SUCCESSIVELY IMPROVING BOUNDS ON PERFORMANCE MEASURES FOR PRODUCT FORM QUEUEING NETWORKS

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ABSTRACT

The use of queueing network models to analyze the performance of computer systems is widespread. Typically the analysis requires certain assumptions to be made. Even under such assumptions, the exact analysis of these models for the performance measures could be quite time consuming, especially when various alternate configurations of the system are to be evaluated. In some situations, bounds on the performance may often be adequate. This issue of obtaining reasonable bounds has hence been the subject of some discussion and a number of bounding techniques have been proposed over the past few years. In this paper we present a bounding technique for networks with a single class of customers that appears to be more effective than techniques previously reported.

1. Introduction:

A computer system is evaluated by estimating its performance under various patterns of demand on the system. Queueing network models are typically used here with a view to obtaining equilibrium analytic solutions for predicting the performance under various configurations. Each node in the network represents a service facility where customers queue up for, and receive, service. The exact solution of these queueing networks is, however, infeasible unless certain assumptions are made. These assumptions give rise to a certain class of networks known as Product Form (PF) networks, or separable networks. For these PF networks, it is possible to obtain equilibrium performance measures with relatively less effort.

Quite often, situations arise where exact solutions of these queueing networks may not be required. This is especially the case when many alternate configurations are to be evaluated. Similarly, if the workload parameters are not known with reasonable accuracy, as is often the case in the design phase of a system, obtaining exact solutions may be unnecessary. In such cases, one would like to obtain approximate solutions fairly quickly. Needless to say, the errors incurred as a result of an approximate solution technique should be bounded, if possible.

1.1 Previous work:

Recent years have witnessed a substantial amount of work on developing techniques for bounding the performance measures of queueing networks for PF networks with a single class of customers [3,4,7,11,13,14,15]. These bounding techniques are applicable when each node in the network is either single server fixed-rate type or is a delay (infinite server) type. The bounds obtained are for the cycle time (the mean system residence time) of a job, and the throughput of the system (number of job completions by the system in unit time). The input parameters required for the calculation of these bounds are the loads at the various nodes. The load at a node is defined as the mean service time demand from the node by a job.

Some results have recently been obtained for networks with multiple customer classes [5,6,7]. Results have also been obtained for networks with load dependent nodes and with a single class of customers [10,12]. The latter results depend, however, on bounds obtained for networks consisiting only of fixed rate and delay nodes, and whose behaviour closely approximates the behaviour of the network with the load dependent nodes. In this paper, we restrict attention to PF networks with fixed rate and delay nodes, and a single class of customers.

A bounding technique which applies to a larger class of networks than those considered here is the Asymptotic Bound Analysis (ABA) [2]. The bounds obtained by this analysis are very loose. The Balanced Job (BJ) bounds analysis [15] obtains tighter bounds than those obtained by the ABA. However, these bounds are also often quite loose. The ABA and BJB techniques do not provide a trade-off between computational effort and accuracy for the bounds. Further, bounds obtained here for networks with delay nodes are poor.

The Performance Bound Hierarchies (PBH) [4] and the Generalized Quick Bounds (GQB) [14] are both based on the Mean Value Analysis (MVA) algorithm [9]. Both these techniques obtain a sequence of improving bounds (referred to as bounds of increasingly higher 'levels') on cycle time and throughput. The GQB technique, however, cannot handle delay nodes, and does not appear to perform significantly better than the BJB technique. A technique based on the MVA algorithm and utilizing some properties of the relation between the throughput and the degree of multiprogramming is presented in [3]. The convolution algorithm of Buzen [1] is used to develop the Convolutional Bound Hierarchies [13]. This bounding technique seems appropriate only for networks with a small number of customers and a relatively large number of nodes. All these techniques are basically recursive in nature.

A technique which presents some closed form expressions for the bounds is given by Kriz [7]. This technique is based on the MVA algorithm. A number of computational formulae are presented by this author. However, these formulae generally perform well only as the network population becomes large (at which stage the network begins to behave like an open network).

Although the PBH technique gives a better set of bounds than these above mentioned schemes, it requires considerable computational effort to produce increasingly tighter bounds.

Exact analysis for the type of networks we consider here takes O(MN) computations, where M is the number of nodes and N the number of customers. The bounds we obtain take O(M) time to compute and use the MVA algorithm to develop a sequence of improving upper and lower bounds with only a marginal increase in computational effort for each set of bounds in the sequence. These bounds are shown to be tighter than the BJ bounds and compare quite favorably with the PBH technique for networks with a single class of customers. In the case of networks consisting of only fixed rate nodes, the first few bounds obtained are shown to be tighter than corresponding PBH bounds. Moreover, in general these successively improving bounds (SIB) can usually be obtained with considerably less computational effort than the PBH bounds.

2. Preliminaries

We first consider networks where all nodes are fixed-rate nodes. The analysis extends directly to include delay nodes, and this is shown in section 3. In the following discussion, unless specified otherwise, the index for any summation is over the range 1 through M. Implicit in the discussion is the understanding that we are considering populations of $N \geq 2$.

From the MVA algorithm, the mean residence time, $w_m(N)$, at a fixed rate node m, with N customers in the system, is obtained as

$$w_m(N) = L_m\big(1 + Q_m(N-1)\big), \tag{2.1}$$

where L_m is the load at node m and $Q_m(K)$ represents the mean queue length that would form at node m with K customers in the system.

The cycle time W(N) for a given request is the sum of the mean residence times at all nodes and is given by

$$W(N) = \sum_m L_m\big(1 + Q_m(N-1)\big) = L\big(1 + \sum_m (L_m/L)Q_m(N-1)\big)$$

$$= L\big(1 + \phi(N-1)\big), \tag{2.2}$$

where

$$L = \sum_m L_m,$$

$$\phi(N-1) = \sum_m \rho_m Q_m(N-1), \tag{2.3}$$

with

$$\rho_m = L_m/L. \tag{2.4}$$

The term ρ_m is defined as the relative utilization at node m.

A straightforward application of Little's rule [8] yields an expression for the mean queue length for any single server fixed-rate center m as (also refer [4] for a similar expression) :

$$Q_m(N) = N \frac{L_m\big(1 + Q_m(N-1)\big)}{L\big(1 + \phi(N-1)\big)}. \tag{2.5}$$

Setting, for notational convenience,

$$D_K = 1 + \phi(K), \tag{2.6}$$

we can rewrite equation (2.5) as

$$Q_m(N) = N\rho_m + N\rho_m\left(\frac{Q_m(N-1) - \phi(N-1)}{D_{N-1}}\right). \tag{2.7}$$

From equations (2.7) and (2.3),

$$\phi(N-1) = (N-1)\sum_m \rho_m^2 + \big((N-1)/D_{N-2}\big)\cdot Y^1(N-2), \tag{2.8}$$

where

$$Y^1(N-2) \quad = \sum_m \rho_m^2 \left(Q_m(N-2) - \sum_n \rho_n Q_n(N-2)\right). \qquad (2.9)$$

As a first step, we obtain a simple set of upper and lower bounds on $\phi(N-1)$ (and hence, on the cycle time) which we show are better than the BJB bounds. We term these bounds as level 1 bounds. These bounds are obtained by noting that the term $Y^1(K)$ is non-negative for any $K \geq 0$. To see this, we make use of a property which we call the 'partial correspondence' property:

<u>Definition 2.1:</u>

Given a sequence $X = \{x_m\}$, $m=1,\ldots,M$, such that

$x_M \geq x_{M-1} \geq \ldots \geq x_1 \geq 0$,

then a sequence $Y = \{y_m\}$, $m=1,\ldots,M$, is said to have a Partial Correspondence with X, denoted as Y(PC)X, if for some k, $1 \leq k \leq M$,

$y_M \geq \ldots \geq y_k \geq 0; \quad y_{k-1},\ldots,y_1 \leq 0.$

<u>Lemma 2.1 :</u>

Given two sequences $X=\{x_i\}$, and $Y=\{y_i\}$, $i = 1,\ldots,m$, such that

$$x_m \geq x_{m-1} \geq \ldots \geq x_2 \geq x_1 \geq 0,$$

Y(PC)X

$$\sum_i x_i \leq 1 ,$$

and

$$\sum_i x_i y_i \geq 0,$$

then,

$$\sum_i x_i^2 y_i \geq (\sum_i x_i y_i)(\sum_j x_j)^2$$

Proof :

An alternate statement of the required inequality is to show that :

$$\sum_i x_i^2\left(\frac{y_i}{\alpha} - 1\right) \geq 0 \qquad \text{(A1)}$$

where $\alpha = \sum_i x_i y_i$. Since $\sum_i x_i \leq 1$, it must be true that

$$\sum_i x_i y_i \geq \left(\sum_i x_i y_i\right)\left(\sum_j x_j\right),$$

So,

$$\sum_i x_i\left(\frac{y_i}{\alpha} - 1\right) \geq 0.$$

Hence, $$\sum_i x_k x_i\left(\frac{y_i}{\alpha} - 1\right) \geq 0 \text{ for all } x_k. \qquad \text{(A2)}$$

In view of (A2) and the fact that $\alpha \geq 0$ and $x_i \geq 0$, there exists some $m \leq n$ such that $\alpha \leq y_i$ for all $i \geq m$.

