ESSAYS ON JOINT OPTIMIZATION OF PRICING AND CAPACITY DECISIONS WITH CUSTOMER BEHAVIOR MODELLING

by

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CHAPTER 1

Overview

This dissertation focuses on joint optimization of pricing and capacity decisions with explicit modelling of customer behavior. In contrast to the traditional Operations-Management literature, which commonly assumes that customer behavior is governed by some exogenous demand profile, this dissertation takes an alternative approach and explicitly models customers decision processes based on their preferences. These decision processes guide customers on where, when, and what to buy, and clearly have a direct impact on firms profits. By endogenizing these decision processes, I focus on optimizing firms operational and marketing decisions in several business scenarios described below.

In the first two essays, I focus on markets where a product can be purchased and consumed at different times (e.g., airline tickets, hotel stays, books, and game consoles). Firms operating in such markets have an option to offer the product prior to consumption time and segment customers into different groups based on the time that customers make the purchase. On the other hand, customers evaluate current and future purchasing opportunities and strategically choose when to buy. I take firms' viewpoint and study whether, when and how they should sell in advance. In the third essay, I switch my focus from advance selling to assemble-to-order systems (e.g., Dell Corporation), where multiple products are jointly produced and marketed from common components. The objective of the research is to improve the match between product demand and component inventory via demand shaping and product overselling.

In the first essay, "Advance Selling – The Effect of Capacity and Consumer Valuation Interdependence," I focus on a firm's optimal advance-selling strategy with various customer valuation interdependence. Although advance selling is widely used in various industries, existing academic literature offers very limited insights on conditions under which firms should sell in advance and how advance selling influences firms pricing and operational decisions. Furthermore, most of the existing work in this area takes a fairly simplistic view of customer behavior, thus results and insights from these models may not hold in practice.

To characterize the interaction between firms and customers, I consider a seller with limited capacity who offers a single product to customers twice, in advance and in spot markets. Customers have uncertain valuations, which are resolved in the spot period, and strategically choose when to purchase the product. Customers' valuations may be inter-dependent in varying degree, from fully independent to perfectly correlated, which creates different markets (different characteristics of aggregate demand) for the seller. Facing these strategic customers, the seller must choose not only price in each period but also what portion of the total capacity to offer in advance.

I find that customer valuation interdependence dramatically influences the firms strategy. For example, in markets where customers preferences are fairly diverse (e.g., hotels), firms typically quote discounted price to provide customers incentives to purchase early, but may limit the capacity available in advance, e.g., a hotel may not release all of the rooms at the discounted price. On the other hand, when customers preferences are highly correlated (e.g., hot Broadway show tickets), firms with limited capacity can exploit customers uncertainty about the product availability and charge premium price in advance. In addition, I also examine how the benefit of advance selling depends on customer valuation interdependence and key problem parameters including capacity, marginal cost, and variability in customer valuation.

In addition to specific lessons and interpretations for various markets, the first essay demonstrates how firms can improve their profitability by jointly considering operational limitations and the nature of customer behavior in the market they operate.

The second essay, "Rationing Capacity in Advance to Signal Quality," extends the analysis of advance selling to incorporate asymmetric information regarding product quality. It was motivated by the observation that customers are often uncertain about products' quality when they decide to buy in advance. For example, the functionality, usefulness, and reliability of many new technical devices (e.g., IPhone 3GS) are quite uncertain before the products are unveiled to the market. Firms, however, usually have better information about product quality than customers. While high-quality firms have a natural incentive to signal to customers its superior quality, it is also in the interest of low-quality firms to mimic such signals. Not surprisingly, it may be not easy to persuade customers wary of quality uncertainty to buy in advance.

Applying Bayesian game theory, I characterize the equilibrium strategies for firms with different quality and find that quality uncertainty always hurts high-quality firms. These firms always have to sacrifice a portion of their potential profit, by lowering price and/or by limiting the amount offered in advance (if rationing is feasible), in order to signal to customers their commitment to high quality. I prove the efficacy of rationing as a signal of quality for both costless and costly quality. I also show that signalling through rationing can be more efficient than some marketing signals such as pricing and advertising. Nevertheless, signalling by capacity rationing can be very costly for the seller. By comparing the two cases when the firm has different flexibility in capacity rationing, I demonstrate with numerical examples that rationing flexibility sometimes can make both types of sellers worse off. Besides, I also identify the conditions under which asymmetric information about quality makes advance selling not beneficial at all.

The second essay contributes to the signalling literature by showing that capacity rationing in advance can be used to effectively convey a signal about product quality.

In the third essay, "Demand Shaping and Product Overselling to Better Match Supply and Demand for Assemble-to-Order Firms," I focus on jointly producing and marketing multiple products for assemble-to-order firms (e.g., Dell Corporation). Assemble-to-Order is a special manufacturing strategy where components are acquired (or produced) to stock, while the assembly of final products is delayed until product demand is realized. By postponing final assembly and pooling component inventory, assemble-to-order can help firms achieve mass customization and quick response at a low cost. In spite of its increasing popularity and wide application in various industries (e.g., computer, automobile, jewelry), assemble-to-order systems have received very limited attention from academic research due to the analytical difficulty, which is rooted in the nature of an assemble-to-order system: it is essentially a special case of multi-product multi-resource system in which products' demand can be inter-correlated and common components are shared among products. Hence, it is necessary to manage the products and components jointly and the main challenge is to match limited component inventory with product demand.

Motivated by the practical problems experienced by assemble-to-order firms, I consider the joint pricing and order-fulfillment decisions for an assemble-to-order firm offering multiple substitutable products. Specifically, for pricing decision, I allow the firm to shape demand by adjusting prices according to component availability, and for order-acceptance decision, I focus on order-rationing policy and allow product overselling. Product overselling can benefit the firm by effectively reducing the losses from high-yield spill and component inventory spoilage.

The third essay contributes to the literature by incorporating the overselling strategy into firms' joint pricing and order-fulfillment decisions. In particular, I characterize the optimal joint decisions for firms with different flexibilities in pricing and overselling, and evaluate the value of pricing and overselling flexibilities both individually and jointly. I also show how the optimal decisions and value of flexibilities depend on the key operational parameters such as levels of component inventory and compensation to customer for unfulfilled orders. Furthermore, I find that, in addition to enhancing firms' profitability, product overselling can also improve consumer and social surpluses.

CHAPTER 2

Advance Selling – The Effect of Capacity and Consumer Valuation Interdependence

2.1 Introduction

Advance selling has become a standard practice in the service industry, including airlines and travel. Due to its success, it has been adopted in retail, including toys, books, and other media products. The crux of advance selling is a separation in the time that consumers purchase a product from the time that the product is consumed, which allows sellers to offer the products both in advance (before the product is consumed) and in spot (consumption time). There are several possible reasons for which a seller could offer advance selling. Advance selling could help the seller to reduce demand variability and hence plan logistics better. Risk-averse customers can benefit from advance selling since it reduces the risk of not getting the product, for which the seller could ask for a premium in exchange of guaranteed availability. Although these reasons provide clear rationale behind advance selling, another reason for advance selling that covers a broad range of applications is consumer's uncertainty in valuation (Xie and Shugan, 2001).

Prior to consumption, customers can be uncertain about their own valuation of the product or service because the valuation may heavily depend on situation, circumstance, or the state at time of consumption. Belk (1975) refers to these factors as "situational variables" and categorized them into five groups: physical surroundings (e.g., weather), social surroundings (e.g., other persons present at consumption time), temporal perspectives (e.g., time of purchase), task definitions (e.g., self-use or gift giving), and antecedent states of customers (e.g., mood, health, financial condition). By Belk's classification, some of these factors are customer-dependent and contingent on an individual customer's state at time of consumption, while others are environment-dependent and determined by state of the nature or the occurrence of some exogenous events. The effect of these situational factors results in different levels of interdependence among consumers' valuations at consumption time.

