last year or so, however, he has discovered that it is not. Accordingly, this note has been written to bring it to the attention of all. Some remarks as to its significance in the solution of nonlinear differential equations have been added.

The procedure is most easily understood by following its application to a relatively simple case. Consider the second-order differential equation

$$Y'' + f_1(x)Y' + f_2(x)Y = f_3(x) \qquad (\text{at } x = a,\ Y = b;\ \text{at } x = c,\ Y = d) \quad (1)$$

where the functions, f_1, f_2, and f_3, are given, known functions and a, b, c, and d are known values. It is desired to find the value of Y' at $x = a$ [written as $Y'(a)$].

We first obtain the value of Y and its derivatives at the first few steps taken, starting at one boundary toward the other. This may be done by expanding the unknown function, Y, as a Taylor series near the starting boundary.

$$\left.\begin{aligned} Y &= Y(a) + \Delta x Y'(a) + \frac{(\Delta x)^2}{2!} Y''(a) + \frac{(\Delta x)^3}{3!} Y'''(a) + \frac{(\Delta x)^4}{4!} Y''''(a) + \ldots +, \text{etc.} \\ Y' &= Y'(a) + (\Delta x)Y''(a) + \frac{(\Delta x)^2}{2!} Y'''(a) + \frac{(\Delta x)^3}{3!} Y''''(a) + \ldots +, \text{etc.} \end{aligned}\right\} \quad (2)$$

The values of the $Y^n(a)$'s are found as follows: One boundary condition states that $Y(a)$ is b. $Y'(a)$ is unknown. From Eq. (1) we obtain $Y''(a)$ as

$$f_3(a) - f_1(a)\ Y'(a) - f_2(a)\ Y(a)$$

or

$$Y''(a) = F_2(a) + Y'(a)\ F_{22}(a)$$

The value of $Y'''(a)$ is obtained from Eq. (1) after the latter has been differentiated. Thus,

$$Y'''(a) = F_3(a) + Y'(a)\ F_{33}(a)$$

Similarly, the higher derivatives are found by successive differentiations of Eq. (1) as

$$Y^n(a) = F_n(a) + Y'(a)\ F_{nn}(a)$$

Substitution back into the Eqs. (2) yields

$$Y^m(x) = F_m(x) + Y'(a)F_{mm}(x) \quad (3)$$

for Y and its derivatives for the first few steps of integration.

We are now ready to apply any one of the available methods of numerical integration (the author prefers the Milne method[2]) to a step-by-step approach to the outer boundary. Once there, we require Eq. (4) to be satisfied.*

$$d = F_0(c) + Y'(a)\ F_{00}(c) \quad (4a)$$

or

$$Y'(a) = [d - F_0(c)]/F_{00}(c) \quad (4b)$$

The procedure just described is the one used by the author many times in manual numerical integrations of boundary-layer equations. It is, of course, directly applicable to digital machine computations. It is not useful, however, in connection with analog computer determinations. For the latter, the following related approach has been developed:

Suppose that, instead of carrying $Y'(a)$ along as an unknown in our analysis, we assign an arbitrary value, A, to it. Then, at the second boundary, we have Eq. (5) instead of Eq. (4a).

$$Y_A(c) = F_0(c) + AF_{00}(c) \quad (5)$$

* In a boundary-layer type of problem, c is infinite and therefore unattainable. In such a case, however, the value of Y at large values of x is essentially the same as at infinity. We therefore apply Eq. (4b) (in which x has been substituted for (c) at several successive, large values of x. When the values of $Y'(a)$ so determined do not change significantly with x, we have the value of $Y'(a)$ sought.

where $Y_A(c)$ differs from the required boundary value, d, and $F_0(c)$ and $F_{00}(c)$ are unknown.

Now, assign a second arbitrary value, B, to $Y'(a)$ and integrate. Eq. (6) is found applicable at the second boundary.

$$Y_B(c) = F_0(c) + BF_{00}(c) \quad (6)$$

where $Y_B(c)$ differs from the required boundary value, d, and $F_0(c)$ and $F_{00}(c)$ are the unknowns of Eq. (5).

