$$A'_{b\alpha}(K,M,c) = \left(\frac{1-c}{2}\right)\left(1+\frac{j}{k}\right)A'_{bh}(k,M,c) - \left(\frac{1+c}{2}\right)^4\left[A_{a\alpha}\left(\frac{1+c}{2}k,M\right) + A_{c\alpha}\left(\frac{1+c}{2}k,M\right)\right]$$

(c) Eq. (22) should read

$$A_{b\alpha}(k,M,c) = A_{a\alpha}(k,M) + cA_{c\alpha}(k,M) - \left(1+\frac{j}{k}\right)\left(\frac{j+c}{2}\right)\left(\frac{j+c}{2}\right)^3\left[A_{ah}\left(\frac{1+c}{2}k,M\right) + A_{ch}\left(\frac{1+c}{2}k,M\right)\right] - \left(\frac{1+c}{2}\right)^4\left[A_{a\alpha}\left(\frac{1+c}{2}k,M\right) + A_{c\alpha}\left(\frac{1+c}{2}k,M\right)\right]$$

The numerical results given in the paper are rather inaccurate (particularly those for $R_{b\alpha}$ and $I_{b\alpha}$), and more accurate results are referenced in a recent report by Biot, et al.[1]

Reference

[1] Karp, S. N., Shu, S. S., Weil, H., and Biot, M. A., *Aerodynamics of the Oscillating Airfoil in Compressible Flow*, F-TR-1167-ND (October, 1947) and F-TR-1195-ND (June, 1948), Hq., A.M.C. Wright Field, Dayton, Ohio.

A Method for Accelerating the Convergence of an Iteration Procedure

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In most problems in which an iteration or successive approximation procedure is used in order to obtain a solution, the rate of convergence decreases as the iteration proceeds. Consequently, high orders of accuracy become increasingly difficult to achieve, and a large number of iterations may be required. In such problems a great deal of time and labor may be saved if the values in a small number of successive approximations can be used to predict the final correct value.

In the present method the assumption is made that the successive values obtained in an iteration procedure follow closely the form of a simple exponential approaching its asymptote. That is, if one value is considered as being the first in a series, the value obtained after the nth subsequent iteration is given by,

$$x = X + Ae^{-an} \quad (1)$$

where X is the asymptotic value and A and a are constants. If x_0, x_1, and x_2 are three values obtained in successive iterations, we may write,

$$x_0 = X + A \quad (2)$$

$$x_1 = X + Ae^{-a} \quad (3)$$

$$x_2 = X + Ae^{-2a} \quad (4)$$

Subtracting Eq. (3) from Eq. (2) gives,

$$x_0 - x_1 = A\ (1 - e^{-a}) \quad (5)$$

Subtracting Eq. (4) from Eq. (3) gives,

$$x_1 - x_2 = Ae^{-a}(1 - e^{-a}) \quad (6)$$

Dividing Eq. (6) by Eq. (5),

$$(x_1 - x_2)/(x_0 - x_1) = e^{-a} \quad (7)$$

Substituting Eq. (7) into Eq. (5) we obtain,

$$A = \frac{x_0 - x_1}{1 - [(x_1 - x_2)/(x_0 - x_1)]} = \frac{(x_0 - x_1)^2}{x_0 - 2x_1 + x_2} \quad (8)$$

Substituting Eq. (8) into Eq. (2) yields, finally,

$$X = x_0 - [(x_0 - x_1)^2/(x_0 - 2x_1 + x_2)] \quad (9)$$

It is thus seen that the asymptotic value may be simply determined from three successive approximations. This asymptotic value will not, in general, be the true value being sought, but it will frequently lie much closer to it than the three values from which it was derived. If greater accuracy is required, the value obtained in this way may be used as the starting point for subsequent iterations and further acceleration.

It should be noted that the calculation involves small differences between successive approximate values, so that a fairly large number of significant figures must be carried if accuracy is to be achieved.

The method has been applied successfully in the matrix iteration procedure for determining the natural mode shapes and frequencies of vibration of an elastic system. The convergence of the mode coefficients to their correct value is accelerated in this manner, and it has been found that in many cases two or more significant figures are gained in each acceleration cycle. It is frequently necessary to defer the acceleration till five or six iterations have been performed, since the variation in the first two or three iterations is sometimes erratic.

