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THE EFFECT OF NOISE LEVEL UNCERTAINTY ON DETECTION PERFORMANCE: A COMPARATIVE STUDY

by

Ronald L. Spooner

Approved by: Theodore G. Birdsall

for

COOLEY ELECTRONICS LABORATORY

Department of Electrical Engineering
The University of Michigan
Ann Arbor, Michigan

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ABSTRACT

This report is concerned with the problem of detecting signals in noise of uncertain level. The impetus for this study is found in many areas; however, the detection of underwater sound signals in an ocean environment provides the primary motivation. It has been found that ocean noise parameters such as the ambient level can vary widely without any apparent change in local sea conditions. The purpose of this report is to further the theory of signal detectability for underwater acoustics by considering the detection performance of a number of different receiver designs operating in such an uncertain noise level environment.

The receiver designs considered range from the hypothetical externally sensed parameter (ESP) receiver to a standard cross-correlation type of receiver. The ESP receiver is a non-realizable receiver whose performance serves as an upper bound for the detection of certain signals in noise of uncertain level. The remaining receivers may be classed into essentially three groups. The first group consists of receivers which attempt to incorporate a priori information concerning the uncertain noise level in their design. The (Bayes) optimum receiver and an estimation receiver are in this class. The second group contains receivers which attempt to achieve performance somewhat independently of the noise level uncertainty. The clipper cross-correlation receiver is in this category. The last group considered in this report consists of

receivers which make little or no attempt to include a priori noise level information in their design. The standard cross-correlation receiver is a member of this group.

The design of the receivers investigated here is briefly considered. The optimum receiver for the detection of certain signals in noise of uncertain level is considered from a general formulation independent of the a priori noise level information. The primary emphasis of the report, however, is on the analysis and determination of each receiver's detection performance under the constraint of operation in an uncertain noise level environment. The detection performance is displayed by means of receiver operating characteristic (ROC) curves for each receiver.

The results of this study indicate that channel variability in the form of noise level uncertainty does not seriously affect the upper performance bound attainable at low false alarm probabilities for a conditionally Gaussian process. In addition, for a given measure of channel variability, it is shown that performance can be greatly improved by receiver designs which incorporate the variability in a Bayesian or estimation sense in their design. In fact, for long averaging times the performance of such receivers can approach the upper performance bound. On the other end of the spectrum it is found that receivers such as the clipper cross correlator and the standard cross correlator can suffer significant performance loss when operating under the condition of uncertain noise level.

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LIST OF SYMBOLS

Symbol	Definition
k	number of samples (observation time)
t	time
t_o	observation starting time
T	terminal decision time
$x(t)$	observed process
SN	signal and noise hypothesis
N	noise hypothesis
$s(t)$	signal process
$n(t)$	noise process
$f(A \mid B)$	probability density function of A conditional to B
$\ell(\cdot)$	likelihood ratio of $(\cdot)$
d	detectability index
P(DET)	probability of detection
P(FA)	probability of false alarm
p	precision (recriprocal of noise power level)
$\Gamma(\cdot)$	Gamma function
α	parameter of Gamma distribution
β	parameter of Gamma distribution
s_i	signal process sample
E_s	signal energy

LIST OF SYMBOLS (Cont.)

Symbol	Definition
λ	ratio of E_s and α
d_e	mean value of d
d_v	variance of d
x_i	observation sample
X_k	k-dimensional vector of observation samples
Δ	threshold value
ESP	externally sensed parameter
Z_k	decision statistic of the ESP receiver
C_i	constant
$K_\nu(\cdot)$	Bessel function of imaginary arguments
$\tilde{P}$	space of p values
γ	see Eq. 22
$h(\cdot)$	function of .
$Y_{1,k}$	decision statistic of the optimum receiver
$Y_{2,k}$	decision statistic of the optimum receiver
A_i	constant
U_k	transformed auxiliary random variable
V_k	transformed auxiliary random variable
r_k	transformed auxiliary random variable
t_k	transformed auxiliary random variable

LIST OF SYMBOLS (Cont.)

Symbol	Definition
$\beta(\cdot\,,\cdot)$	beta function
$T(\cdot)$	decision statistic of the estimation receiver
ESTIMATE	estimation receiver
OPTIMUM	optimum receiver
CCCR	clipper cross correlator receiver
m	number of polarity agreements in CCCR
$q_i(a, b)$	probability of polarity agreement conditional to a and b
$\Phi(\cdot)$	normal distribution function
ζ	a threshold value
CCR	cross correlation receiver
LCCR	likelihood cross correlation receiver
Δ'	a threshold value (see Eq. 77)
Δ''	a threshold value (see Eq. 79)

CHAPTER 1

INTRODUCTION

The purpose and major objectives of this report are outlined and briefly discussed in this chapter. In addition, for the reader's convenience, the major conclusions of the study are given in a condensed form.

1.1 Purpose

The purpose of this report is to provide a relatively comprehensive evaluation of the effect of noise level uncertainty on detection performance. The basic problem is that of detecting the presence or absence of a signal in the presence of noise. Throughout the report the signal to be detected is assumed to be known or specified exactly at the receiver input. In other words, to investigate the effect of noise level (or noise power density) uncertainty on performance, we consider only the detection of a certain signal in noise of uncertain level.

The emphasis of the study falls naturally into two distinct areas. The first area is the effect of noise level uncertainty on the upper performance bound. When the noise level is known or certain (and the signal is also certain), the upper performance bound is given by the classical result for the detection of a known signal in specified noise as given by Peterson, Birdsall, and Fox (Ref. 1). When there is uncertainty in the noise level, however, the upper performance bound is changed even though the signal at the receiver input may still be certain.

Changes in the performance bound caused by uncertainty in the noise level can be interpreted as an effect of transmission channel variability on performance. That is, for a variable channel in which the noise level is uncertain, the upper performance bound differs from that of a channel for which there is no variation. In the general case, channel variability is the combination of many factors such as signal uncertainty, multipath transmission characteristics, noise uncertainties, etc. However, in this work channel variability is considered to be affected only by noise level uncertainty.

The second and more important area of emphasis in this report is the effect of receiver design on detection performance for a given amount of channel variability (caused by noise level uncertainty). Here we investigate the performance of a number of different receiver designs operating under the constraint of a channel whose variation is caused by uncertainty in the noise level. The performances of the various receivers are compared with one another and with the ultimate performance obtainable for the given amount of channel variation. The receiver designs considered range from the optimum receiver for the given channel (which incorporates the channel variation in its design) to the simple cross-correlation receiver (which makes no attempt to incorporate the varying characteristics of the channel in its design).

The usefulness of performance comparisons for cost-effectiveness compromises is evident. Since, in general, cost and complexity of equipment rise rapidly as attempts are made to describe and incorporate channel variation in receiver design, it becomes desirable to have available quantitative results which indicate performance changes as a function of receiver design.

1.2 Motivation

The impetus for this work is found in many of the familiar applications of the theory of signal detectability. In these applications, which involve making the best possible decisions as to the presence or absence of a signal in a noisy reception, there are many instances in which parameters (such as the level) of the ambient noise process are uncertain.

This is particularly true in problems involving the detection of underwater sound signals in an ocean environment. For example, experiments conducted off the coast of Miami, Florida in a joint effort (Project MIMI, Ref. 2) by The University of Michigan and The University of Miami have indicated a wide variation in ambient noise levels as a function of the existing environmental conditions. This type of result is elaborated upon and partially explained for many different areas of the oceans by Albers in his book (Ref. 3). It is discussed there that ambient-noise levels can vary greatly without any apparent change in sea conditions. The variations appear to be larger at lower frequencies, especially when man-made disturbances are present. In addition, the standard deviation of levels at high frequencies in deep water, where the Knudsen curves represent the average, is about 4 or 5 db. The deviation is less at high sea states than at low. The causes appear to be varied; however, sea swell does not appear to be significant.

The Knudsen curves referred to above are the results of extensive studies of ambient noise conditions made during World War II. In these curves the spectrum of deep water noise is plotted as a function of sea state and frequency, and the curves are still accepted as representative of average ambient levels at frequencies between 1 and 24 kHz. The importance of the curves in relation to receiver design and signal processing is that they provide a significant amount of a priori knowledge to a receiver operator <u>at the time</u> of receiver use. In other words, for the appropriately designed receiver, these curves provide the operator with in-field modifications which can be applied at the time of use.

1.3 Method of Approach

The method used to investigate the effect of noise level uncertainty follows closely the Bayesian approach as utilized extensively in applied statistics (Ref. 4). The assumption is made that consistent a priori opinions concerning the nature of the signals and noise involved in the detection problem are held by the receiver designer and that, furthermore, these opinions can be expressed in terms of probability distributions. Thus, when we speak of uncertain noise level or channel variability in this report, it is assumed that we can express these quantities in terms of probability distributions. This type of approach seems to be particularly useful considering the discussion of the Knudsen curves since these curves already express in a probabilistic manner uncertainty concerning the ambient noise level of the ocean.

1.4 Organization of Material

The two primary phases of this work utilize the underlying Bayesian approach. In Chapter 2 the basic class of a priori distributions utilized to describe the noise level uncertainty and induced channel variability is defined. This class is discussed in the light of receiver design and is found to be a natural conjugate prior set for the Gaussian noise process under consideration. The implications of the result in terms of optimum receiver design with finite memory are discussed. In addition, the various measures used to define uncertainty and channel variability are discussed in this chapter.

In Chapter 3 a hypothetical receiver, the externally sensed parameter (ESP) receiver, is defined and its operation is discussed. The performance of this receiver is determined and is used in conjunction with the uncertainty measure to determine the effect of channel variability caused by noise level uncertainty on the upper performance bound. These results represent the first objective of this work.

In Chapters 4 through 6 various receiver designs ranging from the optimum receiver to the simple cross-correlation receiver are considered. The performance of each of these receivers is developed for the constraint of operation in a channel with noise level uncertainty. In addition, some of the salient aspects of receiver operation are considered. In Chapter 7 the performances of each of the receivers considered in Chapters 4 through 6 are compared as a function of channel variability, and the results are discussed in relation to the upper performance bound. The conclusions arrived at represent the second objective of this work.

1.5 Conclusions

The major conclusions of this report are discussed in Chapter 7. These can be briefly summarized as follows:

(1) Channel variability in the form of noise level uncertainty does not seriously affect the upper performance bound at low false alarm probabilities for a conditionally Gaussian process.

(2) For a given measure of channel variability, performance

can be greatly improved by receiver designs which incorporate the variation (in a Bayesian or estimation sense) in their design. In fact, for long averaging times the performance of these receivers can approach the upper performance bound.

(3) The clipper cross-correlation receiver, although easy to implement and often used, can suffer significant performance loss in varying channels which variation is caused by noise level uncertainty.

(4) Receiver designs which make no attempt to include channel variability in their design suffer extreme performance loss in uncertain noise level channels.

CHAPTER 2

GENERAL CONSIDERATIONS

In this chapter the fundamental detection problem is discussed. The notation utilized throughout this report is developed and the various philosophies and methods of problem solving are considered. The groundwork for the development and analysis which follow in the later chapters is developed.

2.1 General Problem Statement

The general problem of signal detection is illustrated in Fig. 1. The receiver is presented with an observation $x(t)$ during a time interval $t_0 \leq t \leq T + t_0$. The observation may consist of signal and noise or just noise alone and on the basis of the observation the receiver must make a binary decision as to which of the two conditions is present. The two conditions are mutually exclusive since we assume that the signal is either present during the entire interval or absent during the entire interval. From a decision theory viewpoint this problem may be viewed as requiring the receiver to decide which one of two mutually exclusive hypotheses, signal and noise, or noise alone, occurred during the observation interval. In this work, we assume that the signal is added to the noise so that mathematically the detection problem may be represented as

$$\text{hypothesis SN: } x(t) = s(t) + n(t)$$

$$\text{hypothesis N: } x(t) = n(t)$$

Thus for the given problem statement the receiver must decide which one of the mutually exclusive hypotheses, denoted by SN or N, is present during the observation interval.

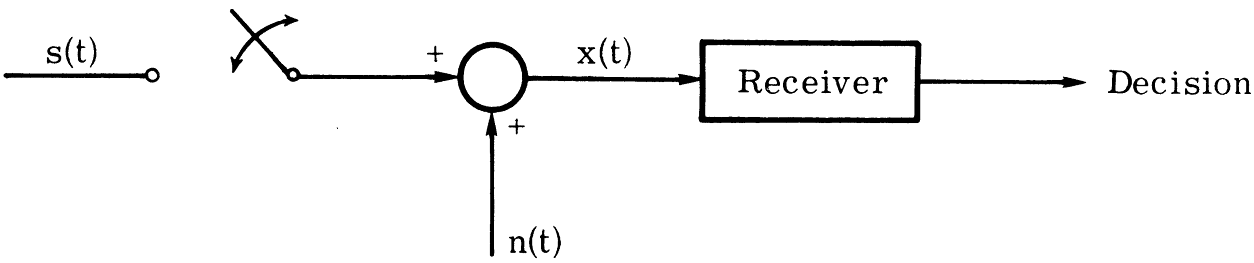

Fig. 1 Illustration of the general detection problem

The classical solution of the detection problem involves the specification of receiver design, the realization of receiver design, and the evaluation of receiver performance. For the situation we have described, two types of solutions are possible. A fixed response time solution in which the receiver is required to make a decision as to which hypothesis is present at a prespecified time $T + t_0$ and an unspecified response time solution in which the time $(T + t_0)$ at which the decision is made remains arbitrary. In general the requirement differences for receiver designs in these two cases lie in the fact that for the case of unspecified response time finite receiver memory considerations require sequential operation of the receiver.

The basic modeling problem required to identify the situation illustrated in Fig. 1 involves specifying in a mathematical sense the various probability distributions associated with the signal-and-noise processes shown in the figure. The specifications should be reasonable approximations to the physical world and at the same time they should allow mathematical tractability. In this work we make the following assumptions for the purposes of modeling the detection problem:

(1) All processes are characterized by 2WT time samples and represented by k-dimensional vectors where $k = 2WT$ with W the bandwidth over which the process is defined, and T the total duration of the observation.

(2) The signal process $s(t)$ is specified or known completely. Thus, the effects of uncertain noise level are studied independently of signal uncertainties.

(3) The noise process $n(t)$ is specified as white Gaussian noise with uncertain level. In other words, conditional to a known level value, the noise samples are normal and independent. (Independence is considered for simplicity since for a known dependence relationship the process may be pre-whitened.) This property is termed conditional independence.

(4) The approach used throughout the work adheres to the Bayesian philosophy. In other words, the assumption is made that consistent a priori opinions concerning the nature of the signals and noise involved in the detection problem are held by the receiver designer and that furthermore, these opinions can be expressed in terms of probability distributions. The modification of these opinions, in the light of evidence gained through observations or otherwise, is made according to Bayes' Rule. [A more comprehensive discussion of the Bayesian approach may be found in the report by Breipolil and Koschmann (Ref. 4) and the paper by Edwards, Lindman, and Savage (Ref. 5).]

2.2 Optimum Receiver Design

In order to consider the specification and comparison of receiver designs which are optimum in some sense, the concept of what is meant by optimum must be included in the work. Since the signal detection problem consists of two mutually exclusive hypotheses, either SN or N is present during the entire observation interval, the choosing of a particular hypothesis by the receiver results in two possible decisions, correct decisions and incorrect decisions. When the actual input consists of signal and noise, these decisions are termed a detection and a miss respectively. Similarly, when the actual input consists of noise alone, there are correct and incorrect decisions.

The situation of responding SN when noise alone is present is generally termed a false alarm. The specification of optimum receiver designs is based on the four possible condition-response pairs. Relative values and costs may be associated with these pairs and for a particular problem, a risk criterion or performance criterion is specified in terms of them and the a priori probabilities. The optimum receiver is then determined with respect to the particular criterion which has been chosen.

In the early 1960's Birdsall (Ref. 6) utilized the methods of statistical decision theory to generalize the proof that optimum receiver design should be specified in terms of the likelihood ratio of the input observation. In this work it was shown that for a wide class of performance criteria, including those described above, receivers which make decisions on the basis of the likelihood ratio of the observation result in optimum performance. In fact, the class of performance criteria for which the likelihood ratio is optimum may generally be thought of as that class for which correct responses are considered "good" and incorrect responses considered "bad" as expressed in terms of any performance goal involving the four possible decisions discussed above.

The likelihood ratio of the input observation $x(t)$ is defined as the ratio of the probability density function of the observation $x(t)$ under the condition that signal and noise are present, to the probability density function of x(t) given that noise alone is present. This function of the input observation x(t) may be expressed as

$$\ell(x) = \frac{f(x|SN)}{f(x|N)}$$

where $\ell(x)$ is the likelihood ratio for x(t), and $f(x|SN)$ and $f(x|N)$ are the two conditional density functions based on the occurrence of the two possible hypotheses, SN or N, respectively. In Chapter 4 of this report the optimum likelihood ratio receiver for the uncertain noise level situation is considered in detail. The receiver design is given, both for sequential and nonsequential operation, and the performance is determined.

In general, suboptimum receiver design is determined on the basis of cost and complexity of equipment, insufficient determination of the detection situation, and numerous additional factors. No matter how the design is achieved, however, we are interested in the resulting performance so we can compare it with the performance of the optimum receiver and with the performance of other suboptimum receivers.

2.3 Receiver Performance Evaluation

In the binary detection problem there are two kinds of errors, a false alarm and a miss, and there are two kinds of correct responses, a detection and a correct rejection. A dependence exists among the four probabilities associated with these responses so that all of the information they contain may be conveyed by considering only two of them. For example, all of the information may be conveyed by a plot of the probability of a detection P(DET) versus the probability of a false alarm P(FA) for all possible decision threshold settings on the output of the receiver. A plot of these two quantities for a given receiver is termed a receiver operating characteristic (ROC) curve and is used in this work to summarize the detection performance of receivers.

An ROC curve is called "normal" if it can be parameterized by the normal probability distribution function. This means that the probability of detection P(DET) and the probability of false alarm P(FA) can be written as

$$P(DET) = \Phi(\xi + d^{\frac{1}{2}}), \text{ when } P(FA) = \Phi(\xi)$$

where

$$\Phi(\xi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\xi} \exp(-\frac{w^2}{2})\, dw$$

and the parameter d is called the normal detectability index.

Normal ROC curves arise whenever the natural logarithm of the likelihood ratio is normally distributed under N and SN with equal variances and means separated by the variance. From the above it follows that an entire normal ROC curve may be characterized by the detectability parameter d. This parameter represents the effective signal-to-noise ratio of the process. Many more of the fundamental properties of ROC curves have been considered in detail by Birdsall (Ref. 7).

In general, the evaluation of the performance of a detection receiver requires the distribution of the detection statistic, or some monotonic function of it, under both signal mixed with noise and noise alone. These distributions are used to determine, as a function of the threshold setting, the P(DET) and P(FA) values needed to plot the ROC curve. For example, if the probability densities under the two hypotheses are those shown in Fig. 2, then for the threshold value shown, P(DET) is given by the striped area and P(FA) is given by the cross-hatched area. The total ROC curve is determined as a function of all possible values of the threshold setting. In practice, the determination of the distribution functions of the detection statistic in an analytic form may be considerably difficult. One is usually able to specify the functions in general, however, the evaluation of the integrals involved frequently becomes difficult. At this point a digital computer is usually used.

Another approach to the evaluation of receiver performance is given by the use of digital computer simulation techniques. For this experimental approach the given detection situation, including receiver

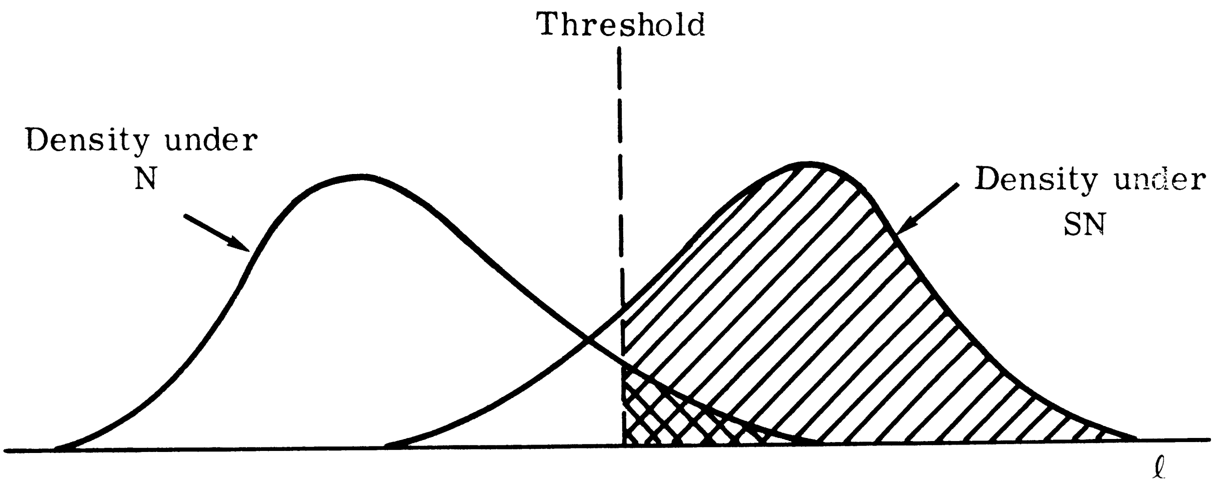

Fig. 2 Probability density functions of the detection statistic under SN and N

operations, is simulated on the digital computer and the signal and noise and noise alone density functions are sampled. The simulation approach, although an approximate one, has been found to provide extremely good accuracy with the results limited only by the number of computer runs feasible. Throughout this report receiver performance is evaluated utilizing both the analytical and computer simulation approaches.

