and as a discontinuity in f' has already been excluded

$$(dM/dM_0)^* = 0$$

This relation, first obtained by Liepmann and Bryson,[4] is satisfied by the experimental results typified in Fig. 2. It is here derived without the necessity of identifying M_0 with M_2, which leads to the incorrect variation of Fig. 1.

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On Almost-Free-Molecule Flow Through an Orifice†

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THE APPLICATION of the principle of almost-free-molecule flow, which is essentially a form of first-order Knudsen iteration of rarefied-gas dynamics, has shown some very encouraging results.[1, 2] The effusion of rarefied gases through an orifice into a vacuum is a very instructive problem for the purpose of studying the basic nature of rarefied-flow phenomena. The object of the present analysis is to provide a microscopic description of the flow parameters pertaining to the steady effusion from an orifice, the diameter (D) of which is of the same order or smaller than the mean free path (λ) of the reservoir gas. A thin diaphragm ($t/D << 1$) which has a small circular orifice separates a large high-pressure (p_1) reservoir from the low-pressure (p_2) region. The pressure ratio will be assumed large enough ($p_1/p_2 > 10^3$) to permit neglect of the back flow from the low-pressure side. This condition distinguishes the present problem from the pitot-pressure problem of reference 1.

Consider first the case of free-molecule effusion, where $\lambda >> D$ and molecules move through the orifice essentially independent of each other. The deviations of the resulting molecular distribution from its equilibrium state will be negligibly small and promptly wiped out by the intermolecular collisions, which always tend to set up and preserve the equilibrium state. The loss of molecules through the orifice, however, develops a trace of mass motion toward the orifice due to absence of those collisions that the lost molecules would have made with the ambient molecules on their return from the wall. This trace of orifice-bound mass motion grows in prominence as λ/D decreases. The principle of almost-free-molecule flow is applied here to calculate the molecular flux of the mass motion within the reservoir as a result of intermolecular collisions or their absence. The following physical model is devised for this purpose.

Imagine the orifice were closed with an imaginary disc of diameter D as the orifice; then equilibrium (Maxwellian) distribution of molecules in the reservoir would be restored through scattering of reflected molecules, from the imaginary disc, with the ambient molecules. It is postulated that the net molecular flux toward the orifice, inhibited by the scattering action of the imaginary reflected molecules, is equal to the difference between the true effusion flux through the orifice and its calculated value based on free-molecule hypothesis. It is assumed that this molecular flux, due to absence of scattering, amounts to only a small fraction of the corresponding free-molecule effusion flux. This implies that the present theory is valid only when D is not much larger than λ, so that the single-collision analysis used here is acceptable. The rate of collisions (N_{id}) between the molecular rays incident on, and reflected from the imaginary disc is calculated for rigid spheres on the basis of classical kinetic theory. The cross section of the spheres is taken from the experimental determination of the mean free path. It is assumed that every collision throws the incident molecule out of the impinging stream on the disc. This assumption can be justified only when λ is not very small compared to D. We assume that the molecules inside the "closed" reservoir are now in thermodynamic equilibrium with Maxwellian distribution. To determine the net contribution, through intermolecular collisions, of the hypothetical reflected molecules, we need to calculate the molecular flux (N_{dd}), originated from the diaphragm surrounding the orifice, that is thrown into the incident ray through their collisions with the reflected molecules from the disc. These molecules (N_{dd}) would otherwise not impinge on the orifice area. Thus, the net impingement inhibition for the imaginary reflection ray is equal to ($N_{id} - N_{dd}$).

Let N_d be reflected molecular flux from the disc, which is equal to $\pi D^2 nc/16$ where n = molecular density, c = mean thermal velocity; $E(E = 0.92/\lambda$ for air) the average collision expectancy per unit distance traveled by a single molecule moving through molecules under equilibrium (Maxwellian) distribution.[2]