$Y_A(c)$, $Y_B(c)$, A, and B being definite numbers, Eqs. (5) and (6) constitute a set of simultaneous equations from which the values of $F_0(c)$ and $F_{00}(c)$ can be determined.† These values of $F_0(c)$ and $F_{00}(c)$ are then substituted into Eq. (4b), and the proper value for $Y'(a)$ is determined.

The advantage of these methods is more fully appreciated when applied to more complex problems. For example, in a current application, two simultaneous fourth-order equations in two dependent variables with four boundary values specified at each boundary are handled. Because of the four simultaneous unknown conditions at each boundary, solution by means of a cut-and-try procedure is a herculean task—even when using an analog computer. The methods described, on the other hand, simplify the solution procedure to essentially five integrations—done simultaneously, if desired—and the subsequent solution of four simultaneous linear algebraic equations.

It is stimulating to consider the application of the same ideas to the solution of nonlinear total differential equations. A little thought on the matter reveals that, in the strictest sense, the above linear algebraic equations at the outer boundary are replaced by equations of infinite series. (If the step-by-step integration were actually carried out, however, we would obtain equations of finite, high-powered, polynomials instead because of the approximation involved in using steps of finite size.) Thus, it is apparent that there are a large number of *mathematical* solutions to a given nonlinear problem. Of course, in general, most of these solutions are complex or imaginary, but there is no guarantee that only one *real* solution exists. In other words, whereas we can be sure that the solution obtained to a linear differential equation will agree—barring numerical errors—with the physical solution, in the case of nonlinear equations, additional considerations (such as the stability of different solutions) must be brought to bear, in general, before we can be sure of such agreement.

Because of the relative directness of the numerical solution of linear differential equations, it would appear advantageous to solve complex nonlinear differential equations by a procedure involving linearization followed by successive perturbations. The relative merits of this and related procedures are to be studied shortly.

† In more complex problems having many unknown initial boundary conditions, a judicious choice of values for A, B, . . . , etc., greatly simplifies the set of simultaneous equations.

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Concluding Remarks on the Speed of Sound

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REFERENCES 1 AND 3 POINTED OUT the mathematical difficulties in the customary textbook derivations of the formula for the speed of sound

$$a^2 = (dp/d\rho) \tag{1}$$

based on the steady one-dimensional equations of nonviscous fluids. The difficulties center around the interpretations of differentials du, dp, $d\rho$, etc., occurring in the momentum, energy, and continuity equations. The usual interpretation employed in the derivations of these equations of motion—namely,

$$du = \left(\frac{du}{dx}\right) dx, \quad dp = \left(\frac{dp}{dx}\right) dx, \quad d\rho = \left(\frac{d\rho}{dx}\right) dx$$

etc., with the limiting process $dx \rightarrow 0$ in mind does not appear rigorously applicable. References 1 and 2 (as well as older references such as reference 4) have indicated that these difficulties disappear when the sound wave is considered as the limit of a shock wave as its strength approaches zero. Thus, if a shock-strength parameter is considered as the independent variable in the limiting process (rather than x) and the increments across the shock waves $\Delta u, \Delta p$, etc., are related to this parameter, the algebraic manipulation of the usual derivations become meaningful and rigorous. (It is believed that this limiting process is implied, though without explanation, in some of the textbooks, such as reference 5.)

One purpose of this note is to present clearly the details of this approach which are not usually available to the novice in the field of compressible flows. A second purpose is to present to the same public (which may have been somewhat confused by some of the disagreement between references 1, 2, 3, and, more recently, 6) the wide area of agreement reached through these discussions. It is felt that the Readers' Forum will further increase its present high usefulness to the general public if the positive results of any discussion or controversy are stated and emphasized. It is hoped that a precedent may thus be established.