2.4 Noise Level Uncertainty

In this work we are applying the general theory of signal detectability to the specific problem of the detection of a certain signal in noise of uncertain level. We are trying to determine the effect of noise level uncertainty on the ultimate performance bound, and to determine the effect of this uncertainty on the performance of a number of different receivers.

2.4.1 A Priori Distribution. In previous discussions it was stated that the Bayesian approach is followed in our work. Thus, a priori opinions concerning the uncertain noise level are expressed in terms of probability distribution functions. Here we actually choose to work with the reciprocal of the noise power level, and we call this quantity the precision or precision level and denote it by the symbol p.

The probability distribution function used to describe the variation in the precision level and used for the performance evaluation of all of the receivers considered in the succeeding chapters is the Gamma distribution function. The density associated with this distribution function is given below where α and β are arbitrary parameters.

$$g(p) = [\alpha^2 / \Gamma(\beta)] \; p^{\beta-1} \exp(-\alpha p), \quad 0 \leq p \leq \infty \qquad (1)$$

A plot of the Gamma density function is given in Fig. 3.

Three important and well-founded reasons for choosing the Gamma distribution function are the following:

(1) The Gamma distribution is a natural conjugate prior distribution for the conditionally normal observations we are considering and this result insures some mathematical tractability (Ref. 8). This statement is verified in Chapter 4.

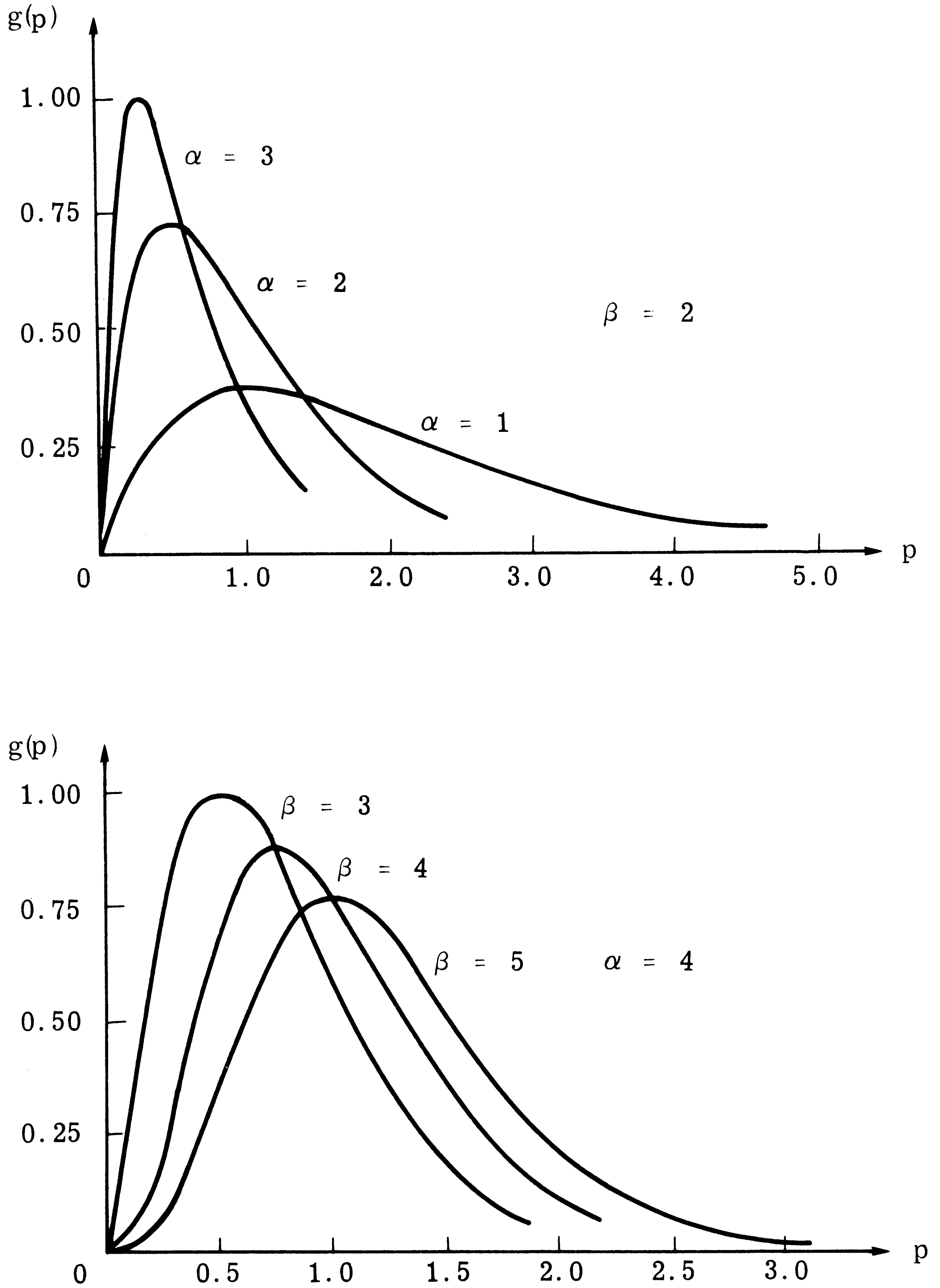

Fig. 3 The Gamma probability density function parameterized by α and β

(2) The observation processing operations of the optimum receiver are not influenced by the choice of a different distribution function since the Gamma distribution is a natural conjugate prior. Indeed, only the manner in which the processed variables are combined is altered by the choice of a different distribution function. This fact is considered in detail in Chapter 4.

(3) The two parameters α and β associated with the Gamma distribution allow a wide range of uncertainty to be modeled. This is illustrated in Fig. 3.

The mean and variance of the Gamma distribution on p can be expressed in terms of the α and β parameters. These quantities are given by

$$\begin{aligned} \text{mean} &= \beta/\alpha \\ \text{variance} &= \beta/\alpha^2 \end{aligned} \tag{2}$$

<u>2. 4. 2 Channel Variability</u>. Throughout this report the signal waveshape is considered as known or certain. Thus, if we define the signal energy of the sampled signal vector as

$$E_s = \sum_1^k s_i^2 \tag{3}$$

then for a given value of the precision p the detectability d or effective signal-to-noise ratio (Sec. 2. 3) can be expressed as

$$d = E_s p \tag{4}$$

Since p is a random variable, d is a random variable. Using the Gamma distribution for p we have the following density function for the detectability or signal-to-noise ratio.

$$g(d) = C_o \, d^{\beta-1} \exp(-\alpha d/E_s), \quad 0 \leq d \leq \infty \tag{5}$$

where

$$C_o = \left(\frac{\alpha}{E_s}\right)^{\beta} [\Gamma(\beta)]^{-1}$$

If we define

$$\lambda = \alpha/E_s \tag{6}$$

then Eq. 5 can be written as

$$g(d) = \frac{\lambda^{\beta}}{\Gamma(\beta)} \, d^{\beta-1} \exp(-\lambda d), \qquad 0 \leq d \leq \infty \tag{7}$$

If we denote the expected value and the variance of d by

$$\begin{aligned} \text{expected value} &= d_e \\ \text{variance} &= d_v \end{aligned} \tag{8}$$

then from Eq. 7 these quantities are given by

$$
\begin{aligned}
d_e &= \beta / \lambda \\
d_v &= \beta / \lambda^2
\end{aligned}
\tag{9}
$$

One objective of this study is to investigate the upper performance bound as a function of channel variability. An objective of this type implies that one must decide upon or define a measure of this quantity. In this report channel variability is measured by the variance of d, d_v, for a given expected value, d_e. This is a reasonable method of measurement since in essence we are measuring the spread or uncertainty in the signal-to-noise ratio of the channel. When d_v is zero, there is no channel variability, and as d_v approaches infinity, the variation in the channel (signal-to-noise ratio) also approaches infinity.

The use of d_v as the channel variability measure was arrived at after investigation of a number of other methods. For example, the Lindley-Shannon information measure (Ref. 9) and the Fisher information measure (Ref. 10) were considered. However, after investigation the use of d_v appeared to be the simplest and most direct measure and appeared to provide as much information as the others for the detection problem we are considering.

2.5 Summary

In this chapter the basic detection problem considered in this report has been outlined and much of the notation used in the solution presented. In addition, the basic approach and assumptions to be used have been discussed. Also, the distributions for the noise level and the measure of channel variability have been given. In the succeeding chapters the various receiver designs will be considered.

CHAPTER 3

THE ESP RECEIVER

In this chapter the externally sensed parameter (ESP) receiver for the case of detecting a certain signal in noise of uncertain level is considered. The performance of the ESP receiver is determined for this case and is used to determine the effect of channel variability caused by noise level uncertainty on the upper performance bound.

3.1 The ESP Receiver Definition

The method used in this report to determine the effect of channel variability on the upper performance bound is to use the performance of a hypothetical receiver termed the externally sensed parameter or ESP receiver for comparison purposes. The ESP receiver is defined as a receiver that functions in a random parameter (noise level) environment and yet, for each possible trial run, first, has perfect or certain knowledge of the particular parameter value in question and second, is optimum for that parameter value. In other words, one can think of the ESP receiver as having an associated teacher which allows it to determine the exact value of the random or uncertain parameter in question prior to a sample run. This is in contrast to a realizable receiver which has only information concerning the distribution of the random parameter prior to each sample run.

Thus, for the uncertain noise level case, the ESP receiver knows prior to the start of a detection trial the value of the noise level present during the trial. Realizable receivers on the other hand are only aware of the a priori probability of a particular noise level value occurring during the run. The ESP receiver therefore has perfect parameter information for the parameter in question even though the parameter is a distributed random variable. Note that the addition of perfect parameter information implies that the performance of the ESP receiver is at least as good and usually better than the performance of the optimum Bayesian receiver for the case in question. In addition, its performance will usually, but not always, be less than that of the optimum receiver for the case in which the parameter is known exactly (non-distributed).

The usefulness of the ESP receiver for performance comparison purposes is twofold. First, even though it functions with perfect parameter information on each trial run, its average performance is independent of the particular uncertain parameter in question. This follows since the parameter is still treated as a random variable with a known a priori distribution and in the determination of average performance it is "averaged out." Thus, one is able to compare receivers which have an average performance independent of any particular value of the uncertain parameter with the appropriate ESP receiver performance. Second, the ESP receiver performance provides an upper bound on performance with respect to uncertainty in a particular parameter value such as noise level. In other words, given a distributed parameter over which one has no control, one can do no better than to know its value on any given trial run. This is exactly the operation of the ESP receiver.

3.2 ESP Performance Computations

In general there are a number of methods which can be used to determine ESP receiver performance for detecting a certain signal in noise of uncertain level. For example, the techniques of ROC characters as developed by Birdsall (Ref. 7) can be employed or the methods of Monte Carlo simulation can be used. In addition, the direct method using the definition of the ESP receiver can be applied. For this latter method the performance of the ESP receiver is computed by first considering the uncertain noise level as a known quantity and then computing the conditional performance as a function of its value. This conditional performance, which is optimum for each value of noise level, is then averaged with respect to the noise level a priori distribution to arrive at the average ESP receiver performance.

For a certain or known value of noise level the optimum receiver forms the likelihood ratio of the input observation $X_k = (x_1 \ldots x_k)$. In terms of the notation defined in Chapter 2 this ratio can be expressed as

$$\ell(X_k) = \frac{(2\pi)^{-\frac{1}{2}} p^{k/2} \exp\left[-\frac{1}{2}\sum_1^k (x_i - s)^2 p\right]}{(2\pi)^{-\frac{1}{2}} p^{k/2} \exp\left(-\frac{1}{2}\sum_1^k x_i^2 p\right)} \tag{10}$$

In this expression we have assumed that the noise process is Gaussian and that the samples are independent conditional to a known value of the precision level p .

If we reduce Eq. 10 to its simplest form and take the logarithm of the result, we have

$$\ell n \ell(X_k) = \left(\sum_1^k x_i s_i - \frac{1}{2} E_s\right) p \tag{11}$$

This latter equation reduces to the following expression

$$\ell n \ell(X_k) = Z_k \tag{12}$$

when we define

$$Z_k = \left(\sum_1^k x_i s_i - \frac{1}{2} E_s\right) p \tag{13}$$

Conditional to the occurrence of the noise hypothesis and a particular value of the precision p, it is easily shown that the random variable Z_k is normal (since it is the sum of independent normal random variables) with mean $-\frac{1}{2} E_s p$ and variance $E_s p$. Conditional to the hypothesis signal and noise and a known value of p, one can show that Z_k is normal with mean $\frac{1}{2}E_s p$ and variance $E_s p$.

The logarithm is a monotone function of its argument, and so the optimum decision action of the ESP receiver is to compare the expression given in Eq. 11 or 12 with a threshold. Thus, to determine the probabilities of detection and false alarm for a particular value of p, we must determine the probabilities that the logarithm of the likelihood ratio is greater than the given threshold conditional to the hypothesis signal and noise, and noise alone for the known value of p. We can express these probabilities symbolically as

$$P\left[\begin{matrix}\text{DET}\\ \\ \text{FA}\end{matrix}\middle| p\right]_{\text{ESP}} = P\left[\ell n \ell(X_k) \geq \Delta \middle| p, \begin{matrix}\text{SN}\\ \\ \text{N}\end{matrix}\right] \tag{14*}$$

where the symbol Δ denotes the threshold value. Substituting in Eq. 14 we have

$$P\left[\begin{matrix}\text{DET}\\ \\ \text{FA}\end{matrix}\middle| p\right]_{\text{ESP}} = P\left[Z_k \geq \Delta \middle| p, \begin{matrix}\text{SN}\\ \\ \text{N}\end{matrix}\right] \tag{15}$$

---

*This style of "equation pair" notation is unusual but extremely convenient in two hypothesis work. Equation 14 should be read as

$$P[\text{DET} \mid p]_{\text{ESP}} = P[\ell n \ell(X_k) \geq \Delta \mid p, \text{SN}] \tag{14 upper}$$

and

$$P[\text{FA} \mid p]_{\text{ESP}} = P[\ell n \ell(X_k) \geq \Delta \mid p, \text{N}] \tag{14 lower}$$

Using the results above which state that the conditional distributions of Z_k are normal, Eq. 15 can be expressed as

$$P\begin{bmatrix} \mathrm{DET} \\ \mathrm{FA} \end{bmatrix} p \Bigg]_{ESP} = \int_{\Delta}^{\infty} (2\pi E_s p)^{-\frac{1}{2}} \exp\left[-(v \mp \tfrac{1}{2}E_s p)^2 / 2E_s p\right] dv \tag{16*}$$

where we have used the appropriate mean and variance values.

To obtain the average performance of the ESP receiver Eq. 16 must be averaged with respect to the a priori distribution of the precision p. Denoting this distribution by g(p) as in Chapter 2 we have

$$P\begin{bmatrix} \mathrm{DET} \\ \mathrm{FA} \end{bmatrix}_{ESP} = \int_P \left\{ \int_{\Delta}^{\infty} (2\pi E_s p)^{-\frac{1}{2}} \exp\left[-(v \mp \tfrac{1}{2}E_s p)^2 / 2E_s p\right] dv \right\} g(p)\, dp \tag{17}$$

In Chapter 2 we discussed the fact that the simplest form for the natural conjugate prior distribution for the precision p is given by the Gamma distribution. If we use this form for g(p) (see Eq. 1) in Eq. 17 and interchange the order of integration (this is permissible since we are dealing with density functions), we obtain

---

*Equation 16 should be read

$$P[\mathrm{DET} \mid p]_{ESP} = \int_{\Delta}^{\infty} (2\pi E_s p)^{-\frac{1}{2}} \exp\left[-(v - \tfrac{1}{2}E_s p)^2/2E_s p\right] dv \quad \text{(16 upper)}$$

$$P[\mathrm{FA} \mid p]_{ESP} = \int_{\Delta}^{\infty} (2\pi E_s p)^{-\frac{1}{2}} \exp\left[-(v + \tfrac{1}{2}E_s p)^2/2E_s p\right] dv \quad \text{(16 lower)}$$

$$P\begin{bmatrix}\mathrm{DET}\\ \mathrm{FA}\end{bmatrix}_{\mathrm{ESP}}$$

$$= C_1 \int_{\Delta}^{\infty} \left\{ \int_0^{\infty} E_s^{-\frac{1}{2}} p^{\beta-\frac{3}{2}} \exp\left[-\left(\left[v \mp \tfrac{1}{2}E_s p\right]^2/2E_s p\right) - \alpha p\right] dp \right\} dv \tag{18}$$

where

$$C_1 = (2\pi)^{-\frac{1}{2}} \alpha^{\beta} \Gamma(\beta)$$

Integrating Eq. 18 we obtain the following expression for the probabilities of detection and false alarm of the ESP receiver for uncertain noise level (Ref. 11):

$$P\begin{bmatrix}\mathrm{DET}\\ \mathrm{FA}\end{bmatrix}_{\mathrm{ESP}} = C_2 \int_{\Delta}^{\infty} |v|^{\beta-\frac{1}{2}} e^{\pm v/2} K_{\beta-\frac{1}{2}}\left(\tfrac{1}{2}\sqrt{1+\frac{8\alpha}{E_s}}\ |v|\right) dv \tag{19}$$

where K_{ν} is the Bessel function of imaginary arguments given by

$$K_{\nu}(y) = \frac{(y/2)^{\nu}\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_1^{\infty} e^{-yt}(t^2+1)^{\nu-\frac{1}{2}}\, dt$$

and

$$C_2 = \frac{(2\alpha/E_s)^{\beta}}{\sqrt{\pi}\,\Gamma(\beta)} \left(1+\frac{8\alpha}{E_s}\right)^{-(\beta-\frac{1}{2})/2}$$

In Chapter 2 the concept of channel variability was considered in terms of the uncertainty or distribution function of the measure d, the detectability or signal-to-noise ratio. The distribution of this quantity was found (for the certain signal case) to be expressible in terms of the power β and the quantity

$$\lambda = \alpha / E_s \tag{20}$$

If we substitute Eq. 20 in Eq. 19, then Eq. 19 can be written in a form more applicable to our definition of channel variability.

$$P \begin{bmatrix} \text{DET} \\ \text{FA} \end{bmatrix}_{\text{ESP}} = C_3 \int_{\Delta}^{\infty} |v|^{\beta - \frac{1}{2}} e^{\pm v/2} K_{\beta - \frac{1}{2}} \left(\tfrac{1}{2} \sqrt{1 + 8\lambda}\, |v| \right) dv \tag{21}$$

where

$$C_3 = \frac{(2\lambda)^{\beta}}{\sqrt{\pi}\,\Gamma(\beta)} (1 + 8\lambda)^{-(\beta - \frac{1}{2})/2}$$

Since we have defined channel variability in terms of the parameters λ and β, we can use Eq. 21 to determine the effect of channel variability caused by noise level uncertainty on the upper detection performance bound.

The determination of the ROC curves for the ESP receiver appears to be a somewhat formidable task when one first considers Eq. 21. However, for the rather general case of integer values of β all integrations can be performed and Eq. 21 reduces to the more convenient computations below.

$$P\begin{bmatrix}\mathrm{DET}\\ \\ \mathrm{FA}\end{bmatrix}_{\mathrm{ESP}} = C_4(\Delta,\gamma,\beta)\exp\left[-(\gamma \mp \tfrac{1}{2})\Delta\right], \qquad \begin{array}{l}\Delta \geq 0\\ \beta \text{ integer}\end{array} \tag{22}$$

where

$$C_4(\Delta,\gamma,\beta) = C_5\left[\sum_{\ell=0}^{2(\beta-1)} (\gamma \mp \tfrac{1}{2})^{-(\ell+1)}\right.$$

$$\left.\left(\sum_{r=0}^{2(\beta-1)-\ell} \frac{[2(\beta-1)-r]\,!}{[2(\beta-1)-r-\ell]\,!} \cdot \alpha_r \gamma^{2(\beta-1)-r} \Delta^{2(\beta-1)-r-\ell}\right)\right]$$

with

$$\alpha_r = \sum_{m=0}^{(\beta-1)-r/2} (-1)^m \binom{\beta-1}{m} \frac{[2(\beta-1)-2m]\,!}{[2(\beta-1)-2m-r]\,!}$$

and

$$C_5 = \frac{(\frac{1}{2})^{\beta-1}}{[\Gamma(\beta)]^2 \lambda^\beta \gamma^{2\beta-1}}, \qquad \gamma = \tfrac{1}{2}\sqrt{1+8\lambda}$$

The derivation of Eq. 22 from Eq. 21 is straightforward since for integer values of β all integrations can be performed and Eq. 22 is obtained by simply reordering the terms in a power series. The ESP performance curves are easily shown to be symmetric about $\Delta = 0$. This result can be used with Eq. 22 to obtain the performance for $\Delta < 0$.

3.3 ESP Performance Comparison

In this section we wish to present the ESP curves for the uncertain noise level case. The curves were obtained from Eqs. 21 and 22 using the digital computer. The purpose in presenting the curves is to investigate the effect of noise level uncertainty on the upper performance bound.

When there is no uncertainty in the noise level, the upper performance bound is given by normal ROC curves, the result of the problem of detecting a certain signal in certain noise (Ref. 7). This normal ROC curve is parameterized by the existing detectability value or the effective signal-to-noise ratio d and so it depends on a particular value of the noise level. On the other hand, the ESP performance for the uncertain noise level case is independent of the noise level and therefore comparison of this performance with the certain noise level performance is difficult. The method used here, as outlined in Chapter 2, is to look at ESP performance for constant values of expected detectability d_e as a function of the variance in the detectability d_v. In conjunction with these curves, the certain signal-certain noise level normal ROC curves with appropriate values of $d = d_e$ are displayed.

