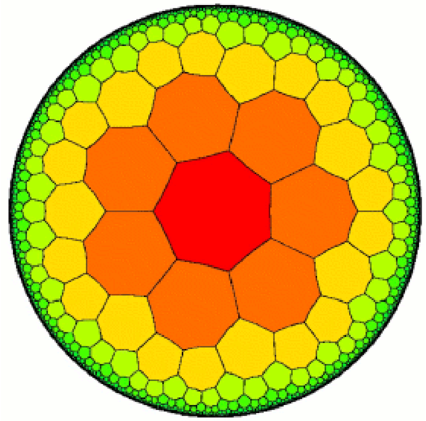

Figure 3.b. Tiling of the hyperbolic plane, realized as the Poincaré Disk Model, by heptagons (7-sided polygons). Figure appeared originally in Series and Wright,
http://plus.maths.org/issue43/features/serieswright/index.html

Simple enumeration of bands of polygons surrounding a chosen central polygon reveals a numerical compression of pattern. Of particular interest, note that once again the Fibonacci sequence gives a systematic basis for the greater numerical compression available in the hyperbolic plane (Series and Wright). Figure 4.a shows the simple linear number pattern that emerges from the hexagonal tiling of the Euclidean plane while Figure 4.b captures systematically the pattern in the hyperbolic plane.

n	Tiles in nth layer
1	6
2	12
3	18
4	24
5	30
6	36
7	42
8	48
9	54
10	60
11	66
12	72

Figure 4.a. Tiling of the Euclidean plane by hexagons. Sequence of tiles in nth layer is the number in a polygonal cell (six) times n. Thus, for example, when n=4, there are 6*4=24 tiles in the fourth layer out from the center.

n	Tiles in nth layer	2nth Fibonacci Number
1	7	1
2	21	3
3	56	8
4	147	21
5	385	55
6	1008	144
7	2639	377
8	6909	987
9	18088	2584
10	47355	6765
11	123977	17711
12	324576	46368

Figure 4.b. Tiling of the hyperbolic plane, realized as the Poincaré Disk Model, by heptagons (7-sided polygons). Sequence of tiles in the nth layer is the number of sides in a polygonal cell (seven) times the 2*n*th Fibonacci number. Thus, for example, when n=4, the 2*n*th Fibonacci number is 21 so that there are 7*21 = 147 tiles in the 4th layer out from the center.

Pattern compression in the hyperbolic plane is more than 4500 fold greater than in the Euclidean plane at the level of *n*=12 and this compression rate appears to increase as *n* increases (Figure 5).

n	Compression fold
1	1.166666667
2	1.75
3	3.111111111
4	6.125
5	12.83333333
6	28
7	62.83333333
8	143.9375
9	334.962963
10	789.25
11	1878.439394
12	4508

Figure 5. Demonstration of the increase in compression as *n* increases.

It remains to offer formal proof of this observation; however, it seems clear that the observation is true since the Fibonacci sequence itself, let alone the 2*n*th Fibonacci sequence, becomes large faster than does the sequence of natural numbers.

Central Place Hierarchies in the Hyperbolic Plane

Possible tilings

While the observation by Series and Wright concerning the Fibonacci sequence in association with an heptagonal tiling of the hyperbolic plane appears both interesting and important, one might ask about other tilings of the plane and such association with the Fibonacci sequence, or other ideas, as a compression tool. An article in Wikipedia is useful in classifying possible tilings of the hyperbolic plane as realized in the Poincaré Disk Model. The reader is encourage to take a look at the link above and study the beautiful and highly symmetric patterns that emerge.

Central Place Theory

One of the "other ideas" that seems natural to consider as a compression tool is that of the fractal. Thus, a logical next step might be to try to merge the fractal compression from central place theory (and hexagonal tilings of the Euclidean plane) with a corresponding construction in the hyperbolic plane (Poincaré Disk Model) in order to assign fractal dimension to these patterns as ways to analyze the extent to which space is "filled" by them in the limiting process.

Tiling by heptagons

In the spirit of the example from Series and Wright, begin by considering the heptagonal tiling of the hyperbolic plane. Figure 6.a shows selection of a generator, similar to the generator for *K*=7 in the Euclidean case above (Figure 1.a) that will generate, when scaled appropriately and applied throughout the Disk, the entire network mesh. Here, the generator has four sides; in the corresponding Euclidean case the generator had 3 sides.

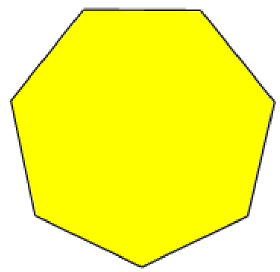

Figure 6.a. Fractal generation of pattern shown in Figure 6.b. The generator has four sides.