Consider a discontinuity (stationary with respect to an observer) separating downstream flow characterized by u_2, p_2, ρ_2, and T_2, from upstream flow characterized by u_1, p_1, ρ_1, and T_1. The fact that the discontinuity is stationary implies that its speed of upstream propagation with respect to the fluid, a, is numerically equal to u_1. The feasibility of this one-dimensional flow system rests completely on the consistency of the conservation laws and the equation of state.

$$\rho_1 u_1 = \rho_2 u_2 \tag{C}$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{M}$$
$$(1/2)u_1^2 + C_p T_1 = (1/2)u_2^2 + C_p T_2 \tag{E}$$
$$p_1 = R\rho_1 T_1; \quad p_2 = R\rho_2 T_2 \tag{S}$$

Setting $u_2 = u_1 + \Delta u$, $p_2 = p_1 + \Delta p$, $\rho_2 = \rho_1 + \Delta\rho$, and eliminating T from Eq. (E) by the use of Eq. (S), the conservation laws can be written in the useful nondimensional incremental form

$$\left(\frac{\Delta u}{u_1}\right) + \left(\frac{\Delta \rho}{\rho_1}\right)\frac{1}{1 + (\Delta\rho/\rho_1)} = 0 \tag{C$'$}$$

$$(\Delta u/u_1) + (p_1/\rho_1 u_1^2)(\Delta p/p_1) = 0 \tag{M$'$}$$

$$\left[1 + \frac{\Delta\rho}{\rho_1}\right]\left[2\frac{\Delta u}{u_1} + \left(\frac{\Delta u}{u_1}\right)^2\right] + \frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1 u_1^2}\right)\left[\frac{\Delta p}{p_1} - \frac{\Delta\rho}{\rho_1}\right] = 0 \tag{E$'$}$$

The conservation laws thus form a system of three equations in the four nondimensional quantities: $p_1/\rho_1 u_1^2$, $\Delta u/u_1$, $\Delta p/p_1$, and $\Delta\rho/\rho_1$, all four of which are measures of the strength of the discontinuity. The system of equations is consistent, validating the original assumptions. Furthermore, one can arbitrarily assign values to one of these quantities (i.e., choose it as the strength parameter), say $(\Delta\rho/\rho_1)$, and expect that the other three will be determined from Eqs. (C′), (M′), and (E′). This is indeed the case; $(\Delta u/u_1)$ is immediately determined from Eq. (C′). Substituting for $(\Delta u/u_1)$ and $(\Delta p/p_1)$ from Eqs. (C′) and (M′) into Eq. (E′) yields, for $\Delta\rho/\rho_1 \neq 0$,

$$\frac{\Delta\rho}{\rho_1} = \left[1 - \gamma\left(\frac{p_1}{\rho_1 u_1^2}\right)\right] \Big/ \left[\frac{\gamma - 1}{2} + \gamma\left(\frac{p_1}{\rho_1 u_1^2}\right)\right] \tag{2}$$

Eq. (2) determines the quantity $(p_1/\rho_1 u_1^2)$ for every chosen value of $\Delta\rho/\rho_1$. Finally, with $(p_1/\rho_1 u_1^2)$ and $(\Delta u/u_1)$ known, Eq. (M′) determines $(\Delta p/p_1)$. Thus, as the strength parameter $\Delta\rho/\rho_1$ varies, the nondimensional characteristics of the flow system under consideration are completely and uniquely determined. Mathematically, a sound wave can be defined as a limit of shock waves for which the strength parameter approaches zero, $\Delta\rho/\rho_1 \rightarrow 0$. This limiting process applied to Eq. (2) results in the determination of the speed of propagation of sound

$$u_1^2 = a^2 = \gamma(p_1/\rho_1) \tag{3}$$

Eq. (3) leads to a more familiar interpretation of the nondimensional quantity $p_1/\rho_1 u_1^2$—namely, $1/\gamma M_1^2$—where M_1 is the upstream Mach Number. Incidentally, Eq. (2) now provides a useful formula for the compression $\Delta\rho/\rho_1$ in a finite shock wave

$$\Delta\rho/\rho_1 = (M_1^2 - 1)/\{1 + [(\gamma - 1)/2]M_1^2\}$$

Subtracting Eq. (C′) from (M′), one obtains

$$u_1^2 = (\Delta p/\Delta\rho)\,[1 + (\Delta\rho/\rho_1)]$$

Considering, as before, the limit of a weak shock wave and thus letting $\Delta\rho/\rho_1 \rightarrow 0$, it is seen that for a sound wave $u_1^2 = \lim_{\Delta\rho \rightarrow 0} (\Delta p/\Delta\rho)$. If, as explained previously, the quantities p_2, T_2, and u_2 in Eq. (C), (M), (E), and (S) are considered as functions of a shock-strength parameter $\Delta\rho/\rho_1$, then the limit of the ratio can be interpreted as an ordinary derivative. Hence,