The ROC curves are given in Fig. 4. The solid curves are the ESP performance curves and the dashed curves are the normal curves with the appropriate value of detectability, i.e., $d = d_e$. In addition, in Fig. 5 the various (Gamma) distributions of the detectability d

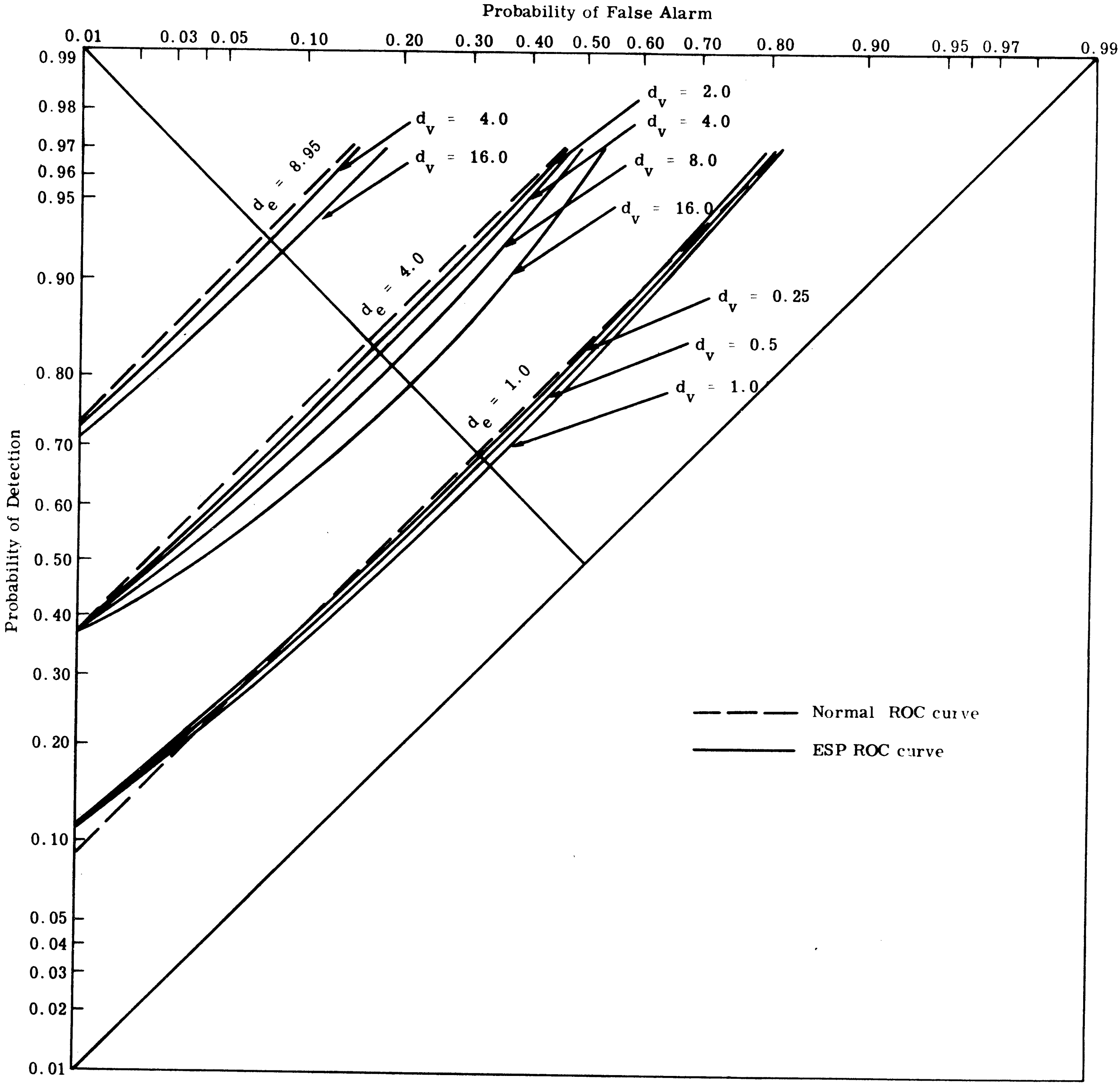

Fig. 4 Comparison of the ROC curves for the uncertain noise level ESP receiver and the corresponding normal ROC curves

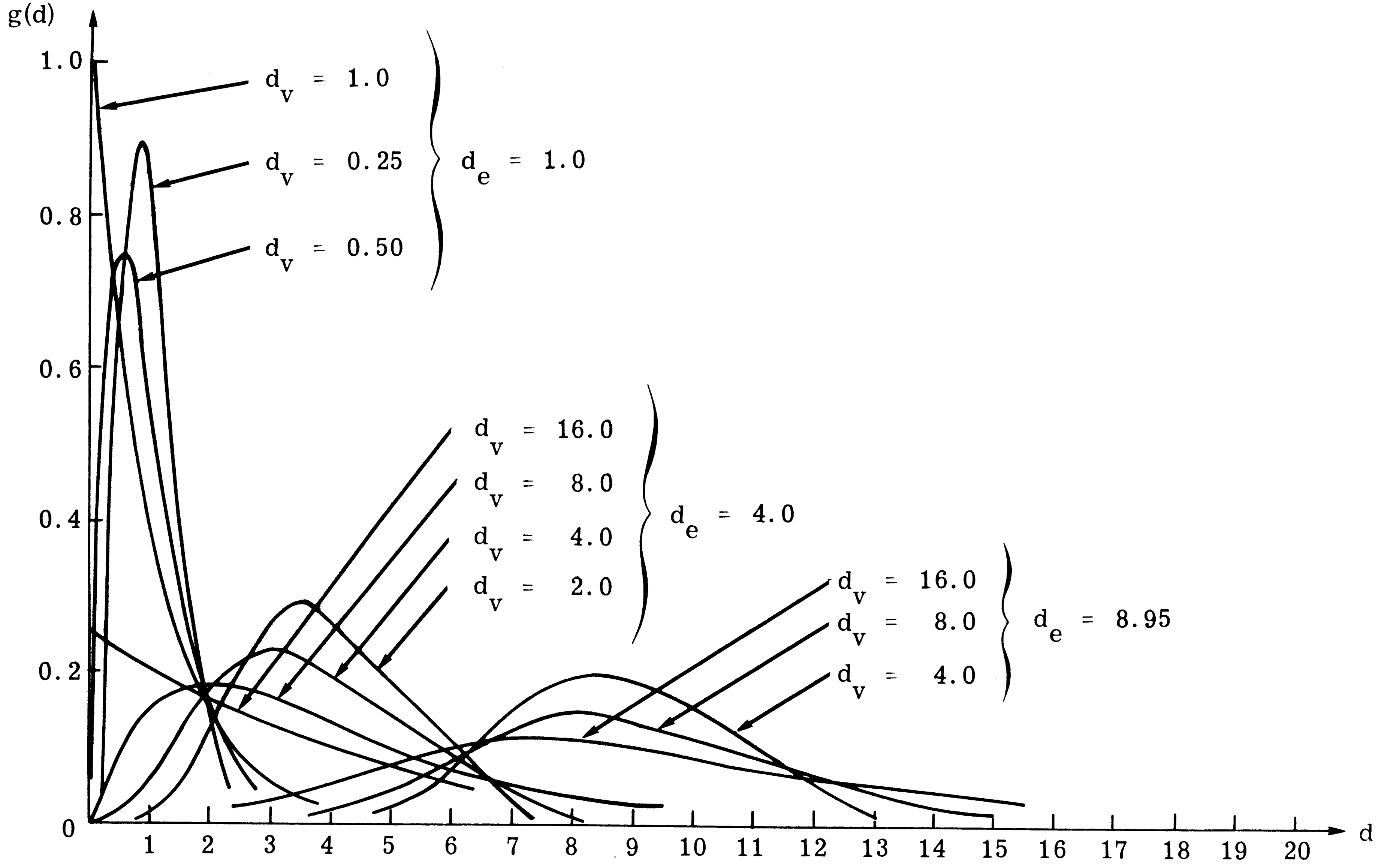

Fig. 5 The distributions of the detectability d parameterized by the expected value d_e and the variance d_v

used for the ESP curves in Fig. 4 are given. In Fig. 4, the ESP curves indicate that an increase in the channel variation (as measured by d_v) causes a decrease in the upper performance bound for the uncertain noise level situation. Thus, for highly variable channels in which the variability is caused by noise level uncertainty the maximum performance achievable is reduced when compared to less variable channels.

By comparing the ESP and normal curves with $d = d_e$ we can obtain some idea of the total effect of channel variability on the upper performance bound. The normal curves represent the case of $d_v = 0$, in other words no channel variation for the given value of d_e. In Fig. 4 it is apparent that the ROC curves for the same value of d_e are reasonably clustered about the equivalent normal ROC curve for a wide range of d_v values. This is especially true at lower values of signal-to-noise ratio and at lower false alarm probabilities. Thus, we have the result that <u>channel variability in the form of noise level uncertainty does not seriously affect the upper performance bound at low probabilities for a conditionally Gaussian process.</u>

CHAPTER 4

OPTIMUM AND ESTIMATION RECEIVERS

In this chapter two receivers that incorporate channel variability caused by noise level uncertainty in their design are considered. The first receiver is the optimum (Bayes) receiver for the certain signal uncertain noise level case. The second receiver is an estimation type receiver for this same case. The design and performance of these two receivers are considered.

4.1 The Optimum Bayes Receiver

4.1.1 Receiver Design. As discussed in Chapter 2 the optimum (Bayes) receiver (OPTIMUM) for a given detection problem is a receiver which forms the likelihood ratio as the test statistic and then compares this quantity with an appropriate threshold. In this section the optimum receiver design for the certain signal uncertain noise level case is considered. The work is largely a condensation of previous work performed by the Author (Ref. 12), however, some new concepts are included here.

For situations in which the noise level is uncertain each hypothesis, signal and noise and noise alone, is composite. For these double composite situations the likelihood ratio of the observation $X_k = (x_1, \ldots x_k)$ assumes the form

$$\ell(X_k) = \frac{\int_{\widetilde{P}} f(X_k | p, SN)\, g(p) dp}{\int_{\widetilde{P}} f(X_k | p, N)\, g(p) dp} \tag{23}$$

where g(p) is the a priori density of the random precision p. Note that Eq. 23 is a ratio of integrals and not an integral of a ratio. This implies that the resulting optimum receiver will have the form of two channels or branches, each branch acting on a single hypothesis.

In this work we are considering a noise process which is conditionally Gaussian as discussed in Chapter 2. Incorporating this fact in Eq. 23 we have the following form for the likelihood ratio:

$$\ell(X_k) = \frac{\int_{\widetilde{P}} (p/2\pi)^{k/2} \exp\left[-\frac{1}{2} \sum_{1}^{k} (x_i - s_i)^2 p \right] g(p) dp}{\int_{\widetilde{P}} (p/2\pi)^{k/2} \exp\left[-\frac{1}{2} \sum_{1}^{k} x_i^2 p \right] g(p) dp} \tag{24}$$

In Chapter 2 we discussed the fact that the simplest form for the natural conjugate prior distribution for the precision p is given by Gamma distribution. It was also mentioned that the use of this type of distribution leads to an optimum receiver which requires a finite memory when used in a time-sequential manner. We wish to validate these two statements.

Let us consider an a priori distribution for the precision p that is of the form of a Gamma distribution multiplied by an arbitrary function of p, $h(p)$, such that the product remains a valid probability density function.

$$g(p) = h(p)p^{\beta-1} e^{-\alpha p} , \quad 0 \leq p \leq \infty \tag{25}$$

Using this form of $g(p)$ in Eq. 24 we have

$$\ell(X_k) = \frac{\int_0^\infty h(p)\, p^{\beta-1+k/2} \exp\left\{-\left[\frac{1}{2}\sum_1^k (x_i - s_i)^2 + \alpha\right] p\right\} dp}{\int_0^\infty h(p)\, p^{\beta-1+k/2} \exp\left[-\left(\frac{1}{2}\sum_1^k x_i^2 + \alpha\right) p\right] dp} . \tag{26}$$

If we define the following two random variables

$$Y_{1,k} = \frac{1}{2}\sum_1^k (x_i - s_i)^2$$

and (27)

$$Y_{2,k} = \frac{1}{2}\sum_1^k x_i^2$$

then we can substitute in Eq. 26 to obtain the expression below for the likelihood ratio.

$$\ell(X_k) = \frac{\int_0^\infty h(p)\, p^{\beta-1+k/2} \exp\left[-(Y_{1,k} + \alpha)p\right] dp}{\int_0^\infty h(p)\, p^{\beta-1+k/2} \exp\left[-(Y_{2,k} + \alpha)p\right] dp} \qquad (28)$$

From Eq. 28 it is evident that $Y_{1,k}$ and $Y_{2,k}$ are the statistics formed from the observations by the optimum receiver. This is true independent of the actual form of $g(p)$ resulting from the form of the modifier $h(p)$. The modifier $h(p)$ only changes the manner in which the two statistics $Y_{1,k}$ and $Y_{2,k}$ are combined to form the likelihood ratio; it does not change or have any influence on the statistics that are formed. From this result we have the following design procedure for the optimum receiver: (1) compute the statistics $Y_{1,k}$ and $Y_{2,k}$, (2) combine $Y_{1,k}$ and $Y_{2,k}$ in the manner prescribed by Eq. 28 in conjunction with the appropriate modifier $h(p)$, and (3) compare the resulting statistic with a predetermined threshold. A receiver diagram indicating these operations is shown in Fig. 6. The receiver design given in this diagram is for time-sequential operation (see below).

The decomposition of the optimum receiver into the steps of computation, combination, and comparison has been discussed and elaborated upon by Birdsall in conjunction with adaptive optimum receiver design (Ref. 13). Our main concern with this fact is that it enables us to see that the simplest form for the a priori density for the precision p is given by the Gamma distribution.

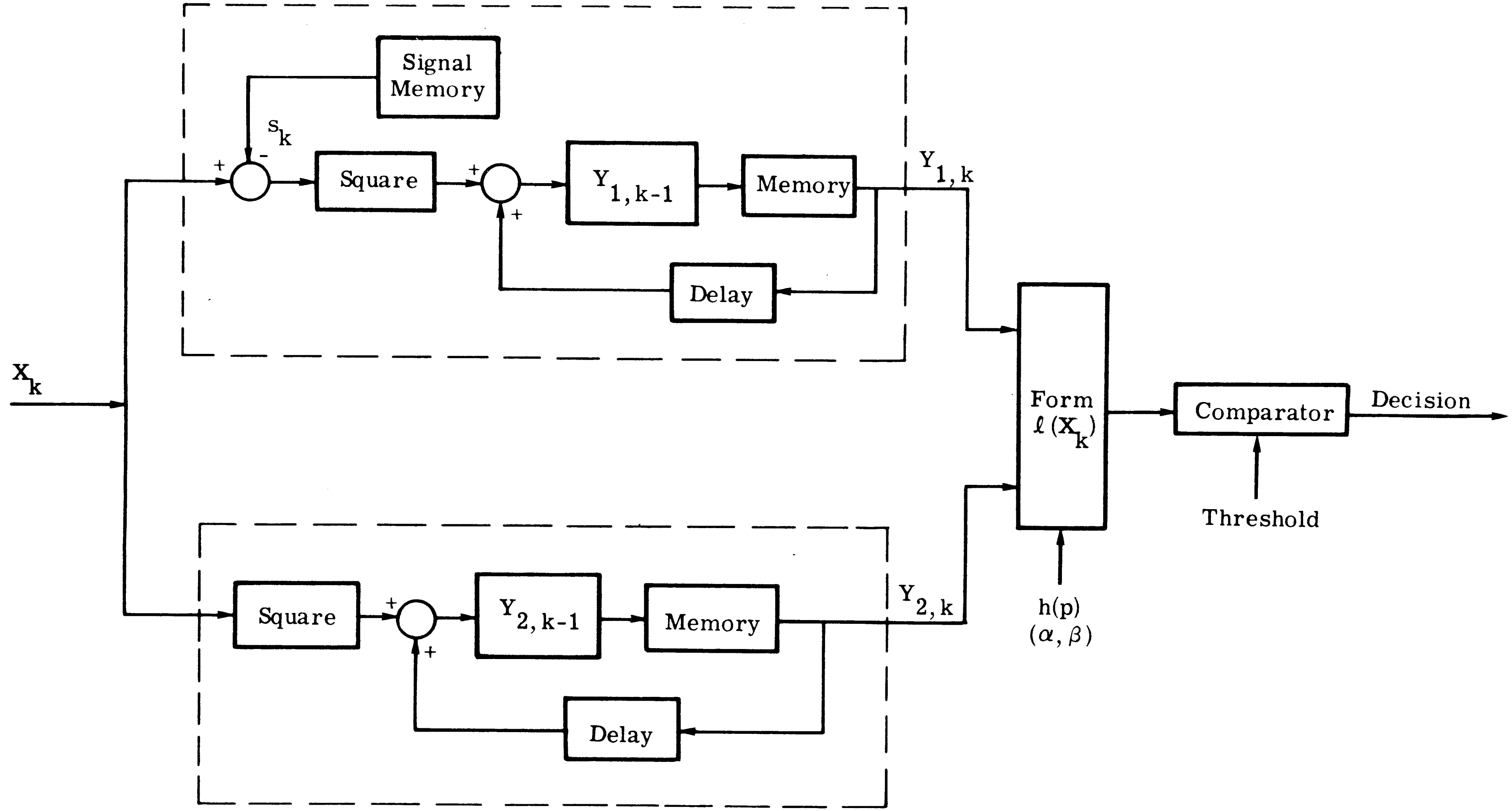

Fig. 6 A sequential realization of the generalized optimum (Bayes) receiver for the detection of a certain signal in noise of uncertain level

In addition it is evident that the use of a distribution of this type results in a receiver which requires a finite memory when used in a time-sequential mode. This follows since the two test statistics required can be written sequentially as

$$Y_{1,k} = Y_{1,k-1} + \tfrac{1}{2}(x_i - s_i)^2$$

and (29)

$$Y_{2,k} = Y_{2,k-1} + \tfrac{1}{2}x_i^2$$

From Eq. 29 it follows that a current value of the particular test statistic can be realized from the previous value and the current observation, the result determining a sequential mode of operation. The memory requirement utilized by this sequential mode is obviously finite since we need store only the current values of the statistics $Y_{1,k}$ and $Y_{2,k}$.

If we use the Gamma distribution for the precision p, then Eq. 28 becomes

$$\ell(X_k) = \frac{\int_0^{\infty} p^{\beta-1+k/2} \exp\left[-(Y_{1,k} + \alpha)p\right] dp}{\int_0^{\infty} p^{\beta-1+k/2} \exp\left[-(Y_{2,k} + \alpha)p\right] dp} \tag{30}$$

Performing the integrals in this latter expression we have the following result

$$\ell(X_k) = \left(\frac{Y_{2,k} + \alpha}{Y_{1,k} + \alpha}\right)^{\beta + k/2} \tag{31}$$

The optimum receiver design indicated by the operations given in Eq. 31 can be realized by the operations of square law detection and cross-correlation between the received observation and the known signal waveshape. A sequential receiver of this type is illustrated in Fig. 7. From this result we conclude that the optimum receiver design for uncertain noise level includes an energy measurement $(Y_{1,k})$ as well as the usual correlation measurement $(Y_{2,k})$. In a sense, one can visualize the energy measurement as an estimation of the noise level or an adaptation of the preset threshold value. These facts will be more evident in a later section of this chapter.