Customer-dependent factors are idiosyncratic, personal, and affect different customers in a different way. Examples of customer-dependent factors include customer's taste, mood, health, scheduling conflict, consumption occasion, accompanying person, and so on. When these factors prevail, customers' valuations tend to be independent with each other. The effect of customer-dependent factors have been studied in marketing and economics literature. For example, Koopmans (1964), Kreps (1979), and Walsh (1995) consider customers' uncertain future preference due to "uncertainty about future tastes." Hauser and Wernerfelt (1990) suggest that the utility a customer obtains from a wine varies by the customer's consumption occasion (e.g., the meal or the guests). Shugan and Xie provide numerous examples, including cruise, vacation package, Broadway show tickets, and conference registration, where a customer's valuation depends on the consumption state of the customer, "including health, mood, finances, work schedule, and family situation" (Shugan and Xie 2000, 2004, Xie and Shugan 2001). More generally, some of these factors may have big impacts on a group of customers, but not on the others. For example, family situation (e.g., an unexpected visitor or a sick child) affects the valuation of a summer holiday valuation package for all members of the given family (Shugan and Xie 2000), and an unexpected change in work schedule influences all the co-workers' valuation of a happy-hour cruise party. In such a case, valuations are uncorrelated for customers belonging to different groups and yet interdependent among those in the same group.

On the other hand, environment-dependent factors are exogenous and include physical surroundings like weather, and social surroundings like economy, government policy, and the presence of a celebrity. These exogenous factors affect all customers in a similar way and result in highly interdependent valuations. Phillips (2005) shows that weather significantly affects customer valuation (and hence total demand) of baseball game tickets. Ng (2007) suggests that "knowing a certain celebrity will be patronizing a club increases its appeal." Png and Wang (2009) quote postings from travel forum to show that weather is a major uncertainty for travellers to beach resorts.

For most products and services, customer valuation is influenced by mixtures of these state-dependent factors. For example, consider a potential attendee of an outdoor Jazz concert on a specific date (Ng 2007). The valuation of the outdoor concert depends on both the state of the customer (e.g., being healthy or sick) and the state of the nature (e.g., raining or shining) on the day of concert. Nevertheless, for some products, customer-dependent idiosyncratic factors might dominate other factors. For example, the valuation of a Chinese dinner buffet on a given evening may largely depend on whether the customer is in the mood of a Chinese buffet and how hungry the customer is on that specific night (Shugan and Xie 2004). Environmental factors play a less dominant role in this case. On the other hand, nature-dependent exogenous factors might prevail in some other cases. For example, the valuation of a ticket to a specific major league baseball game is highly dependent on whether a superstar player plays in that game (Phillips 2005). Likewise, the value of a skiing season pass is largely determined by the quality of snow and is unknown in advance of the snow season.[1] Hence, for products and services with different nature, there can exhibit a range of scenarios where valuation interdependence varies.

This paper primarily examines the effect of the interdependence among consumer valuations on seller's advance selling decisions, e.g., whether the seller should use advance selling, what terms of trade the seller should choose in advance selling, and how they depend on the nature of the seller. We identify the condition under which seller would engage in advance selling and characterize the optimal pricing and capacity allocation policies of the seller. Specifically, we characterize when the seller will offer a discount in advance selling and when he will charge a premium instead. We also characterize when the seller will ration a portion of its capacity for advance selling.

To our knowledge, this paper is the first paper that studies the effect of consumer valuation interdependence on a seller's optimal policy. To examine this, we studies four different valuation models-namely, deterministic, heterogeneous, homogeneous-k, and homogenous-1 models. These four models represent increasing degrees of valuation interdependence and, equivalently, decreasing level of predictability of demand for a given price after valuation uncertainty is resolved. Our analytical and numerical results show that the degree of valuation interdependence critically influences the seller's pricing policy. The seller's capacity, however, plays an important role. We prove that the seller with sufficient capacity always offers a discount if advance selling is offered. In this case, we find that the exact depth of a discount depends on marginal cost and the valuation distribution, but, it is independent of valuation interdependence. However, when the seller's capacity is limited, we show that the valuation interdependence significantly not only affects the pricing policy, but also the seller's decision on how much of its limited capacity to sell in advance. Specifically, when the degree of valuation interdependency is extremely low (as in our deterministic model), advance selling will be deployed only when the seller's capacity is not very tight, and it will be deployed with a discounted advance price. Furthermore, the seller may offer only a fraction of capacity in advance. On the other hand, the degree of valuation interdependence is high (as in our homogenous-1 model), advance selling can be profitable to the seller even when the capacity is very tight. In this case, the seller could charge a premium when offering the product in advance. Furthermore, we show that it is never optimal for the seller to offer only a fraction of capacity in advance. If advance selling is offered, the seller should make its entire capacity available. In between these two extreme cases, we

[1] source:http://seattlepi.nwsource.com/othersports/209260_skiseason25.html

show that the seller's policy gradually changes in valuation interdependence.

2.2 Literature Review

The papers in revenue (yield) management are related to our work. Some papers in this area focus on quantity-based revenue management (e.g., setting booking levels). Examples include Littlewood (1972), Weatherford and Pfeiffer (1994), and Robinson (1995). Other papers consider price-based revenue management (dynamic pricing). These papers include Gallego and van Ryzin (1994, 1997), Feng and Gallego, G. (1995), Bitran and Mondschein (1997), Dana (1998), and You (1999). McGill and van Ryzin (1997) and Elmaghraby and Keskinocak (2003) provide a very comprehensive review of the literature in the revenue management. In most of the papers in the revenue management (including all of the aforementioned papers), consumers do not act strategically. Consumers do not strategically choose when to buy. Instead, they decide to buy or not when they enter the market. The assumption that consumers act myopically simplifies the analysis, but allows to capture additional features that are relevant to particular industry settings, for examples, leisure passengers make purchase decisions before business passengers who need to buy a ticket in short notice. In contrast to these papers, the core of our model is strategic behavior of consumers.

Our paper is more closely related to the ones that consider the seller's pricing policy in the presence of strategic consumers. The papers in this category include Stokey (1979), Landsberger and Meilijson (1985), Besanko and Winston (1990), Su (2005), Elmaghraby et al. (2006), Aviv and Pazgal (2005), Zhang and Cooper (2006), Levin et al. (2006). Although these papers explicitly capture the strategic response of consumers to the seller's pricing policy, none of the papers considers valuation interdependency. In contrast, our paper explicitly models consumers valuation interdependence and examines its effect on the seller's pricing policy.

Our paper is also related to the advance selling literature. The existing literature on advance selling identifies a number of situations in which the seller wants to offer advance selling. When facing risk-averse consumers, advance selling exploits consumer's desire for guaranteed availability (Png, 1989; Liu and van Ryzin, 2009). If advance selling conveys a demand signal, it could help the seller to appropriately plan for the selling season (e.g., inventory) ahead of time (Tang et al. 2004; Song and Zipkin, 2007; Zhao et al. 2006, Prasad et al. 2008). Offering a discount on the less desirable flight enables the seller to balance the load (Gale and Holmes, 1993). When facing heterogeneous consumers (in terms of their willingness to pay, price sensitivity), advance selling serves as a tool of price discrimination (Desiraju and Shugan, 1999, and Dana, 1998). The seller may also offer advance selling as a competitive response: Not offering advance selling allows the competitor to capture a bigger portion of the market (McCardle et al, 2004). These papers establish intuitive rationale behind advance selling.

However, a few papers found that advance selling may be profitable for the seller even when none of the reasons above is present, as long as consumers are uncertain about their future valuation. Gallego and Sahin (2006) examine a consumer option (i.e., buying in advance a right to buy the product in spot) when the seller faces consumers who are uncertain about their valuation for the product. They show that offering an option (along with judicious rationing) can increase the seller's profit as the option induces consumers to buy options (as a form of insurance) before their valuation is realized. Koenisberg et al. (2006) examine the airplane pricing policy and show when offering a last minute deal is optimal when the airline already offers advance selling. A paper by Shugan and Xie (2001) is closely related to our paper. They assume that each consumer can either have high or low valuation in the spot period, but does not know its valuation in the advance period. They find that the seller may offer a different strategy depending on its marginal cost and capacity. All of these papers on the valuation uncertainty use a particular functional form for modelling spot demand, namely, spot demand for a given price is equal to its expected demand (in some literature, this model is referred to as a fluid model). Such spot model is justified when (i) consumer valuation is predominantly determined by idiosyncratic factors, (ii) one consumer's valuation is independent form that of another consumer, and (iii) the population of consumers is sufficiently large. Instead of analyzing its exact spot demand (typically modelled as a binomial random variable), the fluid model utilizes the law of large number and ignores the variation of demand around the mean. As a result, the spot demand at a given price is a deterministic function of the spot price. However, as illustrated in earlier examples, there are several situations where the valuation of one consumer is highly correlated to that of other consumers. Our paper shows that the interdependence of consumer valuation critically affects whether and how the seller uses advance selling.

