

**ON THE ANALYSIS OF APPROXIMATION ALGORITHMS
FOR CLOSED QUEUEING NETWORKS**

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ABSTRACT

The Linearizer and the Schweitzer algorithms are well known approximation algorithms for solving product form queueing networks. For a network consisting only of fixed rate servers and delay servers and with a single class of customers, it has been shown that the system of equations defining the algorithm of Schweitzer have a unique solution, and that the resulting cycle time estimate is an upper bound.

The purpose of this paper is to complement the work on analysis of these approximate algorithms. We provide simple proofs of the existence and uniqueness of a solution for both the Schweitzer algorithm and the Linearizer, in the case of a network having only fixed rate and delay servers and with a single customer class. A simpler proof of the fact that the cycle time estimate obtained by the Schweitzer algorithm is an upper bound, is also presented. Some other properties of these heuristics are also discussed.

1. Introduction:

The use of closed queueing networks to model computer systems and flexible manufacturing system is widespread. A class of these networks, known as product form (PF) networks is often used in these models for reasons of tractability. Quite often, however, the computational effort required to analyze even these products form networks exactly may be either quite high, or unnecessary. This has motivated research in approximation algorithms/bounds for obtaining the performance measures. The Schweitzer algorithm [5] and the Linearizer [1] are well known heuristic algorithms which obtain approximate performance measures for such networks. The Linearizer in turn uses the Schweitzer heuristic to obtain its performance measures. We shall henceforth refer to the Schweitzer heuristic as the first order heuristic, and the Linearizer as the second order heuristic.

In the ensuing discussion, we restrict our attention to PF networks consisting only of fixed rate and delay (infinite) servers, and with a single class of customers. It has been shown [2] that for such networks, the first order heuristic defines a system of equations which have a unique solution. Moreover, the resulting cycle time estimate was shown to be an upper bound on the exact cycle time.

In this paper, we provide simple proofs which show the existence and uniqueness of solutions for both the first order heuristic as well as the second order heuristic. A simple proof of the fact that the cycle time estimate of the first order heuristic is an upper bound, is presented. In addition, it is shown that the estimate of the mean queue length at the bottleneck node obtained by the first order heuristic is an upper bound on the exact mean queue length at this node. Under some restrictive assumptions, lower bounds for the cycle times are obtained.

2. Preliminaries:

For the class of networks considered here, the only input parameters required are the mean service time demands (loads) at the nodes, and the customer population, N . For computing the exact performance measures, all delay nodes can be replaced by a single delay node with a suitably chosen load. Thus, let this delay node, if present, be indexed by 0, and let there be M fixed rate nodes, indexed 1 through M . Let L_m , $m = 0, \dots, M$ be the loads at the nodes, and without loss of generality assume that the nodes are indexed so that $L_M \geq \dots \geq L_1$. Denote the exact mean residence time at node m as $W_m(N)$, the exact mean queue length as $Q_m(N)$, and the exact cycle time as $C(N)$. The corresponding estimates for the first order heuristic (respectively, second order heuristic) will be denoted as $WF_m(N)$, $QF_m(N)$, and $CF(N)$ (respectively, $WS_m(N)$, $QS_m(N)$, and $CS(N)$). Unless otherwise specified, the index for summation is henceforth assumed to be over the range 1 through M . We also assume that $M \geq 2$, $N > 1$.

The Mean Value Analysis (MVA) algorithm [4] obtains exact values of the mean residence times by iteratively solving the system of equations

$$W_m(N) = L_m \left(1 + Q_m(N-1) \right), \quad m = 1, \dots, M. \quad (2.1a)$$

$$= L_m, \quad m = 0, \quad (2.1b)$$

with

$$Q_m(1) = L_m / \sum_{n=0}^M L_n, \quad m = 1, \dots, M. \quad (2.2)$$

Then, the cycle time is given by

$$C(N) = \sum_{m=0}^M W_m(N) = L_0 + \sum_m W_m(N). \quad (2.3)$$

The rest of this paper is organized as follows:

In section 3, some properties of the first order heuristic are obtained. Specifically, it is shown here that (i) a unique solution exists for the first order heuristic, (ii) the cycle time estimate obtained is an upper bound on the exact cycle time, and (iii) the mean queue length estimate for the bottleneck node is an upper bound on the exact mean queue length at this node. It is also shown how a simple relation exists between the expressions for the mean queue length estimates and the exact mean queue lengths. Finally, under some restrictive conditions, a simple lower bound on the exact cycle time is obtained.

Section 4 presents some results for the second order heuristic. It is shown (i) that a unique solution exists for the second order heuristic, and (ii) that it returns a lower cycle time estimate compared to the first order heuristic. A lower bound is also presented on this cycle time estimate under some restrictive conditions.