We therefore have to consider two cases: (i) where all terms in the sum in Equation (A2) are non-negative, that is, $\alpha \leq y_1$, and (ii) not all terms are non-negative, that is $y_1 < \alpha \leq y_n$.

Case (i) is trivial: all the terms in the summation in Equation (A1) are non-negative. We consider Case (ii). Here we choose k in Equation (A2) such that $y_k \leq \alpha \leq y_{k+1}$. Then, from (A2),

$$\begin{aligned} 0 \leq \sum_i x_k x_i(y_i/\alpha - 1) &= \sum_{i=1}^{k} x_k x_i(y_i/\alpha - 1) + \sum_{i=k+1}^{n} x_k x_i(y_i/\alpha - 1) \\ &\leq \sum_{i=1}^{k} x_i x_i(y_i/\alpha - 1) + \sum_{i=k+1}^{n} x_k x_i(y_i/\alpha - 1) \\ &\leq \sum_{i=1}^{k} x_i x_i(y_i/\alpha - 1) + \sum_{i=k+1}^{n} x_i x_i(y_i/\alpha - 1) \end{aligned}$$

where the first (respectively second) inequality follows from the fact that $(y_i/\alpha - 1) \leq 0$ (respectively ≥ 0) for all $i \leq k$ (respectively $i > k$). This final inequality is the same as (A1), so we are done.

From the lemma, we have the following corollary which shows that the term

$Y^1(K) \geq 0$, for $K \geq 0$.

Corollary 2.1 :

$$\sum_m \rho_m^2 Q_m(K) \geq (\sum_m \rho_m Q_m(K))(\sum_i \rho_i^2); \; K \geq 0 \qquad (2.10)$$

Proof:

The mean queue length at node i with K customers in the system can be expressed as [2]:

$$Q_i(K) = \sum_{n=1}^{K} \rho_i^n G(K-n)/G(K); \; K > 0, \qquad (2.11)$$

where G( ) is a normalizing constant.

In Lemma 2.1, let $x_i=\rho_i$, and $y_i=Q_i(K)$, i=1,2,...,M. The nodes can be ordered so that $0 \leq \rho_1 \leq \rho_2 \leq ... \leq \rho_M$, and from equation (2.11), we then get $Q_1(K) \leq ... \leq Q_M(K)$, in which case, (2.10) follows from Lemma 2.1.

□

Let

$$S_i = \sum_m \rho_m^i, \qquad (2.12)$$

and define

$$\lambda_N = N/D_{N-1}. \qquad (2.13)$$

Here, λ_N can be thought of as the throughput of a system with total load L = 1, and with population N. Note that $\lambda_1 = 1$, and $\lambda_N \geq 1$; $N \geq 1$.

Theorem 2.1 :

Let u = node with maximum load. Then, the term $\phi(N-1)$ is bounded by :

$$(N-1)S_2 \leq \phi(N-1) \leq \Psi = (T1 + T2)\cdot 0.5 \qquad (2.14a)$$

where

$$T1 = ((N-1)\,\rho_u - 1), \quad \text{with} \quad \rho_u = L_u/L, \qquad (2.14b)$$

and

$$T2 = SQRT(T1^2 + 4(N-1)S_2) \qquad (2.14c)$$

Proof:

From equations (2.8), (2.12) and (2.13),

$$\phi(N-1) = (N-1)S_2 + \lambda_{N-1} Y^1(N-2). \tag{2.15}$$

Clearly, $\lambda_{N-1} \geq 0$. From Corollary 2.1, $Y^1(N-2)$ is non-negative. Hence the term $(N-1)S_2$ is a lower bound on $\phi(N-1)$. For the upper bound, note that

$$\begin{aligned} Y^1(N-2) &= \sum_m \rho_m^2 \left(Q_m(N-2) - \sum_n \rho_n Q_n(N-2)\right) \\ &\leq (\rho_u - S_2)\, \phi(N-2). \end{aligned} \tag{2.16}$$

It is clear that $\rho_u \geq S_2$. Noting that $D_{N-2} = 1 + \phi(N-2)$, it is seen that the term $\phi(N-2)/D_{N-2} \leq \phi(N-1)/D_{N-1}$. Hence we can write

$$\phi(N-1) \leq (N-1)S_2 + (N-1)\left(\phi(N-1)/D_{N-1}\right) \cdot (\rho_u - S_2). \tag{2.17}$$

The above equation is a quadratic in $\phi(N-1)$, whose solution gives the desired bound.

□

It can be seen from equations (2.14a,b,c) that the solution of this quadratic requires about 11 operations, given that S_2 and ρ_u have already been computed.

It is easy to show that the bounds developed above are tighter than the BJ bounds. This is expressed as:

<u>Corollary 2.2</u>:

The level one SI bounds on cycle time and throughput are tighter than the corresponding BJ bounds.

Proof :

We only need to show that the level 1 bounds on cycle time obtained by the SIB technique are tighter than the BJ bounds. A simple application of Little's rule will then show that the level 1 bounds on the throughput obtained by the SIB technique are tighter than the BJ bounds.

The bounds on cycle time obtained by the BJB analysis are:

$$L + (N-1)\cdot L_a \leq W(N) \leq L + (N-1)\cdot L_u, \tag{2.19}$$

where L_u is the load at the node with the highest loading, and L_a is the average load among all the nodes, i.e., $L_a = \sum_m L_m/M$.

In view of equations (2.17) and (2.19), we need to show that

(i) $L\xi \geq (N-1)L_a$, and (ii) $L\psi \leq (N-1)L_u$.

A Lemma proved in [10] gives the following inequality with two increasing sequences $\{x_i\}$, $\{y_i\}$, i=1,..,n:

$$\sum_i x_i y_i \geq \bar{x} \sum_i y_i \text{ where } \bar{x} = (1/n) \sum_i x_i$$

Now, to show (i), let $x_i = y_i = \rho_i$. Then

$$\sum_i \rho_i^2 \geq \rho_a \sum_i \rho_i = \rho_a$$

where $\rho_a = (1/M) \sum_m \rho_m$. Hence $L\xi = L\cdot(N-1)\sum_m \rho_m^2 \geq L\cdot(N-1)\cdot\rho_a = (N-1)\cdot L_a$.

(ii) From Equation (2.14) we have,

$$2\psi = ((N-1)\rho_u - 1) + \mathrm{SQRT}[((N-1)\rho_u - 1)^2 + 4\xi]$$

Since, $\rho_u \geq \sum_m \rho_m^2$, hence we have $(N-1)\rho_u \geq \xi$. So we can write

$$2\psi \leq ((N-1)\rho_u - 1) + \mathrm{SQRT}[((N-1)\rho_u - 1)^2 + 4(N-1)\rho_u]$$

$$= ((N-1)\rho_u - 1) + \mathrm{SQRT}[((N-1)\rho_u + 1)^2] = 2(N-1)\rho_u.$$

Hence,

$$L\psi \leq (N-1)L\rho_u = (N-1)L_u.$$

□

In concluding this section, it is remarked that the paper by Kriz has used the idea of obtaining 'square root' bounds on throughputs for networks with delay nodes. When delay nodes are absent, however, these square root bounds reduce to the BJ bounds.

3. Successively Improving Bounds:

In the above analysis, the level one bounds required the computation of the terms S_2 and L, and these computations needed about 3M operations. We now express the term $Y^1(N-2)$ as the sum of a set of non-negative terms. This will enable a sequence of improving bounds to be obtained with some additional computational effort. These improving bounds are termed bounds of increasing levels. Each higher level requires an additional S_i term to be computed. Thus, the level 2 SI bounds require the terms S_2 and S_3, the level 3 bounds require the terms S_i, i=2,..,4, and so on. Each of these terms takes about 2M operations to compute. Given that the terms S_j, j=2,..,i, have been computed for some i, our objective here is to compute as tight a bound as possible with these terms.