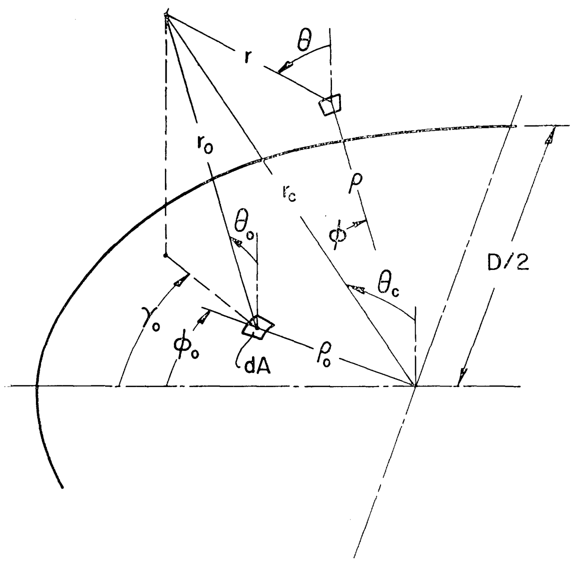

FIG. 1. Geometry of scattering analysis.

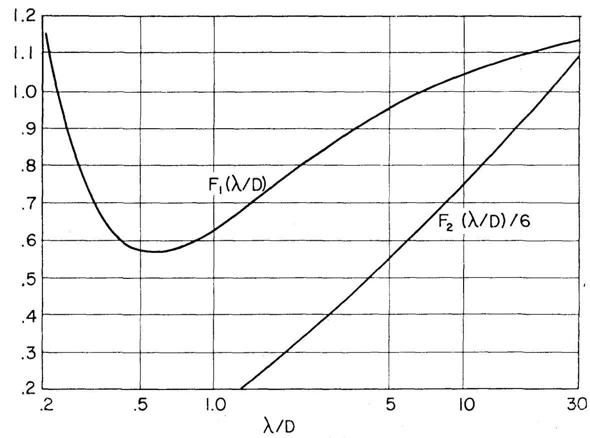

FIG. 2. Functions F_1 (λ/D) and $F_2(\lambda/D)$.

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Referring to the coordinate system shown in Fig. 1, we obtain

$$\frac{N_{id}}{N_d} = \frac{E}{4\pi^2}\int_0^{2\pi} d\phi_0 \int_0^{2\pi} d\gamma_0 \int_0^{\pi/2} d\theta_0 \int_0^{D/2} d\rho_0 \int_0^{\infty} \frac{\rho_0 r_0 \sin\theta_0 \cos^2\theta_0 e^{-2r_0/\lambda}}{[r_0^2 + \rho_0^2 + 2r_0\rho_0 \sin\theta_0 \cos(\gamma_0 - \theta_0)]^{3/2}} dr_0 \quad (1)$$

Let ρ and ϕ be the polar coordinates of an area element of the diaphragm outside the orifice area; r the distance from the point r_0 to this elemental area; θ, the angle between r and the normal to this elemental area (see Fig. 1). We obtain

$$\frac{N_{dd}}{N_d} = \frac{E}{8\pi^3}\int_0^{2\pi} d\phi_0 \int_0^{2\pi} d\gamma_0 \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta_0 \int_{D/2}^{\infty} d\rho \int_0^{D/2} d\rho_0 \int_0^{\infty} \frac{\rho\rho_0 r_0^2 \sin\theta_0 \cos^3\theta_0 e^{-2r_0/\lambda}}{r_0^3 r^3} dr_0 \quad (2)$$

After the exponential functions in the integrands are approximated with appropriate parabolic representations (these approximations remain close when D is not much larger than λ), we obtain from Eqs. (1) and (2)

$$N_{id}/N_d \simeq 0.153\ (D/\lambda)\ F_1(\lambda/D) \quad (3)$$

$$N_{dd}/N_d \simeq 0.019\ (D/\lambda)\ F_2(\lambda/D) \quad (4)$$

where $F_1(\lambda/D)$ and $F_2(\lambda/D)$ are given in Fig. 2.

From the hypothesis that the intermolecular-collision effect on the effusion rate is equal to the molecular flux inhibited from hitting the orifice area when closed—namely $(N_{id} - N_{dd})$, we have

$$N/N_F = 1 + [(N_{id} - N_{dd})/N_F] \quad (5)$$

where N_F is the free-molecule effusion rate through the orifice (note that $N_d = N_F$ as a first approximation). A graph of N/N_F as a function of λ/D is shown in Fig. 3, in which are also shown the experimental results[3] which were taken with $p_1/p_2 > 10^3$. It is felt that with refined analysis—e.g., taking into account the contribution to the impinging flux from molecules, emerging from collisions (N_{id}), that are deflected toward the orifice area, the validity of the theory can be extended to lower values of λ/D. The present analysis appears fruitful in illustrating the functional dependency of the effusion-flow rate on the intermolecular collisions.