3.0 Properties of the first order heuristic:

The first order heuristic approximates the mean residence times by solving the system of equations:

$$WF_m(N) = L_m \left(1 + \frac{N-1}{N} QF_m(N) \right),$$

and using Little's law [3], this gives

$$WF_m(N) = L_m + \frac{N-1}{N} WF_m(N) \cdot \frac{N}{CF(N)}.$$

Simplifying the above expression, we have

$$WF_m(N) = \frac{L_m}{1 - (N-1)L_m/CF(N)}. \quad (3.1)$$

Hence,

$$CF(N) = L_o + \sum_m \frac{L_m}{1 - (N-1)L_m/CF(N)}. \quad (3.2)$$

The other desired performance measures, namely, the throughput and mean queue length estimates are now obtained easily using Little's rule [3].

For example, a simple application of Little's rule to equation (3.1) gives

$$QF_m(N) = \frac{NL_m}{CF(N) - (N-1)L_m}$$

3.1 Existence and uniqueness of a solution:

We can immediately observe that the solution to equation (3.2), if it exists, is unique. To see this, suppose equation (3.2) has two solutions CF_1 and CF_2 such that $CF_2 < CF_1$. Then

$$\begin{aligned} CF_2 &= L_o + \sum_m \frac{L_m}{1 - (N-1)L_m / CF_2} \\ &> L_o + \sum_m \frac{L_m}{1 - (N-1)L_m / CF_1} = CF_1, \end{aligned}$$

which is a contradiction. Hence, equation (3.2) provides a unique solution, if it exists. Note then, that from equation (3.1), the mean residence times, $WF_m(\cdot)$ are uniquely determined. It is also clear from equations (3.1) and (3.2) that $CF(N) > 0$. Lemma 1 now demonstrates the existence of a solution for $CF(N)$ such that all mean residence times are non-negative. This would then complete the proof of existence and uniqueness for the performance measure estimates provided by the first order heuristic.

Lemma 1:

There exists a unique solution to the system of equations for the mean residence times, $WF_m(\cdot)$, $m = 1, \dots, M$, such that

$$WF_m(\cdot) \geq 0, \quad m = 1, \dots, M.$$

Proof:

Consider the behavior of equation (3.2) where we replace $CF(N)$ by a parameter γ . We need to find a solution to the equation

where

$$\gamma = g(\gamma),$$

$$g(\gamma) = L_o + \sum_m \frac{\gamma L_m}{\gamma - (N-1)L_m}. \quad (3.3)$$

Differentiating $g(\gamma)$ with respect to γ , we have

$$g'(\gamma) = \sum_m \frac{\left(\gamma - (N-1)L_m \right) L_m - \gamma L_m}{\left(\gamma - (N-1)L_m \right)^2}$$

$$= - \sum_m \frac{(N-1)L_m^2}{\left(\gamma - (N-1)L_m \right)^2}, \quad (3.4)$$

i.e. $g'(\gamma) < 0$ for all γ .

Now, consider the function, $g(\gamma)$, only in the region where $\gamma \geq (N-1)L_M$.

Equation (3.4) shows that $g(\gamma)$ is a monotonically decreasing function of γ in this region with $g(\gamma) \rightarrow \infty$ as $\gamma \rightarrow (N-1)L_M$. (refer to Figure 3.1).. To complete the proof, we now only need to obtain a point $\gamma_2 > \gamma_1$, such that $g(\gamma_2) < \gamma_2$. For this, note that as

$$\gamma \rightarrow \infty, \quad g(\gamma) \rightarrow \sum_{m=0}^M L_m < \infty.$$

Thus, we can choose γ_2 , to be arbitrarily large, to satisfy $g(\gamma_2) < \gamma_2$.

Hence, there exists a unique point, γ_0 , $\gamma_1 < \gamma_0 < \gamma_2$, such that $\gamma_0 = g(\gamma_0)$. Noting that $\gamma_0 > (N-1)L_M$, it is seen that every term in the summation in equation (3.3) is positive. Hence, setting $CF(N) = \gamma_0$ in equation (3.1) gives the desired result. \square

3.2 The cycle time estimate of the first order heuristic is an upper bound:

Here we present a simple proof that the cycle time estimate $CF(N)$, provided by the first order heuristic, is an upper bound on $C(N)$. This requires the following:

Lemma 2:

Consider the function

$$\Theta(K) = \sum_m \rho_m Q_m(K),$$

where $\rho_M \geq \dots \geq \rho_1 \geq 0$. Then

$$\frac{\Theta(N)}{N} - \frac{\Theta(N-1)}{N-1} \geq 0.$$

A proof of Lemma 2 is given in [6].

Lemma 3

The cycle time estimate provided by the first order heuristic is an upper bound on the exact cycle time.