One additional point should be mentioned concerning the (Bayes) optimum receiver for uncertain noise level. In Chapter 1 we discussed the use of such items as the Knudsen curves for providing a priori knowledge at the time of receiver use. We can now see how this knowledge is incorporated in the optimum receiver design. For the case of the receiver shown in Fig. 7, the design is based on the Gamma distribution and the Knudsen curves could be used to establish appropriate values of the parameters α and β at the time of receiver use. For the more general receiver illustrated in Fig. 6 the curves could be used

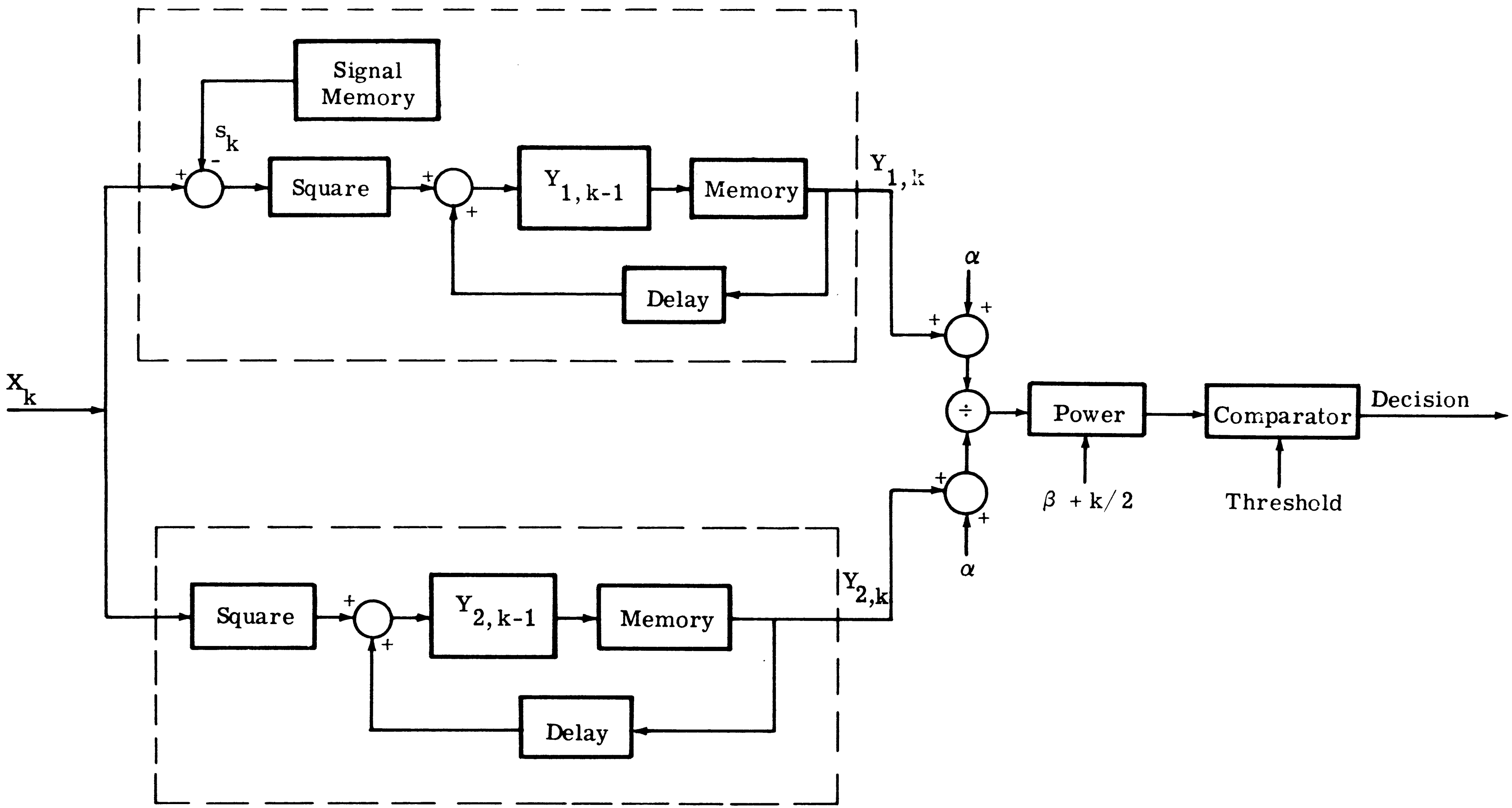

Fig. 7 A sequential realization of the optimum (Bayes) receiver for the detection of a certain signal in noise of uncertain level with a Gamma distribution

not only to establish values for α and β but also to determine any parameters occurring in the chosen expression for $h(p)$. The primary point is that the optimum receiver allows, even requires, the inclusion of a priori knowledge (no matter how diffuse) at the time of receiver use.

4.1.2 Receiver Performance. To determine the performance of the optimum receiver we must determine the probabilities of detection and false alarm for a given threshold value. Since no direct method for doing this has been found, we begin by first considering the following transformation:

$$
\begin{aligned}
U_k &= \frac{1}{E_s} \sum_1^k x_i s_i \\
V_k &= \sum_1^k (x_i - s_i U_k)^2 = \sum_1^k x_i^2 - E_s U_k^2
\end{aligned}
\tag{32}
$$

For a given or fixed value of the precision p, the transformed variables U_k and V_k act very much like the mean and variance of a normal or Gaussian sample. It is easy to show, using the results of Cramér (Ref. 14), that for a given value of p, U_k and V_k are independent and that U_k is normal while V_k is chi-square with $k-1$ degrees of freedom. Thus, in the notation of Chapter 2,

$$f\left(U_k, V_k \,\middle|\, p, \begin{matrix} SN \\ N \end{matrix}\right)$$

$$= A_1 \, p^{k/2} \, V_k^{(k-3)/2} \exp\left[- \left(U_k - \begin{matrix} 1 \\ 0 \end{matrix}\right)^2 E_s p/2 - V_k p/2 \right] \tag{33}$$

where

$$A_1 = \left[E_s / 2\pi\right]^{\frac{1}{2}} \Big/ 2^{\frac{k-1}{2}} \, \Gamma\left(\frac{k-1}{2}\right)$$

To determine the joint distribution of U_k and V_k conditional to only the hypotheses SN and N, Eq. 33 must be averaged over the a priori distribution of p. Using the Gamma distribution for p, Eq. 33 becomes

$$f\left(U_k, V_k \,\middle|\, \begin{matrix} SN \\ N \end{matrix}\right) = A_2 \, V_k^{\frac{k-3}{2}} \left[V_k + \left(U_k - \begin{matrix} 1 \\ 0 \end{matrix}\right)^2 E_s / 2 + 2\alpha \right]^{-\beta - k/2} \tag{34}$$

where

$$A_2 = \left[E_s / 2\pi\right]^{\frac{1}{2}} \alpha^{\beta} \, \Gamma(\beta + k/2) / 2^{(k-3)/2} \, \Gamma\left(\frac{k-1}{2}\right) \Gamma(\beta)$$

Note that U_k and V_k are no longer independent. For the evaluation procedure which follows, it becomes extremely convenient to have independent random variables. In this light we consider the following transformation,

$$
\begin{aligned}
r_k &= 2\alpha(V_k + 2\alpha)^{-1} \\
t_k &= E_s\left(U_k - \frac{1}{0}\right)^2 \left[V_k + E_s\left(U_k - \frac{1}{0}\right)^2 + 2\alpha\right]^{-1}
\end{aligned}
\tag{35}
$$

We now use the rules of probability theory to find the distribution of r_k and t_k. From Eqs. 34 and 35 we have

$$
f\left(r_k, t_k \,\middle|\, \begin{matrix} SN \\ N \end{matrix}\right) = A_3 r_k^{\beta-1} (1 - r_k)^{\frac{k-3}{2}} t_k^{-\frac{1}{2}} (1 - t_k)^{\beta + \frac{k-3}{2}} \tag{36}
$$

where

$$
A_3 = \left[B\left(\beta, \frac{k-1}{2}\right) B\left(\tfrac{1}{2}, \beta + \frac{k-1}{2}\right)\right]^{-1}
$$

The quantity $B(\cdot, \cdot)$ is the Beta function (Ref. 15). Equation 36 gives the distribution of the two random variables r_k and t_k conditional to the hypotheses signal and noise and noise alone. Using the transformations defined by Eqs. 35 and 32 one can work backward to express these two random variables in terms of the quantities

$$
\sum_1^k x_i s_i, \text{ and } \sum_1^k x_i^2
$$

From Eqs. 29 and 31 these latter quantities are the functions of the observations which appear in the likelihood ratio; thus it follows that the likelihood ratio can be expressed as a function of the <u>independent</u> random variables r_k and t_k. The joint distribution (in reality the marginal distributions since they are independent) of r_k and t_k can be programmed on the digital computer along with the likelihood ratio expressed as a function of r_k and t_k. In this way one can determine the distributions of the likelihood ratio conditional to signal and noise alone using Monte Carlo techniques. From these two distributions the ROC curves follow directly.

The interesting trick employed in the above procedure was the method of transforming to auxiliary random variables until a set (r_k and t_k) was found which was independent. The independence property makes programming a great deal easier and, in fact, to some extent permits a solution of the problem.

The ROC curves for the optimum receiver obtained in the manner outlined above are not displayed in this section but rather are deferred to Chapter 7 where the performance of all of the receivers considered are compared. However, investigation of Eq. 36 indicates that the performance of the optimum receiver is a function of the 2WT product or observation time k. The curves given in Chapter 7 are for a 2WT product of 100, i. e., $k = 100$, and the natural question arises as to the performance as a function of observation time. We now wish to show that in the limit of infinite observation time the performance of the optimum receiver for the case of detecting a certain signal in noise of uncertain level approaches the upper performance bound attainable in the absence of noise level uncertainty. In other words, as the observation time increases, the performance of the optimum receiver approaches the ESP receiver performance for the uncertain noise level case. Thus, for large averaging times the optimum receiver learns or adapts to the actual value of noise level present and the initial uncertainty in the noise level causes no loss in performance on a single trial.

To show the asymptotic approach of the optimum receiver performance to the upper bound let us first express the likelihood ratio given by Eq. 31 in terms of the random variables U_k and V_k. Using Eqs. 32 and 27 we have,

$$\ell(X_k) = \left[1 + \frac{\frac{1}{2}E_s - E_s U_k}{\frac{1}{2}V_k + \frac{1}{2}E_s U_k^2 + \alpha} \right]^{-(\beta+k/2)} \tag{37}$$

For each value of p we now define the new random variable

$$w_k = \frac{V_k p - (k - 1)}{\sqrt{2(k - 1)}} \tag{38}$$

so that

$$\ell(X_k) = \left[1 + \frac{\frac{1}{2}E_s p - E_s U_k p}{\sqrt{2(k - 1)}\; w_k + (k - 1) + \frac{1}{2}E_s p U_k^2 + \alpha p} \right]^{-(\beta+k/2)}$$

This latter expression can be written as

$$\ell(X_k) = \left[1 + \frac{E_s p - 2E_s U_k p}{(k - 1)\left(1 + \frac{2\sqrt{2}}{\sqrt{k - 1}} w_k + E_s pU_k^2 + \frac{2\alpha p}{k - 1}\right)} \right]^{-(\beta + k/2)} \tag{39}$$

For a given value of the precision p, we have already seen (Eq. 33) that U_k and V_k are independent with U_k a normal random variable and V_k a chi-square random variable with $(k - 1)$ degrees of freedom. The moments of the conditional distribution of U_k are independent of k under both hypotheses SN and N as Eq. 33 indicates. On the other hand the results of Cramér show that the chi-square distribution with n degrees of freedom approaches normality with mean n and variance $2n$ as $n \to \infty$ (Ref. 14). Hence, from Eq. 38 the conditional distributions of w_k conditional to SN and N approach normality with unity means and variances that are <u>independent</u> of k as $k \to \infty$.

We now return to the expression for the likelihood ratio given by Eq. 39. As $k \to \infty$ the conditional distributions of U_k and w_k approach normality with parameters independent of k. Thus for fixed values from these distributions we have,

$$\ell(X_k) \underset{k \to \infty}{\to} \left[1 + \frac{E_s p - 2E_s U_k p}{k} \right]^{-k/2} \tag{40}$$

But from Cramér we have the result that for every fixed x

$$\lim_{k \to \infty} \left[1 + \frac{x^2}{k}\right]^{k/2} \to e^{\frac{x^2}{2}} \tag{41}$$

Using this relationship, Eq. 40 becomes

$$\ell(X_k)_{k \to \infty} \to e^{(E_s U_k - \frac{1}{2} E_s)p} \tag{42}$$

From the definition of U_k given in Eq. 32 we can write Eq. 42 as

$$\ell(X_k)_{k \to \infty} \to \exp\left(\sum_1^k x_i s_i - \frac{1}{2} E_s\right) p \tag{43}$$

Thus, for each value of the precision p, as $k \to \infty$ the likelihood ratio approaches the value the likelihood ratio would take if p were known a priori. Since p is a distributed parameter, the average performance is the average of these conditional performances. However, this is just the ESP performance as given in Chapter 3. So, for extensive observation time the optimum receiver performance approaches the ESP receiver performance as an upper bound. This is the result we wished to determine.

4.2 The Estimation Receiver

4.2.1 Receiver Design. The design of the estimation receiver (ESTIMATE) is a two-fold process. The first step is to design the receiver as if all parameters that occur are known or certain. The criterion which this initial design is based on may vary somewhat; however, the Bayesian or likelihood ratio method is a widely used one. After this first step has been completed the second step is to determine those parameters about which uncertainty exists. One then constructs estimators or uses estimation techniques to determine these uncertain parameters and the estimated values of the parameters are plugged into the original system as if they were the correct values. This method of receiver design is sometimes termed the "estimate-and-plug" method for obvious reasons.

One of the fundamental problems associated with the estimate-and-plug method is the lack of a uniform or standard method for determining the estimators. One generally tries to pick estimators which are unbiased and provide minimum variance estimates; however, the method for doing this is not always clear. The fundamental property common to all estimation receivers is that they make no use of a priori information other than the fact that parameters are uncertain. In other words, no information provided by a priori distributions is utilized by the estimate-and-plug type receiver. In essence these receivers operate with fixed a priori opinions concerning the state of the environment. In general these opinions are diffuse, perhaps even unrealizable in the sense of a distribution function, and they do not change during operation.

The estimation receiver considered for the certain signal, uncertain noise level case in this report is arrived at from the optimum Bayes' receiver considered in the previous section. In essence, the receiver is designed by considering a nonrealizable diffuse prior distribution function for the precision p. This prior distribution function is maintained independent of the actual prior on p. In other words, there is no mechanism by which one modifies the estimation receiver to account for various environmental conditions as was possible with the optimum receiver using the Knudsen curves previously discussed.

The method used to design the estimation receiver considered is to choose the Bayes' optimum receiver for the completely diffuse nonrealizable prior density on p. To obtain this density we use the Gamma distribution in the limit as

$$\beta \to 1 \quad \text{and} \quad \alpha \to 0 \tag{44}$$

This limiting process yields the nonrealizable uniform distribution over the entire range of p. If we invoke this limiting process in the equation for the optimum Bayes' receiver, i. e., the likelihood ratio, given in Eq. 31, we obtain

$$T(X_k) = \left(\frac{Y_{2,k}}{Y_{1,k}}\right)^{k/2} \tag{45}$$

where $T(X_k)$ denotes the test statistic of the estimation receiver. Note that Eq. 45 does not contain any parameters relevant to the a priori distribution of p.

A diagram illustrating an estimation receiver realizing the operations given by Eq. 45 is given in Fig. 8. In the next section we consider the performance of this receiver.

4.2.2 Receiver Performance. The performance of the estimation receiver designed in the previous section is quite easy to determine once the initial groundwork of the evaluation procedure has been performed for the optimum receiver. The procedure replaces the likelihood ratio $\ell(X_k)$ for the optimum receiver by the estimation ratio $T(X_k)$ for the estimation receiver in the evaluation program. In this manner one determines the distribution functions of $T(X_k)$ conditional to the hypotheses SN and N. From these distributions the ROC curves follow directly.

As with the optimum receiver, the ROC curves for the estimation receiver are not displayed here, but are deferred to Chapter 7 where the performances of all the receivers are compared. The curves given in Chapter 7 are for an observation time or 2WT product of 100 as with the optimum receiver. The ROC curves for the estimation

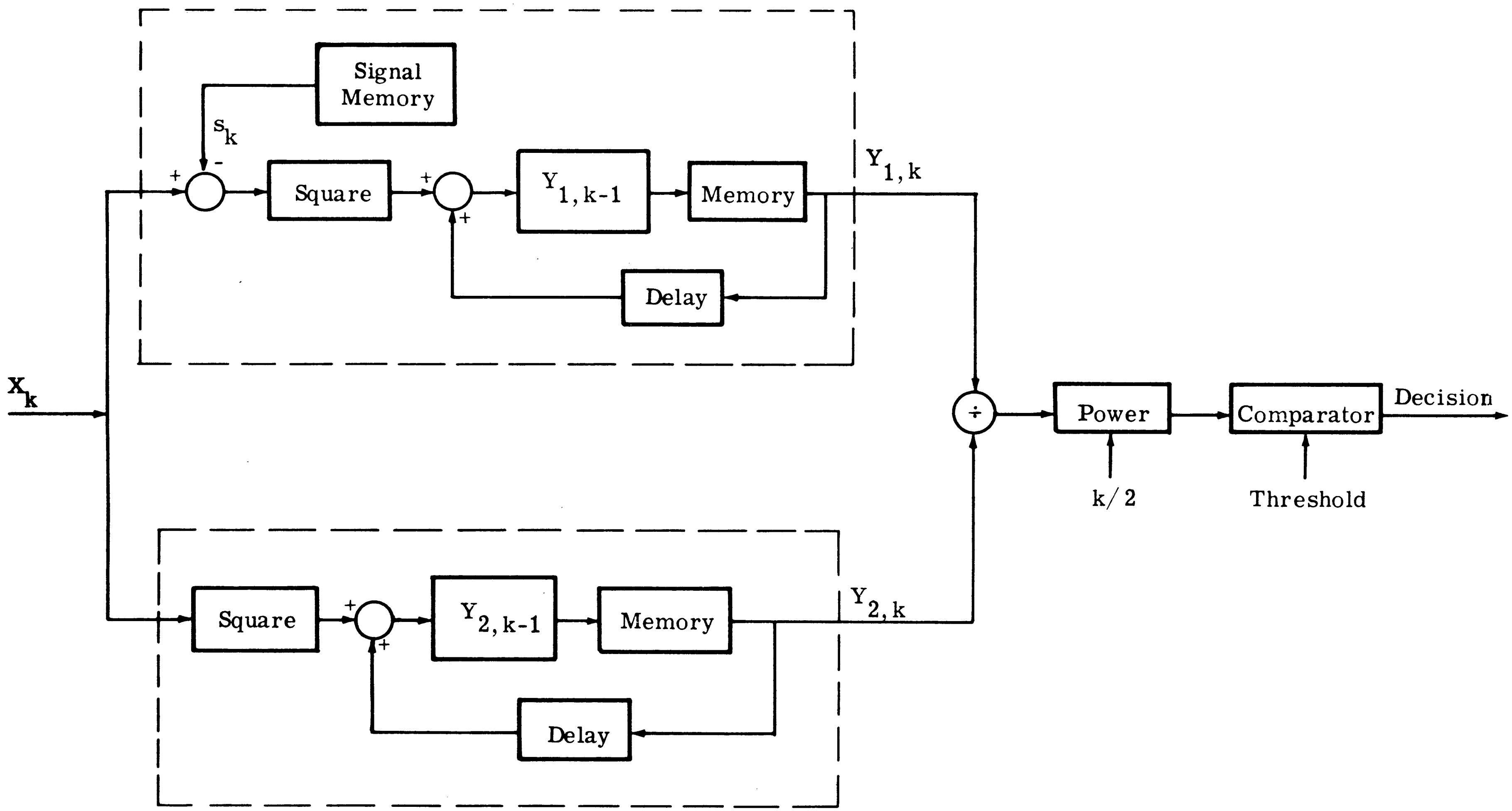

Fig. 8 An estimation receiver for the detection of a certain signal in noise of uncertain level

receiver depend on the observation time, and from the results for the optimum receiver, in the limit of infinite observation time, the performance of the estimation receiver approaches the ESP receiver performance. In other words, for the case of a certain signal in noise of uncertain level, the use of the estimation receiver results in a performance which becomes independent of the uncertainty in the noise level on any given trial. It is important to note, however, that for any given observation time the performance of the estimation receiver will be less than or equal to the performance of the optimum receiver.

4.3 Summary

In this chapter we have considered the optimum (Bayes) receiver and an estimation receiver for the detection of a certain signal in noise of uncertain level. The general receiver designs consisting of two branch receivers have been given and more detailed designs for the case of a Gamma distributed level have also been considered for both receivers. The performance of the receivers has been evaluated and it has been shown that in the limit of long observation time, there is no performance loss caused by noise level uncertainty.

CHAPTER 5

THE CLIPPER CROSS-CORRELATOR RECEIVER

In this chapter a receiver which is often used because it is easy to implement and because it is relatively insensitive to level changes is considered. The design of this receiver, the clipper cross-correlator receiver (CCCR), is considered briefly and its performance is evaluated for the case of detecting a certain signal in noise of uncertain level. (This is a "stored reference" or "replica" type correlator.)

5.1 CCCR Design

In the previous chapter receivers which directly incorporate the channel variability in their design were considered. The receiver considered in this section accounts for this variability in a more indirect manner by deliberately ignoring all but the polarity of the reception. The receiver, the clipper cross-correlator receiver (CCCR), is considerably less complex than the optimum receiver and is therefore easier to implement, especially with the level of digital technology present today. The CCCR has been studied extensively by numerous other authors (Ref. 16, 17, 18), however, the work presented here is felt to be new in that, to the Author's knowledge, no one has actually carried out an evaluation of the CCCR performance for the case of detecting a certain signal in noise of uncertain level.

The primary advantage of the CCCR is that the processing mechanism can be digitalized and the operations carried out on either a general or special purpose digital computer. The initial digitalizing operation of the CCCR consists of sampling and clipping or hard limiting the input observation. The median of the noise is usually chosen as a convenient clipping level since the noise is generally assumed to be symmetric and the median is generally an easily determined level, such as a transducer ground. The use of the initial clipping process in the CCCR destroys all of the information in the input reception other than that of signal polarity. At this point it is evident that the operation of this receiver is suboptimum. However, from the clipping operation forward the processing method can be optimum for certain types of signals, namely, those with samples of equal magnitude but not necessarily equal sign. This statement is verified in Appendix A.

After the clipping operation the CCCR cross-correlates the clipped input observation with the known signal waveshape. Since the clipped observation contains only polarity information, only the polarity of the reference signal is generally used for the cross-correlation operation. This implies that the known signal waveshape is transformed into a binary waveform with only zero crossing information retained to perform the cross-correlation. In addition, when the clipping is done about the median of the noise, cross-correlation may be implemented by simply multiplying the clipped input observation by the binary version of the known signal waveshape. The decision statistic is then obtained by accumulating the pulses obtained from the cross-correlator. This operation can be performed digitally by a preset counter as shown in the digital implementation of the CCCR illustrated in Fig. 9. We again remark that the CCCR operations performed after the clipping operation can be optimum for the detection of signals whose samples have the same magnitude but not necessarily the same sign as shown in Appendix A.

