A Note on the Velocity of Sound

M. V. Morkovin
University of Michigan, Ann Arbor, Mich.
October 17, 1949

Reference 1 called attention to a certain lack of rigor found in most of the elementary derivations of the formula for the speed of sound. The main point of the criticism was that the ratio $dp/d\rho$ was indeterminate in the sound wave and should not be treated as a derivative. While agreeing about the lack of rigor in the customary derivations, the present note concludes that the key to the confusion is not the alleged indeterminateness of $dp/d\rho$ but the lack of clarity in specifying the variables involved, especially the independent variables. In his counterexample, the author of reference 1 unfortunately forgets which quantities are variable and which are constant under the original assumptions of his problem.

(1) The assumptions of the problem are those of steady, continuous, adiabatic flow, without viscosity, in a duct of constant area. Then and only then Eqs. (1), (2), and (3) of reference 1—namely, $\rho\, du + u\, d\rho = 0$, $\rho u\, du + dp = 0$, and $u\, du + C_p\, dT = 0$—are valid simultaneously. These assumptions imply a flow at constant entropy, s, as the elimination of the term $C_p\, dT$ [which equals $T\, ds + (dp/\rho)$] between Eqs. (2) and (3) demonstrates. As the author of reference 1 remarks, these equations admit only the "trivial" solution u = constant, p = constant, ρ = constant; in an isentropic flow through a one-dimensional duct, there is simply no agency that could cause a change in these quantities.

The counterexample of reference 1 considers the particular solution of these equations, $u^2 = \gamma p/\rho$. This is one of the family of the constant solutions above, for which the constant values of the quantities u, p, and ρ are numerically related. Under the assumptions of the duct problem $du = 0$, $dp = 0$, and $d\rho = 0$, Eqs. (1), (2), and (3) are satisfied *without any contradiction.*

In order to allow a continuous change of u, p, and ρ, in steady flow through a constant area duct, as reference 1 assumes without explicit statement, one of the other assumptions must yield. The usage of Eq. (2) in the counterexample implies that the flow is still assumed nonviscous but that heat may be added steadily. This and the usage of $u^2 \equiv \gamma p/\rho$ *as an identity* for all flows in the given duct of necessity leads to nonisentropic processes characterized by Eq. (7) of reference 1. There is no contradiction, merely a misunderstanding.

(2) When more complicated flows are considered, such as viscous and diabatic flows, a more careful specification of the thermodynamic process corresponding to the phenomenon of propagation of sound is in order. The properties of a homogenous medium that is in thermodynamic equilibrium depend upon two independent variables, say ρ and s. In particular, $p = f(\rho, s)$. Following reference 2, a complete and rigorous definition of the speed of sound is obtained through the identity

$$a^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s \equiv f_\rho(\rho, s) \tag{4}$$

It is immediately recognized that the use of the ratio $dp/d\rho$ from Eqs. (1), (2), and (3) is covered by definition (4).

(3) In order to dispose of the question of indeterminateness of the ratio $dp/d\rho$, it is merely necessary to use exactly the same reasoning as reference 1 did in its derivation of the limit $u_1^2 \to \gamma p_1/\rho_1$ as a shock wave approaches a sound wave. A somewhat tedious manipulation of the three conservation laws valid across a shock wave[3] leads to

$$\frac{p_2 - p_1}{\rho_2 - \rho_1} = \gamma \frac{p_1 + [(p_2 - p_1)/2]}{\rho_1 + [(\rho_2 - \rho_1)/2]} \tag{5}$$

As the strength of the shock wave vanishes, the ratio in question approaches not only a determinate value but also the "correct" one, $\gamma p_1/\rho_1$. Since, furthermore, $(s_2 - s_1)$ approaches zero, the concept of a sound wave as a limit of a shock wave checks with definition (4). Incidentally, Eq. (5) is valid for unsteady flows, and our deductions are general.

(4) Finally, it is proper to remark that Lamb's "rigorous" derivation of the equation for the speed of sound, recommended by reference 1, is subject to exactly the same criticism as the elementary derivations originally criticized. Lamb[4] assumes the determinateness of $dp/d\rho$ and a specific thermodynamic process with ρ as independent variable when he replaces $\partial p/\partial x$ by $(dp/d\rho)(\partial\rho/\partial x)$. The further assumption of infinitesimal pressure and density changes identifies the process as isentropic.

References

[1] Morduchow, M., *A Note on the Velocity of Sound,* Readers' Forum, Journal of the Aeronautical Sciences, Vol. 16, No. 10, p. 635, October, 1949.
[2] Courant, R., and Friedrichs, K. O., *Supersonic Flow and Shock Waves,* p. 5; Interscience Publishers, New York, 1948.
[3] Sauer, R., *Introduction to Theoretical Gas Dynamics,* English Ed., pp. 99–101; Edwards Bros., Ann Arbor, Mich., 1947.
[4] Lamb, H., *Hydrodynamics,* 6th Rev. Ed., pp. 476–477; London, 1932; reprinted by Dover Publications, New York, 1945.

Further Remarks on the Velocity of Sound

Morris Morduchow
Department of Aeronautical Engineering and Applied Mechanics, Polytechnic Institute of Brooklyn, Brooklyn, N.Y.
November, 1949

I should like, if possible, to eliminate any confusion that may arise from a comparison of references 1 and 2.

In reference 1, item (1), it is at first agreed that Eqs. (1), (2), and (3) of reference 2 imply u, p, ρ, and T to be constants (with the space variable x). This, however, appears to be forgotten in the second part of item (1), reference 1, where it is attempted to explain the contradiction pointed out by Eqs. (4)–(7) in reference 2 with the statement that, if the equation

$$u^2 = \gamma(p/\rho) \tag{a}$$

is treated as an identity, then this merely replaces Eq. (3) of reference 1 and implies a "nonisentropic process." The fact that is overlooked in reference 1 is that the set of Eqs. (1) and (2) of reference 2, and Eq. (a) will also imply only the "trivial" type of solution: p, ρ, u constant. Consequently, these equations cannot imply any "process" at all in which p may be considered as varying as a function of ρ. It is implied in reference 1 that Eq. (7) of reference 2 represents such a "process." However, a different and contradictory "process" could have been mathematically arrived at by substituting $u\, du = (1/2)\gamma d(p/\rho)$ into Eq. (1), instead of Eq. (2), of reference 2. Then one would obtain $p = k/\rho$, contradicting Eq. (7). Moreover, contrary to the belief in reference 1, even an "isentropic process" could be mathematically arrived at by substituting Eq. (a) into Eq. (4) [which was obtained from Eqs. (1) and (2)] of reference 2. As implied in reference 2, such apparent contradictions are bound to occur when equations like Eqs. (1), (2), and (3) of reference 2—or Eqs. (1), (2), and (a)—are incorrectly treated as if the quantities p and u could be considered as dependent variables with ρ as independent variable, instead of remembering that all of the quantities u, p, T, and ρ are really constants in these equations, with the space variable x as the independent variable. (If there does exist any confusion with respect to dependent and independent variables, it is thus in reference 1.)