Proof:

Equation (3.2) for $CF(N)$ can be rewritten as

$$CF(N) = L_o + \sum_m L_m + (N-1) \sum_m \frac{L_m^2}{CF(N) - (N-1)L_m}$$

Without loss of generality, assume that

$$L_o + \sum_m L_m = 1.$$

Hence

$$CF(N) = 1 + (N-1) \sum_m \frac{L_m^2}{CF(N) - (N-1)L_m}. \quad (3.5)$$

We now prove, by contradiction, that $CF(N) > C(N)$. Suppose, otherwise, that $C(N) > CF(N)$. Then, from equation (3.5),

$$CF(N) > 1 + (N-1) \sum_m \frac{L_m^2}{C(N) - (N-1)L_m}. \quad (3.6)$$

From equations (2.1a) and (2.3), we have

$$C(N) = 1 + \phi(N-1), \quad (3.7)$$

where we have set

$$\phi(K) = \sum_m L_m Q_m(K).$$

From equations (3.6) and (3.7), after some straightforward algebra we get,

$$CF(N) > 1 + (N-1) \sum_m L_m^2 + (N-1) \sum_m L_m^2 \left\{ \frac{(N-1)L_m - \phi(N-1)}{C(N) - (N-1)L_m} \right\}. \quad (3.8)$$

From Lemma 2, with L_m in place of ρ_m , we get

$$\begin{aligned} C(N) &\leq 1 + (N-1) \sum_m L_m \frac{Q_m(N)}{N} \\ &= 1 + (N-1) \sum_m L_m \frac{W_m(N)}{C(N)} \\ &= 1 + (N-1) \sum_m L_m^2 \left\{ \frac{1 + Q_m(N-1)}{1 + \phi(N-1)} \right\} \\ &= 1 + (N-1) \sum_m L_m^2 + (N-1) \sum_m L_m^2 \left\{ \frac{Q_m(N-1) - \phi(N-1)}{1 + \phi(N-1)} \right\}. \end{aligned}$$

Hence,

$$1 + (N-1) \sum_m L_m^2 \geq C(N) - (N-1) \sum_m L_m^2 \left\{ \frac{Q_m(N-1) - \phi(N-1)}{1 + \phi(N-1)} \right\}. \quad (3.9)$$

From equations (3.8) and (3.9),

$$CF(N) > C(N) + (N-1) \sum_m L_m^2 \left\{ \frac{(N-1)L_m - \phi(N-1)}{C(N) - (N-1)L_m} - \frac{Q_m(N-1) - \phi(N-1)}{C(N)} \right\},$$

and this simplifies to the inequality

$$CF(N) > C(N) + (N-1) \sum_m L_m^2 \left\{ \frac{(N-1)C(N) \left\{ W_m(N)/C(N) - Q_m(N-1)/(N-1) \right\}}{\left\{ C(N) - (N-1)L_m \right\}} \cdot C(N) \right\}.$$

Hence, noting that $W_m(N)/C(N) = Q_m(N)/N$, we have

$$CF(N) - C(N) > (N-1)^2 \sum_m g_m \left\{ \frac{Q_m(N)}{N} - \frac{Q_m(N-1)}{N-1} \right\},$$

where

$$g_m = \frac{L_m^2}{C(N) - (N-1)L_m}.$$

It is clear that $g_M > g_1$. Hence setting $g_m = \rho_m$ in Lemma 2 gives

$$CF(N) - C(N) > 0.$$

This is a contradiction to our original premise that $C(N) > CF(N)$, and hence the result is proved. □

3.3 The mean queue length estimates:

An expression for the exact mean queue length at any fixed rate node, m , in a network \mathbb{N} can be obtained as follows: For each such fixed rate node, m , consider an alternate network \mathbb{N}' which is identical to network \mathbb{N} except that it has an additional fixed rate node with load L_m . Let $C^{(m)}(N)$ denote the cycle time for this network. Then it can be shown that [7]:

$$Q_m(N) = \frac{NL_m}{C^{(m)}(N) - NL_m} \quad (3.10)$$

Comparing equations (3.3) and (3.10), the similarity of the two equations is quite striking. We use equation (3.10) to prove that $QF_M(N) \cong Q_M(N)$. This fact, expressed as Lemma 4, had been stated without proof in [2].

Lemma 4:

$$QF_M(N) \geq Q_M(N). \quad 3.11$$

Proof:

Since $L_M \geq \dots \geq L_1$, it can be easily shown that a direct application of the results on monotonicity of throughputs in such networks [8] gives

$$C^{(M)}(N) \geq \dots \geq C^{(1)}(N). \quad (3.12)$$

It is now shown that there exists a $k \leq M$ such that

$$C^{(k)}(N) \geq CF(N), \quad k \leq m \leq M.$$

Suppose otherwise that $CF(N) > C^{(M)}(N)$. This implies that for all m , we must have

$$\begin{aligned} Q_m(N) &> \frac{NL_m}{CF(N) - NL_m} > \frac{NL_m}{CF(N) - (N-1)L_m} \\ &= QF_m(N). \end{aligned}$$

Noting that

$$\sum_{m=0}^M Q_m(N) = \sum_{m=0}^M QF_m(N) = N,$$

we must then have

$$QF_0(N) = \lambda F(N) \cdot L_0 > \lambda(N) \cdot L_0 = Q_0(N).$$

This is clearly a contradiction to the result shown in Lemma 3 on the cycle time estimate being an upper bound.

