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Vibrations of Skew Cantilever Plates

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References 1 and 2 describe the results of calculations of frequencies and nodal lines of vibrating rectangular cantilever plates. These calculations have been extended to vibrating skew cantilever plates. An IBM Fortran program is available. The plate is assumed to be vibrating transversely, in a single harmonic. Figure 1 gives the geometry of the plate.

In Refs. 1 and 2, the solution was obtained by a Fourier-sine-series method. Here, it was found convenient to use the Rayleigh-Ritz method. The mathematical development is the same as that of Ref. 3. The result of applying the Rayleigh-Ritz method is an infinite, real, symmetric matrix, for which the eigenvalues and eigenvectors are to be calculated. In the calculations the matrix is truncated in the usual fashion to successive finite-order matrices, and the limits of the eigenvalues and eigenvectors are evaluated numerically. The method used for calculating the eigenvalues and eigenvectors of the finite matrices is developed in Ref. 4.

In order to obtain any degree of accuracy in calculating the eigenvectors, and therefore the nodal lines, it has been found absolutely essential that the same finite-order matrix be used as was used for the eigenvalues. In fact, it is desirable to calculate the eigenvalues to several more significant figures than the number required for the nodal lines. This is in direct contrast to the Fourier-series method described in Refs. 1 and 2, where an estimated limit for the frequencies was used in calculating the nodal lines.

References 1 and 2 describe the variations of the frequencies and the nodal lines as functions of the ratio of sides a/b for a rectangular plate. Thus, the different harmonics were thought of as "frequency curves." It is now possible

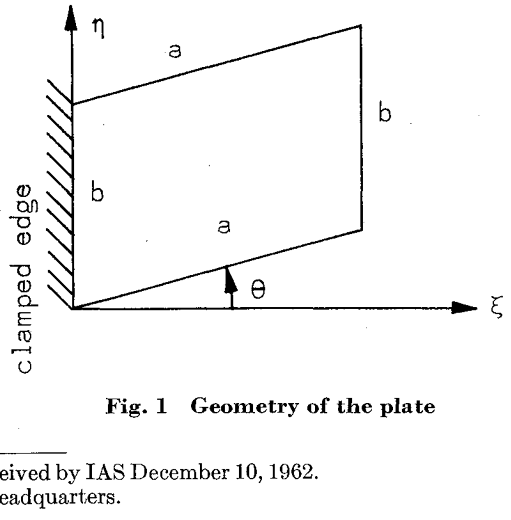

Fig. 1 Geometry of the plate

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to consider the frequencies and nodal lines as functions of two independent variables, a/b and θ. Instead of referring to "frequency curves," it is now possible to refer to "frequency sheets."

References 1 and 2 referred to "transition points," points at which the frequency curves should have crossed each other but actually refused to do so, markedly changing their curvature instead. It is now possible to state the existence of "transition curves," curves along which the frequency sheets refuse to cross each other but instead markedly change their curvature. However, along different segments of a transition curve a wide variation is possible in the distance between two frequency sheets. In fact, the sheets can actually touch (become tangent to) each other at isolated points.

As in Refs. 1 and 2, the nodal lines rotate about one or several points as the frequency sheets change their curvature. Reference 5 gives a detailed description of the mathematical development and the results, as well as the Fortran program used in calculating the eigenvalues and eigenvectors.

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An Example of Boundary Layer Formation

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A major difficulty in the teaching of fluid dynamics is the lack of a simple exact solution of the Navier-Stokes equations in which both the viscous and inertial forces are active. Viscous forces only are involved in Poiseuille and Couette flows, and consequently the velocity fields are independent of the Reynolds number. There are two exact solutions that depend on viscous and inertial forces, namely the von Kármán flow produced by a rotating disk and the Jeffrey-Hamel flow in a converging or diverging channel. Interesting as these solutions are, they suffer from the disadvantage of requiring the solution of nonlinear differential equations, and the velocity fields cannot be expressed in simple terms. For teaching purposes, a solution is required which can be expressed in simple functions, is exact, and involves a balance between viscous and inertial forces, so that the dependence on the Reynolds number can be exhibited and the formation of a boundary layer as the Reynolds number increases demonstrated. It also would be helpful if a class of solutions rather than a single solution were known, to provide examples for the student to find for himself. It does not seem to be recognized widely that a class of solutions satisfying these requirements exists, a particular case

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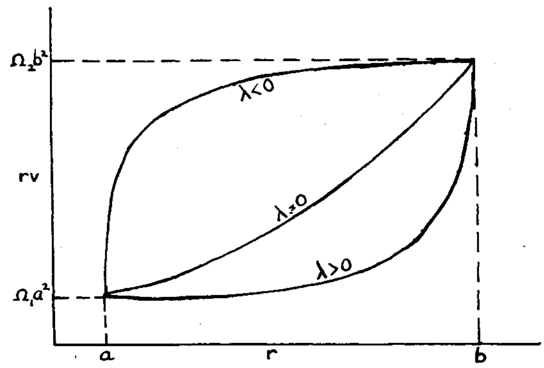

Fig. 1 The circulation for various λ, showing the presence of a boundary layer for large $|\lambda|$

having been discussed by Preston.[1] The purpose of this note is simply to bring this class of solutions to the attention of those engaged in the teaching of fluid dynamics.