5.2 CCCR Performance Computations

From Fig. 9 it is evident that the decision statistic used by the CCCR is the number of polarity agreements (or disagreements) in k observations between the clipped input observation and the binary reference signal. To determine the probabilities of detection and false alarm for the CCCR we need to know the distribution functions of this polarity statistic conditional to the hypotheses SN and N.

Let us denote the number of polarity agreements by the symbol m. Using this notation the probabilities we seek are given by

$$P\begin{bmatrix}\mathrm{DET}\\ \mathrm{FA}\end{bmatrix}_{\mathrm{CCCR}} = P\left[m \geq \Delta \;\middle|\; \begin{matrix}\mathrm{SN}\\ \mathrm{N}\end{matrix}\right] \tag{46}$$

where Δ is an arbitrary threshold value. We note that the random

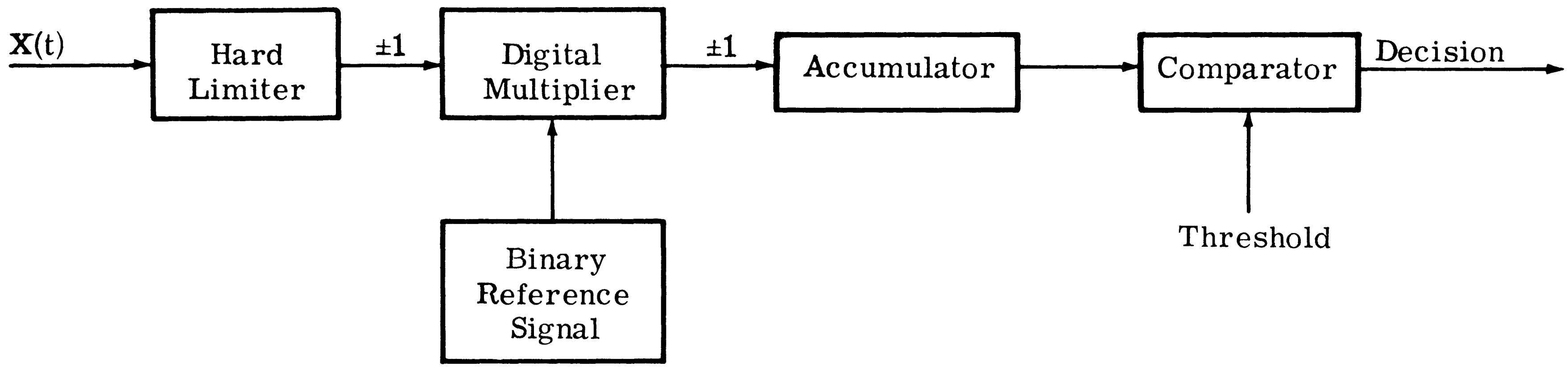

Fig. 9 A digital implementation of the clipper cross-correlation receiver

variable m is a Bernoulli random variable characterized by the binomial distribution function. Hence, conditional to a known value of the precision p and the hypothesis SN we have

$$P[m \text{ "agreements" in } k \text{ observations } | \, p, SN]$$

$$= \binom{k}{m} [q_1(p, s)]^m \; [1 - q_1(p, s)]^{k-m} \tag{47}$$

In this expression $q_1(p, s)$ is the probability of a polarity agreement for a signal sample s conditional to a known value of p and hypothesis SN. To determine the probability conditional only to SN, Eq. 47 must be averaged over the a priori distribution of the precision p. Symbolically this latter probability is given by

$$P[m \text{ "agreements" in } k \text{ observations } | \, SN]$$

$$= \binom{k}{m} \int_{\tilde{P}} [q_1(p, s)]^m \; [1 - q_1(p, s)]^{k-m} g(p)dp \tag{48}$$

To evaluate Eq. 48 for a given g(p) we must determine the form of $q_1(p, s)$. As defined above this is the probability of polarity agreement between the clipped observation and the binary reference signal. In Appendix A it is shown that when the median of the noise is used as the clipping threshold, the probabilities associated with the clipper cross-correlation output are the same as those associated with the input observation $x(t)$. Hence,

$$q_1(p, s) = P[x(t) \geq 0 | p, SN] \tag{49}$$

The median of conditionally Gaussian noise is zero so for this case we have

$$q_1(p, s) = \Phi(sp^{\frac{1}{2}}) \tag{50}$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\, dx$$

and we have chosen the signal samples to all have the same magnitude which is denoted by

$$s_i = s \qquad i = 1, 2, \ldots, k$$

The choice of equal magnitude signal samples is made for two reasons. First, this choice greatly facilitates the resulting mathematical computations and second, the resulting CCCR performance is a maximum over the class of inputs since the CCCR is an optimum processor given the clipped input for this case (Appendix A). Thus, the CCCR performance which we calculate is the best attainable. The reason for the desirability of determining the best attainable will be evident in Chapter 7.

If we substitute Eq. 50 in Eq. 48 and use the Gamma distribution for the precision p for $g(p)$, then Eq. 50 can be written as

$$P\left[\begin{matrix}\text{m agreements in} \\ \text{k observations}\end{matrix} \;\middle|\; SN\right]$$

$$= \binom{k}{m} \frac{\alpha^{\beta}}{\Gamma(\beta)} \int_0^{\infty} [\Phi(sp^{\frac{1}{2}})]^m [1 - \Phi(sp^{\frac{1}{2}})]^{k-m} \cdot p^{\beta-1} \exp(-\alpha p)\, dp \tag{51}$$

By substituting Eq. 51 in Eq. 46 we can determine the detection probability of the CCCR.

$$P[DET]_{CCCR}$$

$$= \sum_{m=\Delta}^{m=k} \binom{k}{m} \frac{\alpha^{\beta}}{\Gamma(p)} \int_0^{\infty} [\Phi(sp^{\frac{1}{2}})]^m [1 - \Phi(sp^{\frac{1}{2}})]^{k-m} \cdot p^{\beta-1} \exp(-\alpha p)\, dp \tag{52}$$

To determine the probability of false alarm we need to investigate the distribution of the number of polarity agreements m conditional to the precision p and hypothesis N. Since m is a Bernoulli random variable with the binomial distribution, we have

$$P\left[\begin{matrix}\text{m agreements in} \\ \text{k observations}\end{matrix} \;\middle|\; p, N\right] = \binom{k}{m} [q_2(p,s)]^m [1 - q_2(p,s)]^{k-m} \tag{53}$$

In this expression $q_2(p, s)$ is the probability of a polarity agreement for a signal sample s conditional to a known value of p and hypothesis N. For the case in which the median of the noise is used as the clipping threshold, the probabilities associated with the clipper cross-correlator output are the same as those associated with the input observation $x(t)$. Hence

$$q_2(p, s) = P\,[x(t) \geq 0 \mid p, N] \tag{54}$$

There is no signal present under hypothesis N and so for the median clipped conditionally Gaussian noise we are considering

$$P[x(t) \geq 0 \mid p, N] = \tfrac{1}{2} \tag{55}$$

which implies that

$$q_2(p, s) = \tfrac{1}{2}\,. \tag{56}$$

Substituting the result of Eq. 56 in Eq. 53 we have

$$P\begin{bmatrix} \text{m agreements in} \\ \text{k observations} \end{bmatrix} \mid p, N\Bigg] = \binom{k}{m} (\tfrac{1}{2})^k \tag{57}$$

Using Eqs. 57 and 46 we determine that the false alarm probability for the CCCR is given by

$$P[FA]_{CCCR} = \sum_{m=\Delta}^{m=k} \binom{k}{m} (\tfrac{1}{2})^k \tag{58}$$

It is interesting to note that <u>the false alarm probability for the CCCR is independent of the distribution of the noise level.</u> This result occurred because the clipping level was set at the median of the noise and therefore caused the probabilities associated with $x(t)$ to be independent of the noise level. This independence of the false alarm probability of the noise level is often considered as a reason for using the CCCR in situations for which the noise level is uncertain.

Using Eqs. 52 and 58 we can find the performance of the CCCR for the case of uncertain noise level (with a Gamma distribution). The false alarm probability, Eq. 58, is particularly easy to determine while the detection probability, Eq. 52, is a great deal more complicated. Also, we note that the ROC curves are a function of the number of observations or observation time k.

The fact that the ROC curves for the CCCR are a function of the observation time (for fixed signal energy) creates somewhat of a problem in terms of data display. For the optimum receiver considered in the previous chapter, somewhat the same problem existed; however, it was found that the performance of the optimum receiver was asymptotic to the ESP receiver performance. Thus since the ESP receiver performance was already natural to display, one could display the optimum performance for some other specified value of k (e.g., $k=100$) and have a complete picture. For the CCCR the asymptotic bound is not the ESP performance and so one must judiciously choose the amount and type of data to display to develop a reasonable presentation. In this report only the asymptotic (i.e., $k \to \infty$) ROC curves for the CCCR are displayed. The motivation for displaying only these curves is that they represent the optimum performance for a given set of parameters; thus, if this performance is judged insufficient, one would not be interested in the performance for less averaging time.

To determine the asymptotic behavior of the CCCR we must examine Eqs. 52 and 58 for large values of k. As k becomes large the binomial portions of these expressions approach the normal density function with mean np and variance npq (Ref. 9). (Here the notation n represents the number of Bernoulli trials, and p and q are the standard binomial probabilities.) Using the normal approximation in Eqs. 52 and 58 we have

$$P[\mathrm{DET}]_{\mathrm{CCCR}}^{\infty} \cong \int_0^{\infty} \int_{\Delta}^{\infty} \frac{\alpha^{\beta}}{\Gamma(\beta)} (2\pi k \sigma_a^2)^{-\frac{1}{2}} e^{-(m-km_a)^2/2k\sigma_a^2} \tag{59}$$

$$p^{\beta-1} e^{-\alpha p}\, dm\, dp$$

where

$$m_a = \Phi(sp^{\frac{1}{2}}), \quad \sigma_a^2 = \Phi(sp^{\frac{1}{2}})\,[1 - \Phi(sp^{\frac{1}{2}})] \tag{60}$$

and

$$P[\mathrm{FA}]_{\mathrm{CCCR}}^{\infty} \cong \int_{\Delta}^{\infty} 2(2\pi k)^{-\frac{1}{2}} e^{-(m-k/2)^2\, 4/k}\, dm \tag{61}$$

If for convenience we make the following substitution for the threshold Δ,

$$\Delta = (k^{\frac{1}{2}} \zeta + k) / 2$$

and then perform the integration with respect to m in Eqs. 59 and 61 we have

$$P[\text{DET}]^{\infty}_{\text{CCCR}} \cong \int_0^{\infty} \frac{\alpha^{\beta}}{\Gamma(\beta)}\, p^{\beta-1} \left\{ 1 - \Phi\left[\frac{\zeta - k^{\frac{1}{2}}(2m_a - 1)}{2\sigma_a}\right]\right\} e^{-\alpha p}\, dp \tag{62}$$

and

$$P[\text{FA}]^{\infty}_{\text{CCCR}} \cong 1 - \Phi\,(\zeta) \tag{63}$$

For simplicity we previously assumed a constant signal wave-shape. Using this fact Eq. 60 can be rewritten in terms of k and E_s as

$$m_a = \Phi\left[(E_s p/k)^{\frac{1}{2}}\right] \qquad \sigma_a^2 = \Phi\left[(E_s p/k)^{\frac{1}{2}}\right]\left\{1 - \Phi\left[(E_s p/k)^{\frac{1}{2}}\right]\right\} \tag{64}$$

Equation 63 indicates that for a given threshold value ζ, the false alarm probability for large k is unaffected by further increases in k. However, this is not true of the detection probability as given by Eq. 62. As $k \to \infty$, Eq. 62 indicates that the detection probability reaches an upper bound. This bound can be determined by using the results of Appendix B. Using L'Hospital's rule and Eq. 64 it is shown in Appendix B that

$$\lim_{k \to \infty} \left\{ 1 - \Phi\left[\frac{\zeta - k^{\frac{1}{2}}(2m_a - 1)}{2\sigma_a}\right]\right\} \to 1 - \Phi\left[\zeta - (2E_s\, p/\pi)^{\frac{1}{2}}\right] \tag{65}$$

Using the result of Eq. 65 in Eq. 62 we have

$$P[\mathrm{DET}]^{\infty}_{\mathrm{CCCR}} = \int_0^{\infty} \frac{\alpha^{\beta}}{\Gamma(\beta)} p^{\beta-1}\{1 - \Phi[\zeta - (2E_s p/\pi)^{\frac{1}{2}}]\} e^{-\alpha p}\, dp \tag{66}$$

The factor $2E_s/\pi$ in Eq. 66 is recognized as the 63 percent efficiency factor which is widely quoted for the CCCR performance when all parameters are certain. In other words the $2/\pi = 0.63$ factor in front of the signal energy E_s implies that the performance of the CCCR for the known parameter case is that of the optimum receiver at 2db less signal energy (since the performance equations are the same except for the $2/\pi$ factor).

To express Eq. 66 in terms of the channel variability parameters λ and β which we adopted in Chapter 2 we make the following substitutions.

$$y = (2E_s p/\pi)^{\frac{1}{2}}, \quad \lambda = E_s/\alpha \tag{67}$$

Using this substitution, Eqs. 66 and 63 can be written as

$$P[\mathrm{DET}]^{\infty}_{\mathrm{CCCR}} = 1 - 2\int_0^{\infty} [(2\lambda/\pi)^{\beta}/\Gamma(\beta)]\, y^{2\beta-1} e^{-y^2/(2\lambda/\pi)} \Phi(\zeta - y)\, dy \tag{68}$$

$$P[\mathrm{FA}]^{\infty}_{\mathrm{CCCR}} = 1 - \Phi(\zeta) \tag{69}$$

These latter equations express the performance of the CCCR (in terms of the equivalent threshold ζ) for large averaging times. The ROC curves obtained from a computer evaluation of these equations are presented in Chapter 7 where the performances of the various processors considered in this report are compared.

CHAPTER 6

CORRELATION RECEIVERS

In this chapter two receivers which incorporate little or no noise level uncertainty information in their design are considered. The design of these receivers, termed the cross-correlator and the likelihood cross-correlator is considered briefly and their performance is evaluated for the case of detecting a certain signal in noise of uncertain level. Some of this work has been previously published by the Author but is included in this report for completeness (Ref. 19).

6.1 The Cross-Correlator Receiver

6.1.1 Receiver Design. As stated we are concerned here with receivers that essentially ignore any uncertainty that may be present in parameters, such as the noise level, and process the input reception as if these parameters were known. By definition then the receivers considered are suboptimum for the uncertain noise level situation under consideration. We wish to consider their performance even though they are suboptimum since this provides a useful method for determining the tradeoffs between equipment complexity, cost, and performance loss.

Perhaps the simplest and most natural receiver to consider is the cross-correlation receiver (CCR) illustrated in Fig. 10. The

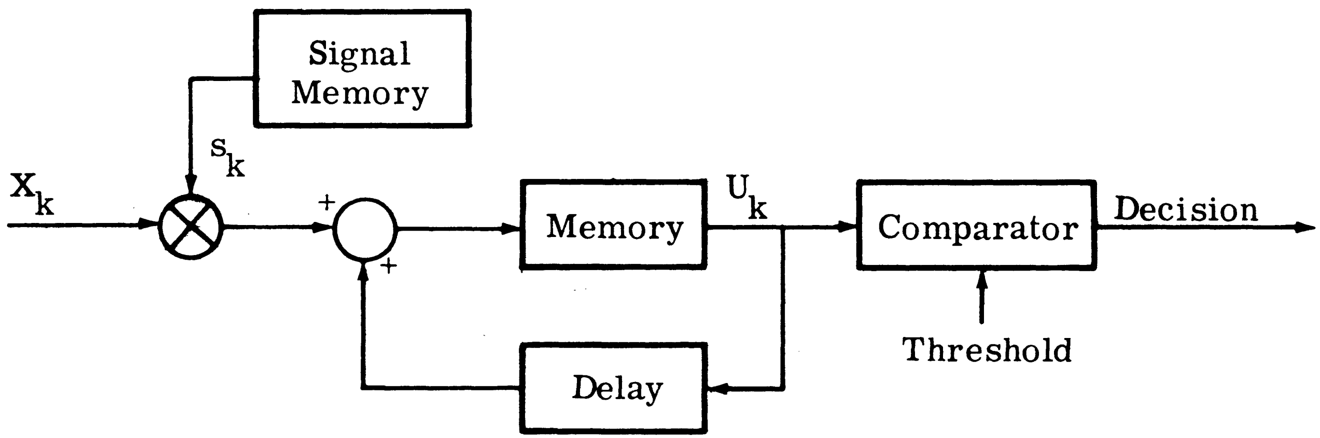

Fig. 10 A sequential realization of the cross-correlation receiver

decision statistic formed by this receiver is the result of cross correlating between the input observation and the certain or known signal waveform. This statistic is optimum for the case of detecting a cerain signal in certain noise, and this result makes it a logical choice for study and comparison in the uncertain noise level case. The operation of the CCR has been considered extensively in the literature (Ref. 1) and so our primary goal is to evaluate its performance for the uncertain noise level case.

6.1.2 Receiver Performance. To determine the performance of the CCR for the uncertain noise level case we must determine the probabilities of detection and false alarm. This is most easily done by referring to the work in Chapter 4. In Eq. 34 the joint distribution of U_k and V_k is given where,

$$U_k = \frac{1}{E_s} \sum_{1}^{k} x_i s_i$$

and

$$V_k = \sum_{1}^{k} x_i^2 - E_s U_k^2$$

The variable U_k above is recognized as the decision statistic formed by the CCR (except possibly for normalization). Hence, to determine the performance of the CCR we need only determine the conditional marginal distributions of U_k. These distribution functions can be found by integrating Eq. 34 over the range of the variable V_k. Performing the integration we obtain

$$f\left(U_k \middle| \begin{matrix} SN \\ N \end{matrix}\right) = \left(\frac{\lambda}{2\pi}\right) \frac{\Gamma(\beta + \frac{1}{2})}{\Gamma(\beta)} \left[1 + \lambda\left(U_k - \begin{matrix} 1 \\ 0 \end{matrix}\right)^2\right]^{-(\beta+\frac{1}{2})} \tag{70}$$

where

$$\lambda = E_s / \alpha \tag{71}$$

It is interesting to note that the two conditional distribution functions for U_k are independent of the number of observations k. This result implies that the performance of the CCR is independent of the number of observations and only depends on the noise level distribution and the signal energy (via β, λ, and E_s). The performance of the CCR is obtained from Eq. 70 by integration with respect to U_k using various threshold values Δ. We note that Eq. 70 is a Student-t or t-distribution function (Ref. 15). Therefore, for the CCR operating in an uncertain noise level environment we have

$$P\begin{bmatrix} DET \\ FA \end{bmatrix}_{CCR} = \int_{\Delta}^{\infty} \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\beta + \frac{1}{2})}{\Gamma\beta} \left[\Gamma + \lambda\left(U_k - \begin{matrix} 1 \\ 0 \end{matrix}\right)^2\right]^{-(\beta+\frac{1}{2})} dU_k \tag{72}$$

The ROC curves obtained for the CCR from Eq. 71 are displayed with the other receiver performance curves in Chapter 7.

6.2 The Likelihood Cross-Correlator Receiver

6.2.1 Receiver Design. In the previous section the cross-correlator receiver was considered for the case of detecting a certain signal in noise of uncertain level. The performance of this receiver is suboptimum for the uncertain noise situation but it represents a logical and easily implemented receiver. A natural method to preserve the basic correlation processing of the CCR and at the same time to improve its performance is to incorporate a small amount of the noise level uncertainty in the receiver design by forming the likelihood ratio of the correlation variable (U_k) using this quantity as the decision statistic. The resulting receiver is termed the likelihood cross-correlator receiver (LCCR) and is illustrated in Fig. 11. The basic processing of this receiver is the cross-correlation of the input reception with the stored reference waveform, the difference between the LCCR and the CCR appearing in the processing of this correlation statistic.

The method used to obtain the processing equation for U_k is to form the likelihood ratio of this variable.

$$\ell(U_k) = \frac{f(U_k|SN)}{f(U_k|N)} \tag{73}$$

From Eq. 70 we can express Eq. 73 as

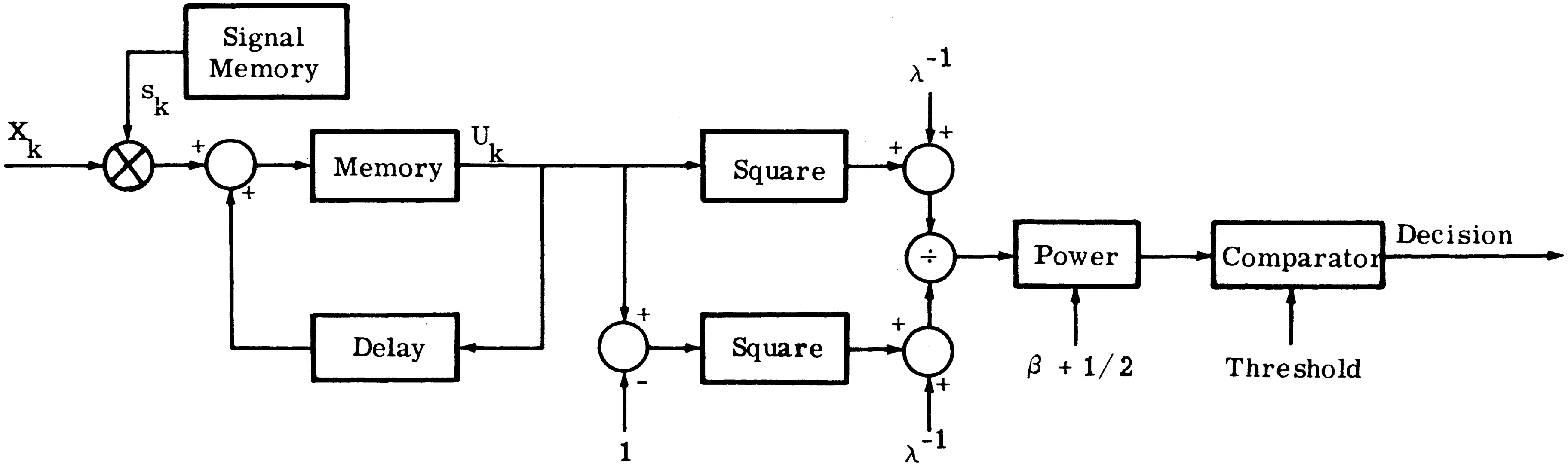

Fig. 11 A sequential realization of the likelihood cross-correlation receiver

$$\ell(U_k) = \left[\frac{1 + \lambda\, U_k^2}{1 + \lambda \left(U_{k-1}\right)^2} \right]^{\beta + \frac{1}{2}} \tag{74}$$

Equation 74 expresses the basic processing method to obtain the improved performance of the LCCR. A receiver design based on this method is illustrated in Fig. 11.