□

As pointed out in [2], in general, the mean queue length estimate at nodes with higher loads is an overestimate while it is an underestimate at nodes with lower loads.

Now, suppose that for some K , we have $Q_M(K) \geq K/2$. Under this somewhat restrictive condition, Lemma 5 obtains a lower bound on $C(N)$. It is noted, however,

that at large population values, the mean queue length at the bottleneck node dominates the mean queue lengths at all other nodes to a considerable extent, and that this restriction is then satisfied, at least asymptotically with N .

Lemma 5:

$$C(N) \geq CF(N-1), \text{ if } Q_M(N-1) \geq \frac{N-1}{2}.$$

Proof: By contradiction. Suppose otherwise, that $C(N) < CF(N-1)$. Then from equation (3.5), setting $L_o + \sum_m L_m = 1$ as before, we have

$$CF(N-1) < 1 + (N-2) \sum_m \frac{L_m^2}{C(N) - (N-2)L_m},$$

and so

$$\begin{aligned} C(N) - CF(N-1) &> \sum_m L_m Q_m(N-1) - (N-2) \sum_m \frac{L_m^2}{C(N) - (N-2)L_m}, \\ &= \sum_m \frac{L_m Q_m(N-1)}{C(N) - (N-2)L_m} \left[C(N) - (N-2)L_m \frac{(1 + Q_m(N-1))}{Q_m(N-1)} \right] \\ &= \sum_m \frac{L_m}{C(N) - (N-2)L_m} \left[C(N)Q_m(N-1) - (N-2)W_m(N) \right] \\ &= \frac{1}{\lambda(N)} \sum_m \frac{L_m}{C(N) - (N-2)L_m} \left[NQ_m(N-1) - (N-2)Q_m(N) \right]. \end{aligned} \tag{3.13}$$

Now, it is easy to show that $Q_M(N) \geq \dots \geq Q_1(N)$. Hence a lower bound on the expression on the right hand side of (3.13) is given by setting

and

$$Q_m(N) = Q_m(N-1), \quad m=1, \dots, M-1,$$

$$Q_m(N) = Q_m(N-1) + 1, \quad m=M,$$

to give

$$\begin{aligned}
C(N) - CF(N-1) &> \frac{1}{\lambda(N)} \sum_{m=1}^{M-1} \frac{L_m}{C(N) - (N-2)L_m} 2Q_m(N-1) \\
&\quad + \frac{1}{\lambda(N)} \frac{L_M}{C(N) - (N-2)L_M} \left[NQ_M(N-1) - (N-2) \left(Q_M(N-1) + 1 \right) \right] \\
&> \frac{L_M}{N - \lambda(N)(N-2)L_M} \left[2 \frac{(N-1)}{2} - (N-2) \right] \\
&\geq 0,
\end{aligned}$$

which is a contradiction to our earlier assumption, proving the result. \square

4. The second order heuristic:

In this section, we derive some properties of the second order heuristic. These properties are expressed as Lemmas 6 through 9. These Lemmas require a few preliminary results expressed below as Propositions 1 through 4. Proof of these Propositions are given in the Appendix.

Proposition 1:

$$\lambda F(N) \geq \lambda F(N-1).$$

Proposition 2:

Let

$$\Delta WF_m(N) = WF_m(N) - WF_m(N-1), \quad m=1, \dots, M. \quad (4.1)$$

Then

$$\Delta WF_M(N) \geq \dots \geq \Delta WF_1(N) \geq 0. \quad (4.2)$$

Proposition 3:

$$CF(N) - CF(N-1) \leq L_M. \quad (4.3)$$

Proposition 4:

For $m > n$,

$$\begin{aligned}
\text{(a)} \quad & \frac{\Delta WF_m(N)}{\Delta F_n(N)} \geq \left(L_m / L_n \right)^2, \\
\text{(b)} \quad & \Delta WF_m(N) \leq \left(L_m^2 / L_M \right), \\
\text{(c)} \quad & \Delta QF_m(N) \leq \frac{L_m}{(N-1)L_M} - \frac{QF_m(N-1)}{(N-1)^2}. \tag{4.4}
\end{aligned}$$

where

$$\Delta QF_m(N) = \left\{ \frac{QF_m(N)}{N} - \frac{QF_m(N-1)}{N-1} \right\}. \tag{4.5}$$

4.1 Existence and uniqueness of a solution for the second order heuristic:

The second order heuristic obtains estimates of the mean residence times by solving the following system of equations; for $m=1, \dots, M$:

$$WS_m(N) = L_m \left\{ 1 + \frac{N-1}{N} QS_m(N) - (N-1) \Delta QF_m(N) \right\}, \tag{4.6}$$