$$u_1^2 = (dp/d\rho) \tag{4}$$

To show, by the weak-shock-wave approach, that a sound wave is isentropic, the entropy change,

$$\Delta S = S_2 - S_1 = C_v \log\left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}\right)^\gamma \tag{5}$$

across the discontinuity can be calculated in terms of the strength parameter $\Delta\rho/\rho_1$. Elimination of $\Delta u/u_1$, and $p_1/\rho_1 u_1^2$ between Eq. (C′), (M′), and (E′) leads to the well-known Hugoniot relation

$$\frac{p_2}{p_1} = \left(1 + \frac{\gamma + 1}{2}\frac{\Delta\rho}{\rho_1}\right) \Big/ \left(1 - \frac{\gamma - 1}{2}\frac{\Delta\rho}{\rho_1}\right) \tag{6}$$

Since ΔS in Eq. (4) can now be expressed as a sum of terms of the form $\log(1 + \text{const.}\ \Delta\rho/\rho_1)$, it can be easily expanded into a series that starts with the cubic term

$$\Delta S = \frac{C_v}{12}\cdot\gamma(\gamma^2 - 1)\left(\frac{\Delta\rho}{\rho_1}\right)^3 + 0\left(\frac{\Delta\rho}{\rho_1}\right)^4 \tag{7}$$

Hence,

$$\lim_{\Delta\rho \rightarrow 0} (\Delta S/\Delta\rho) = 0$$

and a sound wave viewed as the limit of a weak shock is isentropic.

It is thus seen that the familiar expressions (3) and (4) for the velocity of sound, as well as isentropy, can be derived by a simple means without any of the apparent mathematical inconsistencies which a direct differential-equation approach would contain. It should be noted, moreover, that the approach indicated here—namely, a shock wave becoming weak—seems to present a rather clearer physical picture than the situation implied by the use of the differential equations.

It is clear that a full presentation of the preceding lengthy analysis in a classroom may, if desired, be omitted if the essence of the argument is outlined.

Professor Clark B. Millikan has indicated to us that the preceding treatment is conceptually equivalent to the one he uses and that it is implied in essence in his recent note.[6]

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On the Calculation of the Potential Flow Around Airfoils in Cascade

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In order to improve the aerodynamic properties of airfoils in cascade, it is necessary to have a convenient method for the calculation of the potential velocity field. The method of conformal transformation, in our opinion, is much too complicated for practical purposes; only the "singularity method," where the airfoil is replaced by a certain distribution of vortices and sources and sinks, seems adequate. In some recent papers,[1–3] the singularity method has been applied to airfoils in cascade. The calculations on these lines can still be simplified considerably, as we have shown in a paper[7] to be published shortly in detail and an extract of which may be given here. Our method has already been applied with good success to the calculation of boundary layers on airfoils in cascade.[8]

According to the singularity method, each airfoil of the cascade is replaced by a continuous distribution of vortices and sources (and sinks), situated on the mean line of each profile. The main problem is to calculate the field of induced velocities due to these singularities.

In the complex $z = x + iy$-plane (Fig. 1), an infinite row of elements of singularity of strength $g \cdot dz'$ may be given situated on a straight line through the origin inclined at the angle λ against the y-axis. The distance of the airfoils being t, these elements have the complex coordinate

$$z_\nu = i\nu t\, e^{-i\lambda} \quad (\nu = 0, \pm 1, \pm 2, \dots)$$

In the general case, the strength g is complex, $g(z) = q(z) + i\gamma(z)$, $q(z)$ meaning the distribution of sources (and sinks) and $\gamma(z)$ meaning the vorticity. The field of induced velocities due to a single element $g(z')\cdot dz'$, situated at z_ν, is

$$dw_\nu(z) = du_\nu - i\, dv_\nu = g\, dz'/2\pi(z - i\nu t e^{-i\lambda})$$

Summing the contributions of all elements of the infinite row yields

$$dw(z) = \frac{g\, dz'}{2te^{-i\lambda}} \sum_{\nu=-\infty}^{+\infty} \frac{1}{(\pi z/te^{-i\lambda}) - i\pi\nu} \tag{1}$$