Equation (2.5) is first rewritten, using (2.13) as

$$Q_m(N) = \rho_m\lambda_N + \rho_m\lambda_N\, Q_m(N-1). \qquad (3.1)$$

Define, for N ≧ 0,

$$f_m^0(N) = Q_m(N), \qquad (3.2a)$$

and

$$f_m^i(N) = \rho_m\Big(f_m^{i-1}(N) - \sum_n \rho_n f_n^{i-1}(N)\Big);\ i > 0. \qquad (3.2b)$$

To obtain the expansion of the term $Y^1(N-2)$, we use Lemmas 3.1 and 3.2.

Lemma 3.1:

Consider the function $f_m^i(K)$ as defined by equations (3.2a and 3.2b). Let $f^i(K) = \{f_m^i(K)\}$, and $\rho = \{\rho_m\}$, m = 1,...,M. Then, for i, K ≧ 0,

(a) $f^i(K)$ (PC) ρ, (3.3a)

(b) $Y^i(K) = \sum_m \rho_m\, f_m^i(K) \geq 0.$ (3.3b)

A proof of Lemma 3.1 is given in Appendix A.

The terms $Y^i(K)$ are functions of the mean queue lengths at population K. Note that $Y^i(0) = 0$. Lemma 3.2 now expresses $Y^i(K)$ as comprising of (i) a term involving only the relative utilizations ρ_m, m=1,...,M, and (ii) a term involving the mean queue lengths at population K-1, and the throughput at population K. A proof of Lemma 3.2 is given in Appendix B.

Lemma 3.2:

The term $f_m^i(K)$ can be recursively defined, for $i \geq 0$, $K \geq 1$, as

$$f_m^i(K) = K\, f_m^i(1) + \lambda_K\, f_m^{i+1}\,(K-1),$$

and hence,

$$Y^i(K) = \sum_m \rho_m f_m^i(K) = K\,\alpha_i + \lambda_K\, Y^{i+1}(K-1),\ K \geq 1; \qquad (3.4)$$

where

$$\alpha_i = Y^i(1) \geq 0. \qquad (3.5)$$

From Lemma 3.1 and equation (3.2), noting that $Q_m(1) = \rho_m$, we can derive an expression for α_i as follows:

$$\alpha_0 = \sum_m \rho_m f_m^0(1) = S_2$$

$$\alpha_1 = \sum_m \rho_m f_m^1(1) = \sum_m \rho_m^2\, f_m^0(1) - S_2 \sum_m \rho_m f_m^0(1) = S_3 - S_2\,\alpha_0,$$

and in general,

$$\alpha_i = S_{i+2} - \sum_{j=0}^{i-1} S_{i+1-j}\alpha_j,\quad i \geq 1. \qquad (3.6)$$

Now, define

$$\Lambda_{K,i} = \prod_{n=0}^{i} \lambda_{K-n},\quad K \geq i \geq 0. \qquad (3.7a)$$

$$= 1,\ i < 0. \qquad (3.7b)$$

Theorem 3.1 obtains an expression for D_K in terms of the α_i's and the Λ's.

Theorem 3.1:

The term D_K can be written, for $K > i \geq 0$, as

$$D_K = 1 + KS_2 + \sum_{j=1}^{i} (K-j)\,\alpha_j\,\Lambda_{K,j-1} + Y^{i+1}(K-i-1)\,\Lambda_{K,i}. \qquad (3.8)$$

which can also be written, using equation (3.7b), as

$$D_K = 1 + \sum_{j=0}^{i} (K-j)\,\alpha_j\,\Lambda_{K,j-1} + Y^{i+1}(K-i-1)\,\Lambda_{K,i}. \qquad (3.8a)$$

Proof:

From equations (2.6) and (2.15), we have

$$D_K = 1 + KS_2 + \lambda_K Y^1(K-1). \quad (3.9)$$

The proof now follows by a repeated application of equation (3.4) in equation (3.9). □

Theorem 3.2:

$$\phi(K)/K \geq \phi(K-1)/(K-1). \quad (3.10)$$

Proof:

From equation (3.8), noting that $\Lambda_{K,j-1} > \Lambda_{K-1,j-1}$, for $K \geq j \geq 0$,

$$\phi(K) = KS_2 + \sum_{j=1}^{K-1} (K-j)\, \alpha_j\, \Lambda_{K,j-1} \geq KS_2 + \sum_{j=1}^{K-2} (K-j)\, \alpha_j\, \Lambda_{K-1,j-1}.$$

Hence,

$$\frac{\phi(K)}{K} \geq S_2 + \sum_{j=1}^{K-2} \left(\frac{K-j}{K}\right) \alpha_j\, \Lambda_{K-1,j-1}. \quad (3.11)$$

Now, it is clear that for $K > j > 0$, we must have $\frac{K-j}{K} \geq \frac{K-j-1}{K-1}$. Hence the inequality in equation (3.11) gives

$$\frac{\phi(K)}{K} \geq S_2 + \sum_{j=1}^{K-2} \left(\frac{K-j-1}{K-1}\right) \alpha_j\, \Lambda_{K-1,j-1} = \frac{\phi(K-1)}{K-1},$$

where the last equality follows from equation (3.8), with K-1 in place of K, and with i taken as K-2.

□

A lower bound on the term $Y^{i+1}(K-i-1)$ in equation (3.8) is given by Theorem 3.3.

Theorem 3.3:

The term $Y^{j+1}(K) \geq Y^j(K)\, \alpha_{j+1}/\alpha_j$; $j, K \geq 0$. (3.12)

A proof of Theorem 3.3 is given in Appendix C. From this theorem, we have the following:

Corollary 3.1:

$$\alpha_{j+1}/\alpha_j \ \geq \ \alpha_{i+1}/\alpha_i, \quad j \geq i \geq 0. \tag{3.13}$$

Proof:

From Theorem 3.3 for K = 2, and using equation (3.4), we get

$$\begin{aligned} 0 \ &\leq \ Y^{j+1}(2)\ \alpha_j - Y^j(2)\ \alpha_{j+1} \\ &= \left(2\alpha_{j+1} + \frac{2}{D_1} Y^{j+2}(1)\right) \alpha_j - \left(2\ \alpha_j + \frac{2}{D_1} Y^{j+1}(1)\right) \alpha_{j+1} \\ &= (2/D_1)\ (\alpha_{j+2}\ \alpha_j - \alpha_{j+1} \cdot \alpha_{j+1}). \end{aligned}$$

Since $D_1 \geq 0$, this implies

$$\alpha_{j+2}/\alpha_{j+1} \quad > \quad \alpha_{j+1}/\alpha_j. \tag{3.14}$$

A repeated application of the above inequality gives the desired result. □

Theorem 3.4:

For K > i > 0,

$$D_K \geq 1 + KS_2 + \sum_{j=1}^{i} (K-j)\alpha_j\ \Lambda_{K,j-1} + (K-i)\ \Lambda_{K,i-1} \left(\frac{K-i-1}{1+(K-i-1)\alpha_0}\right) \frac{\alpha_i^2}{\alpha_{i-1}}. \tag{3.15}$$

Proof:

From equations (3.8) and (3.12), we get for $K > i \geq 0$,

$$D_K \geq 1 + KS_2 + \sum_{j=1}^{i} (K-j)\alpha_j\ \Lambda_{K,j-1} + \Lambda_{K,i}\ Y^0(K-i-1) \frac{\alpha_{i+1}}{\alpha_0}.$$

Noting that (a) $Y^0(K) = \phi(K) = D_K - 1$, and (b) $\Lambda_{K,i} = \Lambda_{K,i-1} \cdot \lambda_{K-i}$, we can rewrite the above equation, for K > i, as

$$D_K \geq 1 + KS_2 + \sum_{j=1}^{i} (K-j)\alpha_j\ \Lambda_{K,j-1} + \Lambda_{K,i-1}\ \lambda_{K-i}D_{K-i-1}\left(1 - \frac{1}{D_{K-i-1}}\right) \frac{\alpha_{i+1}}{\alpha_0}.$$

Now, using a level one lower SI bound on D_{K-i-1} in the denominator of the last term, and noting that $\lambda_{K-i}D_{K-i-1} = K-i$,

$$D_K \geq 1 + KS_2 + \sum_{j=1}^{i} (K-j)\alpha_j\ \Lambda_{K,j-1} + (K-i)\ \Lambda_{K,i-1} \left(\frac{K-i-1}{1+(K-i-1)\alpha_0}\right)\alpha_{i+1}; \quad i\geq 0.$$

From equation (3.14), for $i > 0$, $\alpha_{i+1} \geq \alpha_i^2/\alpha_{i-1}$. Using this in the inequality obtained above gives the desired result. □

3.1 The sequence of improving upper and lower bounds:

In general, let $\phi_n^u(N-1)$ (respectively, $\phi_n^l(N-1)$) denote the level n upper SI bound (respectively the level n lower SI bound) on $\phi(N-1)$. We first obtain the sequence of upper SI bounds.