In some iteration solutions successive values oscillate about an asymptote. In such cases the acceleration cannot be applied directly to successive values but may be applied to alternate values. A simple rough test for the applicability of the method is that the difference between the second and third values should be of the same sign and somewhat smaller than the difference between the first and second values in the sequence.

It is interesting to note that Aiken's δ^2 process,[1] developed specifically for application to matrix iteration in the solution of characteristic value problems, yields the same expression as Eq. (9).

Reference

[1] Aitken, A. C., *Studies in Practical Mathematics*, Part III, p. 269: Proceedings of the Royal Society of Edinburgh, 1936–1937.

On the Compressibility Correction Factor for Axially Symmetric Bodies

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Several papers have appeared dealing with the problem of the application of the Prandtl-Glauert method to subsonic compressible flow over three-dimensional bodies.[1–4] Attempts have been made using this notion to establish a compressibility correction factor for three-dimensional bodies similar to the familiar compressibility correction factor $(1 - M^2)^{-1/2}$ applied to two-dimensional bodies. It is generally agreed that the Prandtl-Glauert method will not yield a universal correction factor for three-dimensional bodies which is independent of the shape of the body. It is also generally agreed that the effects of compressibility on the surface pressures of a body in a subsonic stream become less pronounced as the body is made more slender. This latter behavior has been illustrated by examining the maximum surface velocity in the field of flow about ellipsoids while varying thickness ratios, aspect ratios, and free-stream subsonic Mach Number.

The statement is made by Sauer[5] that, to a first approximation, the surface pressures over a slender axially symmetric body of revolution situated in a compressible stream remain unchanged from the surface pressures experienced in an incompressible stream at the same velocity. This statement is supported by an analysis employing appropriate source-sink distributions to approximate the presence of a body in the stream.

The mechanisms of this particular analysis remain somewhat obscure. The purpose of this note is to present a clearer illustration of the statement that to a first approximation there is no effect of compressibility on the surface pressures of a slender axially symmetric body in a subsonic stream.

The partial differential equation of continuity for the perturbation velocity potential in an incompressible stream in cylindrical coordinates is, for axially symmetric flow:

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial r'^2} + \frac{1}{r'}\frac{\partial\phi'}{\partial r'} = 0 \tag{1}$$

where ϕ' = antigradient of perturbation velocities. A general solution of Eq. (1) is

$$\phi' = -\frac{1}{4\pi}\int_{-c/2}^{c/2} \frac{f(\xi)d\xi}{\sqrt{(x' - \xi)^2 + r'^2}} \tag{2}$$

where $f(\xi)$ is a function representing the strength distribution of sources and sinks distributed continuously along the x-axis between the vertex of any given body at $x' = -c/2$ and the end of the body at $x' = c/2$.

Prandtl and Glauert are credited with introducing the following affine transformation relating the incompressible space x', r', ω' with a compressible space x, r, ω

$$\left.\begin{aligned} x &= x' \\ r &= {}^1/_\beta\, r' \\ \omega &= \omega' \\ \phi &= K\phi' \end{aligned}\right\} \tag{3}$$

where K is any multiplying factor.

Employing this transformation in conjunction with Eq. (1), there results the linearized partial differential equation of continuity for the perturbation velocity potential in a compressible stream for axially symmetric flow.

$$\beta^2\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0 \tag{4}$$

where $\beta = \sqrt{1 - M^2}$ and M = free-stream Mach Number. Corresponding to Eq. (2), a solution to Eq. (4) using transformation (3) is

$$\phi = -(K/4\pi)\int_{-c/2}^{c/2} f(\xi)\, d\xi/\sqrt{(x - \xi)^2 + \beta^2 r^2} \tag{5}$$

Eq. (5) for slender bodies is subject to the boundary condition that the flow at the body surface must be tangent to the body surface.

$$\phi_r/(U + \phi_x) = (\phi_r/U)_{r\to 0} = \tan\theta \tag{6}$$

where θ = angle between tangent to surface of body and x-axis and U = free-stream velocity.