This infinite sum is identical with the series expansion of hyperbolic cotangent with the argument $e^{i\lambda}\pi z/t$, so that Eq. (1) can be written

$$dw(z) = \frac{e^{i\lambda}}{2t}\, g(z') \coth\left(\frac{\pi z}{t} e^{i\lambda}\right) dz' \tag{1a}$$

Summing up overall elements of the mean line, $z' = -c/2$ to $+c/2$, finally gives the induced velocity field of the whole cascade

$$w(z) = \frac{e^{i\lambda}}{2t} \int_{z'=-c/2}^{+c/2} g(z') \coth\left(\pi e^{i\lambda} \frac{z - z'}{t}\right) dz' \tag{2}$$

Here, λ is identical with the angle between the direction of the mean flow U_∞* along the x-axis and the normal to the cascade axis (Fig. 1).

For airfoils with small camber ($z' \approx x'$), and taking the induced velocity at the chord only (x-axis), one gets from Eq. (2)

$$w(x) = \frac{e^{i\lambda}}{2t} \int_{x'=-c/2}^{+c/2} g(x') \coth\left(\pi e^{i\lambda} \frac{x - x'}{t}\right) dx' \tag{3}$$

in analogy to the downwash on an isolated airfoil according to the theory of Birnbaum and Glauert.[4] In the limit $t \to \infty$, Eq. (3) simply gives

$$w(x)_{\text{isol.}} = \frac{1}{2\pi} \int_{x'=-c/2}^{+c/2} \frac{g(x')}{x - x'}\, dx' \tag{3a}$$

the well-known formula of an isolated airfoil. In the limit $t \to 0$, Eq. (3) can be put into the form

$$w(x) = \frac{\Gamma}{2t} - \int_{x'=-c/2}^{+c/2} \frac{g(x')}{t}\, dx' \tag{3b}$$

where Γ is the total circulation of one airfoil and Γ/t is the total circulation per unit length of the cascade axis.

For the numerical evaluation of Eq. (3), the singularity of the integrand at $x = x'$ is inconvenient. This singularity is a matter of the isolated airfoil only, and therefore it is reasonable to split off the induced velocity field of the isolated airfoil from the velocity field of the cascade. Thus, one gets from Eq. (3)

$$w(x)_{\text{casc.}} - w(x)_{\text{isol.}} = \frac{1}{2t} \int_{x'=-c/2}^{+c/2} g(x') \left[e^{i\lambda} \coth\left(\pi e^{i\lambda} \frac{x - x'}{t}\right) - \frac{t}{\pi(x - x')} \right] dx' \tag{4}$$

The numerical evaluation of Eq. (4) is convenient because the bracket [] is regular in the whole range of integration and is a *universal function* of $(x - x')/t$ and λ only, not dependent on the special airfoil. This universal function has been tabulated in reference 7. The evaluation of the velocity field of the isolated airfoil, Eq. (3a), can be done very simply by the method of Allen.[5] In this way, for a prescribed distribution of vorticity and sources, the induced velocity along the chord of the cascade airfoils can be easily calculated.

From $w_{\text{casc.}} = u_{\text{casc.}} - iv_{\text{casc.}}$ and with the superposed transverse velocity U_∞ along the x-axis, one gets the total velocity on the chord c of the cascade airfoil

$$W_c(x) = \sqrt{(U_\infty + u_{\text{casc.}})^2 + v_{\text{casc.}}{}^2} \tag{5}$$

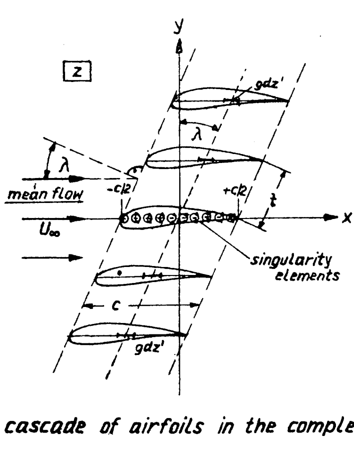

Fig. 1: A cascade of airfoils in the complex plane $z = x + iy$.

* This is the geometric mean value of the directions of flow far in front and behind the cascade.