The rest of the paper is organized as follows. In section 2.3, we formulate the problem and state assumptions. In section 2.4, we consider a basic model where the seller has sufficient capacity to satisfy all demand, characterize the optimal advance and spot prices and analyze how the optimal strategy changes with respect to the seller's cost. Section 2.5 is the main focus of this paper: the effects of capacity and consumer valuation interdependence. We fully characterize the seller's optimal strategy under deterministic and homogeneous-1 valuation model, respectively. For heterogeneous and homogeneous-k valuation models, we report the structure of optimal policy based on numerical studies. In section 2.6, we numerically study the impact of different aspects of consumer valuation distribution and seller's operational capability on the benefit of advance selling. Subsequently, we discuss some extensions of the models in section 2.7 and conclude the paper in section 2.8.

2.3 Model and Assumptions

We consider a seller who has an option to sell in advance as well as in spot. The seller can sell the product in both periods at (possibly) different prices. If the seller decides to sell in advance, she chooses price $p_1 \geq c$ and decides the amount of capacity available for advance sales. Having observed the sales in advance, the seller then decides spot price $p_2 \geq c$, in the second period. We assume the seller has total capacity T and incurs variable cost c for each unit sold.

We assume that consumers arrive over the two periods and that all customers learn their valuations only in the spot period, independently of the time of arrivals. Let N_1 be the size of consumer population who arrive in the advance period and N_2 be the size of consumer population who arrive in the spot period. Customers are risk-neutral, strategic, and forward-looking. In the advance period, customers compare the expected utility of purchasing right away at price p_1 with the expected utility of deferring the decision to the spot period. In the spot period, all remaining customers (new customers arriving in the spot period plus the remaining customers carried over from the advance period) decide whether to buy the product or not after they learn their exact valuation and the price for the product, p_2.

We assume that each consumer's individual action does not influence other consumers' behavior. Thus, each consumer chooses the action that maximizes his/her utility. We assume $U_i = \alpha_i - p$ to be consumer i's utility, where p is the price customer pays and α_i is consumer i's valuation, which will be revealed only in the spot period. We assume that all consumers' spot valuations are drawn from an identical, but not necessarily independent, distribution with *c.d.f.* $G(\cdot)$ and *p.d.f.* $g(\cdot)$. Throughout the paper, we assume that the valuation distribution, $G(\cdot)$, and its density, $g(\cdot)$, satisfy the following conditions:

(1) $G(\cdot)$ is defined on a finite support $[L, H]$ and is twice continuously differentiable.[2]

(2) $g(\cdot) = G'(\cdot) > 0$ on (L, H).

(3) $g(\cdot)$ is log-concave.

Many distributions and their truncated versions satisfy these conditions. For example, uniform, exponential, logistic, normal, extreme-value, as well as power distributions, Weibull, beta, gamma, and χ with shape parameter greater than or equal to 1, and χ^2 with shape parameter greater than or equal to 2, satisfy these assumptions. Truncation of the above distributions to $[L, H]$ is defined

[2]For ease of presentation, we assume a finite support, but all of the results that we derive in the paper hold also when $L = -\infty$ or $H = \infty$ as long as the first two moments are finite.

as a conditional distribution. Note that Condition (3) implies that both $G(\cdot)$ and $\overline{G} = 1 - G(\cdot)$ are log-concave, which then implies that $G(\cdot)$ has an increasing failure rate (Bergstrom and Bagnoli, 2007).

To isolate the effect of valuation uncertainty and its interdependence across customers from other intuitive reasons for advance selling (such as risk aversion or customer heterogeneity), we choose a risk neutral setting and assume that the distribution of consumer valuation is ex ante identical. Hence, in our model, both the seller and customers maximizes their expected utilities. Identical distribution assumption eliminates the possibility of segmenting customers a priori based on their willingness to pay and/or arrival time.[3] The absence of risk aversion and heterogeneous customers makes the valuation uncertainty a primary driver of advance selling. We assume that customers arrive in both advance and spot periods. Our two-period model captures a key difference between customers in advance and in spot periods: In the advance period, customers make a decision to buy in the presence of uncertainty, but in the spot period, customers make the decision after such uncertainty is resolved. We note that our two-period model admits multiple scenarios that may arise. For instance, if all customers have valuation uncertainty, such case can be modelled by setting $N_2 = 0$. Likewise, if no customer experiences valuation uncertainty, it is represented by $N_1 = 0$. One could embellish the the model by adding more periods before the valuation is revealed, but the main insight will remain the same.

Consumers' valuations can be influenced by both idiosyncratic (i.e., specific to each consumer) and exogenous (i.e., affecting all consumers) factors. If exogenous factors such as weather and hype predominantly determine their valuations, then one customer's valuation is closely related to others'. On the other hand, if idiosyncratic factors dominate, customers' valuations are likely to be independent of each other. To examine how valuation uncertainty and its interdependence across consumers affect the seller's optimal pricing policy and resultant profit, we consider four different models with varying degrees of valuation interdependency, namely, heterogenous, deterministic, homogeneous-k, and homogeneous-1 valuation models. We study the seller's and buyer's behaviors under each of the four models, and, more importantly, show that the seller's optimal strategy gradually changes in valuation interdependence. We now define each of the four valuation interdependence models as follows.

Heterogeneous valuation model

[3] Desiraju and Shugan (1999) note that customers who arrive late may have a great need for the service and it is also possible that customers who book the service far in advance may have an equal need for the service as he/she wants the service on a specific date or time.

In this case, each customer's valuation in the spot period is independently drawn from distribution $G(\alpha)$. This model is appropriate when customer valuation is primarily idiosyncratic. In such case, the valuation of one customer is independent from the valuation of another customer. For example, the value of a prepaid fuel option offered by a rental car company depends on the mileage that an individual renter will accumulate, thus its value to one renter is independent of that to another renter (Shugan and Xie, 2004). Since the valuation of an individual customer is independent from that of another customer and is drawn from the distribution $G(\cdot)$, for a given spot price p_2, the demand (i.e., the number of consumers who are willing to pay p_2 or more) in the spot period follows a binomial distribution.

Deterministic (fluid) valuation model

In the deterministic model, a randmon variable representing the number of consumers who buy at a given spot price p_2 is replaced by its expected value. Thus, the spot demand becomes a deterministic function of the spot price. One could consider the deterministic model as an asymptotic version of the heterogeneous model. When the size of consumer population is sufficiently large, the coefficient of variation of the spot demand in the heterogeneous model approaches to 0. This asymptotic model is called fluid model and it replaces a hard-to-analyze binomial model with a more tractable one. Because of its analytical tractability, it has been widely used in operations and pricing models (for example, Gallego and Sahin, 2006, Cachon and Swinney, 2009, Gallego and Hu, 2006). Specfically, in our model, the spot demand for a given price p_2 is simply the proportion of consumers whose valuation exceeds the price p_2 (i.e., $G(p_2)$) and the size of the consumer population in the spot period. This is the same assumption used in several papers on advance selling including Gallego and Sahin (2006) and Xie and Shugan (2001).

Homogeneous-1 Model

In the homogeneous model, all customers will have the same valuation in the spot period, drawn from the distribution $G(\cdot)$. Specifically, for a given price p_2, either all customers in the spot period want to buy the product (this evenet happens with probability $1 - G(p_2)$ or none of them wants to buy (happens with probability $G(p_2)$). This interdependence model is appropriate when exogenous factors (e.g., weather, government policy) or homogeneously evaluated attributes (e.g., hype, fad) dominantly influence consumer valuation.