6.2.2 Receiver Performance. To determine the ROC performance curves for the LCCR we must determine the detection and false alarm probabilities for this receiver as a function of the threshold setting. If a particular threshold value is denoted by Δ, then these two probabilities are given by

$$P\begin{bmatrix}DET\\FA\end{bmatrix}_{LCCR} = P\left[\ell(U_k) \geq \Delta \mid \begin{matrix}SN\\N\end{matrix}\right] \tag{75}$$

Using Eq. 74 we can write Eq. 75 in the form

$$P\begin{bmatrix}DET\\FA\end{bmatrix}_{LCCR} = P\left[\left(\frac{1 + \lambda\, U_k^2}{1 + \lambda(U_k - 1)^2}\right)^{\beta + \frac{1}{2}} \geq \Delta \mid \begin{matrix}SN\\N\end{matrix}\right] \tag{76}$$

Applying algebraic manipulation to Eq. 76 we can express it as

$$P\begin{bmatrix}DET\\FA\end{bmatrix}_{LCCR} = P\left[1 + \lambda\, U_k^2 > \Delta' \lambda \left(U_{k-1}\right)^2 + \Delta' \mid \begin{matrix}SN\\N\end{matrix}\right] \tag{77}$$

where we have defined

$$\Delta' = \Delta^{1/(\beta+\frac{1}{2})}$$

If we complete the square on U_k in Eq. 77 and perform some additional algebraic manipulations, we have

$$P\begin{bmatrix}DET\\FA\end{bmatrix}_{LCCR} = P\left\{[U_k + \Delta'/(1-\Delta')]^2 \geq \Delta'/(1-\Delta')^2 - 1/\lambda \mid \begin{matrix}SN\\N\end{matrix}\right\} \tag{78}$$

This latter expression is equivalent to

$$P\begin{bmatrix}DET\\FA\end{bmatrix}_{LCCR} = P\left[U_k \geq \Delta'' \mid \begin{matrix}SN\\N\end{matrix}\right] \tag{79}$$

where

$$\Delta'' = [\Delta'/(1-\Delta')^2 - 1/\lambda]^{\frac{1}{2}} - \Delta'/(1-\Delta')$$

The probability distributions of U_k conditional to SN and N were determined above and are given by Eq. 72. These distributions are in the form of the t-distribution function with different mean values. To obtain the ROC curves for the LCCR for the case of uncertain noise level these distributions are used with Eq. 79. They are obtained by determining points from the distributions at various values of Δ'' (as a function of Δ) for fixed values of λ and β. The digital computer was used to determine the points using a specially developed subroutine for the t-distribution function. (This computer routine was also used in a slightly different fashion to obtain the performance of the CCR.) The ROC curves for the LCCR are displayed in Chapter 7 along with the other receivers.

CHAPTER 7

COMPARISON OF RECEIVER PERFORMANCE

In the previous chapters the design calculations and performance computations for a number of receivers operating in an uncertain noise level environment were carried out. In this chapter the results of this work are displayed in the form of ROC curves. From these ROC performance curves a number of conclusions are reached concerning the detection of certain signals in noise of uncertain level.

7.1 Receiver Review

The receivers studied in this report are naturally separated into four classes. The first class contains the single receiver termed the ESP receiver (Chapter 3). The predominate design characteristic of this class (or receiver) is that perfect parameter information is incorporated in the receiver design even though the channel in question may be variable. In other words, for any given trial run the hypothetical ESP receiver has ideal information about the value of any uncertain parameter. The performance of this receiver is limited only by the stochastic nature of the detection problem and the variability of the uncertain parameters but not by a lack of knowledge of parameter values on any given trial run. In terms of the uncertain parameters, the ESP receiver performance is the upper performance bound.

The second class of receivers considered in this report consists of realizable receivers which attempt to include in a direct manner the effect of parameter uncertainty in their design. The goal of the receivers in this class is to attain as great a performance as possible for the given situation by incorporating in some manner all of the available knowledge in the receiver design. The two receivers of this class investigated in Chapter 4 were the optimum (Bayes) receiver for the uncertain noise level case and an estimation or an estimate-and-plug type of receiver.

The clipper cross-correlator receiver is the only receiver in the third class of receivers studied (Chapter 5). This class includes receivers which attempt to incorporate the available environmental knowledge in the receiver design and simultaneously to reduce system cost and complexity and to increase ease of implementation. The result is a compromise receiver which suffers some performance loss but at reduced cost and complexity.

The last class of receivers studied is characterized by receivers which make little or no attempt to incorporate parameter uncertainty in their design. In a sense, these receivers are similar to the receivers in class three described above in that they are less complex and easier to implement and have reduced performance compared to the optimum. The receivers in this final class were presented in Chapter 6 and are the cross-correlator receiver and the likelihood cross-correlator receiver.

7.2 Receiver Performance Comparison

An analysis of the performance of the various receivers studied was described in the previous chapters. The results of the analyses consisted (in explicit or implicit form) of expressions for the probability of detection and the probability of false alarm as a function of the threshold for each receiver. In most cases the complexity of the expressions required evaluation on the digital computer.

The results of evaluating the performance expressions for each receiver studied appear as ROC curves in Figs. 12 through 21. The curves are parameterized by the expected value of the detectability (or signal-to-noise ratio) d_e and the variance of the detectability d_v. In addition, the probability density function of the signal-to-noise ratio expressed in decibels corresponding to the values of d_e and d_v is also shown on the performance graph. This plot is included to give the reader an intuitive grasp in terms of familiar units of the relative amount of signal-to-noise ratio uncertainty (channel variability) represented by the performance curves. Note that the abscissa scale can be read directly in decibels. Figure 5 shows plots of the signal-to-noise ratio densities not in decibels. Finally, we have chosen to use the detectability or signal-to-noise ratio as the uncertain parameter. The relationship existing between this quantity and the uncertain noise level is given in Chapter 2.

To contrast the performance of the various receivers three values of d_e were chosen as representative and are presented in the figures. The abbreviations used in labeling the performance curves are repeated here for convenience:

ESP -- Externally Sensed Parameter receiver

OPTIMUM -- Optimum (Bayes) receiver

ESTIMATE -- Estimation or estimate-and-plug receiver

CCCR -- Clipper Cross-Correlation Receiver

LCCR -- Likelihood Cross-Correlation Receiver

CCR -- Cross-Correlation Receiver

The performance curves for the ESP, CCR, and LCCR receivers are independent of the 2WT product (k) as discussed in the previous chapters. The performance curves for the OPTIMUM and ESTIMATE receivers are not independent of k as discussed in Chapter 4. The curves presented in Figs. 12 through 21 represent ROC curves for a value of $k = 100$ for these receivers. As $k \to \infty$, for long averaging times, the performances of these two receivers approach the ESP performance for the corresponding values of d_e and d_v. This attainment of the upper performance bound as a function of k was shown analytically in Chapter 4. The reason for including the curves for $k = 100$ is to illustrate the effect of practical averaging times on the performances of the OPTIMUM and ESTIMATE receivers.

The value of 2WT or averaging time has an effect on the CCCR similar to its effect on the OPTIMUM and ESTIMATE receivers (Chapter 5). However, in the performance plots we have chosen to present the performance for the CCCR corresponding to $k = \infty$. The CCCR performance plots given in Figs. 12 through 21 represent the asymptotic performance of the CCCR for the given values of d_e and d_v. The reasons for presenting the CCCR performance in this manner are detailed in Chapter 5. Basically, we wished to display the best performance obtainable with the CCCR since its performance usually is poor compared to the other receivers. (A more complete discussion of this concept involves singular detection and will not be considered here. See Ref. 20.)

7.3 Conclusions

Investigation of the performance curves presented in Figs. 12 through 21 yields numerous results. One interesting aspect of the receiver performance curves is that the performances of the ESP, OPTIMUM, ESTIMATE, LCCR, and CCR receivers are identical along the negative diagonal. The processing operations of all of these receivers become simple correlation processing along the negative diagonal since the threshold value is unity at this point. Thus the performance of these receivers becomes identical. The implication is that uncertainty in the noise level has no effect on the performance of symmetric binary communication systems. This result is a consequence of the fact that operation of this type of communication system is equivalent to operation along the negative diagonal of the ROC curves where the performance of the receivers is the same.

The usefulness of the performance curves for evaluating tradeoffs between equipment cost and complexity versus performance is apparent. The use of the receivers, such as the OPTIMUM and the ESTIMATE types, that incorporate uncertainty in their design results in performance values that are very close to the upper bound for reasonable averaging times and that can approach the upper performance bound for long averaging times. These results are illustrated in Figs. 12 through 21. The construction and implementation of the OPTIMUM and ESTIMATE receivers are reasonably complex when compared to the digitally implemented CCCR whose performance is not as great. The effects on performance of a tradeoff in the direction of the CCCR is shown by the performance curves. The performance of the CCCR is well below that of the upper performance bound and that of the ESTIMATE and OPTIMUM receivers.

The use of receivers which use little or no information concerning the noise level uncertainty in their design can cause considerable performance loss as the ROC curves for the CCR and LCCR indicate. At low values of false alarm probability the performance of the CCCR is particularly poor. Thus, for detection situations in which the noise level is uncertain, the use of simple correlation receivers can result in extreme performance losses. These losses in performance are magnified by increasing variability in the channel characteristic. In addition, Fig. 14 indicates that for low expected signal-to-noise ratios and relatively high channel variability caused by noise level uncertainty the performance of the simple correlator approaches the chance diagonal.

When the CCCR is compared to the correlation receivers (CCR and LCCR), we see that for operation at very small false alarm probabilities the performance of the CCCR is better. However, there is a cross-over point which is a function of the process parameters at which the correlation receivers provide an advantage. For the case of a binary communication system where operation is along the negative diagonal the correlation receivers yield a decided advantage in performance.

Perhaps the primary message of the performance curves is to illustrate the effect of neglecting to include the effect of noise level uncertainty in receiver design when the situation warrants such considerations. This fact is dramatically illustrated by a comparison of the OPTIMUM or ESTIMATE performance curves with the CCR performance curve in Fig. 18.

We can consider this loss in performance more quantitatively by comparing Figs. 15 through 18. In each of these figures the expected value of d, the signal-to-noise ratio, is 4. For each of the figures the probability of less than a 3 db swing in the signal-to-noise ratio (i. e., $2 \leqq d \leqq 6$) is given by

$$P(\pm 3\text{db}) = \int_2^6 g(d)\, dd$$

where g(d) is the appropriately chosen distribution for d. A tabulation of these values is given below along with the detection probabilities of the ESP receiver and the cross-correlator receiver (CCR) at a false alarm probability of 0.02.

d_e	d_v	P(±3db)	P_{ESP}(DET)	P_{CCR}(DET)	P(F.A.)
4	2	0.859	0.48	0.41	0.02
4	4	0.706	0.47	0.33	0.02
4	8	0.537	0.45	0.20	0.02
4	16	0.383	0.43	0.05	0.02

In terms of underwater sound propagation in an ocean environment a 3db swing in the signal-to-noise ratio is rather common (Ref. 3) and yet from the tabulation above we see that for reasonable probabilities of this event occurring there is a significant loss in detection probability (for a reasonable false alarm probability of 0.02) for the cross-correlation receiver when compared to the upper bound probability $P_{ESP}(DET)$. Since with reasonable averaging times, the OPTIMUM and ESTIMATE receivers approach the upper bound probability, there is considerable difference between their performances and the CCR performance in the uncertain noise level environment.

7.4 Summary

In this report the problem of detecting a certain signal in noise of uncertain level is considered in detail. A number of different receiver designs ranging from the hypothetical ESP receiver to a correlation receiver are considered and their performances evaluated. The major emphasis in the study has been to show the effect of noise level uncertainty on the performance of the various receiver designs. The results, as summarized by ROC curves, appear to be very useful in providing a basis for making compromises between equipment cost and complexity and average equipment performance. In particular, the results of this study as applied to transmissions of underwater sound in an ocean environment indicate that some procedure for acquiring noise level should be incorporated in detection receiver designs. This need for acquisition of the noise level arises because of the random nature of the ocean ambient noise level discussed in Chapter 2.

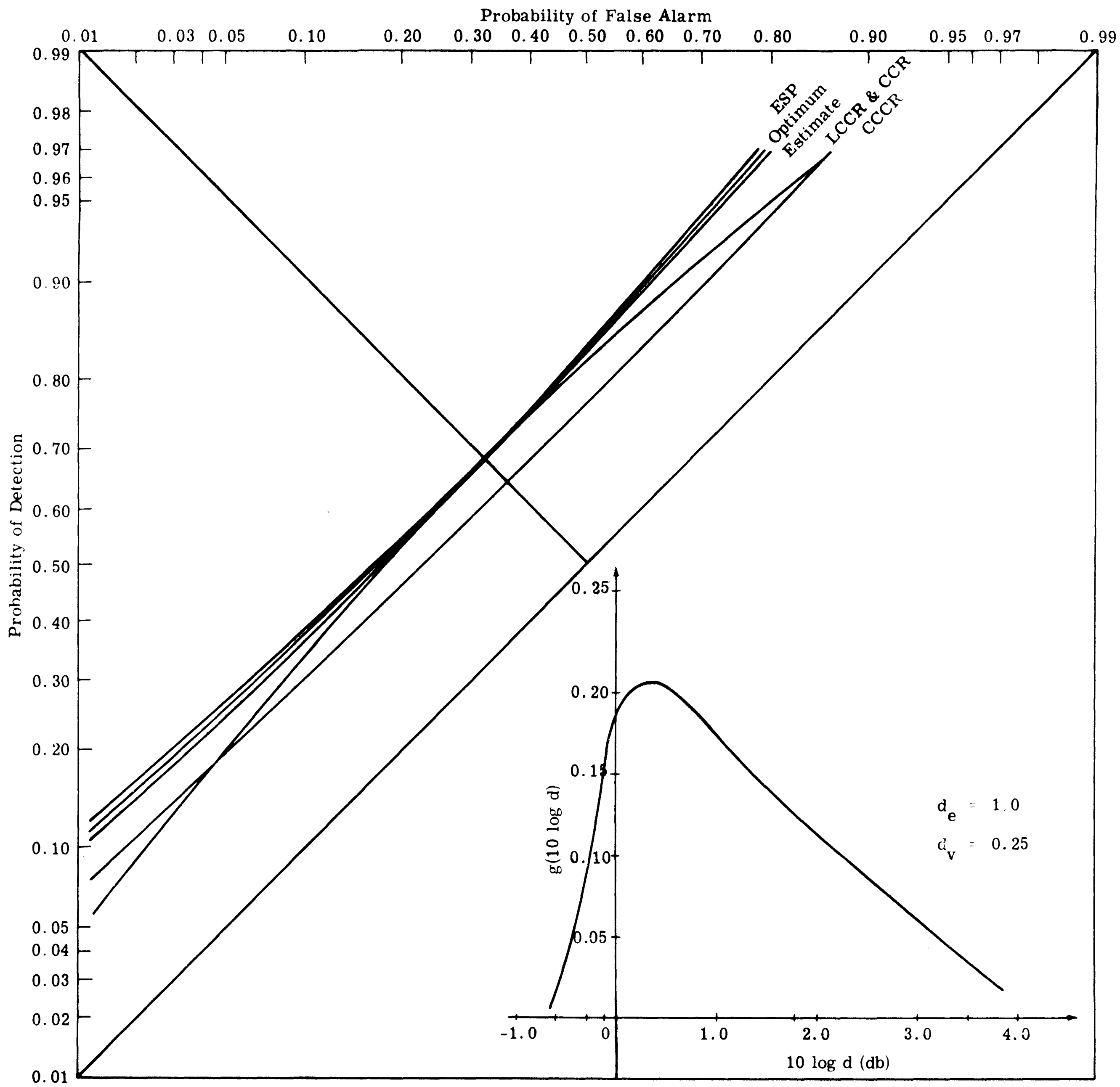

Fig. 12 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 1.0$, $d_v = 0.25$.

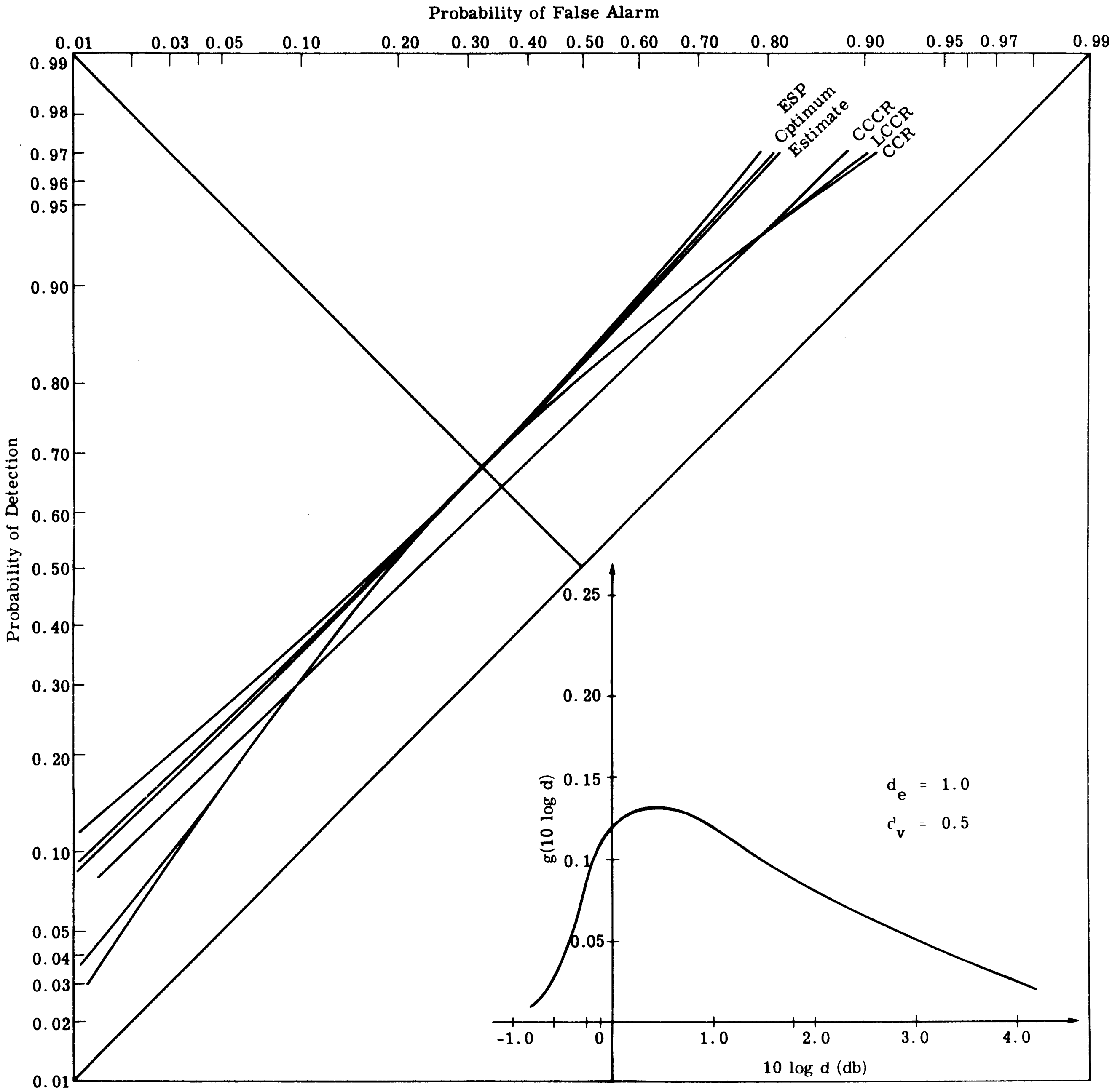

Fig. 13 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 1.0$, $d_v = 0.5$.