This, therefore requires solutions to the first order heuristic at population N and $(N-1)$. Equation (4.6) can be rewritten, using Little's rule, as

$$WS_m(N) = L_m \left\{ 1 + \frac{N-1}{N} WS_m(N) \cdot \frac{N}{CS(N)} - (N-1) \Delta QF_m(N) \right\},$$

from which we get

$$WS_m(N) = \frac{L_m \left\{ 1 - (N-1) \Delta QF_m(N) \right\}}{1 - (N-1) L_m / CS(N)}. \tag{4.7}$$

The cycle time estimate is then obtained from

$$CS(N) = L_o + \sum_m \frac{L_m \left(1 - (N-1) \Delta QF_m(N) \right)}{1 - (N-1) L_m / CS(N)}. \tag{4.8}$$

Noting that the values of $QF_m(N)$ and $QF_m(N-1)$ exist and are unique, it is clear that the $\Delta QF_m(N)$ s as defined by equation (7) exist and are unique. Now, Lemma 6

establishes existence and uniqueness for the system of equations defining the second order heuristic:

Lemma 6:

There exists a unique solution to the system of equations for the mean residence times, $WS_m(\cdot)$, $m = 1, \dots, M$, such that

$$WS_m(\cdot) \geq 0, \quad m = 1, \dots, M.$$

Proof:

Let

$$K_m(N) = L_m \left(1 - (N-1)\Delta Q F_m(N) \right), \quad m = 1, \dots, M. \quad (4.9)$$

Note that from proposition (4), $K_m(N) \geq 0$, for all m .

Then we can rewrite equation (4.8) as

$$CS(N) = L_o + \sum_m \frac{CS(N) K_m(N)}{CS(N) - (N-1)L_m}. \quad (4.10)$$

As before, replacing $CS(N)$ by a parameter γ , we need a solution to the equation

$$\gamma = g(\gamma),$$

where

$$g(\gamma) = L_o + \sum_m \frac{\gamma K_m(N)}{\gamma - (N-1)L_m}. \quad (4.11)$$

Differentiating $g(\gamma)$ with respect to γ , we have

$$g'(\gamma) = \sum_m \frac{\left\{ \gamma - (N-1)L_m \right\} K_m(N) - \gamma K_m(N)}{\left\{ \gamma - (N-1)L_m \right\}^2},$$

which is negative for all values of γ .

The remainder of the proof is now identical to that of Lemma 1. □

It is next shown in Lemma 7 that the second order heuristic returns a cycle time estimate that is less than that returned by the first order heuristic.

Lemma 7:

$$CS(N) \leq CF(N)$$

Proof:

By contradiction. Assume that $CS(N) > CF(N)$.

Then, from equations (3.2) and (4.8),

$$\begin{aligned} CF(N) - CS(N) &= \sum_m \frac{L_m}{1 - (N-1)L_m / CF(N)} - \sum_m \frac{L_m \left(1 - (N-1)\Delta QF_m(N) \right)}{1 - (N-1)L_m / CS(N)} \\ &\geq \sum_m \frac{L_m}{1 - (N-1)L_m / CS(N)} - \sum_m \frac{L_m \left(1 - (N-1)\Delta QF_m(N) \right)}{1 - (N-1)L_m / CS(N)} \\ &= \sum_m \alpha_m \Delta QF_m(N), \end{aligned}$$

where

$$\alpha_m = \frac{(N-1)L_m}{1 - (N-1)L_m / CS(N)} \quad (4.12)$$

It is clear from equation (4.12) that $\alpha_M \geq \dots \geq \alpha_1$. Further from Lemma 6, $\alpha_m \geq 0$ for all m .

Now, the function $\Delta QF_m(N)$ can be rewritten, after some algebra, as

$$\begin{aligned} \Delta QF_m(N) &= \frac{L_m \left(CF(N-1) + L_m - CF(N) \right)}{\left(CF(N) - (N-1)L_m \right) \left(CF(N-1) - (N-2)L_m \right)} \\ &= \beta_m \left(L_m - \Delta CF(N) \right), \end{aligned}$$

where

$$\beta_m = L_m / \left(CF(N) - (N-1)L_m \right) \left(CF(N-1) - (N-2)L_m \right),$$

and

$$\Delta CF(N) = CF(N) - CF(N-1).$$

Note that $\beta_M \geq \dots \geq \beta_1 \geq 0$. Since $\sum_m \Delta QF_m(N) \geq 0$, it is thus clear that there exists a $k \leq M$, such that

$$\Delta QF_M(N) \geq \dots \geq \Delta QF_k(N) \geq 0.$$

Hence

$$\begin{aligned} 0 \leq \alpha_k \sum_m \Delta QF_m(N) &= \sum_{m=1}^{k-1} \alpha_k \Delta QF_m(N) + \sum_{m=k}^M \alpha_k \Delta QF_m(N) \\ &\leq \sum_{m=1}^{k-1} \alpha_m \Delta QF_m(N) + \sum_{m=k}^M \alpha_k \Delta QF_m(N) \\ &\leq \sum_{m=1}^{k-1} \alpha_m \Delta QF_m(N) + \sum_{m=k}^M \alpha_k \Delta QF_m(N) \\ &= CF(N) - CS(N). \end{aligned}$$