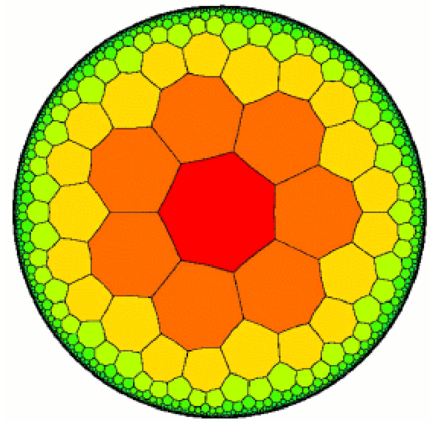

Figure 6.b. Figure 3.b is repeated here for ease in visual reference with Figure 6.a.

The simplest hierarchy to consider is the one in which the natural ring of surrounding polygons needs to be produced: in the Euclidean case with hexagonal tiling of the plane, that is *K*=7 in which a central hexagon is surrounded by 6 others (for 7=6+1). In the hyperbolic case with heptagonal tiling of the plane, a natural extension is to refer to the pattern as *K*=8 (for 8=7+1). One of the reasons to capture central place pattern as a fractal involved a desire to be able to assign a single number, a fractal (Hausdorff-Besicovitch) dimension *D*, to characterize the entire hierarchy. For the Euclidean case, the formula based on work of Mandelbrot, used log(number of generator sides) / log (square root of *K*). Thus, when *K*=7 in the Euclidean case, the generator with 3 sides yielded a value for *D* of 1.1291501... For the generator of 4 sides (Figure 6.a) in the *K*=8 hierarchy, the value of *D = LOG10(4)/LOG10(SQRT(8))* = 1.333333..., if one extends this formula to the hyperbolic plane. Under this latter assumption, clearly more space is filled in the hyperbolic case than it is in the associated Euclidean space: again, compression is greater in the hyperbolic than in the Euclidean world.

To continue the central place theory extension, one might ask about the cases for *K*=4 and *K*=3. A natural appoach, for *K=4* for example, is to consider using as a generator one that surrounds half the area of a heptagon in the hyperbolic plane. Then, apply this generator in an "in and out" fashion to eventually generate a set of polygons that will cover 7*0.5 + 1 amount of heptagonal cells. The problem with this approach is that the "in and out" approach requires an even number of sides in the basic cell; otherwise, two "outs" are adjacent and the inner central cell becomes the wrong shape. Thus, the hierarchy cannot be produced. Figure 7 shows the beginning of this application of a natural generator, with reference to the underlying base so that the difficulty with iteration is clearer.

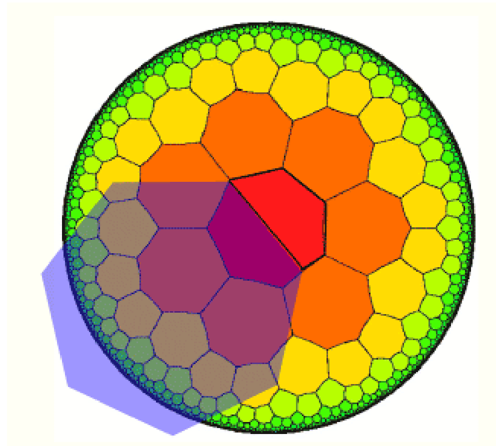

Figure 7. Lack of iterative capability of naturally selected fractal generator.

Prime numbers and polygon shape

Clearly one cannot examine each individual case, as with Figures 6 and 7, to determine when a particular tiling might or might not be generated through fractal generation. One easy approach to this problem is to look at the number-theoretic properties of the base cell of the tiling. In the case of hexagons, the unique factorization of 6 is 3*2. Thus, one can look to generating 1/3 (*K=3*) and 1/2 (*K=4*) polygonal pieces in association with fractal iteration. In the case of a prime number, such as 7, one can find only the natural surrounding pattern of full tiles--no fractional tiles.

With this idea in mind, move now to considering other tilings of the hyperbolic plane as candidates for calculation of fractal dimension. A natural place to begin is with tiling by hexagons.

Tiling by hexagons

One of the characteristics that a central place style of tiling has is that cells on each side of a central cell touch other cells with a side in common with that central cell. Thus, the tiling by hexagons shown in Figure 8 is not this sort of tiling: tiles 1 and 2 are in correct position in relation to the central cell and to each other. So also are the polygon pairs 3 and 4, as well as 5 and 6. However 2 and 3 are separated by an x cell, as are 4 and 5, as well as 6 and 1. The required adjacency pattern is only partially present and so this case is discarded (for now) as not a true related central place case. Thus, one sees why Series and Wright chose the heptagonal tiling, rather than an hexagonal tiling, of the hyperbolic plane as the one naturally associated with the hexagonal tiling of the Euclidean plane. One could, of course, choose other tilings (Woldenberg, 1968). The point here is to indicate what tiling of the hyperbolic plane is, or is not, similar to a tiling of the Euclidean plane by hexagons.