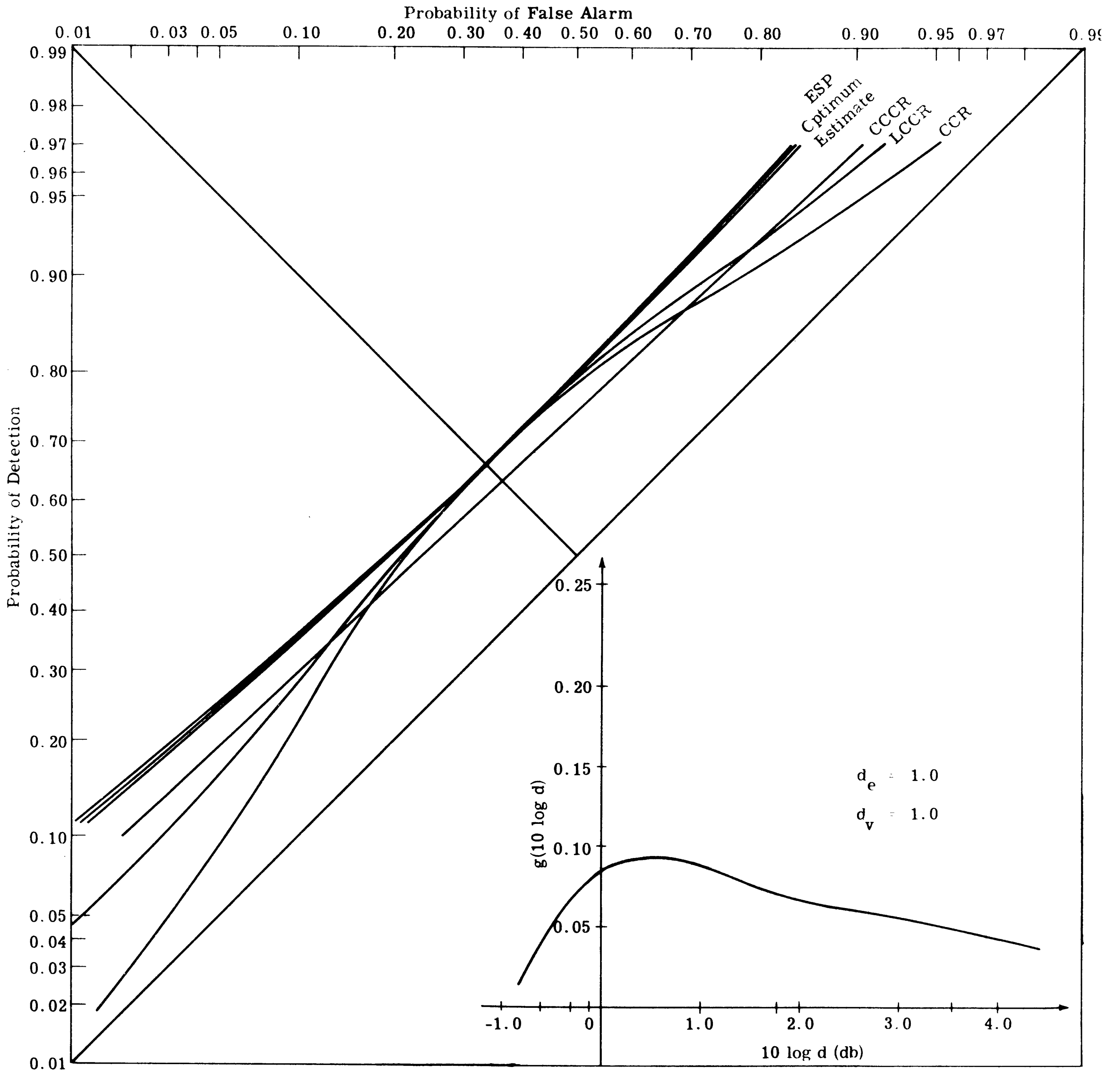

Fig. 14 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 1.0$, $d_v = 1.0$.

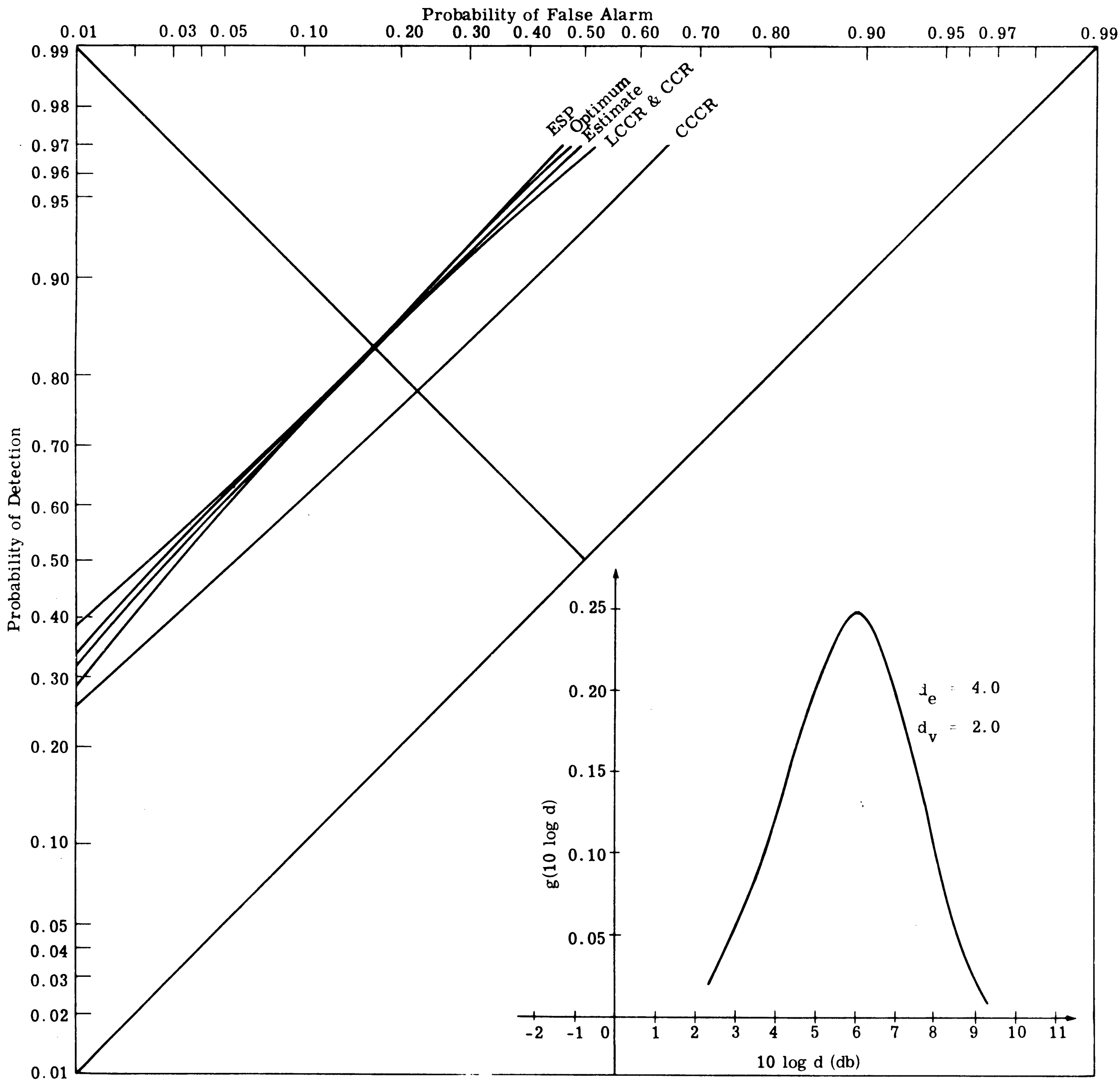

Fig. 15 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 4.0$, $d_v = 2.0$.

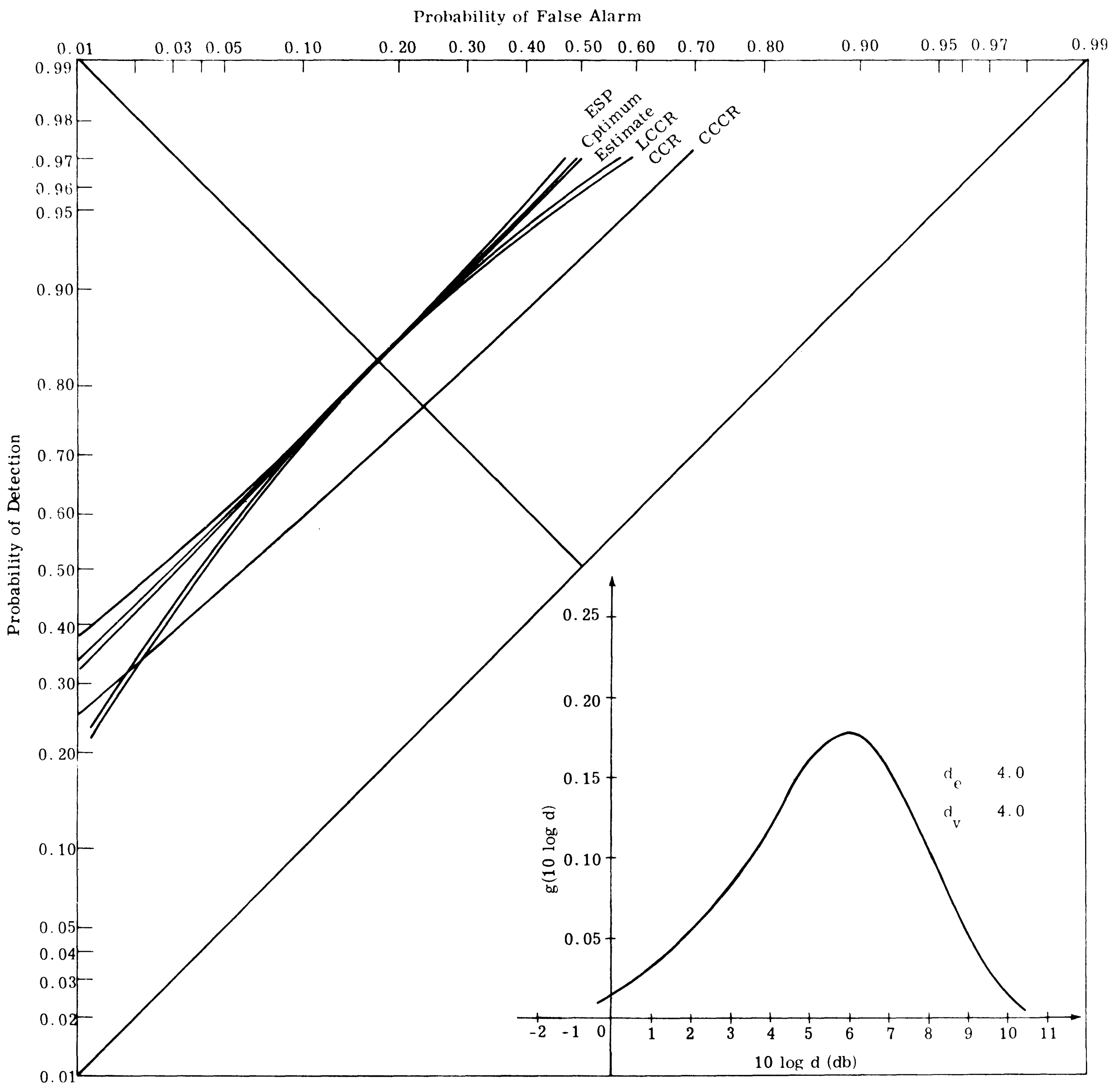

Fig. 16 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 4.0$, $d_v = 4.0$.

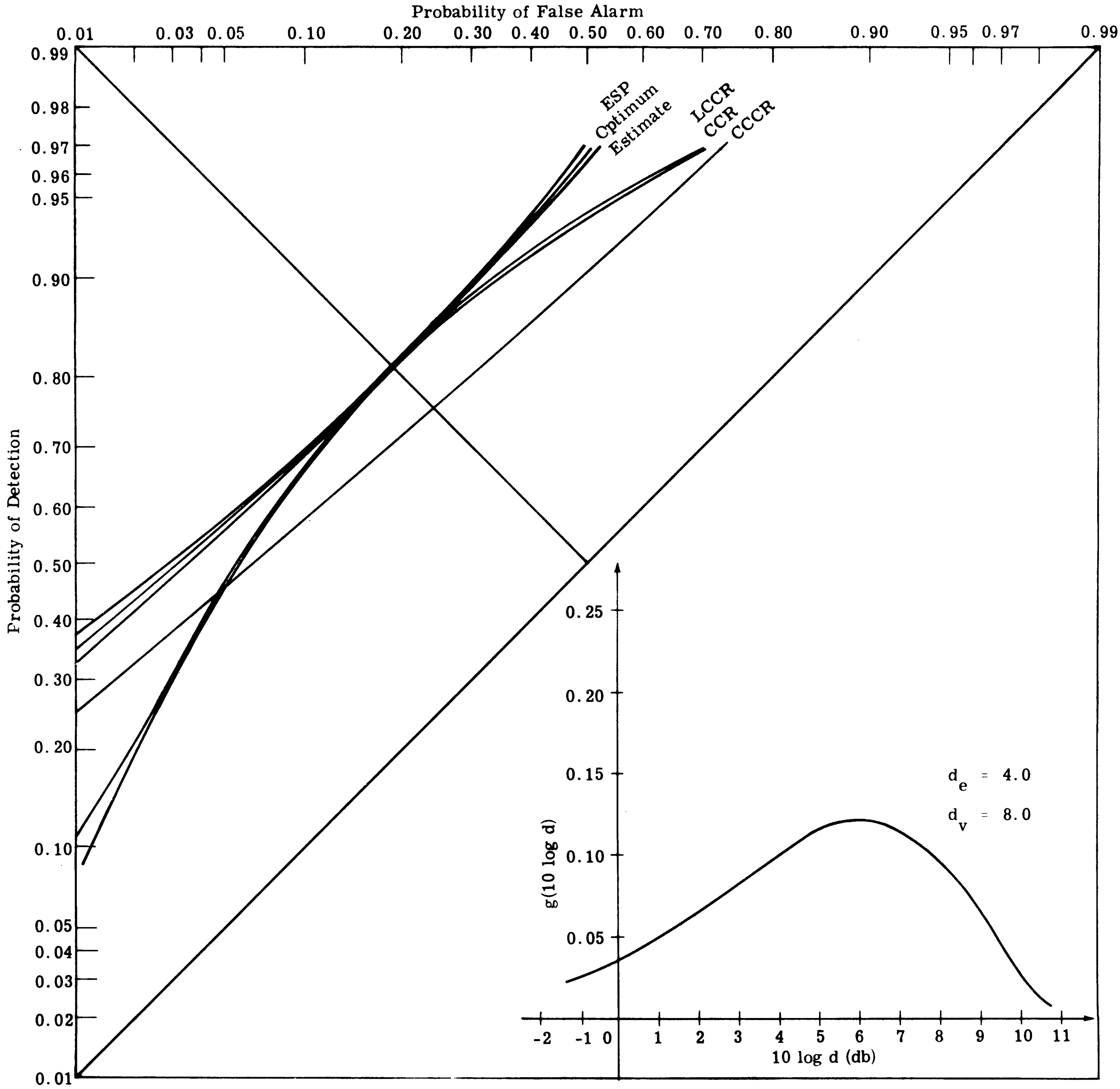

Fig. 17 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 4.0$, $d_v = 8.0$.

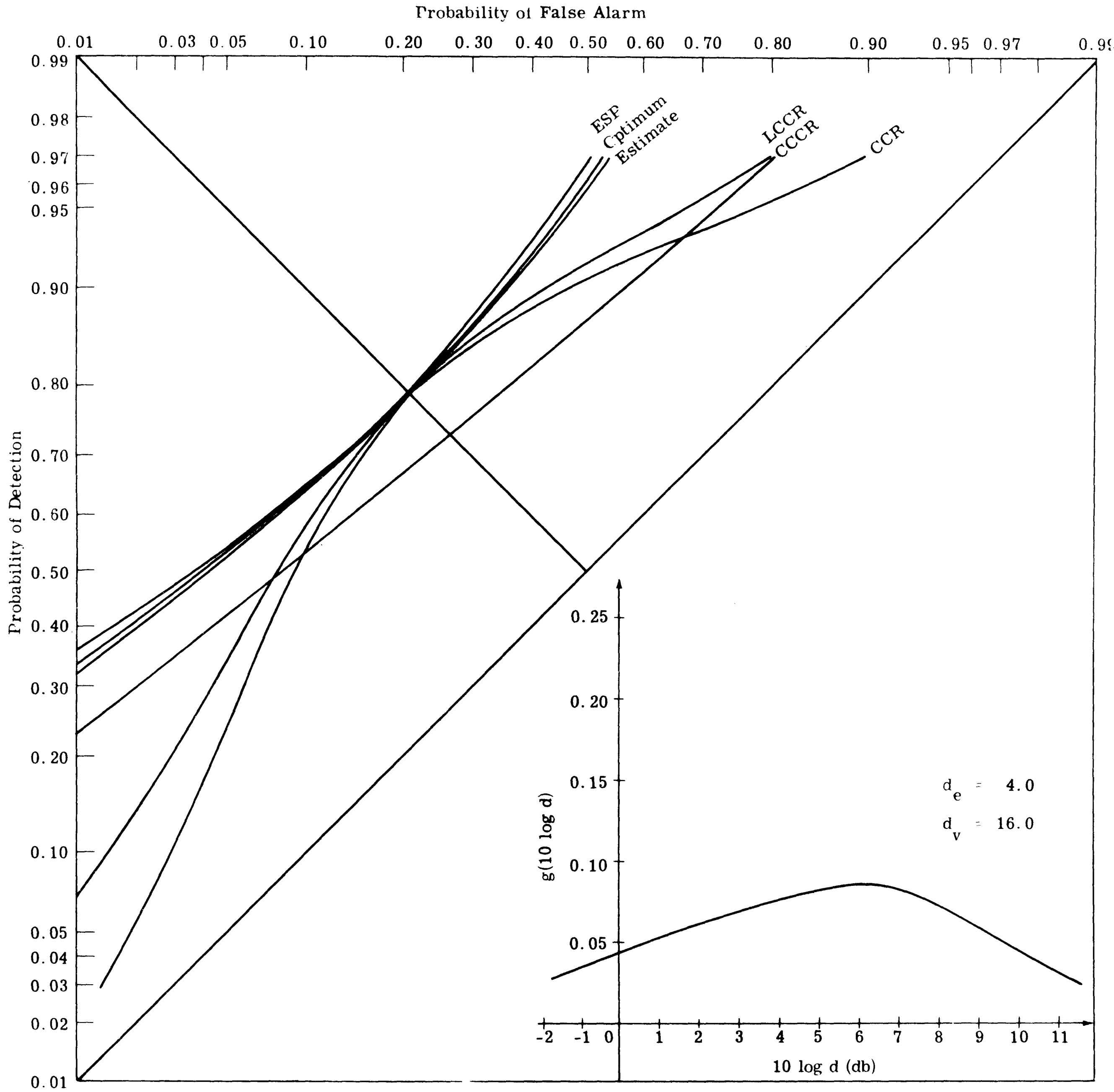

Fig. 18 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 4.0$, $d_v = 16.0$.

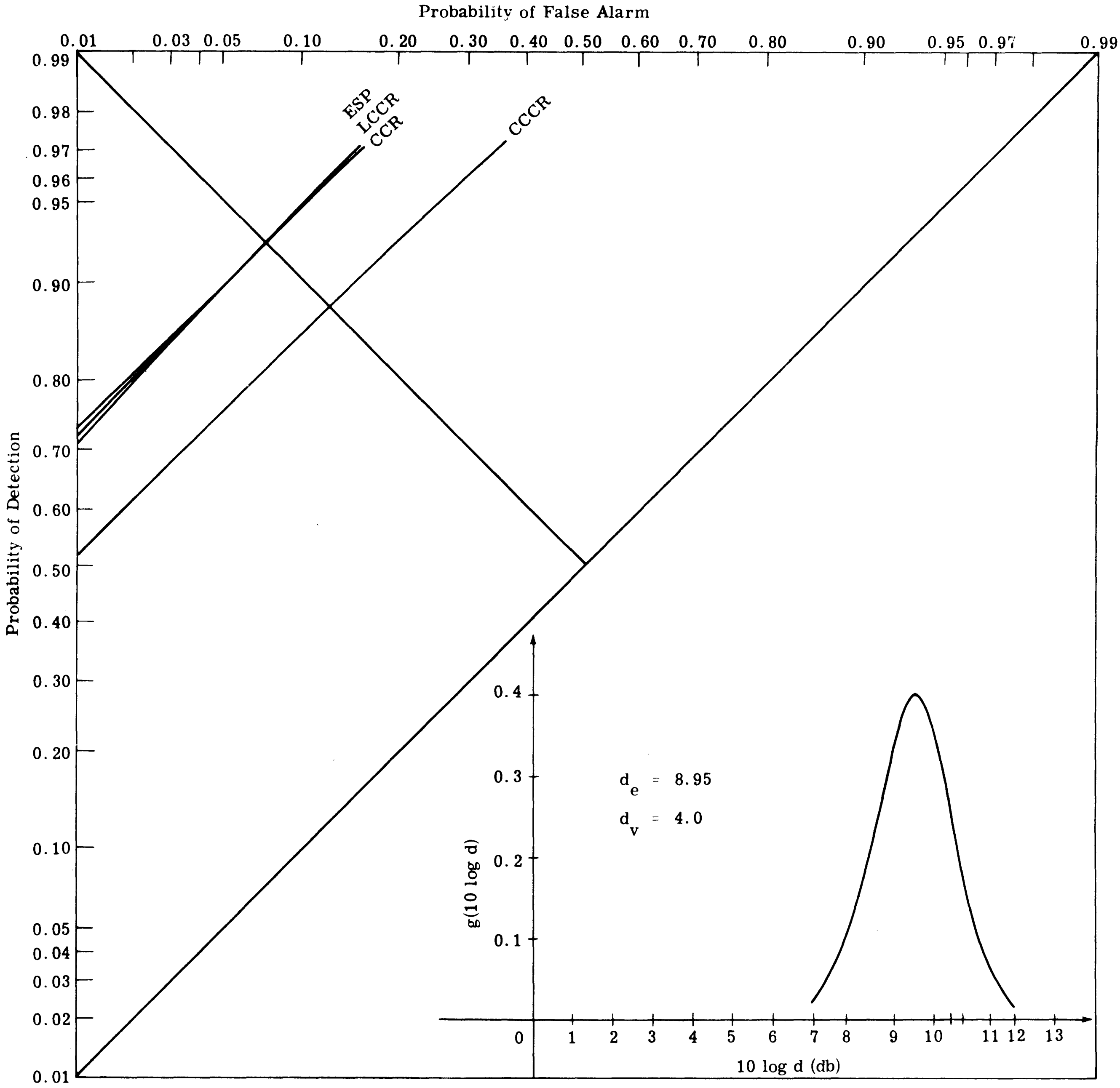

Fig. 19 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 8.95$, $d_v = 4.0$.