This is a contradiction to our earlier assumption, and so the proof is complete. \square

To conclude this section, it is shown that under some restrictive conditions, a lower bound on the cycle time estimate provided by the second order heuristic can be given:

Lemma 8:

$$CS(N) \geq CF(N-1), \quad \text{if } QF_M(N-1) \geq \frac{N-1}{2}.$$

Proof: By contradiction. Assume, otherwise, that $CS(N) < CF(N-1)$. Then

$$\begin{aligned} CS(N) &= L_o + \sum_m \frac{K_m(N)CS(N)}{CS(N) - (N-1)L_m} \\ &> L_o + \sum_m \frac{K_m(N)CF(N-1)}{CF(N-1) - (N-1)L_m}. \end{aligned}$$

Hence,

$$CS(N) - CF(N-1) \geq \sum_m \left| \frac{K_m(N) CF(N-1)}{CF(N-1) - (N-1)L_m} - \frac{L_m CF(N-1)}{CF(N-1) - (N-2)L_m} \right|.$$

Substituting for $K_m(N)$ using equation (4.9), the above inequality reduces to

$$\begin{aligned} CS(N) - CF(N-1) &\geq \sum_m L_m CF(N-1) \left| \frac{L_m - (N-1)\Delta QF_m(N) \left(CF(N-1) - (N-2)L_m \right)}{\left(CF(N-1) - (N-1)L_m \right) \left(CF(N-1) - (N-2)L_m \right)} \right| \quad (4.13) \\ &= \sum_m L_m \left| \frac{WF_m(N-1)/CF(N-1) - (N-1)\Delta QF_m(N-1)}{CF(N-1) - (N-1)L_m} \right| \\ &= \sum_m \frac{L_m}{CF(N-1) - (N-1)L_m} \left| \frac{N-1}{N} QF_m(N) - \frac{N}{N-1} QF_m(N-1) \right|. \end{aligned}$$

The rest of the proof is now very similar to the proof of Lemma 5. \square

For the general case, a looser bound is provided by Lemma 9. The proof involves some straightforward albeit tedious algebra, and is omitted. here

Lemma 9:

$$CS(N) \geq CF(N) \left| 1 - \frac{M-1}{4M} \right| + \frac{L_o}{M} \left| \frac{L_o}{CF(N)} + \frac{1}{M-1} \right| \quad (4.14)$$

5. Conclusions

This paper has extended previous work on the analysis of approximation algorithms for closed queueing networks. Simple proofs have been provided on the existence and uniqueness of a solution for both the Schweitzer heuristic as well as the Linearizer heuristic. Simple proofs have also been obtained showing that the cycle time estimate returned by the Schweitzer heuristic is an upper bound on both the exact cycle time as well as the cycle time estimate returned by the Linearizer heuristic. It has been shown that under some restrictive conditions, the cycle time estimate provided by the Schweitzer heuristic at population $N-1$ serves as a lower bound for both the exact cycle time as well as the Linearizer estimate of cycle time

at population N . The restriction, specifically, is that the mean queue length at the bottleneck node account for at least half the network population. This condition is usually observed to be satisfied for large values of N .

The similarity of the expressions for the exact mean queue length and the corresponding Schweitzer estimate was exhibited. This similarity was used to show that the Schweitzer estimate of the mean queue length at the bottleneck node is an upper bound of the exact mean queue length at this node.