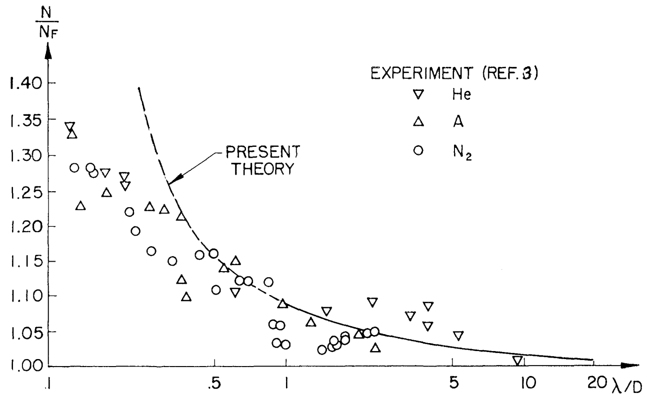

FIG. 3. Effusion rate vs. λ/D.

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An Equivalence Principle for Water-Exit and -Entry Problems†

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THERE EXISTS an extensive body of literature concerned with the impact forces exerted on a body entering water, a review of which was recently made by Szebehely,[1] while a smaller, but growing, number of analyses deals with problems of water exit.[2–6] However, it does not seem to have been recognized in these previous studies that, outside of viscous effects, there is a definite equivalence between problems of water-exit and -entry (provided that there is no cavitation).

To simplify the demonstration of this equivalence principle, we shall consider the vertical exit or entry of a symmetric body in uniform axial motion, as shown in Fig. 1. The concept is readily generalized to other situations, as will be noted below.

As usual, we assume the flow to be incompressible and irrotational. We may, therefore, work in terms of a potential φ, whose gradient is the fluid velocity and which satisfies Laplace's equation by continuity:

$$\nabla^2\varphi = 0 \quad (1)$$

An integration of the Eulerian equations of motion yields Bernoulli's equation, which in the space-fixed coordinates (x', y', t') defined in Fig. 1 is

$$\partial\varphi/\partial t' + (1/2)(\nabla\varphi)^2 + (p - p_s)/\rho + gy' = 0 \quad (2)$$

where p denotes the pressure, g the acceleration due to gravity, and the constant of integration has been evaluated on the water surface, $y' = y_s'$, far from the body creating the motion.

Neglecting the density of the air above the surface relative to that of the water below, we require that the pressure on the surface, p_s, be a constant. Thus, from Eq. (2)

$$gy_s' = -[\partial\varphi/\partial t' + (1/2)(\nabla\varphi)^2]_{y'=y_s'} \quad (3)$$

It is somewhat more convenient to work in the dimensionless body-fixed coordinate system (x, y, Z) defined in Fig. 1 and related to (x', y', t') by

$$x = x'/L, \quad y = Z - y'/L, \quad Z = \pm Ut/L,$$
$$\phi(x, y, Z) = \varphi(x', y', t')/UL \quad (4)$$

Where a double sign is indicated, the upper sign refers to the exit problem and the lower to the entry situation. The free-surface boundary condition, Eq. (3), now transforms to

$$gy_s'/U^2 = -[\pm D\phi/Dt + (1/2)(\nabla\phi)^2]_{y=Z-y_s'/L} \quad (5)$$

where $D(\ )/Dt$ denotes $\partial(\ )/\partial y + \partial(\ )/\partial Z$.

The boundary condition of no flow through the body may be written

$$dX(y)/dy = \pm\left[\frac{\partial\phi/\partial x}{U \pm \partial\phi/\partial y}\right]_{x=X(y)} \quad (6)$$

Now taking note of the homogeneity of Laplace's equation, Eq. (1), and the way in which the signs change in the boundary

† Based on a portion of reference 8 in which the principle was demonstrated for slender bodies. The author is indebted to Dr. W. R. Sears for suggesting the problem and supervising the research, and also to Dr. S. H. Lam, for pointing out that the existence of the principle does not depend on the slenderness of the body.

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