Homogeneous-k Model

The homogeneous-k model is a hybrid between heterogenous and homogeneous-1 models. In this case, the consumer population in the spot period is divided into k equal-sized subgroups. Consumers within the same subgroup will have the identical valuation while the valuation of each subgroup is independently and identically drawn from the distribution $G(\cdot)$. For a given spot price p_2, the number of subgroups with valuation p_2 or higher is binomidally distributed, thus the spot demand is the size of the population in a subgroup times the number of subgroups with valuation p_2 or higher. This case is appropriate when there are multiple segments within the consumer population.

The four aforementioned models represent a varying degree of valuation interdependence, which corresponds to how predictable the spot demand is as a function of the spot period price. Among the four models, the homogeneous-1 model exhibits the strongest valuation interdependent as all customers have identical valuation in the spot period. The next one is the homogeneous-k model where the valuation is identical only in the same subgroup. As the number of subgroup, k, increases, the correlation among consumer valuations gets weaker. In the extreme form (k equals to the size of consumer population), this model converges to the heterogeneous model. Notice that as the valuation interdependency weakens (from homogeneous-1 to homogeneous-k to heterogeneous models), the demand in the spot price becomes more predictable as a function of the spot price. In the deterministic model, which is the asymptotic version of the heterogenous model, the spot demand becomes a deterministic function of the spot price and completely predictable. Figure 2.1 positions each of the four models with respect to the valuation interdependence.

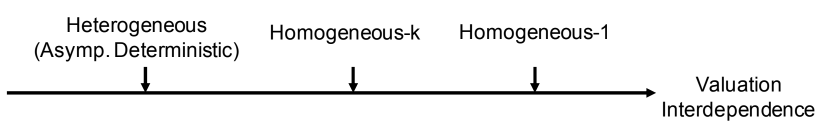

Figure 2.1. Four Models with Various Degree of Valuation Interdependence

To illustrate the differences among the four valuation models, suppose that the consumer population is 100 customers., arriving in advance. In advance all of the customers do not know their valuation, but know that their valuation in the spot period will be either high with probability 0.6 or low with probability 0.4. Suppose that the seller's advance price is very high and, as a result, no one buys in the advance period. Then, in all four models, there are 100 customers in the spot period. The realized valuations of 100 customers in the spot period, however, are different in all four models. In the deterministic model, exactly 60 customers will have high valuation and 40 will have low valuation in the spot period. In the heterogeneous model, the number of customers with high valuation follows a binomial distribution with $n = 100$ and $p = 0.6$. In the homogeneous-1 model, either all 100 customers will have high valuation with probability 0.6 or all of them will have low valuation with probability 0.4. In the homogeneous-2 model, the valuation of each of two subgroups of 50 customers will be high with probability 0.6 or low with probability 0.4. Thus, the number of customers with high valuation will be zero (with probability 0.16), 50 (with probability 0.48), or 100 (with probability 0.36).

In what follows, we characterize the seller's optimal strategy for each of the four cases. Section 2.4 examines the case where the seller has enough capacity to satisfy all potential demands, i.e., $T \geq N_1 + N_2$. Then, in section 2.5, we consider the case with limited capacity, $T < N_1 + N_2$.

2.4 Advance Selling with Unlimited Capacity

When the seller has sufficiently large capacity i.e., $T \geq N_1 + N_2$, the sales will never be bounded by capacity and it is optimal for the seller to sell to all customers who want to buy in either period. Consequently, we focus on the seller's pricing strategy. We first characterize the (remaining) customers' optimal strategy in the spot period which then allows us to determine the optimal spot price. We roll back this to the advance period, and characterize the maximum price at which customers will buy in advance instead of delaying purchase decision to the spot period. Finally, we determine whether the seller should offer advance selling and, if so, at what price.

2.4.1 Spot Period

In the spot period, customers buy the product when their *ex-post* utility from a purchase exceeds the reservation utility, which is assumed to be zero. In other words, a customer with valuation $\alpha \geq p_2$ will buy the product.

Since the prior distribution of valuation is identical for all customers, customers in advance choose the same action: Either all want to buy or all prefer to wait. Depending on whether advance customers buy or not, the remaining customer population in the spot period, D_2, is either N_2 (if N_1 customers purchase in the advance period), or $N_1 + N_2$, otherwise. Let $\pi_2(p_2)$ be the seller's expected profit in the spot period for a given spot price, p_2. The following theorem shows that, for a given spot price, p_2, and the remaining customer population, D_2, the seller's expected spot profit is the same for all four valuation models and is expressed as follows:

$$\pi_2(p_2) = (p_2 - c)D_2\overline{G}(p_2). \tag{2.1}$$

Theorem 1 *If the seller's capacity is sufficiently large ($T \geq N_1 + N_2$), the seller's optimal strategy in the spot period and the resultant expected spot profit are the same for the four valuation models.* ■

Proof: All proofs are relegated to the Appendix.

The theorem implies that the spot price and the seller's expect profit in the spot period do not depend on the valuation interdependence. We also show that the optimal spot price is unique.

Lemma 1 [The optimal spot price in the unlimited-capacity model]
The expected spot profit function $\pi_2(p_2)$ is quasi-concave in p_2 and has a unique maximizer $p_2^U \in [L, H]$:

$$p_2^U = \begin{cases} H & \text{for } c \geq H \\ \in [L,H] \text{ and is a solution to } p_2 = c + \frac{\overline{G}(p_2)}{g(p_2)} & \text{for } \underline{c} < c < H \\ L & \text{for } c \leq \underline{c} \end{cases} \tag{2.2}$$

where $\underline{c} = L - \frac{1}{g(L)}$.

Lemma 1 implies that the optimal spot price depends on the marginal cost. The higher the marginal cost, the higher the spot price. Since this result is quite intuitive, we suppress this dependence throughout the paper except when we evaluate the effect of the marginal cost.

2.4.2 Advance Period

The theorem implies that the spot price and the seller's expect profit in the spot period do not depend on the valuation interdependency. This result has a profound impact on the customers' behavior in the advance period. in the advance period, customers see the same spot price, resulting in the same customer behavior in advance and the same optimal advance price by the seller.

Theorem 1 implies that the spot price and the seller's expect profit in the spot period do not depend on the valuation interdependency. Consequently, customers in the advance period anticipate the same spot price in all four models: The seller's optimal pricing strategy and resultant customer behavior in the advance period are also identical for the four models.

Corollary 1 *If the seller's capacity is sufficiently large ($T \geq N_1 + N_2$), the seller's optimal strategy and expected profit in both periods are the same for the four valuation models.*

We now examine whether the seller should offer advance selling or not. Note that customers in advance have two options: purchase in the advance period or defer decision to the spot period. The expected utility from purchasing in the advance period is

$$U_A = \mathrm{E}[\alpha - p_1] = \mathrm{E}[\alpha] - p_1.$$

When they decide to defer, they will buy only if their valuation exceed the spot price, p_2^U. Thus, the expected utility from deferring the decision is

$$U_W = \mathrm{E}\left[\max(\alpha - p_2^U, 0)\right]$$

Customers buy in the advance if and only if $U_A \geq U_W$, i.e.,

$$p_1 \leq p_1^{\max,U} = \mathrm{E}[\alpha] - \mathrm{E}[\max(\alpha - p_2^U, 0)] = \mathrm{E}[\min(p_2^U, \alpha)]$$

where $p_1^{\max,U}$ is the maximum price that the seller can induce customers to buy in advance. Clearly, if the seller offers advance selling, the optimal advance price must be $p_1^{\max,U}$ as stated in the following proposition.

Proposition 1 [The optimal advance price in the unlimited-capacity model]

If the seller's capacity is sufficiently large ($T \geq N_1 + N_2$), the optimal advance price never exceeds optimal spot price, i.e., $p_1^* = p_1^{\max,U} = \mathrm{E}[\min(p_2^U, \alpha)] \leq p_2^U$.

With unlimited capacity, the seller must offer a discount in the advance period to induce the customers to buy in advance. The discount compensates for customers' commitment to buy before they learn their valuation. To examine whether the seller should offer advance selling, we first compare two pricing strategies - the one that offers advance selling and the other that does not.

(i) **A** (mandatory advance selling): The seller must offer an advance price which (weakly) induces customers to buy in advance.

(ii) **S** (spot only): The seller does not offer advance selling.

