should be divided and not multiplied by Ω. (3) Eqs. (14) and (15), the denominator of the last term should read $2(1 - \xi)^2 l_R$ and not $2(1 - \xi^2) l_R$.

Page 462: (1) Integral for P, Insert R before dr. (2) In the expressions for H_0, H_1, etc., λ should read λ_R. (3) In the expression for H_4, $C_{D_0}(1 - \xi^2)$ should read $(C_{D_0}/a)(1 - \xi^2)$. (4) Eq. (16), $\left(\dfrac{\Delta I}{I}\right)$ should read $\left(\dfrac{\Delta I}{I}\right)^2$.

Page 463: (1) In the expressions for A_{22} multiply the first term, ν^2, by $[1 + \mathfrak{M}(e/d)]$ and replace $\mathfrak{M}_3$ by $[\mathfrak{M}_3 + 2\mathfrak{M}(e/d)]$. With this modification the qualification "neglecting the last two terms of Eqs. (14) and (15). . . ." may be eliminated. (2) In the expression for R_{22}, ζ should read $\zeta + \omega_b{}^2 + \delta B$.

Page 464: (1) In the expression for J, M_4 should read $\mathfrak{M}_4$.

Page 465: (1) In the stability determinant, $\tau C_{D_0}/4$ should read $\tau C_{D_0}/4a$.

Page 466: (1) In text, after "Laplace transform," $\mathcal{L}(\alpha_3)$ should read $\mathcal{L}(\alpha_1)$. (2) The expressions for $G(\nu)$ and $g(\nu)$ should read

$$G(\nu) = X\nu + (HX - JZ)$$
$$g(\nu) = Y\nu + (HY - JV)$$

where

$$H = -\left\{ H_4 + \frac{H_1}{1 + (\bar{\zeta}/A)^2}\left[\frac{D}{A} + \frac{\bar{\zeta}\beta_0 C}{A^2}\right] + \frac{H_9}{1 + (\bar{\zeta}/A)^2} \times \left[\frac{\bar{\zeta}D}{A^2} - \frac{\beta_0 C}{A}\right]\right\}$$

Page 467: (1) The statement after the simplified expressions for X and Y should read: "In determining initial response, H and J may always be assumed zero." (2) Eq. (20), X and Y should be multiplied by Ω^2.

Page 468: (1) Equation before Eq. (22), the fourth λ should be squared. (2) Eq. (22), $-[\Omega^2 kX - JH_0\Omega^3 l]$ should read $-[\Omega^2 kX - JH_0\Omega^3 l]\lambda$. (3) In the expression for D', multiply $2(\theta_0 - \delta\beta_0)$ by rR.

Page 469: (1) Eq. (24), Γ should be multiplied by 2 throughout. (2) Following Eq. (24), the first roots given should be $\lambda = -0.585 \pm 3.53 i$, etc. (3) First column, fifth line from bottom, $\alpha_1/\theta_1 = 4.5$, not 1.5. (4) In the expressions for $(\alpha_1 - \beta_1)$ and β_0, the terms containing C_{m_0} should not contain a.

Page 470: (1) First column, fourth line from bottom, Eq. (23) should read Eq. (25).

Page 471: (1) In expressions for m_q and z_α, C_i should read C_L. (2) In expression for α/δ_e, u should read μ. (3) In line following Eq. (2A) "right" should read "left."

Page 472: (1) $pb/2u$ should read $pb/2U$. (2) ϕ/δ_a is negative.

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Concerning Linearized Supersonic Flow Solutions for Rotationally Symmetric Bodies

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NUMEROUS AUTHORS[1-4] have formulated solutions to the linearized partial differential equation of compressible potential flow applicable to rotationally symmetrical bodies when the incident Mach Number exceeds 1. The solution of von Kármán and Moore for such bodies aligned at zero angle of attack, and the Tsien[2] or Ferrari[3] solution for bodies aligned at a small angle of attack with the incident flow can be shown to have a common origin in certain solutions appearing in the theory of wave mechanics. Unfortunately, the Tsien or Ferrari solution has

been misrepresented to some extent in certain widely distributed texts.[5, 6] The purpose of this note is to clarify this misrepresentation by examining the origin of the solutions formulated by the above-mentioned authors.

The linearized partial differential potential equation governing these solutions in cylindrical coordinates is

$$(M^2 - 1)\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}. \tag{1}$$

where M = free stream Mach Number and (x, r, θ) are cylindrical coordinates.