If the given body is to retain its shape no matter what the variation of free-stream Mach Number, then

$$\tan\theta = (\phi_r/U)_{r\to 0} = \text{constant} \tag{7}$$

Returning to Eq. (5)

$$\phi_r = \frac{K\beta^2 r}{4\pi}\int_{-c/2}^{c/2} \frac{f(\xi)d\xi}{[(x - \xi)^2 + \beta^2 r^2]^{3/2}} \tag{8}$$

This integral is to be evaluated as r approaches a small quantity hereafter designated as $r \to \epsilon$. This can be done in a more lucid manner than the process employed by Sauer.[5]

$$\phi_r = \frac{K}{4\pi r}\int_{-c/2}^{c/2} \frac{f(\xi)\,(d\xi/\beta r)}{\{[(x - \xi)/\beta r]^2 + 1\}^{3/}} \tag{9}$$

let

$$\frac{x - \xi}{\beta r} = t, \qquad \frac{x + (c/2)}{\beta r} = a, \qquad \frac{x - (c/2)}{\beta r} = b$$

then

$$\phi_r = -\frac{K}{4\pi r}\int_a^b f(x - \beta r t)\,\frac{dt}{(t^2 + 1)^{3/2}} \tag{10}$$

Eq. (10) is evaluated now as $r \to \epsilon$ by taking advantage of the mean value theorem. As $r \to \epsilon$, $a \to \infty$, $b \to -\infty$, and $f(x - \beta r t) \to f(x_1)$, where x_1 is some value of x between $x = c/2$ and $x = c/2$

$$(\phi_r)_{r\to\epsilon} = \frac{-Kf(x_1)}{4\pi r}\int_{\infty}^{-\infty} \frac{dt}{[t^2 + 1]^{3/2}} = \frac{Kf(x_1)}{2\pi r} \tag{11}$$

Eq. (11), with eq. (7), yields

$$Kf(x_1)/2\pi r = U\tan\theta \tag{12}$$

Eq. (12) illustrates that no compensating factor K containing Mach Number is needed in the equation to preserve the required independence of Mach Number.

Within the rigor of the linearization implied in Eq. (4), the expression for the ratio of the compressible stream pressure coefficient to the incompressible stream pressure coefficient at any subsonic free stream velocity U is

$$(C_{pm})_{r\to\epsilon}/(C_{p0})_{r'\to\epsilon'} = u/u' \tag{13}$$

From Eqs. (2) and (5) for slender bodies with $K = 1$,

$$(\phi'_{x'})_{r'\to\epsilon'} = (\phi_x)_{r\to\epsilon} = \frac{1}{4\pi}\int_{-c/2}^{c/2} \frac{\pm f(\xi)d\xi}{(x - \xi)^2} \tag{14}$$

and therefore,

$$(C_{pm})_{r\to\epsilon} = (C_{p0})_{r'\to\epsilon'} \tag{15}$$

The practical importance of this knowledge lies in the consideration of the effects of thickness ratio or aspect ratio on the compressibility phenomena. Simply stated, the more slender the body, the less apparent are the effects of compressibility. Thus the airflow over fuselages will be less influenced by the consideration of the compressibility effects than will the airflow over wing and tail surfaces with higher aspect ratio.

References

[1] Goethert, B., *Plane and Three-Dimensional Flow at High Subsonic Speeds*, N.A.C.A. T.M. No. 1105, Reprint, October, 1946.

[2] Goethert, B., *Several Remarks on Prandtl's Rule With Regard to Two- and Three-Dimensional Flow*, U.S.A.A.F. Translation No. F-TS-1514-RE April, 1947.

[3] Lees, Lester, *A Discussion of the Application of the Prandtl-Glauert Method to Subsonic Compressible Flow over a Slender Body of Revolution*, N.A.C.A. T.N. No. 1127, September, 1946.

[4] Hess, Robert V., and Gardner, Clifford S., *Study by the Prandtl-Glauert Method of Compressibility Effects and Critical Mach Number of Ellipsoids of Various Aspect Ratios and Thickness Ratios*, N.A.C.A. T.N. No. 1792, January, 1949.

[5] Sauer, R., *Theoretical Gas Dynamics*, pp. 36–51; reprinted by Edward Brothers, Ann Arbor, 1947.

Structural Design for Minimum Weight

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I have read with great interest Mr. Shanley's paper (in the March, 1949, issue of the Journal of the Aeronautical Sciences) on the "Principles of Structural Design for Minimum Weight." The conclusions regarding the most economical de-