Let π_A^U and π_S^U be the corresponding optimal expected profits over two periods under (A) and (S). Applying Proposition 1 and Lemma 1, we get

$$\begin{aligned}\pi_A^U &= N_1(p_1^{\max,U} - c) + N_2(p_2^U - c)\overline{G}(p_2^U), \text{ and} \\ \pi_S^U &= (N_1 + N_2)(p_2^U - c)\overline{G}(p_2^U)\end{aligned}$$

Comparing π_S^U and π_A^U, the seller will sell in advance if and only if

$$p_1^{\max,U} - c - (p_2^U - c)\overline{G}(p_2^U) \geq 0. \tag{2.3}$$

Although we suppress dependence on c, clearly the marginal cost c affects both $p_1^{\max}$ and p_2^U. However, it is not immediately obvious how the marginal cost c will influence the efficacy of advance selling. The following lemma characterizes seller's optimal strategy with respect to cost c.

Lemma 2 [Advance selling and the marginal cost: lower and upper bounds]

Consider a seller with sufficient capacity.

a) If $c \leq \underline{c}$, $p_1^{\max,U} = L$ and $\pi_A^U(c) = \pi_S^U(c)$. As a result, the seller is indifferent between advance selling and spot-only.

b) There exists a $\bar{c} \in [\underline{c}, H)$ such that $\pi_A^U(\bar{c}) = \pi_S^U(\bar{c})$, and $\pi_A^U(c) < \pi_S^U(c)$ for all $c > \bar{c}$ (selling only in the spot period is optimal).

Lemma 2 implies that advance selling can be strictly optimal if c is between $\underline{c}$ and $\bar{c}$. For some specific valuation distributions, there can exist multiple disjoint regions between $\underline{c}$ and $\bar{c}$, where advance selling is strictly optimal. To guarantee that advance selling is strictly optimal for the entire interval $(\underline{c}, \bar{c})$, we need an additional technical assumption. We say that a real-valued function $a(\cdot)$ is *positive-negative* if $a(x_0) < 0$ implies that $a(x) < 0$ for all $x > x_0$.

Assumption (A) $\left(\frac{G(x)\overline{G}(x)}{g(x)}\right)' + \overline{G}(x) - k$ *is positive-negative for any* $k \in [0, 1]$.

Assumption (A) is not restrictive. In fact, it can be shown that all (full and truncated) distributions listed in section 2.3 satisfy the assumption. For the remainder of the paper we impose Assumption (A).

Theorem 2 [Advance selling and the marginal cost in the unlimited-capacity model]
If Assumption (A) holds, advance selling is strictly optimal only for $c \in (\underline{c}, \bar{c})$. *In other words,*

$$\begin{cases} \pi_A^U = \pi_S^U & \text{for } c \leq \underline{c}, \\ \pi_A^U > \pi_S^U & \text{for } \underline{c} < c < \bar{c}, \\ \pi_A^U \leq \pi_S^U & \text{at } c \geq \bar{c} \text{ (equality holds at } c = \bar{c}). \end{cases}$$

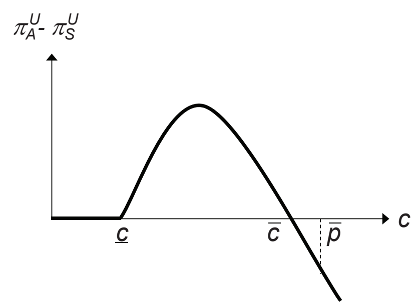

Figure 2.2. Illustration of $\pi_A^U - \pi_S^U$ as a function of marginal cost, c.

As one can observe from Figure 2.2 and Theorem 2, mandatory advance selling is worse than spot-only when the marginal cost is very high. When cost is very high, the seller wants to set a high spot price and sell only to a small portion of customers (i.e., those with very high valuations). Advance selling could increase the sales, but the discount the seller needs to offer in order to induce advance selling is so large so that the increased sales volume will not make up for the decrease in margin. On the other hand, both mandatory advance selling and spot-only strategies result in the same profit when the marginal cost is very low. When the spot price is so low that all customers will buy (thus, the sales quantities are the same whether the seller offers advance selling or not), the seller does not gain anything from offering a discount in the advance period. The intermediate area is the one where advance selling benefits the seller: a relatively modest price discount can induce all N_1 customers to buy in advance, and this improves the sales quantity and profit.

2.5 Advance Selling with Limited Capacity

Building on the results of Section 2.4, we now examine the case where the seller does not necessarily have the capacity to satisfy all customers, i.e., $T < N_1 + N_2$. In contrast to the sufficient capacity case, the seller with limited capacity, in addition to price decision, needs to determine how much of its capacity to be allocated in the advance period. Obviously, with limited capacity, some customers may not be able to purchase the good even if they want to in advance or spot period. We will show that unlike the sufficient capacity case, all decisions – pricing and how the seller will allocate its capacity – critically depend on valuation interdependence.

In the advance period, the seller with capacity T decides the amount of capacity it will allocate (ration) for advance selling, which we denote by S, $0 \leq S \leq \min(N_1, T)$. Note that $S = 0$ corresponds to the spot-only strategy (*no advance selling*) and $S = \min(N_1, T)$ corresponds to the case where the seller decides to sell as much as it can in the advance period (*full advance selling*). If $0 < S < \min(N_1, T)$, the seller purposefully reserves some of its capacity to sell it in the spot period, which we label as *limited advance selling.*

The sequence of events and all assumptions remain the same, except that in advance period, the seller announces an advance price p_1, now along with the rationed capacity, S. We assume that the seller credibly commits S. It can be shown that it is a (weakly) dominating strategy. To see why, note that customers in advance period make decisions based on their expected utility. Thus, for any given advance price, the seller can perfectly predict how many customers will buy in the advance period. As a result, the seller can set the capacity rationed equal to capacity sold and no capacity offered in advance will be left unused.

We first characterize the seller's optimal policy in the two anchoring cases in our spectrum of valuation interdependence: deterministic and homogenous-1 models. Then, we examine the seller's optimal policy for the remaining two cases numerically.

2.5.1 Deterministic Valuation Model

As we described earlier, the aggregate demand in the spot period is expressed as a deterministic function of a spot price. We examine the seller's decision in the spot period, and then in the advance period.

Spot Period

Consider a subgame where the seller sold S units in the advance period (therefore, the seller has the remaining capacity $T - S$ and faces $N_1 + N_2 - S$ customers in the spot period.) For a given spot price p_2, the spot demand is $(N_1 + N_2 - S)\overline{G}(p_2)$.

Therefore, the sales quantity in the spot period is either the spot demand or the seller's remaining capacity, whichever is smaller. The seller's profit in the spot period is

$$\pi_2(p_2, S) = (p_2 - c) \min \left[T - S, (N_1 + N_2 - S)\overline{G}(p_2)\right]$$

Notice that the seller's spot profit and optimal spot price depend on the remaining capacity $T - S$. To see how optimal spot price varies in the remaining capacity, consider the case where the remaining capacity is sufficiently large, thus will not bind the sales quantity in the spot period. In this case, the seller's optimal spot price will be the same as in the unlimited capacity case (i.e., p_2^U). On the other hand, when the seller's remaining capacity is very tight, the seller will raise the price to clear the market and sell only to customers with high valuations. Let $p_2^B(S)$ be the market clearing price for the seller with the remaining capacity, $T - S$. By definition, $p_2^B(S)$ can be found by solving $T - S = (N_1 + N_2 - S)\overline{G}(p_2)$. The following lemma shows that the optimal spot price is indeed the maximum of p_2^U and $p_2^B(S)$.

Lemma 3 [The optimal spot price in the deterministic valuation model]

$$p_2^*(S) = \max(p_2^U, p_2^B(S)) = \begin{cases} p_2^U & \textit{if } \frac{T-S}{N_1+N_2-S} \geq \overline{G}(p_2^U) \\ p_2^B(S), & \textit{otherwise} \end{cases} \tag{2.4}$$

where p_2^U *is defined in Lemma 1 and* $p_2^B(S)$ *is a solution to* $\overline{G}(p_2) = \frac{T-S}{N_1+N_2-S}$.