For the axially symmetric flow about a rotationally symmetric body, the flow is independent of θ and Eq. (1) becomes

$$(M^2 - 1)\frac{\partial^2 \phi_0}{\partial x^2} = \frac{\partial^2 \phi_0}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_0}{\partial r} \tag{2}$$

von Kármán and Moore employed the simple substitution $t = x/V$ to change Eq. (2) into the familiar form of the two-dimensional wave equation for infinitesimal disturbances. They formulated the solution below arising from a solution to this wave equation.

$$\phi_0 = \frac{1}{4\pi}\int_{\cosh^{-1}(x/\beta r)}^{0} f(x - \beta r \cosh u)\,du \tag{3}$$

where $\beta = \sqrt{M^2 - 1}$.

If the substitution $\xi = x - \beta r \cosh u$ is used in Eq. (3) this solution takes the form of the source-sink solution, appearing in incompressible flow solutions.

$$\phi_0 = -(1/4\pi)\int_0^{x - \beta r} \frac{f(\xi)\,d\xi}{\sqrt{(x - \xi)^2 - \beta^2 r^2}} \tag{4}$$

This solution represents the potential at a field point (x, r) due to a distribution of sources and sinks along the x-axis at points ξ between 0 and $x - \beta r$.

Following in form the procedure outlined by Lamb,[8] the general solution to Eq. (1) when the Mach Number exceeds 1 is

$$\phi = \sum_{s=1}^{n} \phi_s r^s \cos s\theta + \psi_s r^s \sin s\theta \tag{5}$$

where ϕ_s and ψ_s must satisfy the equation below obtained by substituting Eq. (5) into Eq. (1).

$$\frac{\partial^2}{\partial x^2}\left\{\phi_s, \psi_s\right\} = \frac{1}{\beta^2}\left[\frac{\partial^2}{\partial r^2}\left\{\phi_s, \psi_s\right\} + \frac{2s + 1}{r}\frac{\partial}{\partial r}\left\{\phi_s, \psi_s\right\}\right] \tag{6}$$

A known solution to Eq. (6) is

$$\phi_s = [(1/r)/(\partial/\partial r)]^s \phi_0 \tag{7}$$

where ϕ_0 is identically the solution to Eq. (2) given by von Kármán and Moore.

By taking the index s equal to 1 in Eq. (5), Eq. (7) yields the solution below formulated by Tsien and Ferrari

$$\phi = (\partial \phi_0/\partial r)\cos\theta \tag{8}$$

The differentiation of ϕ_0 is performed on the integral in the form of Eq. (3) because the integral in Eq. (4) is improper since the denominator vanishes at the upper limit. The solution to Eq. (1) is then

$$\phi = -\frac{\beta \cos\theta}{4\pi}\int_{\cosh^{-1}(x/\beta r)}^{0} f'(x - \beta r \cosh u)\cosh u\,du \tag{9}$$

if $f(0)$ can be taken as 0, a reasonable assumption since these solutions within the rigor of the linearization are restricted to sharp pointed bodies.

Calling $\xi = x - \beta r \cosh u$ and $f'(\xi) \equiv g(\xi)$, solution (9) becomes

$$\phi = (\cos\theta/4\pi r)\int_0^{x - \beta r} g(\xi)(x - \xi)\,d\xi/\sqrt{(x - \xi)^2 - \beta^2 r^2} \tag{10}$$

This solution has been incorrectly represented in references 5 and 6 as

$$\phi = (\beta^2 r \cos\theta/4\pi)\int_0^{x-\beta r} g(\xi)d\xi/[(x-\xi)^2 - \beta^2 r^2]^{3/2} \quad (11)$$

The integrand of this incorrect solution resembles in form the integrand of the subsonic doublet solution obtained by differentiating the source-sink solution integral (4) with respect to r under constant limits. When the Mach Number is less than 1, this procedure is valid because the limits of integration are constant for a body of finite extent. When Mach Number exceeds 1, the upper limit becomes $x - \beta r$ as a consequence of Eq. (2) changing from the elliptic type to the hyperbolic type characteristic of supersonic flow fields. Only when Mach Number exceeds 1 can solutions of the two-dimensional wave equation be adapted to Eqs. (1) and (2) in the manner discussed herein. As such, solution (10) is the proper form of the solution to Eq. (1) when Mach Number exceeds 1 and should be used instead of the form of Eq. (11) whenever Eq. (11) appears.