3.1.1 The upper bounds:

We first develop a tighter level one upper bound on $\phi(N-1)$ than was obtained in Section 2. This is obtained as follows: From equations (2.6), (2.8), (2.16) and (3.10),

$$\phi(N-1) \leq (N-1)S_2 + (N-1) \cdot (\rho_u - S_2) \cdot \frac{\eta\phi(N-1)}{1 + \eta\phi(N-1)}, \tag{3.16}$$

where

$$\eta = (N-2)/(N-1). \tag{3.17}$$

Equation (3.16) is a quadratic in $\phi(N-1)$. Its solution is given by $\phi_1^u(N-1)$, where

$$\phi_1^u(N-1) = (T1 + T2) \cdot \frac{0.5}{\eta}, \tag{3.18}$$

where

$$T1 = (N-2)\,\rho_u - 1, \tag{3.18a}$$

and

$$T2 = SQRT\,(T1^2 + 4(N-2)S_2), \tag{3.18b}$$

with η as defined by (3.17).

It is clear that since $\eta < 1$,

$$\frac{\eta\,\phi(N-1)}{1 + \eta\,\phi(N-1)} < \frac{\phi(N-1)}{1 + \phi(N-1)}. \tag{3.19}$$

Hence use of equation (3.16) gives a tighter bound on $\phi(N-1)$ than the bound obtained using equation (2.17). Corollary 3.2 shows that this bound is tighter than the level 2 PBH bound. (Note that both bounds need to compute the term S_2).

Corollary 3.2:

The level 1 upper SI bound given by equation (3.16) is tighter than the level 2 PBH upper bound for networks without delay nodes.

Proof:

The level one upper SI bound, $\phi_1^u(N-1)$, is obtained using equation (3.16). This equation can be rewritten after some elementary algebra as

$$\phi_1^u(N-1) = (N-1) \left(\frac{S_2 + (N-2)\rho_u \, \phi_1^u(N-1)/(N-1)}{1 + (N-2) \, \phi_1^u(N-1)/(N-1)}\right) . \qquad (3.20)$$

Let $\theta_2^u(N-1)$ denote the level two PBH bound on $\phi(N-1)$. When delay nodes are absent, this is obtained as (equation (18) in [4])

$$\theta_2^u(N-1) = (N-1) \left(\frac{S_2 + (N-2)\rho_u^2}{1 + (N-2)\rho_u}\right) . \qquad (3.21)$$

Comparing equations (3.20) and (3.21), it is easily seen that $\phi_1^u(N-1) \leq \theta_2^u(N-1)$ if $\phi_1^u(N-1) \leq (N-1)\rho_u$. Noting that $(N-1)\rho_u$ is the BJ bound on $\phi(N-1)$, this is obvious from Corollary (2.2). □

The sequence of upper SI bounds on $\phi(N-1)$, is given by Theorem 3.5.

Theorem 3.5:

The level n upper bound on $\phi(N-1)$ is given, for $N-1 \geq n$, by

$$\phi_n^u(N-1) = \frac{0.5}{n} (T1 + T2(n)), \qquad (3.22)$$

where

$$T1 = (N-2)\rho_u - 1, \qquad (3.22a)$$

$$T2(n) = SQRT\left(T1^2 + 4(N-2)(S_2 - \sigma(N-1,n-1))\right), \qquad (3.22b)$$

and

$$\sigma(N-1,i) = \sum_{j=1}^{i} (\rho_u S_{j+1} - S_{j+2}) \prod_{m=0}^{j-1} \frac{N-2-m}{\bar{D}_{N-3}}. \qquad (3.22c)$$

with $\bar{D}_{N-3}$ being a level one upper bound on D_{N-3}.

A proof of Theorem 3.5 is given in Appendix D.

For example, a level three upper bound is given using $\sigma(N-1,2)$ in equation (3.22b) where

$$\sigma(N-1,2) = \left((\rho_u S_2 - S_3) + (\rho_u S_3 - S_4)\frac{N-3}{\bar{D}_{N-3}} \right) \frac{N-2}{\bar{D}_{N-3}}.$$

3.1.2 The lower bounds:

Theorem 3.4 is now used to develop a sequence of lower bounds on $\phi(N-1)$. Theorem 3.6 develops the bounds. A proof of Theorem 3.6 is given in Appendix E.

Theorem 3.6:

The level n lower bound on $\phi(N-1)$ is given, for $N-1 \geq n \geq 1$, by

$$\phi_n^1(N-1) = (T1 + T2(n))/2n, \tag{3.23}$$

where

$$T1 = \left((N-2)S_2 - 1\right), \tag{3.23a}$$

$$T2(n) = \mathrm{SQRT}\left[T1^2 + 4(N-2)\left(S_2 + (N-2)\beta(N-1,n-1)\right)\right], \tag{3.23b}$$

and

$$\beta(N-1,i) = 0; \quad i=0,$$

$$\beta(N-1,i) = \sum_{j=1}^{i-1} \alpha_j \prod_{m=2}^{j} \frac{N-1-m}{\bar{D}_{N-1-m}} + \alpha_1 \left(\prod_{m=2}^{i} \frac{N-1-m}{\bar{D}_{N-1-m}} \right) \left(1 + \frac{\alpha_i}{\alpha_{i-1}} \frac{(N-i-2)}{1+(N-i-2)\alpha_0}\right); \quad i>0, \tag{3.23c}$$

with $\bar{D}_{N-1-m} = 1+\bar{\phi}(N-1-m)$ where $\bar{\phi}(N-1-m)$ is an upper bound on $\phi(N-1-m)$.

Empty products are assumed here to have a value of unity in equation (3.23c). The level two lower SI bound obtained by setting i=1 in equation (3.23c) is tighter than the level three PBH bound as Corollary 3.3 shows (note that both bounds require the terms S_2 and S_3):

Corollary 3.3:

For networks without delay nodes, the level two lower SI bound on D_{N-1} is tighter than the level three PBH bound on D_{N-1}, for $N > 3$.

A proof of Corollary 3.3 is given in Appendix F.

For example, a level three lower bound is given by using $\beta(N-1,2)$ in equation (3.23b). Here, $\beta(N-1,2)$ is given from equation (3.23c) as

$$\beta(N-1,2) = \alpha_1 + \alpha_2\left((N-3)/\overline{D}_{N-3}\right)\left(1 + \frac{(N-4)}{1 + (N-4)S_2}\,\frac{\alpha_2}{\alpha_1}\right),$$

with $\overline{D}_{N-3}$ being given by a suitable upper bound on D_{N-3}.

Operationally, we can first calculate the upper bound on $\phi(N-1)$ using equation (3.22); call it $\overline{\phi}(N-1)$. Now, to calculate the lower bound on $\phi(N-1)$ using equation (3.23), we need the terms $\overline{\phi}(N-1-m)$. These are easily calculated using Theorem 3.2 which gives $\overline{\phi}(N-1-m) \leq \overline{\phi}(N-1)\cdot(N-1-m)/(N-1)$.

The performance of the SIB technique is now illustrated with an example, comparing the level three SI bounds with the level four PBH bounds. In addition, a comparison is made with the bounds developed by Kriz: the formulae used here are those recommended by Kriz, and use equation 26(a) in [7] for the upper bound on cycle time (this is the BJ bound since there are no delay servers), and the tighter of the bounds computed using equations 26(b) and 27 in [7], for the lower bound on cycle time. Table 3.1 tabulates the results. It is seen here that the SIB technique compares favorably with the other techniques.

Example 3-1: This is the example presented in [4] to illustrate the PBH technique. There are 50 nodes with loadings as follows: 1 node at 20/417, 2 at 19/417, 5 at 18/417, 5 at 15/417, 5 at 10/417, 8 at 7/417, 8 at 5/417, 8 at 4/417, and 8 at 2/417. All nodes are single server fixed rate nodes. Table 3.1 presents the upper and lower bounds obtained using the SIB technique (SI up, SI dn), the PBH technique (PB up, PB dn), and the bounds obtained by Kriz (Kr up, Kr dn), at various population levels.