Lemma implies that the spot price is a non-increasing function of the remaining capacity. Furthermore, the larger the initial capacity T is, the larger the region where the unrestricted spot price, p_2^U is optimal. Customers in the advance period not only anticipate the spot price but also the likelihood of shortage if they defer the decision to the spot period. Let $\lambda_2(S, p_2)$ be the probability that a customer who wants to buy the product in the spot period actually obtains it given that the remaining capacity is $T - S$ and the seller's spot price is p_2:

$$\lambda_2(S, p_2) = \min \left[1, \frac{T - S}{(N_1 + N_2 - S)\overline{G}(p_2)}\right] \tag{2.5}$$

After substituting p_2 with the optimal spot price defined in Lemma 2.5.1, we immediately have the following.

Corollary 2 [Spot supply shortage never occurs in the deterministic valuation model]

For any $S < T$, $\lambda_2(S, p_2^*(S)) = 1$.

The result implies that the shortage in supply will never occur in the spot period no matter how tight the remaining capacity is. In this case, the seller gains by increasing the spot price to clear the market instead of leaving some customers to face any shortage.

Advance Period

In the advance period, the seller must decide the advance price p_1 and the portion of capacity rationed, S. We first find the optimal price $p_1^*(S)$ for a given rationing decision, S, and then determine the amount of capacity that should be rationed for the advance period, $S^* \in [0, \min(T, N_1)]$.

Since the seller can limit the quantity sold in the advance period, not necessarily all of N_1 customers can buy in the advance period. For example, if the capacity rationed, S, is less than N_1, only a portion of advance customers are able to buy the product in advance no matter what the advance price is. Let $\lambda_1(S)$ be the probability that a customer obtains the product in the advance period. If a customer buys the good in advance, her expected utility is $\mathrm{E}[\alpha] - Bp_1$. If she does not buy in the advance period, her expected utility is $\mathrm{E}[\max(\alpha - p_2^*(S), 0)]$ from Corollary 2. Hence, the expected utility of a customer who attempts to buy in the advance period is[4]

$$U_A(S) = \lambda_1(S)\mathrm{E}[\alpha - p_1] + \big(1 - \lambda_1(S)\big)\mathrm{E}[\max(\alpha - p_2^*(S), 0)].$$

On the other hand, the expected utility from waiting until the spot period is $U_W(S) = \mathrm{E}[\max(\alpha - p_2^*(S), 0)]$. Buying in the advance period is optimal for a customer if and only if $U_A(S) \geq U_W(S)$, which is equivalent to

$$\lambda_1(S)\mathrm{E}([\alpha - p_1] + \big(1 - \lambda_1(S)\big)\mathrm{E}[\max(\alpha - p_2^*(S), 0)] \geq \mathrm{E}[\max(\alpha - p_2^*(S), 0)]$$

Simplifying the inequality, the maximum (hence the optimal) price that the seller can charge in the advance period is:

$$p_1^{\max,D}(S) = \mathrm{E}[\alpha] - \mathrm{E}[\max(\alpha - p_2^*(S), 0)] = \mathrm{E}[\min(p_2^*(S), \alpha)] \tag{2.6}$$

Proposition 2 [The optimal advance price in the deterministic valuation model]
For any S, $p_1^{\max,D}(S) \leq p_2^*(S)$. Hence, the seller always offers a discount in the advance period.

A conventional intuition suggests that the seller could be better off charging a premium price in the advance period when capacity is very tight. A premium advance price could be justified from customers' point of view, if there was a possibility of shortage in spot. As shown in Corollary 2, however, in the deterministic valuation model, the seller will always raise the spot price and eliminate any shortage. Thus, the effect of tight capacity is primarily manifested in an increase in the spot price as opposed to a premium in the advance period. This phenomenon is due to the fact

[4]Note that instead of considering a consumer who attempts to buy in advance, we can alternatively model the tradeoff faced by an consumer who is offered a guaranteed seat in advance and decides whether to accept the offer. It can be shown that the alterative model is equivalent to the model presented.

that, in the deterministic model, the spot demand is a deterministic and continuous function of the spot price.

Our result differs from the findings of Xie and Shugan (2001), in which the premium pricing can be optimal in advance. As in our deterministic model, their model assumes that the spot demand is a deterministic function of the spot price. The difference is that in their model, the spot demand is a discrete function. Specifically, there are only two possible valuations: L and H. Hence, the spot demand can take only two values- all customers purchase if $p_2 \leq L$ and only customers with valuation H will buy if $p_2 > L$. Notice that for any $p_2 \in (L, H]$, the demand remains constant. Likewise, the demand remains constant for $p_2 \in [0, L]$. In our model, however, between L and H, the spot demand continuously responds to the price change. While each model imposes assumptions on the demand function (two discrete values vs. continuous demand function), there are probably very few examples where the demand from a large population of consumers is nonresponsive to any change in price except for one or two threshold prices.

Capacity Rationing

With the seller's optimal pricing policy characterized (equations (2.4) for spot price and (2.6) for advance price), we now analyze the seller's choices on whether to offer advance selling and how much of the total capacity to be allocated for advance selling if he or she decides to do so. The seller's problem is described as follows:

$$\begin{aligned}&\max_{S} \ \ \pi_{AS}^{D}(S) = (p_1^{\max,D}(S) - c)S + (p_2^*(S) - c)\min(T - S\ , (N_1 + N_2 - S)\overline{G}(p_2^*(S)))\\ &\text{subject to} \quad 0 \leq S \leq \min(N_1, T)\end{aligned}$$

Note that $S = 0$ indicates *spot only* selling, $S = \min(N_1, T)$ represents *full advance* selling, and $S \in (0, \min(N_1, T))$ corresponds to *limited advance* selling. Thus, S not only represents the amount of capacity reserved for advance selling but also the seller's decision to offer advance selling or not (i.e., $S = 0$ vs. $S > 0$).

The amount of capacity that the seller will ration for the advance period is influenced by two forces pulling in the opposite directions. If the seller commits a larger capacity in advance, product availability in spot will drop and resultantly, spot price will increase. However, a larger portion of customers will buy at a discount price in the advance period. We characterize the structure of the seller's optimal capacity rationing as a function of the seller's capacity T and marginal cost c.

In preparation for stating the result formally, we first introduce two thresholds, T_1 and T_2, which are defined as follows:

$$T_1 = (N_1 + N_2)\overline{G}(p_2^U)\big|_{c=\bar{c}} \quad \text{and} \quad T_2 = N_1 + N_2\overline{G}(p_2^U)\big|_{c=\bar{c}}$$

where $\bar{c}$ is the largest marginal cost at which advance selling can be optimal when the seller's capacity is unlimited.

Theorem 3 [The optimal capacity rationing in the deterministic valuation model]
There exist a threshold $T^D \in (T_1, T_2)$ and a function $c^D(T)$ for $T > T_1$ such that
(i) if $T \in (T_1, T^D)$ and $c \le c^D(T)$, then $0 < S^* < \min[T, N_1]$ **[limited advance selling]**,
(ii) if $T \ge T^D$ and $c \le c^D(T)$, then $S^* = \min(T, N_1)$ **[full advance selling]**, *and*
*(iii) Otherwise, $S^* = 0$***[no advance selling]**.
Furthermore, $c^D(T) \ge \bar{c}$ for $T_1 < T < T_2$, and $c^D(T) = \bar{c}$ for $T \ge T_2$.

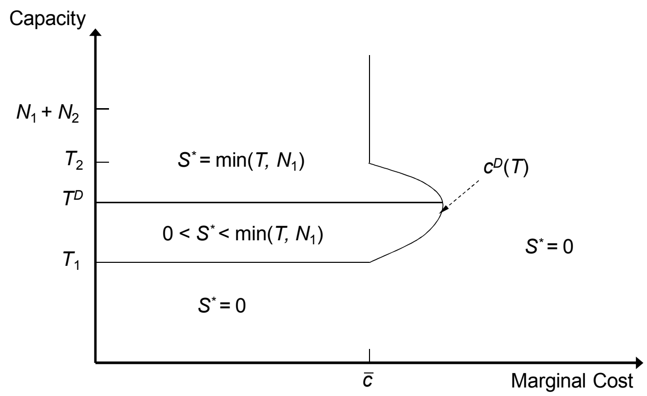

Figure 2.3. Seller's optimal strategy with limited capacity and deterministic valuations

As Theorem and Figure 2.3 illustrate, advance selling is not optimal if the capacity is tight ($T \le T_1$) or the marginal cost is high ($c > c^D(T)$). With tight capacity, the seller will charge a high spot price to sell only to customers with high valuations. Thus, selling the scarce product in advance at a discounted price is not optimal. Likewise, advance selling is not optimal when the marginal cost is very high. While advance selling can increase the sales, but the discount that the seller needs to offer is prohibitively high compared to the small profit margin. Thus, the increased sales volume will not make up for the decrease in margin.