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Quasi-Stationary Airfoil Theory in Compressible Flow

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IT WAS RECENTLY POINTED OUT that the use of steady flow theory could lead to erroneous results in the calculation of stability derivatives, and the correct results for quasi-stationary incompressible flow over a thin airfoil were obtained as limiting results of the previously known flutter results.[1] These results have now been extended to the case of subsonic, compressible flow by obtaining a solution to the Possio integral equation,[2] considering only terms which are first order in frequency.

The lift and quarter-chord moment coefficients on a thin airfoil with leading edge at $x = -1$ and trailing edge at $x = +1$ in a subsonic flow of velocity U and Mach Number M with downwash distribution $w(x) \exp(i\omega t)$ are given by

$$C_l = 2(1 - M^2)^{-1/2}\int_0^\pi \left\{ (1 - \cos\varphi) + \right.$$
$$ik(1 - M^2)^{-1}\sin^2\varphi + ik(1 - M^2)^{-1}(1 - \cos\varphi)\left[\gamma + \frac{i\pi}{2} + \right.$$
$$\left.\left.\log_e\left(\frac{k}{2}\right) - F(M)\right]\right\}\frac{w(-\cos\varphi)}{U}d\varphi + 0(k^2\log k) \quad (1)$$

$$C_m = \frac{1}{2}(1 - M^2)^{-1/2}\int_0^\pi [(\cos\varphi - \cos 2\varphi) - ik(1 - M^2)^{-1} \times$$
$$(1 - \cos\varphi)\sin^2\varphi]\frac{w(-\cos\varphi)}{U}d\varphi + 0(k^2\log k) \quad (2)$$

$$F(M) = M^2 + \log_e\left[\frac{2(1 - M^2)}{M}\right] - (1 - M^2)^{1/2}\log_e \times$$
$$\left[\frac{1 + (1 - M^2)^{1/2}}{M}\right] \quad (3)$$

$$k = \omega c/2U \quad (4)$$

where k is the reduced frequency, based on the semichord $(c/2)$, and γ is Euler's constant. These results reduce to the well-known Munk formulas,[3] together with the Prandtl-Glauert factor $(1 - M^2)^{-1/2}$, for $k = 0$.

For the important case of pitching about the quarter chord—

$$W_\alpha(x) = U\{1 + [(1/2) + (2x/c)]ik\} \quad (5)$$

Eqs. (1) and (2) reduce to

$$C_{l\alpha} = 2\pi(1 - M^2)^{-1/2}\left\{(1 + ik) + ik(1 - M^2)^{-1} \times \right.$$
$$\left.\left[\frac{1}{2} + \gamma + \log_e\left(\frac{k}{2}\right) - F(M)\right]\right\} \quad (6)$$

$$C_{m\alpha} = -(i\pi k/4)(1 - M^2)^{-3/2}(2 - M^2) \quad (7)$$

For plunging of the airfoil—i.e.,

$$w_h(x) = ikU \quad (8)$$

the results are

$$C_{l_h} = 2\pi ik(1 - M^2)^{-1/2} \quad (9)$$

$$C_{m_h} = 0 \quad (10)$$

In interpreting these results, it may be remarked that nonstationary effects need be considered only when the downwash includes a term that is zero order in frequency. In this case, however, the term introduced is logarithmic and will be increasingly important for small k, corresponding to high-speed flight. Moreover, the compressibility correction of this term is approximately $(1 - M^2)^{-3/2}$ rather than $(1 - M^2)^{-1/2}$. It follows that the effects under consideration will be particularly important at high speeds.

As an example, consider the calculation of the damping of a rotary motion of a tail surface about a center five chord lengths ahead of its quarter-chord. Carrying out the calculations with the aid of the above results and also with the results obtained by setting $k = 0$ in Eqs. (1) and (2) and designating the damping derivatives as C_{m_q} and $C^0{}_{m_q}$, respectively, it is found that the latter calculation overestimates the damping by a considerable margin. Typical numbers are

$\dfrac{C_{m_q}}{C^0{}_{m_q}} =$	M^k	0.1	0.01	
	0	0.828	0.618	(11)
	0.7	0.662	0.342	

While these ratios neglect induction effects and wing interference, they are indicative of the errors that may be expected when steady flow theory is used for the calculation of tail stability derivatives. Similar results may be expected for the damping in pitch of a swept wing.

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