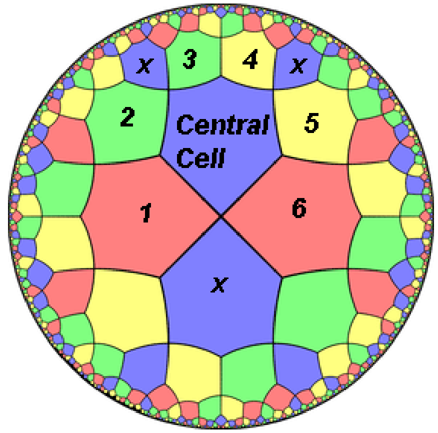

Figure 8. Hexagonal tiling not of central place net type.
http://en.wikipedia.org/wiki/Uniform_tilings_in_hyperbolic_plane

Tiling by pentagons

As in the case above, there are gaps separating tiles. This time, not in pairs but between every tile surrounding the central tile. Figure 9 illustrates this difficulty (which is more extreme than in Figure 8) using notation parallel to that employed in Figure 8. Again, as with the hexagonal tiling, this case which might have appeared attractive at the outset, is discarded as well: its level of violation of a basic premise is even more extreme than is the case with the hexagonal tiling.

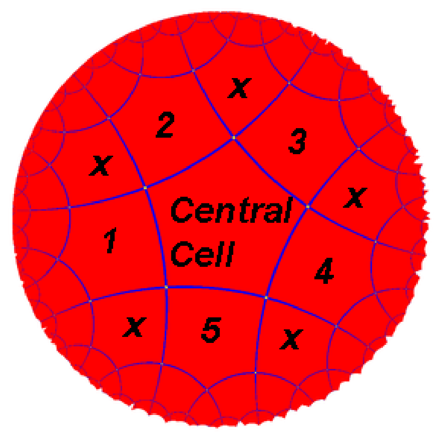

Figure 9. Pentagonal tiling not of central place net type.
http://en.wikipedia.org/wiki/Uniform_tilings_in_hyperbolic_plane

Tiling by octagons

The tiling of the hyperbolic plane shown in Figure 10 is not even as close to a central place net as those in either Figure 8 or Figure 9. Here, the central octagonal cell is surrounding by 8 cells with correct adjacency but they are hexagonal rather than octagonal cells. The next layer out has octagonal cells with no adjacency--each pair of octagonal cells is separated by an hexagonal cell.

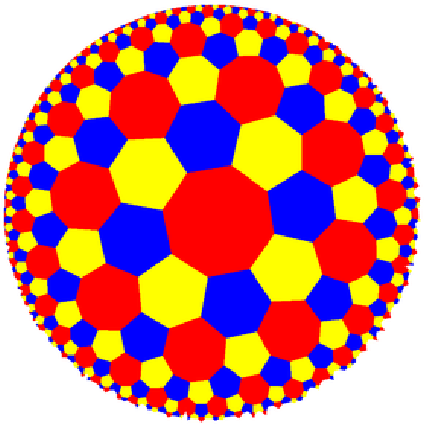

Figure 10. Octagonal tiling not of central place net type.
http://en.wikipedia.org/wiki/Uniform_tilings_in_hyperbolic_plane
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It remains an open question to determine if there are any other true central place net styles of tiling of the hyperbolic plane (there were with the Euclidean plane however there only the case of square tiles (Arlinghaus, 1992) is of interest since a hexagon may be decomposed into triangles). Examination of cases did suffice in the Euclidean case and it may well be sufficient in the hyperbolic case, as well--a quick examination of the visual pattern displayed in the Wikipedia article suggests no others; however, a more systematic search, based on careful notational analysis rather than solely on visual pattern, is necessary to prove there are no others. Indeed, that is one direction for the future--to prove uniqueness of a single central place net for hyperbolic space.

For the Future...

- Uniqueness paves the way for canonical form
- Generating function
- Extensions to different mixed and other tilings (as, conceptually, in earlier work of Woldenberg, 1968; Arlinghaus, 1991)
- Extensions to higher order places
- Graph theory and duality
- Interpretations, focusing perhaps on pattern compression for urban areas, networks, and elsewhere.

Related Materials

General links (last accessed June 15, 2010) and references:

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- Wikipedia, Fibonacci coding, http://en.wikipedia.org/wiki/Fibonacci_coding
- Wikipedia, Uniform tilings in hyperbolic plane, http://en.wikipedia.org/wiki/Uniform_tilings_in_hyperbolic_plane
- Woldenberg, Michael. 1968. Hierarchical Systems, Cambridge, Mass : Laboratory for Computer Graphics and Spatial Analysis, Harvard Center for Environment Desing Studies, The Graduate School of Desing Harvard University. Link

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Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2001 (top 80 out of over 1000 entries worldwide)

One article in Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2003 (Spatial Synthesis Sampler).

Solstice is listed in the Directory of Open Access Journals maintained by the University of Lund where it is maintained as a "searchable" journal.

Solstice is listed on the journals section of the website of the American Mathematical Society, http://www.ams.org/

Solstice is listed in *Geoscience e-Journals*

IMaGe is listed on the website of the Numerical Cartography Lab of The Ohio State University: http://ncl.sbs.ohio-state.edu/4_homes.html

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Congratulations to all *Solstice* contributors.

Remembering those who are gone now but who contributed in various ways to *Solstice* or to IMaGe projects, directly or indirectly, during the first 25 years of IMaGe:

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