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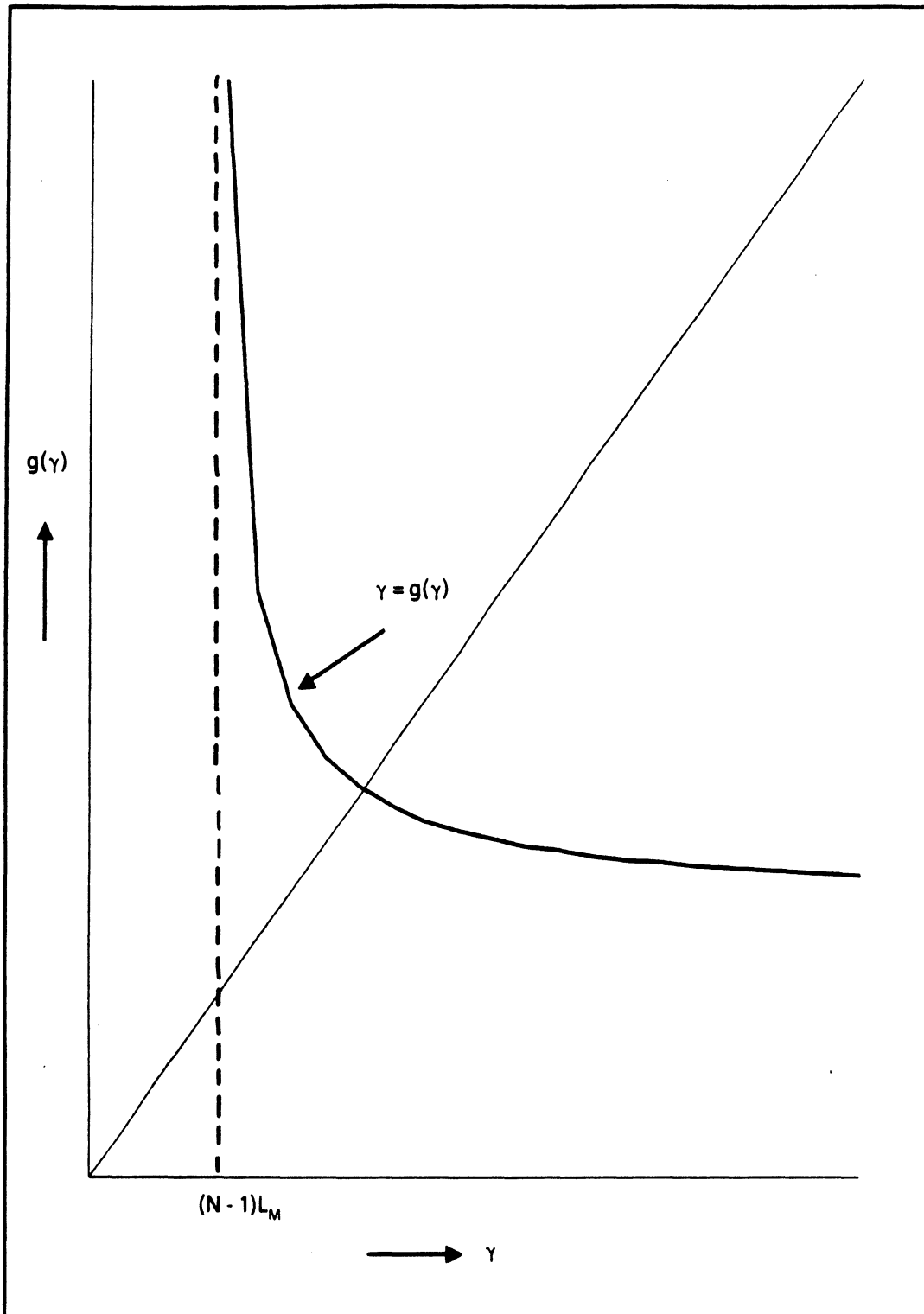


Figure 3.1: Plot of the function $g(\gamma)$

APPENDIX

Proofs of Propositions 1 through 4

Proposition 1:

Consider the throughput estimates $\lambda F(N)$ and $\lambda F(N-1)$, obtained by the first order heuristic at populations N and $N-1$. Then

$$\lambda F(N) > \lambda F(N-1). \quad (\text{A1})$$

Proof:

First, note that for all $m = 1, \dots, M$,

$$\frac{WF_m(N)}{CF(N)} = \frac{L_m}{CF(N) - (N-1)L_m} < 1. \quad (\text{A2})$$

Using equation (3.2) at populations N and $N-1$, we have

$$\begin{aligned} CF(N) - CF(N-1) &= \sum_m \left\{ \frac{L_m}{1 - (N-1)L_m/CF(N)} - \frac{L_m}{1 - (N-2)L_m/CF(N-1)} \right\}, \\ &= \sum_m L_m^2 \frac{(N-1)CF(N-1) - (N-2)CF(N)}{\left(CF(N) - (N-1)L_m \right) \left(CF(N-1) - (N-2)L_m \right)} \\ &< (N-1)CF(N-1) - (N-2)CF(N), \end{aligned}$$

where the last inequality follows from (A2).