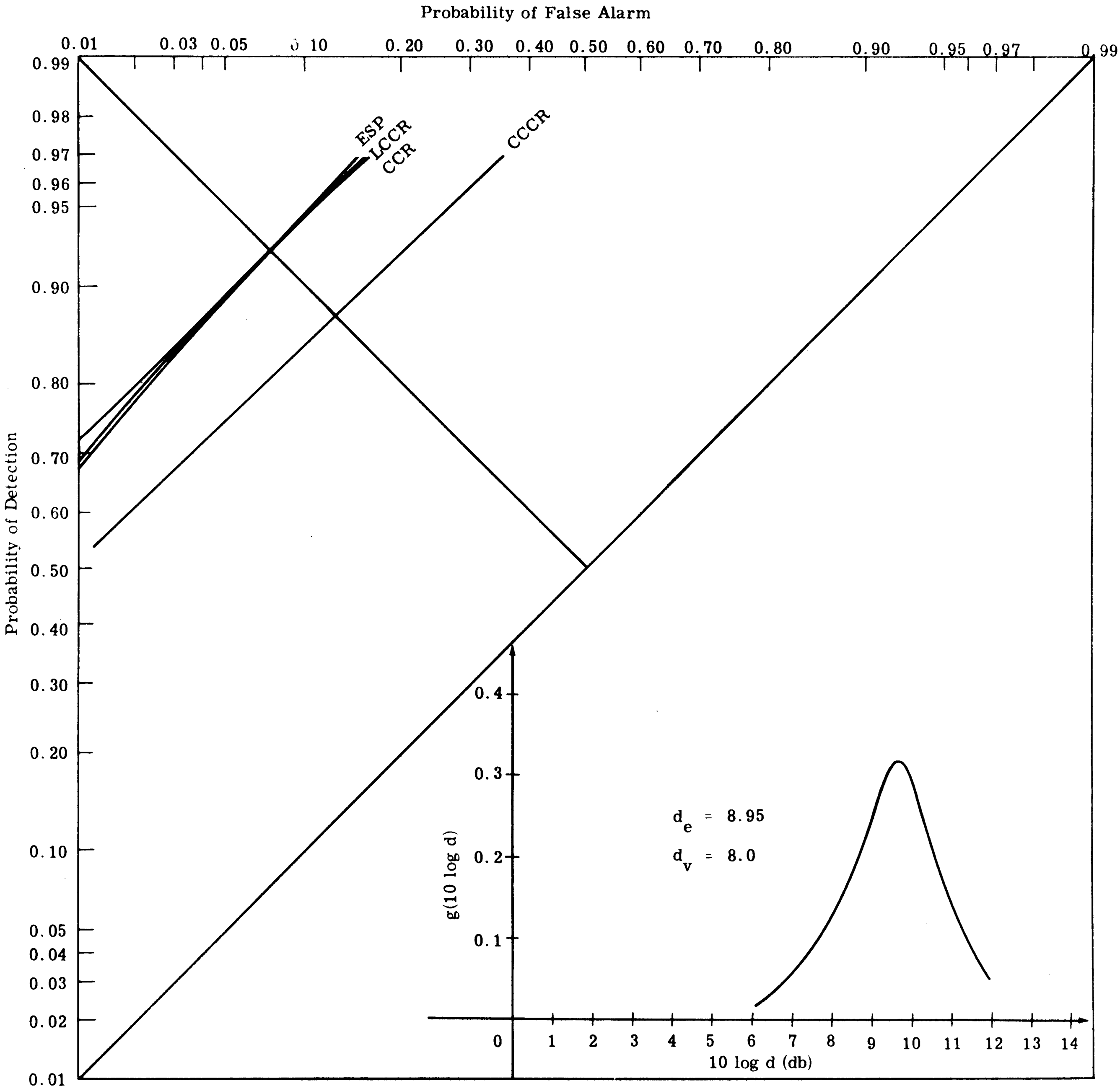

Fig. 20 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 8.95$, $d_v = 8.0$.

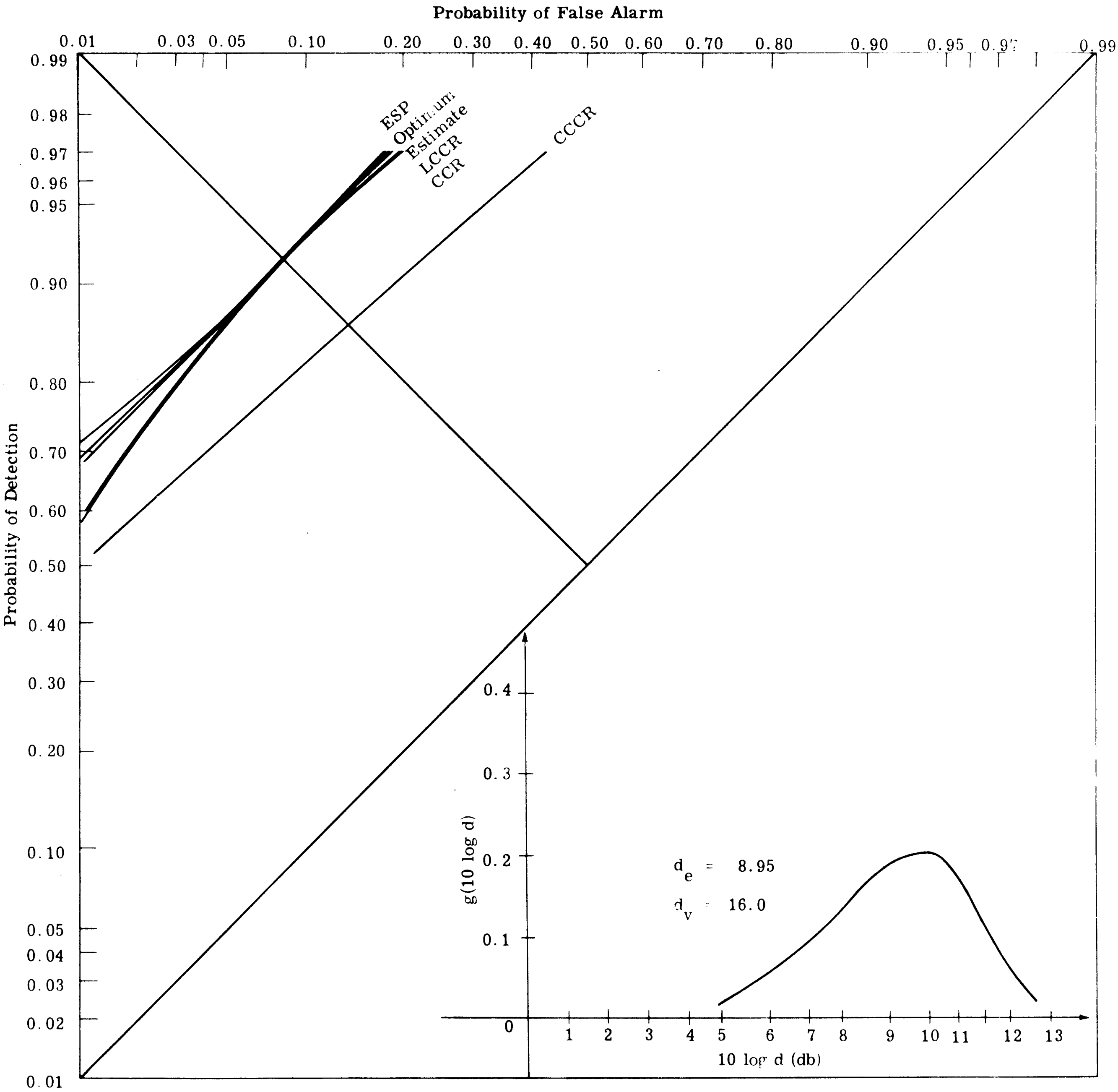

Fig. 21 Comparison of receiver performance for the detection of a certain signal in noise of uncertain level. The distribution of the signal-to-noise ratio expressed in decibels is given. The mean and variance of the Gamma distributed signal-to-noise ratio are $d_e = 8.95$, $d_v = 16.0$.

APPENDIX A

OPTIMUM CHARACTERISTICS OF THE CCCR

In this appendix we wish to show that the operations performed by the CCCR after the initial clipping or hard limiting operation are optimum for the case of equal signal sample (s_i) values. In addition, we wish to investigate the operation of the CCCR in general.

In Chapter 2 we discussed the fact that optimum processing is obtained by forming the likelihood ratio. Following the limiting operation in the CCCR the processor is presented with a vector $Z_k = (z_1, \ldots, z_k)$ where each z_i is a clipped version of the corresponding input observation sample x_i and has a value ± 1. Thus the optimum CCCR processor forms the likelihood ratio of the vector Z_i at this juncture.

$$\ell(Z_k) = \frac{f(Z_k \mid SN)}{f(Z_k \mid N)} \tag{80}$$

Since the precision level p of the input process x(t) is uncertain, Eq. 80 can be written as

$$\ell(Z_k) = \frac{\int_{\tilde{P}} f(Z_k \mid p, SN)\, g(p)\, dp}{\int_{\tilde{P}} f(Z_k \mid p, N)\, g(p)\, dp} \tag{81}$$

Conditional to a value of p the z_i components are independent since we have chosen x(t) to be a conditionally independent process. Using this assumption in Eq. 81 we have the following expression.

$$\ell(Z_k) = \frac{\int_{\tilde{P}} \prod_{i=1}^{k} f(z_k \mid p, SN)\, g(p)\, dp}{\int_{\tilde{P}} \prod_{i=1}^{k} f(z_k \mid p, N)\, g(p)\, dp} \tag{82}$$

The Z_k process is discrete since it is obtained as the result of sampling and hard limiting a continuous process. Hence, we may replace the continuous probability density functions in Eq. 82 with discrete probabilities to yield

$$\ell(Z_k) = \frac{\int_{\tilde{P}} \prod_{i=1}^{k} P(z_k \mid p, SN)\, g(p)\, dp}{\int_{\tilde{P}} \prod_{i=1}^{k} P(z_k \mid p, N)\, g(p)\, dp} \tag{83}$$

Since each z_i is a hard limited version of x_i we have

$$P\left(z_i = \pm 1 \mid p, \cdot\right) = P\left(x_i \gtrless \Delta \mid p, \cdot\right) \tag{84}$$

where Δ is a threshold value. Since x(t) is a Gaussian process with precision level p and either additive signal s(t) or no signal, we have

$$P\left(z_i = \pm 1 \mid p, \cdot\right) = \begin{cases} \left.\begin{array}{l} 1 - \Phi\left[(\Delta - s_i)\, p^{\frac{1}{2}}\right] \\ \Phi\left[(\Delta - s_i)\, p^{\frac{1}{2}}\right] \end{array}\right\} \text{SN} \\ \left.\begin{array}{l} 1 - \Phi(\Delta\, p^{\frac{1}{2}}) \\ \Phi(\Delta\, p^{\frac{1}{2}}) \end{array}\right\} \text{N} \end{cases} \tag{85}$$

If we hard limit at the median of the Gaussian process, then $\Delta = 0$ and

$$P\left(z_i = \pm 1 \mid p, \cdot\right) = \begin{cases} \left.\begin{array}{l} 1 - \Phi(-s_i\, p^{\frac{1}{2}}) \\ \Phi(-s_i\, p^{\frac{1}{2}}) \end{array}\right\} \text{SN} \\ \left.\begin{array}{l} \frac{1}{2} \\ \frac{1}{2} \end{array}\right\} \text{N} \end{cases} \tag{86}$$

where the probabilities conditional to N are independent of p.

We now consider the case for which all the s_i values have the same magnitudes. We denote the magnitude of the s_i values by

$$s = |s_i| \quad , \quad i = 1, 2, \ldots, k \tag{87}$$

Using the symmetry properties of the $\Phi(\cdot)$ function and the definition given in Eq. 87, Eq. 86 for the SN condition can be written as

$$P\left(z_i = \pm 1 \mid p, \text{SN}\right) = \begin{cases} \begin{array}{ll} \underline{s_i \geq 0} & \underline{s_i < 0} \\ \Phi(sp^{\frac{1}{2}}) & 1 - \Phi(sp^{\frac{1}{2}}) \\ 1 - \Phi(sp^{\frac{1}{2}}) & \Phi(sp^{\frac{1}{2}}) \end{array} \end{cases} \tag{88}$$

Investigating Eq. 88 it becomes clear that when all the s_i values have the same magnitude one can reduce consideration to the $s_i > 0$ case by multiplying the obtained z_i value by the correct sign of the s_i sample. This is the form of the CCCR which uses the binary reference signal (retaining sign information only) to cross-correlate with the (clipped) Z_k signal obtained from $x(t)$. When the s_i values do not have the same magnitude, an optimum device must retain the appropriate magnitude values as indicated by Eq. 88. In this more general case the form of the optimum processor is not as simple and is not generally termed a clipper cross correlation receiver.

For the equal magnitude case let us define

$$q_s(p) = \Phi(sp^{\frac{1}{2}}) \tag{89}$$

Using Eqs. 89 and 88 in the general form of the likelihood ratio (Eq. 83) we have for the equal magnitude case

$$\ell(Z_k) = 4 \int_{\tilde{P}} [q_s(p)]^{1-m} [1 - q_s(p)]^{m} g(p)\, dp \tag{90}$$

since conditional to N the probabilities are independent of p. The quantity m is the number of +1's or polarity agreements. Expression 90 is easily shown to be a monotonic function of m and so m may be used as the decision variable by the CCCR.

It follows from the discussion above that the simplified form of the CCCR [using polarity information of $s(t)$ only] is an optimum processor for the clipped input case where all the s_i values have the same magnitude. When this is not the case, the optimum device must include signal waveshape information and the simplified form of the CCCR is a suboptimum processor for the clipped input case.

APPENDIX B

PERFORMANCE BOUND FOR THE CCCR

In this appendix we wish to show that

$$\lim_{k\to\infty} \left\{ 1 - \Phi\left[\frac{\zeta - k^{\frac{1}{2}}(2m_a - 1)}{2\sigma_a}\right]\right\} \to 1 - \Phi\left[\zeta - (2E_s p/\pi)^{\frac{1}{2}}\right] \tag{91}$$

where

$$m_a = \Phi\left[(E_s p/k)^{\frac{1}{2}}\right] \tag{92}$$

and

$$\sigma_a^2 = \Phi\left[(E_s p/k)^{\frac{1}{2}}\right] \left\{1 - \Phi\left[(E_s p/k)^{\frac{1}{2}}\right]\right\} \tag{93}$$

The function $\Phi(\cdot)$ is a probability distribution function and so it is bounded between the values zero and unity with

$$\Phi(0) = \tfrac{1}{2} \tag{94}$$

For fixed values of E_s and p it is clear that

$$\lim_{k\to\infty} \sigma_a \to \tfrac{1}{2} \tag{95}$$

and so we need only concern ourselves with the term

$$k^{\frac{1}{2}}(2m_a - 1) \tag{96}$$

in Eq. 91. If we use Eq. 92 to rewrite Eq. 96, we are concerned with

$$\lim_{k\to\infty} k^{\frac{1}{2}}\{2\Phi\,[(E_s\,p/k)^{\frac{1}{2}}] - 1]\} \tag{97}$$

or

$$\lim_{k\to\infty} \frac{2\Phi\,[(E_s\,p/k)^{\frac{1}{2}} - 1] - 1}{k^{-\frac{1}{2}}} \tag{98}$$

Using 94 we find that this latter limit is indeterminate since it is 0/0. We can use L'Hospital's rule to obtain after differentiation of the numerator and denominator in Eq. 98,

$$\lim_{k\to\infty} \frac{(2/\pi)^{\frac{1}{2}}[-\tfrac{1}{2}(E_s\,p)^{\frac{1}{2}}]\;k^{-\frac{3}{2}}\;e^{-E_s\,p/2k}}{-\tfrac{1}{2}k^{-\frac{3}{2}}} \to (2/\pi)^{\frac{1}{2}}(E_s\,p)^{\frac{1}{2}} \tag{99}$$

which by L'Hospital's rule implies that

$$\lim_{k\to\infty} k^{\frac{1}{2}}\{2\Phi\,[(E_s\,p/k)^{\frac{1}{2}}] - 1\} = (2/\pi)^{\frac{1}{2}}\,(E_s\,p)^{\frac{1}{2}} \tag{100}$$

Using Eq. 95 and Eq. 100 we have the sought-after result.

LIST OF REFERENCES

1. W. W. Peterson, T. G. Birdsall, and W. C. Fox, "The Theory of Signal Detectability," IRE Trans. on Information Theory, IT-4, 1954, pp. 171-212.

2. J. C. Steinberg and T. G. Birdsall, "Underwater Sound Propagation in the Straits of Florida," J. Acoust. Soc. Am., Vol. 39, No. 2, February 1966, pp. 301-315.

3. V. M. Alpers, Underwater Acoustics Handbook II, The Pennsylvania State University Press, University Park, Pennsylvania, 1965.

4. A. M. Breipolil and A. H. Koschmann, Adaptive Communication Systems, Engineering Experiment Station Technical Report EE-98, The University of New Mexico, Albuquerque, New Mexico, August 1963.

5. W. Edwards, H. Lindman, and L. J. Savage, "Bayesian Statistical Inference for Psychological Research," Psychological Review, Vol. 70, No. 4, May 1963.

6. T. G. Birdsall, Random Processes, Engineering Summer Conference, The University of Michigan, Ann Arbor, 1963.

7. T. G. Birdsall, "The Theory of Signal Detectability: ROC Curves and Their Character," Ph. D. Thesis, The University of Michigan, Ann Arbor, 1966.

8. T. S. Ferguson, Mathematical Statistics, A Decision Theoretic Approach, Academic Press, New York, 1967.

9. D. V. Lindley, "On a Measure of the Information Provided by an Experiment," Annals of Mathematical Statistics, Vol. 27, December 1956, pp. 986-1005.

10. H. Raiffa and R. Schlaifer, Applied Statistical Decision Theory, Harvard University Press, Boston, 1967.

11. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1965.

12. R. L. Spooner, Theory of Signal Detectability: Extension to the Double Composite Hypothesis Situation, Cooley Electronics Laboratory Technical Report No. 192, The University of Michigan, Ann Arbor, July 1968.

13. T. G. Birdsall, Adaptive Detection Receivers and Reproducing Densities, Cooley Electronics Laboratory Technical Report No. 194, The University of Michigan, Ann Arbor, July 1968.

14. H. Cramér, Mathematical Methods of Statistics, Princeton University Press, Princeton, 1946.

15. H. W. Alexander, Elements of Mathematical Statistics, John Wiley and Sons, New York, 1961.

16. S. S. Wolf and J. L. Gastwirth, "Robust Two-Input Correlators," J. Acoust. Soc. Am., Vol. 41, No. 5, May 1967, pp. 1212-1219.

17. J. B. Thomas and T. R. Williams, "On the Detection of Signals in Nonstationary Noise by Product Arrays," J. Acoust. Soc. Am., Vol. 31, No. 4, April 1959, pp. 453-462.

18. P. Rudnick, J. Acoust. Soc. Am., Vol. 32, 1960, pp. 871-877.

19. R. L. Spooner, "On the Detection of a Known Signal in a Non-Gaussian Noise Process," J. Acoust. Soc. Am., Vol. 43, No. 7, July 1968.

20. H. L. Van Trees, Detection, Estimators, and Modulation Theory - Part I, John Wiley and Sons, New York, 1968.

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13. ABSTRACT This report is concerned with the problem of detecting signals in noise of uncertain level. The impetus for this study is found in many areas; however, the detection of underwater sound signals in an ocean environment provides the primary motivation. The purpose is to consider the detection performance of a number of different receiver designs operating in an uncertain noise level environment. A number of receiver designs are considered ranging from the hypothetical externally sensed parameter (ESP) receiver to a standard cross-correlation type of receiver and including the clipper cross-correlator. The ESP receiver is a possibly non-realizable receiver whose performance serves as an upper bound while the cross-correlator is often used in practice. The design of the receivers investigated is briefly considered. The detection performance is displayed by means of receiver operating characteristic (ROC) curves for each receiver. The results indicate that channel variability in the form of noise level uncertainty does not seriously affect the upper performance bound attainable at low false alarm probabilities for a conditionally Gaussian process. In addition, for a given measure of channel variability it is shown that performance can be greatly improved by receiver designs which incorporate the variability (in a Bayesian or estimation sense) in their design. In fact, for long averaging times the performance of such receivers can approach the upper performance bound. On the other end of the spectrum it is found that receivers such as the clipper cross-correlator can suffer significant performance loss when operating under the condition of uncertain noise level.

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Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Signal detection for underwater acoustics Bayes receivers Externally sensed parameter receiver Signals in noise of uncertain level						

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