N	SI up	PB up	KR up	Exact	KR dn	PB dn	SI dn
10	1.279	1.280	1.432	1.278	1.180	1.278	1.278
20	1.632	1.639	1.911	1.619	1.380	1.615	1.616
40	2.4543	2.467	2.871	2.376	1.931	2.337	2.346
60	3.349	3.364	3.830	3.197	2.882	3.090	3.106
80	4.274	4.288	4.789	4.057	3.839	3.859	3.878

Table 3-1 Bounds on cycle time for example 3.1

The bounds were computed for population values from 1 to a 100. The total time taken by the SIB and PBH techniques, (including computation of S_2, S_3 and S_4) for obtaining these bounds at these populations, were measured. The calculations were made using an AMDAHL 5860 system running MTS. The SIB technique took a total of 6 milliseconds to obtain the bounds. The PBH technique took 15 milliseconds. In order to avoid unnecessary computation and any overheads in operation, the SIB formulae, as also the PBH formulae given by (3.27) were implemented without the use of recursion. These times appear to conform with the analysis of operation counts which is made in section 4.

3.2 Terminal Driven Systems:

When interactive systems are modeled by queueing networks, typically a delay node is used to represent the time spent by users thinking between interactions with the system. The average think time is represented here as L_0, the load at the delay node. The BJB formulae here are not as simple as for the case with no delay nodes, and require an additional O(N) computation. The PBH technique develops a hierarchy of bounds for terminal driven systems. Bounds are also developed by Kriz for terminal driven systems.

SIB has a direct extension for terminal driven systems: consider the equation for mean queue lengths at the nodes. When a delay center is present, this can be written as

$$Q_m(N) = NL_m \frac{1 + Q_m(N-1)}{L_o + \sum_n (L_n + L_n Q_n(N-1))}$$

$$= N\tilde{\rho}_m \left(\frac{1 + Q_m(N-1)}{1 + \sum \tilde{\rho}_n Q_n(N-1)}\right) , \qquad (3.24)$$

where $\tilde{\rho}_m = L_m/(L_o + L)$, and $L = \sum_{m=1}^{M} L_m$.

Comparing equations (3.24) and (2.5), it is seen that all the bounds we had derived earlier would apply here too, if we replace every occurrence of ρ_m by $\tilde{\rho}_m$, for m = 1,...,M, and replace the S_i terms by $\tilde{S}_i = \sum \tilde{\rho}_m{}^i$. No additional operations are needed here.

Example 3.2 compares the performance of the level 3 SIB technique with the level 4 PBH technique and the Kriz bounds.

Example 3.2:

This is the same example as given in [4] to demonstrate PBH for networks with delay nodes. The network is basically the same as example 3.1 with an additional delay node with a load of 4000/417. Table 3.2 shows upper and lower cycle time bounds at various population levels. The exact cycle times are also given.

N	SIB up	PBH up	Kriz up	Exact	Kriz Down	PBH Down	SIB3B Down
20	10.648	10.648	10.686	10.648	10.611	10.648	10.648
40	10.716	10.720	10.802	10.715	10.652	10.715	10.715
80	10.885	10.959	11.110	10.883	10.744	10.880	10.882
120	11.132	11.418	11.566	11.118	10.851	10.096	11.104
160	11.525	12.156	12.244	11.466	10.976	11.376	11.396
200	12.196	13.162	13.202	12.004	11.123	11.731	11.764

Table 3.2 : Bounds on cycle times for example 3.2

The bounds were computed for a range of population values from 1 to 200, and the total time taken by the SIB and PBH techniques when run on the AMDAHL 5860 machine, were compared. The SIB technique took 14 milliseconds to obtain the bounds, while the PBH technique took 37 milliseconds to obtain the bounds.

It can be observed that the level 3 upper bound of the SIB technique gives noticeably better results than the other two techniques in this case.

4. Computational Effort:

The bounding technique outlined above produces a sequence of increasingly tighter bounds. The computational effort required to produce these bounds is, further, usually less than that required of the PBH bounds. To illustrate this point, we analyze the number of operations required by both techniques. Suppose we compare the level three SI bounds with the level four PBH bounds. Both techniques, here, require computation of the terms upto S_4 as an initial step. Given that these terms have been computed, for each population level, N, the number of arithmetic operations for the two techniques are as follows:

(a) The SIB technique requires about 14 operations to compute a level one upper bound on D_{N-3} (refer equation 3.18), which is used for both the upper and the lower bounds on D_{N-1}. Assuming that (i) the square root operation counts as one operation, and (ii) given N, evaluating N-i for any $i > 0$ counts as one operation, the upper bound on $\phi(N-1)$ then requires about 27 additional operations, (refer equation 3.22) and the lower bound about 22 additional operations (refer equation 3.23) for a total of about 63 operations.

(b) The PBH technique proceeds as follows: the level i PBH upper or lower bound on the cycle time is given by f(i,1,N), where

$$f(j,p,N) = S_p + (N-1) \frac{f(j-1,\ p+1,\ N-1)}{L_0 + f(j-1,\ 1,\ N-1)}, \qquad (4.1)$$

with some boundary conditions when j=0 in both cases. (It may be noted that computing a level i PBH bound on cycle time at population N requries computation of a level i-1 PBH bound at N-2, and so on upto a level 1 PBH bound at population N-i+1). A level i PBH bound requires (i+1) (i+2)/2 function evaluations of the form given by (4.1) for each of the upper and lower bounds at each population value. Of these, i+1 evaluations are at the boundary condition which requires only about 3 arithmetic operations. All other function evaluations require about 4 arithmetic operations at least when delay nodes are present, as can be seen from equation (4.1). This does not however, include the operations necessary to index into the array f, in order to evaluate f(j,p,n). For example, to compute f(j,p,n), it is required to set the indices j-1, p+1, and N-1 to determine the location of the desired functions f(j-1, p+1, N-1) and f(j-1, 1, N-1). Hence, the calculation of the level four bounds requires about 10x4 + 5x3 = 55 operations for each upper and lower bound, for a total of about 110 operations per population value without considering these indexing operations.

Thus, the PBH level four bounds require about twice as many arithmetic operations as the SI level three bounds do, even if the indexing operations, as described above, are disregarded in calculating the bounds. (If the setting of the indices are considered, each function evaluation other than those at the boundary condition, requires about 3 additional operations). Further, the PBH bounds require computation which increase as the square of the level. The SI bounds at higher levels require, on the other hand, about 10 additional operations in total for each additional level (plus some one time operations to calculate the required α terms). It is noted, though, that while the SI bounds are thus more efficient to compute than the PBH bounds, the sequence of PBH bounds ultimately converge to the exact solution while the SI bounds, as described above, do not.

5. <u>An alternate set of lower bounds</u>:

In this section, an alternate means of obtaining a sequence of lower bounds is presented. These bounds require additional computational effort compared to those presented in section 3. The effectiveness of these bounds are demonstrated with a few examples.

From equations (3.8) and (3.12), and using equation (3.7) to set $\Lambda_{K,i} = \lambda_K \Lambda_{K-1,i-1}$,

$$D_K \geq 1 + KS_2 + \sum_{j=1}^{i} (K-j)\alpha_j \Lambda_{K,j-1} + \frac{\alpha_{i+1}}{\alpha_i} Y^i(K-i-1) \Lambda_{K,i}$$

$$= 1 + KS_2 + \sum_{j=1}^{i} (K-j)\alpha_j \Lambda_{K,j-1} + \frac{\alpha_{i+1}}{\alpha_i} \lambda_K Y^i(K-i-1) \Lambda_{K-1,i-1} \qquad (5.1)$$

From equation (3.8a), with K-1 in place of K, and i-1 in place of i, we get for $K > i \geq 1$,

$$\Lambda_{K-1,i-1} Y^i(K-i-1) = D_{K-1} - 1 - \sum_{j=0}^{i-1} (K-j-1)\alpha_j \Lambda_{K-1,j-1}. \qquad (5.2)$$

So, from (5.1) and (5.2), for $K \geq i \geq 0$,

$$D_K \geq 1 + KS_2 + \sum_{j=1}^{i} (K-j)\alpha_j \Lambda_{K,j-1} + \frac{\alpha_{i+1}}{\alpha_i} \lambda_K \left[D_{K-1} - 1 - \sum_{j=0}^{i-1} (K-j-1)\alpha_j \Lambda_{K-1,j-1}\right]. \qquad (5.3)$$

In the right hand side of (5.3), since the term in the square bracket is non-negative for all values of i, it can contribute to a tighter lower bound on D_K. Since $\lambda_K D_{K-1} = K$, we can rewrite (5.3) after some elementary algebra as

$$D_K \geq 1 + K\left(\alpha_0 + \frac{\alpha_{i+1}}{\alpha_i}\right) - \sum_{j=0}^{i-1} (K-j-1) \left(\frac{\alpha_{i+1}}{\alpha_i}\alpha_j - \alpha_{j+1}\right) \Lambda_{K,j} - \lambda_K \frac{\alpha_{i+1}}{\alpha_i}; \quad i \geq 0. \qquad (5.4)$$

Noting, from Corollary 3.1, that the term $(\alpha_{i+1}/\alpha_i)\alpha_j \geq \alpha_{j+1}$ for $j \leq i$, the bounds are obtained now by suitably bounding the terms λ_K and $\Lambda_{K,j}$.