For everywhere else, advance selling is optimal, but the type of advance selling the seller offers depends on the seller's capacity. When the capacity is large (above T^D), the seller will not limit the quantity sold in advance. As in the unlimited-capacity case (studied in Section 2.4), the benefit of advance selling in this region is primarily from the increased total sales. When the seller's capacity is at moderate level (between T_1 and T^D), advance selling is optimal, but the seller intentionally limits the quantity offered in advance (*limited advance selling*). To see why this is the case, compare this strategy to no advance and full advance selling strategies. At capacity level $T \in (T_1, T^D)$, some capacity will be unused when the seller sells only in spot. Hence, offering some quantity in advance increases the sales quantity, which can also raise the spot price as well (by getting rid of some capacity in advance). If the seller uses full advance selling, the seller can guarantee to sell all of its capacity within the two periods. But, a significant portion of customers will buy at a (possibly heavily) discounted advance price. In such a case, the seller's profit can be increased by reducing the capacity ration in advance so that a larger portion of capacity can be sold at a higher spot price. The limited advance selling bears resemblance to the use of booking limits by many airlines. Although there might be several reasons for using the booking limit, our result indicates that one of the benefits is that limiting the quantity available at a discounted price will raise the spot price for the remaining seats as well as the seller's total profit.

2.5.2 Homogeneous-1 Valuation Model

In the homogeneous-1 model, customers' valuations are determined by exogenous factors (e.g., weather) or homogeneously evaluated attributes (e.g., hype). All customers will have the identical valuation in the spot period although it is drawn from a distribution $G(\cdot)$. Thus, for a given spot price, either all remaining customers want to buy or none of them does. In terms of the variation in the spot-period demand, the homogeneous valuation model has the largest variance of the spot demand among all models we consider.

As in the deterministic model, we first analyze the spot period and then the advance period.

Spot Period

Consider a subgame where the seller sold S units in the advance period (therefore, the seller with the remaining capacity $T - S$ faces $N_1 + N_2 - S$ customers in the spot period.) Since all customers' spot valuations are identical, the demand in the spot period is

$$D_2(S) = \begin{cases} (N_1 + N_2 - S) & \text{if } \alpha \geq p_2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

When all remaining customers want to buy, the sales quantity will be bounded by the seller's remaining capacity, $T - S$, and some customers will experience the shortage. Hence, for a given spot price p_2, the seller's expected spot profit is

$$\pi(p_2, S) = E\left[\min(T - S, D_2(S))(p_2 - c)\right].$$

The following lemma characterizes the seller's optimal spot price.

Lemma 4 [The optimal spot price in the homogeneous-1 valuation model]

Regardless of the remaining capacity, it is optimal for the seller to charge the same spot price that the seller with unlimited capacity will charge in the spot period, i.e., $p_2^*(S) = p_2^U$*. Furthermore, for* $c < H$*, there is always a positive probability of shortage.*

At first glance, it is quite surprising that the seller's optimal spot price is independent of the remaining capacity. To see why, first notice that all customers will have the same valuation in the spot period and, as a result, the spot demand is either 0 or $N_1 + N_2 - S$ for any spot price p_2 and any remaining capacity. Thus, the expected profit for the spot period can be rewritten as

$$\pi(p_2, S) = (T - S)\overline{G}(p_2)(p_2 - c).$$

Thus, the spot price that maximizes the seller's profit is the same as as the one used in the unlimited capacity case. Furthermore, notice that this spot price is always less than or equal to the spot price in deterministic case.

Lemma 4 also implies that for all realistic cases ($c < H$) , the shortage is always a possibility. Define $\lambda_2(S) = \frac{T-S}{N_1+N_2-S}$ to be the probability that a customer who wants to buy in the spot period actually obtains the product. Thus, when all customer wants to buy, the chance that a customer experiences the shortage is simply $1 - \lambda_2(S)$.

Advance Period

We now examine the seller's decision in the advance period. As in the deterministic valuation case, we investigate the customer's decision in the advance period first.

Let $\lambda_1(S)$ be the probability that a customer can obtain (assuming he/she wants to buy) the product in the advance period when the seller rations S units of capacity. Recall that all customers in the advance period (who will act according to their expected valuation) will act in unison, thus $\lambda_1(S) = \frac{S}{N_1}$. If no customer wants to buy the product, we assume $\lambda_1(S) = 1$.

For a given advance price p_1 and available capacity S in the advance period, the expected utility of a customer who attempts to buy in advance is

$$U_A(S) = \mathrm{E}[\alpha - p_1]\lambda_1(S) + \mathrm{E}\left[\lambda_2(S)\max(\alpha - p_2^*(S), 0)\right](1 - \lambda_1(S)).$$

On the other hand, the expected utility from deferring to the spot period is

$$U_W(S) = \mathrm{E}\left[\lambda_2(S)\max(\alpha - p_2^*(S), 0)\right],$$

Since $U_W(S) \geq 0$, buying in advance is optimal for a customer if and only if $U_A(S) \geq U_W(S)$, which is equivalent to

$$\mathrm{E}[\alpha - p_1]\lambda_1(S) + \mathrm{E}\left[\lambda_2(S)\max(\alpha - p_2^*(S), 0)\right](1 - \lambda_1(S)) \geq \mathrm{E}\left[\lambda_2(S)\max(\alpha - p_2^*(S), 0)\right].$$

The above inequality shows that the maximum (hence the optimal) price that the seller's can charge in the advance period is:

$$p_1^{\max,H}(S) = \mathrm{E}[\alpha] - \mathrm{E}\left[\lambda_2(S)\max(\alpha - p_2^*(S), 0)\right] = \mathrm{E}[\alpha] - \tfrac{T-S}{N_1+N_2-S}\mathrm{E}[\max(\alpha - p_2^*(S), 0)]. \quad (2.8)$$

Noting that $T < N_1 + N_2$ and $p_2^*(S) = p_2^U$, it follows immiediately that $p_1^{\max,H}(S)$ is increasing and convex in S.

The seller's expected profit over the two periods is

$$\begin{aligned} \pi_{AS}^H(S) &= (p_1^{\max,H}(S) - c)S + (p_2^*(S) - c)\mathrm{E}[\min(T - S, D_2(S))] \\ &= (p_1^{\max,H}(S) - c)S + (T - S)\overline{G}(p_2^U)(p_2^U - c) \end{aligned} \quad (2.9)$$

where the second equality follows from $\mathrm{E}[\min(T - S, D_2(S))] = (T - S)\overline{G}(p_2^*(S))$.

Capacity Rationing

With the seller's optimal pricing policy characterized (Lemma 4 for spot price and equation (2.8) for advance price), we now examine the seller's capacity rationing decision. The seller solves the following problem to maximize its expected profit over the two periods.

$$\max_{0 \le S \le \min(N_1, T)} \pi_{AS}^H(S) = (p_1^{\max,H}(S) - c)S + (T - S)\overline{G}(p_2^U)(p_2^U - c)$$

Recall that $S = 0$, $S \in (0, \min[N_1, T])$, and $S = \min[N_1, T]$ correspond to no advance selling, limited advance selling, and full advance selling, respectively. We first show that that the limited advance selling (allocating only a fraction of capacity to advance selling) is never optimal for the seller in the homogeneous case.