The fluid is incompressible and is bounded by two long cylinders of radii a and b, rotating with angular velocities Ω_1 and Ω_2. The walls of the cylinders are porous, and fluid is emitted by one cylinder and absorbed by the other at equal rates. The motion depends on only one space coordinate, the radial distance r, and there are two velocity components, u radial and v transverse. The reason for the existence of a simple, exact solution is that the radial velocity is unaffected by the transverse velocity; but the reverse is not true, which provides the required interaction between viscous and inertial forces. The equations of motion and continuity are

$$u\frac{du}{dr} - \frac{v^2}{r} = -\frac{dp}{\rho dr} + \nu\left(\frac{d^2u}{dr^2} + \frac{du}{rdr} - \frac{u}{r^2}\right) \qquad (1)$$

$$u\frac{dv}{dr} + \frac{uv}{r} = \nu\left(\frac{d^2v}{dr^2} + \frac{dv}{rdr} - \frac{v}{r^2}\right) \qquad (2)$$

$$(d/dr)(ru) = 0 \qquad (3)$$

These equations have a solution

$$u = m/r \qquad (4)$$

$$V = (A/r) + Br^{1+\lambda} \qquad (5)$$

where m is the strength of the source in the inner cylinder (negative if the inner cylinder is absorbing fluid), and $\lambda = m/\nu$; $|\lambda|$ is the Reynolds number of the radial motion. If $\lambda = -2$, the expression for v becomes

$$v = (A/r) + (B/r)\log r \qquad (6)$$

The boundary conditions are $v = \Omega_1 a$ on $r = a$, and $v = \Omega_2 b$ on $r = b$; these determine the coefficients A and B to be

$$A = \frac{a^2b^2(\Omega_1 b^\lambda - \Omega_2 a^\lambda)}{b^{2+\lambda} - a^{2+\lambda}} \qquad B = \frac{\Omega_2 b^2 - \Omega_1 a^2}{b^{2+\lambda} - a^{2+\lambda}} \qquad (7)$$

and for the exceptional case $\lambda = -2$,

$$A = \frac{\Omega_1 a^2 \log b - \Omega_2 b^2 \log a}{\log b - \log a} \qquad B = \frac{\Omega_2 b^2 - \Omega_1 a^2}{\log b - \log a} \qquad (8)$$

The circulation rv has a typical behavior as indicated in Fig. 1, from which the formation of a boundary layer as the Reynolds number increases may be seen. For large Reynolds numbers, the circulation remains constant as it is convected across the region between the cylinders until it reaches the cylinder that is absorbing fluid, where viscous forces produce a sudden change in the value determined by the boundary condition. The division of the flow into a potential flow region and a boundary layer thus is demonstrated clearly.

Other solutions of a similar type which the reader (or his students) will easily obtain are provided by introducing a pressure gradient around the annulus with the cylinders at rest (flow in a curved channel), or by making the cylinders move at different rates parallel to their axes, or by introducing a pressure gradient in this direction. These four types can all be superimposed. Limiting cases when the outer cylinder tends to infinity, or the inner one shrinks to zero, provide some interesting results.[1] Another set of limiting cases is provided by keeping the difference in the radii fixed and letting the radii tend to infinity so that flow between parallel walls is obtained. All these solutions provide useful material for classroom discussion, but their common feature is the change in velocity profile to a boundary layer type as the Reynolds number increases, which the author feels is important to be able to demonstrate in a simple manner.

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Long Circular-Cylindrical Shells Subjected to Circumferential, Radial Line Loads

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In Ref. 1, expressions for stresses and displacements of a long circular-cylindrical shell subjected to a uniform, external, circumferential, radial line load are derived by using complex Fourier transform of Love's stress function ϕ for an axisymmetrical problem, a technique used by Tranter and Craggs.[2] In this note the solution is derived by a direct procedure similar to the solution given for a solid circular cylinder in the recent handbook edited by Flügge.[3]

Governing Equations

It is required to find a stress function ϕ to satisfy the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \phi = 0 \qquad (1)$$

for the cylinder shown in Fig. 1, with the additional conditions along the internal boundary C_i

$$\int_{C_i} du = 0 \qquad \int_{C_i} dv = 0 \qquad \int_{C_i} dw = 0 \qquad (2)$$

The stresses and displacements are given by

$$\begin{aligned}
\sigma_r &= (\partial/\partial z)[\nu\nabla^2 - (\partial^2/\partial r^2)]\phi \\
\tau_{rz} &= (\partial/\partial r)[(1-\nu)\nabla^2 - (\partial^2/\partial z^2)]\phi \\
\sigma_\theta &= (\partial/\partial z)[\nu\nabla^2 - (1/r)(\partial/\partial r)]\phi \\
\sigma_z &= (\partial/\partial z)[(2-\nu)\nabla^2 - (\partial^2/\partial z^2)]\phi \\
u &= [(1+\nu)/E][2(1-\nu)\nabla^2 - (\partial^2/\partial z^2)]\phi \\
w &= [-(1+\nu)/E](\partial^2\phi/\partial r\partial z)
\end{aligned} \qquad (3)$$

The boundary conditions are, when

$$r = b \qquad \tau_{rz} = 0 \text{ and } \sigma_r = 0 \qquad (4)$$

and when

$$r = a \qquad \tau_{rz} = 0 \text{ and } \sigma_r = -f(z)$$

where $f(z)$ is the applied loading on the outer boundary, as-

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