To illustrate, the level 3 bound is obtained as follows: Equation (5.4) with i=1 gives

$$D_K \geq 1 + K\left(\alpha_0 + \frac{\alpha_2}{\alpha_1}\right) - (K-1)\left(\frac{\alpha_2\alpha_0}{\alpha_1} - \alpha_1\right)\lambda_K - \frac{\alpha_2}{\alpha_1}\lambda_K. \qquad (5.5)$$

Thus, setting $\lambda_{N-1} = (N-1)/D_{N-2}$, a level 3 lower bound on D_{N-1} (and hence W(N)) is given by

$$D_{N-1} \geq 1 + (N-1)\left(\alpha_0 + \frac{\alpha_2}{\alpha_1}\right) - \frac{(N-1)}{D_{N-2}}\left[(N-2)\left(\frac{\alpha_2\alpha_0}{\alpha_1} - \alpha_1\right) + \frac{\alpha_2}{\alpha_1}\right]. \qquad (5.6)$$

Equation (5.6) can use a level 3 SI bound or a level 4 PBH lower bound on D_{N-2}. We now present a few examples which compare the lower bound obtained using equations (5.4) with the PBH lower bound and the bound obtained using the formulae given by Kriz[7]. The bound obtained by the Asymptotic Bound Analysis (ABA) is also presented for comparison.

In the following examples, the bound obtained using (5.4) is termed SIB-nA, where n is the level. This equation is used in conjunction with the PBH technique to illustrate the effect of a hybrid of the two techniques as follows: As we remarked earlier, computing a level n PBH bound at population K in turn computes a level n-1 PBH bound at population K-1, a level n-2 PBH bound at population K-2, and so on till a level 1 PBH bound at population K-n+1. Now, suppose we wish to compute the SIB-nA lower bound on D_K. For this we would compute the S_j terms for j=2,..,n+1. With these S_j terms, we can compute a level n+1 PBH upper bound on λ_K, and this in turn would directly obtain PBH upper bounds on the terms $\lambda_{K-1},\ldots,\lambda_{K-n}$. Some of these bounds may now be used in equation (5.4) to obtain the SIB-nA lower bound. (The use of equation (5.4) in this manner could also be interpreted as a possible extension of the PBH technique.)

Example 5.1:

Example 3.1 is now reworked to illustrate SIB-3A. Table 5.1 shows the comparisons at various population levels.

N	Exact	SIB-3A	PBH Level 4	Kriz	ABA
10	1.278	1.278	1.278	1.180	1.000
20	1.619	1.617	1.615	1.380	1.000
40	2.376	2.354	2.337	1.931	1.918
60	3.197	3.130	3.090	2.882	2.878
80	4.057	3.920	3.859	3.839	3.836
90	4.497	4.324	4.246	4.315	4.314

Table 5.1 Lower Bounds on D_{N-1} for example 5.1.

Example 5.2

There are 25 nodes in this example with loads as follows: 1 at 40/575, 2 at 35/575, 3 at 30/575, 5 at 25/575, 5 at 20/575, 5 at 18/575, and 4 at 15/575. The level 5 SI bound based on equation (5.6) (termed SIB-5A) and the level 6 PBH bound is used here. The comparisons are made in Table 5.2.

N	Exact	SIB-5A	PBH Level 6	Kriz	ABA
10	1.404	1.404	1.404	1.360	1.000
20	1.886	1.885	1.883	1.760	1.391
30	2.408	2.396	3.389	2.089	2.087
40	2.974	2.930	2.913	2.783	2.783
50	3.581	3.479	3.450	3.478	3.478

Table 5.2: Lower Bounds on D_{N-1} for example 5.2

Example 5.3:

There are 15 nodes in this example with loads as follows: 1 node at 20/192, 2 at 18/192, 2 at 17/192, 2 at 15/192, 2 at 14/192, 2 at 10/192, 2 at 7/192, and 2 at 5/192. Table 5.3 tabulates the results. The level 3 SI bound based on equation 5.6, and the level 4 PBH bound is used here.

N	Exact	SIB-3A	PBH Level 4	Kriz	ABA
10	1.716	1.715	1.714	1.60	1.042
20	2.568	2.556	2.546	2.267	2.083
30	3.463	3.416	3.393	3.134	3.125
40	4.392	4.286	4.247	4.170	4.167
50	5.351	5.208	5.208	5.209	5.208

Table 3.3: Lower Bounds on D_{N-1} for Example 3.3

It may be noted from these examples that the Kriz bounds obtained are usually just a little better than the ABA bounds.

6. Conclusion:

A new means of obtaining bounds for closed single chain, separable networks has been developed. These are successively improving bounds in the sense that increasingly tighter bounds can be obtained, although at the expense of increased computation. These bounds are termed as bounds of increasing levels. The bounds obtained are computationally efficient. The SI bounds were compared with those obtained by other techniques through a number of examples to illustrate the effectiveness of the SIB technique. It was shown that for networks without delay nodes, the bounds at lower levels produced by the SIB technique are tighter than those obtained by the PBH technique with about the same degree of computational effort. At higher levels, the SIB technique requires much less computational effort than the PBH technique to obtain bounds. (although it may not produce comparable bounds at higher levels as it does not approach the exact solution.)

Finally, in section 5, it was indicated how the SIB and the PBH technique could be used in conjunction to obtain lower bounds on the cycle times. These cycle time bounds appear to be better than the bounds obtained by either of the techniques in isolation.

Appendix A

Proof of Lemma 3.1:

Lemma 3.1:

Consider the function $f_m^i(K)$ as defined by equation 3.2. Let $f^i(K)=\{f_m^i(K)\}$, and $\rho=\{\rho_m\}$, $m = 1,\ldots,M$. Then, for $i, K \geq 0$,

(a) $f^i(K)$ (PC) ρ, (A1)

(b) $Y^i(K) = \sum_m \rho_m f_m^i(K) \geq 0$ (A2)

Proof:

Proof by induction on i.

For clarity we drop the subscript K since this will not affect the proof.

For i=0, the proof is obvious since $f_m^o = Q_m$. Assume that for some $i \geq 1$, $f^j(PC)\rho$, $j = 1,\ldots,i$, and $Y^i = \sum_m \rho_m f_m^i \geq 0$.

In other words,

$f_m^i \geq \ldots f_k^i \geq 0;\ f_{k-1}^i,\ldots, f_1^i \leq 0$, for some $1 \leq k \leq M$. Hence there exists a number $k1 \geq k$ such that;

$$f_M^i \geq \ldots \geq f_{k1}^i \geq \sum_m \rho_m f_m^i \geq \ldots \geq f_k^i \geq 0;\ f_{k-1}^i,\ldots,f_1^i \leq 0. \qquad \text{(A3)}$$

Lemma 2.1 shows that

$$\sum_m \rho_m^2 f_m^i \geq \sum_m \rho_m^2 \sum_n \rho_m f_m^i.$$

Hence,

$$Y^{i+1} = \sum_m \rho_m f_m^{i+1} = \sum_m \rho_m{}^2 (f_m^i - \sum_n \rho_n f_n^i) \geq 0.$$

Also, from (A3), it is clear that

$$\rho_M(f_M^i - Y^i) \geq \ldots \geq \rho_{k1}(f_{k1}^i) \geq 0,$$

i.e.,

$$f_M^{i+1} \geq \ldots \geq f_{k1}^{i+1} \geq 0;$$

and hence f^{i+1}(PC) ρ. □

Appendix B

Proof of Lemma 3.2:

Lemma 3.2:

The term $f_m^i(K)$ can be recursively defined, for $i \geq 0$, $k \geq 1$, as

$$f_m^i(K) = K\, f_m^i(1) + \lambda_K\, f_m^{i+1}\,(K-1), \qquad \text{(B1)}$$

and hence

$$Y^i(K) = \sum_m \rho_m f_m^i(K) = K\,\alpha_i + \lambda_K\, Y^{i+1}\,(K-1),\ K \geq 1, \qquad \text{(B2)}$$

where

$$\alpha_i = Y^i(1) \geq 0.$$

Proof:

Proof by induction on i.