Hence, we have

$$(N-1)CF(N) < N \cdot CF(N-1). \quad (\text{A3})$$

Noting that $\lambda F(n) = n/CF(n)$, for $n \geq 0$, this gives the desired result. \square

Proposition 2:

Let

$$\Delta WF_m(N) = WF_m(N) - WF_m(N-1), \quad m = 1, \dots, M. \quad (\text{A4})$$

Then

$$\Delta WF_M(N) \geq \dots \geq \Delta WF_1(N) \geq 0.$$

Proof:

Applying equation (3.1) at populations N and (N-1), we can write

$$\begin{aligned} WF_m(N) - WF_m(N-1) &= L_m \left\{ \frac{1}{1 - (N-1)L_m/CF(N)} - \frac{1}{1 - (N-2)L_m/CF(N-1)} \right\} \\ &= L_m \frac{(N-1)L_m/CF(N) - (N-2)L_m/CF(N-1)}{\left\{ 1 - (N-1)L_m/CF(N) \right\} \left\{ 1 - (N-2)L_m/CF(N-1) \right\}}. \end{aligned} \quad (A5)$$

It is clear that the denominator of equation (A5) is positive (refer the proof of Lemma 1). Also, from equation (A3) we see that the numerator of equation (A5) is positive. Thus, we get

$$\begin{aligned} \Delta WF_m(N) &= L_m^2 \left\{ \frac{(N-1)/CF(N) - (N-2)/CF(N-1)}{\left\{ 1 - (N-1)L_m/CF(N) \right\} \left\{ 1 - (N-2)L_m/CF(N-1) \right\}} \right\} \\ &> 0. \end{aligned} \quad (A6)$$

It may easily be observed from equation (A6) that since $L_M \geq \dots \geq L_1$, hence

$$\Delta WF_M(N) \geq \dots \geq \Delta WF_1(N) \geq 0.$$

□

Proposition 3:

$$CF(N) - CF(N-1) \leq L_M.$$

Proof:

The term $\Delta QF_m(N)$ can be expressed as

$$\Delta QF_m(N) = \frac{L_m [CF(N-1) + L_m - CF(N)]}{\left(CF(N) - (N-1)L_m \right) \left(CF(N-1) - (N-2)L_m \right)}. \quad (A7)$$

It is clear that

$$\sum_m \Delta QF_m(N) \geq 0, \quad (A8)$$

since we have

$$\sum_m \Delta QF_m(N) + \Delta QF_o(N) = 0,$$

and

$$\Delta QF_o(N) = \frac{QF_o(N)}{N} - \frac{QF_o(N-1)}{N-1} = L_o \left[\frac{1}{WF_o(N)} - \frac{1}{WF_o(N-1)} \right] \leq 0.$$

It is also clear that the numerator in equation (A7) increases monotonically with L_m . The denominator always remains positive. Hence in order for the inequality in (A8) to hold, the numerator must become non-negative for some value of $m \leq M$. \square

Proposition 4:

For $m > n$,

$$(a) \frac{\Delta WF_m(N)}{\Delta WF_n(N)} \geq \left(L_m / L_n \right)^2, \quad (A9)$$

$$(b) \Delta WF_m(N) \leq L_m^2 / L_M, \quad (A10)$$

$$(c) \Delta QF_m(N) < \frac{L_m}{(N-1)L_M} - \frac{QF_m(N-1)}{(N-1)^2}.$$

Proof:

From equations (3.1) and (4.1),

$$\begin{aligned} \frac{\Delta WF_m(N)}{\Delta WF_n(N)} &= \left(L_m / L_n \right)^2 \frac{[CF(N) - (N-1)L_n] [CF(N-1) - (N-2)L_n]}{[CF(N) - (N-1)L_m] [CF(N-1) - (N-2)L_m]} \\ &= \left(L_m / L_n \right)^2 \Psi_{mn}(N) \Psi_{mn}(N-1), \end{aligned}$$

where

$$\Psi_{mn}(N) = \frac{CF(N) - (N-1)L_m}{CF(N) - (N-1)L_n} = 1 + \frac{(N-1)(L_m - L_n)}{CF(N) - (N-1)L_m}.$$

Hence the inequality (A9) follows.

From Propositions 2 and 3, we have

$$\Delta WF_m(N) \geq 0, m=1, \dots, M,$$

and

$$CF(N) - CF(N-1) = \sum_m \Delta WF_m(N) \leq L_M.$$

Hence it follows that

$$\Delta WF_M(N) \leq L_M.$$

So, from the inequality (A9),

$$\begin{aligned} \Delta WF_n(N) &\leq \left(L_n / L_M \right)^2 \Delta F_M(N) \\ &\leq \frac{L_n^2}{L_M}. \end{aligned}$$

Now,

$$WF_m(N) = L_m \left[1 + \frac{N-1}{N} QF_m(N) \right],$$

and

$$WF_m(N-1) = L_m \left[1 + \frac{N-2}{N} QF_m(N-1) \right].$$

So

$$\begin{aligned} \Delta WF_m(N) &= L_m \left[\frac{N-1}{N} QF_m(N) - \frac{N-2}{N-1} QF_m(N-1) \right] \\ &= (N-1) L_m \Delta QF_m(N) + L_m \frac{QF_m(N-1)}{N-1} \end{aligned}$$

Noting, from (A10), that $\Delta WF_m(N) < L_m^2 / L_M$, it follows that

$$\Delta QF_m(N) < \frac{L_m}{(N-1)L_M} - \frac{QF_m(N-1)}{(N-1)^2}.$$

□