For $i = 0$, the proof is obvious from equations (2.7) and (3.1) (note that $Y^o(K) = \phi(K)$).

Assume that the result holds for some $i \geq 1$. We need to show the result holds for $i + 1$.

From equation 3.2,

$$f_m^{i+1}(K) = \rho_m\, \left(f_m^i(K) - \sum_n \rho_n f_n^i(K)\right).$$

Using (B1) to substitute for $f_m^i(K)$ and $f_n^i(K)$ yields

$$f_m^{i+1}(K) = \rho_m[Kf_m^i(1) + \lambda_K\, f_m^{i+1}(K-1) \;\; - \sum_n \rho_n\, (K\, f_n^i(1) + \lambda_k\, f_n^{i+1}\,(K-1))]$$

$$= K\,\rho_m(f_m^i(1) - \sum_n \rho_n f_n^i(1))$$

$$+ \;\; \lambda_K\,\rho_m\,(f_m^{i+1}(K-1) - \sum_n \rho_n f_n^{i+1}(K-1)),$$

and hence, from equation (3.2),

$$f_m^{i+1}(K) = K\, f_m^{i+1}(1) + \lambda_k\, f_m^{i+2}(K-1). \qquad \text{(B3)}$$

Further, from (B3),

$$Y^{i+1}(K) = \sum_m \rho_m f_m^{i+1}(K) = K\, Y^{i+1}(1) + \lambda_K\, Y^{i+2}\,(K-1). \qquad \square$$

Appendix C

Proof of Theorem 3.3

Theorem 3.3

$$Y^{j+1}(k)\alpha_j \geq Y^j(k)\,\alpha_{j+1} \quad ; \quad j,k\geq 0$$

Proof:

Proof by induction on k:

For k = 1, the result is obvious, since

$Y^j(1) = \alpha_j$

Assume the result holds for same k-1.

$$\therefore \sum_m \rho_m f_m{}^{j+1}(k-1)\alpha_j - \sum \rho_m f_m{}^j(k-1)\alpha_{j+1} \geq 0.$$

Hence, from equation (3.3),

$$Y^{j+1}(k)\alpha_j - Y^j(k)\alpha_{j+1} = \sum_m \rho_m f_m{}^{j+1}(k)\alpha_j - \sum_m \rho_m f_m{}^j(k)\alpha_{j+1}$$

$$= k\sum_m \rho_m f_m{}^{j+1}(1)\alpha_j + \frac{k}{D_{k-1}}\alpha_j\Big(\sum_m \rho_m{}^2\{f_m{}^{j+1}(k-1) - Y^{j+1}(k-1)\}\Big)$$

$$- k\sum_m \rho_m f_m{}^j(1)\alpha_{j+1} - \frac{k}{D_{k-1}}\alpha_{j+1}\Big(\sum_m \rho_m{}^2\{f_m{}^j(k-1) - Y^j(k-1)\}\Big)$$

$$= \frac{k}{D_{k-1}}\Big[\sum_m \rho_m{}^2\big(f_m{}^{j+1}(k-1)\alpha_j - f_m{}^j(k-1)\alpha_{j+1}\big)$$

$$-S_2\sum_m \rho_m\big(f_m{}^{j+1}(k-1)\alpha_j - f_m{}^j(k-1)\alpha_{j+1}\big)\Big]$$

$$= \frac{k}{D_{k-1}}\Big[\sum_m \rho_m{}^2\big(g^j{}_m(k-1) - \sum_n \rho_n g^j{}_n(k-1)\big)\Big],$$

where

$$g^j{}_m(k-1) = f_m{}^{j+1}(k-1)\alpha_j - f_m{}^j(k-1)\alpha_{j+1} \qquad \text{(C1)}$$

$\therefore$ If we can show $g^j(k)(PC)\rho$ for all k, where $g^j = \{g^j_m\}$, m=1,...,M, we are done.

Note that from equations (3.2b) and (C1),

$$g_m^j(k-1) = \rho_m\left(f_m^j(k-1) - Y^j(k-1)\right)\alpha_j - f_m^j(k-1)\alpha_{j+1}$$

$$= \alpha_j\rho_m\left[f_m^j(k-1)\left(1 - \frac{\alpha_{j+1}}{\rho_m\alpha_j}\right) - Y^j(k-1)\right].$$

Also, since

$\sum_n \rho_n g_n{}^j(k-1) \geq 0$ by inductive assumption, at least one of the g_m s ≥ 0.

It is known that f^j(PC)ρ. Here, if for any two nodes m, n, we have both $f_m{}^j$, $f_n{}^j \geq 0$, then $\rho_m \geq \rho_n$ implies $f_m{}^j \geq f_n{}^j$. This in turn implies that

$$f_m{}^j(k-1)\left(1 - \frac{\alpha_{j+1}}{\rho_m\alpha_j}\right) \geq f_n{}^j(k-1)\left(1 - \frac{\alpha_{j+1}}{\rho_n\alpha_j}\right).$$

Hence if there are nodes m_1, n_1, such that $g_{m1}^j(k-1) \geq 0$, and $g_{n1}^j(k-1) \geq 0$, then $\rho_{m1} \geq \rho_{n1}$ implies $g_{m1}^j(k-1) \geq g_{n1}^j(k-1)$. Thus, $g^j(k)$ (PC) ρ, and we are done. □

Appendix D

Proof of Theorem 3.5:

Theorem 3.5:

The level n upper bound on $\phi(N-1)$ is given, for $N-1 \geq n$, by

$$\phi_n^u(N-1) = \frac{0.5}{\eta} (T1 + T2(n)), \quad (D1)$$

where

$$T1 = (N-2)\rho_u - 1, \quad (D1a)$$

$$T2 = SQRT\left(T1^2 + 4(N-2)(S_2 - \sigma(N-1,n-1))\right), \quad (D1b)$$

$$\sigma(N-1,i) = \sum_{j=1}^{i} (\rho_u S_{j+1} - S_{j+2}) \prod_{m=0}^{j-1} \frac{N-2-m}{\bar{D}_{N-3}}, \quad (D1c)$$

with $\bar{D}_{N-3}$ being a level one upper bound on D_{N-3}.

Proof:

We can express the term $\sum \rho_m^2 Q_m(N-2)$ from equation (2.9) as

$$\sum_m \rho_m^2 Q_m(N-2) = \rho_u\phi(N-2) - \sum_m(\rho_u\rho_m - \rho_m^2)Q_m(K),$$

and using equation (3.1) repeatedly, this gives

$$\sum_m \rho_m^2 Q_m(N-2) = \rho_u\phi(N-2) - \lambda_{N-2}(\rho_u S_2 - S_3) - \lambda_{N-2} \sum_m(\rho_u\rho_m - \rho_m^2)\rho_m Q_m(N-3)$$

$$= \rho_u\phi(N-2) - \lambda_{N-2}(\rho_u S_2 - S_3) - \lambda_{N-2}\lambda_{N-3}(\rho_u S_3 - S_4) - \lambda_{N-2}\lambda_{N-3} \sum_m(\rho_u\rho_m - \rho_m^2)\rho_m^2 Q_m(N-4),$$

and so on to get

$$\sum_m \rho_m^2 Q_m(N-2) \geq \rho_u\phi(N-2) - \sum_{j=1}^{i} (\rho_u S_{j+1} - S_{j+2}) \Lambda_{N-2,j-1}, \quad i \leq N-2. \quad (D2)$$

Hence, from equations (2.8) (2.9) and (3.7), noting that for $k > 0$, $\lambda_k = k/D_{k-1}$,

$$\phi(N-1) \leq (N-1)S_2 + \frac{(N-1)}{D_{N-2}}\left((\rho_u - S_2)\,\phi(N-2) - \sum_{j=1}^{i} (\rho_u S_{j+1} - S_{j+2}) \prod_{m=0}^{j-1} \frac{N-2-m}{D_{N-m-3}} \right).$$

Since $\phi(N-2)/D_{N-2} \leq \eta\phi(N-1)/\left(1+\eta\phi(N-1)\right)$, ($\eta$ is given by equation (3.17)),

$$\phi(N-1) \le (N-1)S_2 + (N-1)(\rho_u - S_2)\frac{\eta\,\phi(N-1)}{1+\eta\phi(N-1)} - \frac{N-1}{D_{N-2}} \sum_{j=1}^{i} (\rho_u S_{j+1} - S_{j+2}) \prod_{m=0}^{j-1} \frac{N-2-m}{D_{N-m-3}}.$$

Since $D_{N-2} = 1 + \phi(N-2) \le 1 + \eta\phi(N-1)$, and $D_{N-m-3} \le D_{N-3}$ for $m>0$, we can write

$$\phi(N-1) \le (N-1)S_2 + (N-1)(\rho_u - S_2)\frac{\eta\,\phi(N-1)}{1+\eta\phi(N-1)} - \frac{N-1}{1+\eta\phi(N-1)}\sigma(N-1,i), \quad \text{(D3)}$$

where

$$\sigma(N-1,i) = \sum_{j=1}^{i} (\rho_u S_{j+1} - S_{j+2}) \prod_{m=0}^{j-1} \frac{N-2-m}{D_{N-3}}. \quad \text{(D4)}$$

Equation (D3) is a quadratic in $\phi(N-1)$ if we replace D_{N-3} in (D4) by its level one upper bound.

□

Since the terms $(\rho_u S_{i+1} - S_{i+2})$, $i \ge 0$, are all non-negative, $\sigma(N-1,i)$ is an increasing function of i and hence bounds of higher levels are increasingly tighter.

Appendix E

Proof of Theorem 3.6:

Theorem 3.6:

The level n lower bound on $\phi(K)$ is given, for $K \geq n$, by

$$\phi_n^1(K) = (T1 + T2(n))/2\eta, \qquad (E1)$$

where

$$T1 = \left((K-1)S_2 - 1\right), \qquad (E1a)$$

$$T2(n) = SQRT(T1^2 + 4(K-1)\left(S_2 + (K-1)\beta(K,n-1)\right), \qquad (E1b)$$

and

$$\beta(K,i) = 0; \quad i=0,$$

$$\beta(K,i) = \sum_{j=1}^{i-1} \alpha_j \prod_{m=2}^{j} \frac{K-m}{\bar{D}_{K-m}} + \alpha_1 \left(\prod_{m=2}^{i} \frac{K-m}{\bar{D}_{K-m}}\right)\left(1 + \frac{\alpha_i}{\alpha_{i-1}} \frac{(K-i-1)}{1+(K-i-1)\alpha_o}\right); \quad i>0, \qquad (E1c)$$

and $\bar{D}_{K-m}$ is a suitably chosen upper bound on D_{K-m}.

Proof:

A bound on the term D_K as given by (3.15) is

$$D_K \geq 1 + KS_2 + \sum_{j=1}^{i} (K-j)\alpha_j \Lambda_{K,j-1} + (K-i)\Lambda_{K,i-1} \left(\frac{K-i-1}{1+(K-i-1)\alpha_o}\right) \frac{\alpha_i^2}{\alpha_{i-1}}. \qquad (E2)$$

The term $(K-j)\Lambda_{K,j-1}$ in the above equation is rewritten as follows:

$$(K-j)\Lambda_{K,j-1} = (K-j)\prod_{n=0}^{j-1} \lambda_{K-n} = (K-j)\prod_{n=0}^{j-1} (K-n)/D_{K-n-1}$$

$$= K \prod_{n=0}^{j-1} (K-n-1)/D_{K-n-1} = K \prod_{m=1}^{j} (K-m)/D_{K-m}.$$

$$= \frac{K(K-1)}{D_{K-1}} \prod_{m=2}^{j} \frac{K-m}{D_{K-m}}.$$

Hence, from equation (E2) and (3.10),

$$\phi(K) \geq KS_2 + \frac{K(K-1)}{D_{K-1}} \left(\sum_{j=1}^{i} \alpha_j \prod_{m=2}^{j} \frac{K-m}{D_{K-m}} + \frac{\alpha_i^2}{\alpha_{i-1}} \frac{(K-i-1)}{1+(K-i-1)\alpha_o} \prod_{m=2}^{i} \frac{K-m}{D_{K-m}} \right)$$

$$\geq \; KS_2 + \frac{K(K-1)}{1+\left((K-1)/K\right)\phi(K)} \beta(K,i), \tag{E3}$$

where

$$\beta(K,i) = \sum_{j=1}^{i-1} \alpha_j \prod_{m=2}^{j} \frac{K-m}{\bar{D}_{K-m}} + \alpha_1 \left(\prod_{m=2}^{i} \frac{K-m}{\bar{D}_{K-m}} \right) \left(1 + \frac{\alpha_i}{\alpha_{i-1}} \frac{(K-i-1)}{1+(K-i-1)\alpha_o}\right). \tag{E4}$$

and $\bar{D}_{K-m}$ is a suitably chosen upper bound on D_{K-m}.

Equation (E3) is a quadratic in $\phi(K)$ whose solution gives the desired result.

□

Appendix F

Proof of Corollary 3.3:

Corollary 3.3:

For networks without delay nodes, the level two lower SI bound on D_{N-1} is tighter than the level three PBH bound on D_{N-1} for N>3.

Proof:

Let $\phi_2^1(N-1)$ denote the level two lower SI bound on $\phi(N-1)$ and $\theta_3^1(-1)$ denote the corresponding level three lower PBH bound.

From equation (3.15) with i=1, setting K=N-1, we get

$$\phi(N-1) \geq (N-1)S_2 + \lambda_{N-1}(N-2)\alpha_1 + \lambda_{N-1}\frac{(N-2)(N-3)\alpha_1^2/\alpha_0}{1+(N-3)S_2},$$

and using Theorem 3.2 this gives, with $\eta=(N-2)/(N-1)$,

$$\phi(N-1) \geq (N-1)S_2 + \frac{(N-1)(N-2)\alpha_1}{1+\eta\phi(N-1)}\left(1 + \frac{(N-3)\alpha_1/S_2}{1+(N-3)S_2}\right). \quad \text{(F1)}$$

The solution to this quadratic in $\phi(N-1)$ gives the bound $\phi_2^1(N-1)$. For notational ease, we omit the argument (N-1) on ϕ_2^1 and θ_3^1 to get

$$\phi_2^1 = (N-1)S_2 + \frac{(N-1)(N-2)\alpha_1}{1+\eta\ \phi^S}\left(1 + \frac{(N-3)\alpha_1/S_2}{1+(N-3)S_2}\right). \quad \text{(F2)}$$

The PBH bound is given from section (3.2) of [4] as

$$\theta_3^1 = (N-1)\left(\frac{S_2 + (N-2)S_3}{1+(N-2)S_2}\right). \quad \text{(F3)}$$

We now prove that $\phi_2^1 > \theta_3^1$ by contradiction.

Suppose that $\phi_2^1 \leq \theta_3^1$. Then from (F2), this implies that

$$\theta_3^1 \geq \phi_2^1 \geq (N-1)S_2 + \frac{(N-1)(N-2)\alpha_1}{1 + \eta\theta_3^1}\left(1 + \frac{(N-3)\alpha_1/S_2}{1 + (N-3)S_2}\right), \quad \text{(F4)}$$

Equation (F3) can be rewritten after some elementary algebra, as

$$\theta_3^1 = (N-1)S_2 + \frac{(N-1)(N-2)\alpha_1}{1+(N-2)S_2} \qquad (F5)$$

From equations (F4) and (F5), we get

$$(N-1)S_2 + \frac{(N-1)(N-2)\alpha_1}{1+(N-2)S_2} \geq (N-1)\ S_1 + \frac{(N-1)(N-2)\alpha_1}{1+\eta\ \theta_3^1}\left(1 + \frac{(N-3)\alpha_1/S_2}{1+(N-3)S_2}\right),$$

and cancelling out common items, we get

$$\frac{1}{1+(N-2)S_2} \geq \frac{1}{1+\eta\theta_3^1}\left(1 + \frac{(N-3)\alpha_1/S_2}{1+(N-3)S_2}\right). \qquad (F6)$$

Expressing θ_3^1 in equation (F6) using equation (F5), and after some straight forward manipulation, this gives the inequality

$$(N-2)^2 \geq \frac{(N-3)}{S_2} + 2(N-3)(N-2). \qquad (F7)$$

This last inequality is clearly a contradiction for $N \geq 4$, and hence the result follows.

□

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