Lemma 5 [Limited advance selling is never optimal in the homogeneous-1 valuation case]

$S^* = 0$ *or* $\min(T, N_1)$. *Thus, the seller should either sell only in the spot period or use the full advance selling.*

Although the limited advance selling could be optimal in the deterministic case, this is no longer true when customers' spot valuations are homogeneous. Recall that the seller's optimal spot price does not depend on the remaining capacity. Also recall from equation (2.8) that the seller's optimal advance price is increasing and convex in S. Combining these two facts, it can be shown that the seller's expected profit function,described in equation (2.9), is convex in S, thus the optimal capacity rationed must be either $S^* = 0$ or $S^* = \min[N_1, T]$. Building on Lemma 5, we now characterize the seller's optimal strategy as a function of the seller's capacity and marginal cost.

Theorem 4 [The optimal capacity rationing in the homogeneous-1 valuation model]

There exists a non-increasing function $c^H(T)$ *in* $T \in (0, N_1 + N_2)$ *such that* *(i) if* $c > c^H(T)$, $S^* = 0$ **[no advance selling]** *and*

(ii) if $c \leq c^H(T)$, $S^* = \min(T, N_1)$ **[full advance selling]**.

Furthermore, $c^H(T) \geq \bar{c}$ *for all* $T \in (0, N_1 + N_2)$.

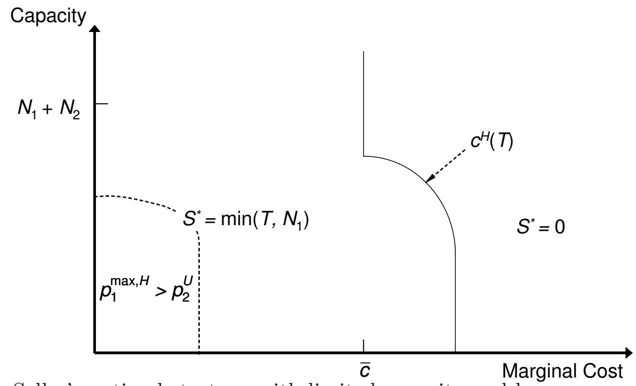

Figure 2.4. Seller's optimal strategy with limited capacity and homogeneous-1 valuations

Notice that the seller's optimal strategy is significantly different from that in the deterministic valuation model (which is illustrated in Figure 2.3): In the deterministic valuation case, advance selling is optimal only when capacity is not too tight, $T > T_1 > 0$, but in the homogeneous case, advance selling is always optimal at any capacity level.. Recall that, as the remaining capacity decreases, customers anticipating possible shortage in spot are willing to pay a higher advance price, which, in turn, increases the seller's incentive to sell in advance. As a consequence, advance selling can be optimal even when the capacity is tight, and the region in which advance selling is offered is larger than that in the deterministic valuations case.

Recall that the optimal advance price was always less than or equal to the spot price in the unlimited capacity case as well as in the deterministic valuation case. The story is different in the homogeneous-1 case: There is a region in which the advance price is higher than the spot price. When the seller's capacity is tight,the seller can exploit the fact that customers are worried about potential shortage in the spot period by raising the advance price, possibly above the spot price (*premium advance price*). The following proposition characterizes the region in which the premium advance pricing is used. (See also Figure 2.4). For ease of exposition, we write the dependence of the seller's optimal prices on the marginal cost c explicitly: $p_1^{\max,H}(c)$ and $p_2^*(c) = p_2^U(c)$.

Proposition 3 [The optimality of a premium advance price in homogeneous-1 valuation model] *Advance selling with a premium advance price is optimal when both cost and capacity level are sufficiently low. Specifically,* $p_1^{\max,H}(c) > p_2^*(c)$ *if and only if* $\mathrm{E}[\alpha] - p_2^U(c) > 0$, $T < N_1 + \frac{\mathrm{E}[\alpha] - p_2^U(c)}{\mathrm{E}[\max(\alpha - p_2^U(c), 0)]} N_2$, *and* $c \leq c^H(T)$.

Proposition 3 highlights the two key components leading to a premium advance price: low marginal cost and tight capacity. When the cost is low, the spot price is also low and the chance that a customer will face shortage is high. Of course, the chance of shortage increases as the capacity becomes tighter. In either case, tight capacity (directly) or low cost (indirectly) lead to significant increase in the chance of shortage, and this will induce customers to buy at a premium advance price. Likewise, the region where a premium advance price is used expands as the number of customers (either in spot or in advance) increases (since shortage is more likely).

On the other hand, the premium pricing region shrinks as customers' valuation distribution becomes more variable. This result can be formally shown when valuation follows a uniform distribution and is also observed for other distributions in our numerical study. As the variance of valuation increases, both spot and advance price tend to increase. However, the advance price increases at a much slower rate than the spot price.

Proposition 4 [The premium advance price in homogeneous-1 valuation model]

When customer valuation follows uniform distribution, the region where a premium advance price is optimal contracts as the variance increases.

2.5.3 Homogeneous-k and Heterogeneous Valuation Models

We now focus on the remaining two models: homogeneous-k and heterogeneous valuation models. In the homogeneous-k model, the consumer population in the spot period is divided into k equal-sized subgroups. Each subgroup behaves like the consumers in the homogeneous-1 model, but the valuation of each subgroup is independently and identically drawn from the distribution $G(\cdot)$. Let $D_2^k(S, p_2)$ be the demand in the spot period when the seller with the remaining capacity $T - S$ charges a spot price p_2 in the homogeneous-k model. For a given spot price p_2, the number of subgroups with valuation p_2 or higher is binomially distributed, thus the spot demand is the size of the population in a subgroup times the number of subgroups with valuation p_2 or higher. Hence, the spot demand $D_2^k(S, p_2)$ is a random variable defined as follows:

$$D_2^k(S, p_2) = \frac{N_1 + N_2 - S}{k} Y$$

where Y is a binomial random variable with $n = k$ and $p = \overline{G}(p_2)$. In the heterogeneous case, each customer's valuation is independently drawn from distribution $G(\alpha)$. This case corresponds to a special case of the homogenous-k model where $k = N_1 + N_2 - S$.

Let $\lambda_2(S, p_2)$ be the probability that a customer who wants to buy the product in the spot period actually obtains it at a spot price p_2. Since the number of customers who want to buy the product in the spot is $D_2^k(S,p_2)$, $\lambda_2(S,p_2) = \mathrm{E}\left[\min\left(1, \frac{T-S}{D_2^k(S,p_2)}\right)\right]$. Notice that $\lambda_2(S,p_2)$ corresponds to the expected fill ratio in the spot period. Following similar steps used in the previous subsections, we find the expressions for the optimal spot and advance prices (denoted by $p_2^*(S)$ and $p_1^{\max,k}(S)$ when S units are sold in the advance period:

$$p_2^*(S) = \arg\max_{p_2 \geq c} \left\{(p_2 - c)\mathrm{E}[\min(T-S, D_2^k(S,p_2))]\right\}, \text{ and}$$
$$p_1^{\max,k}(S) = \mathrm{E}[\alpha] - \lambda_2(S, p_2^*(S))\mathrm{E}\big[\max[\alpha - p_2^*(S), 0]\big]$$

Substituting these expressions, the seller's problem of choosing the optimal amount of capacity that should be sold in the advance period can be writte as follows:

$$\max_{0 \leq S \leq \min(N_1,T)} \quad \pi_{AS}^k(S) = [p_1^{\max,k}(S) - c]S + (p_2^*(S) - c)\mathrm{E}\left[\min(T-S, D_2^k(S, p_2^*(S)))\right]$$

Solving this problem analytically for a general k is very difficult. Instead, we solve it numerically and test it over a range of scenarios.

Figure 2.5 illustrates a typical structure of the seller's optimal strategy in the heterogeneous case. Advance selling is optimal only when the marginal cost is not too high. With very tight capacity, the seller will only sell the product to consumers with high valuation in the spot period, thus advance selling is not optimal. At moderate capacity levels, advance selling is optimal, but the seller limits the quantity sold in the advance period. This is because disposing some of the seller's capacity in advance will raise the spot price and increase the likelihood that the seller sells all of its capacity after the two periods. At high capacity level, the seller offers advance selling without limiting the quantity sold in advance. However, In this region, for relatively low cost, the seller's optimal strategy resembles that in homogeneous valuation model and advance selling with a premium advance price ($p_1^* > p_2^*$) emerges to be optimal.

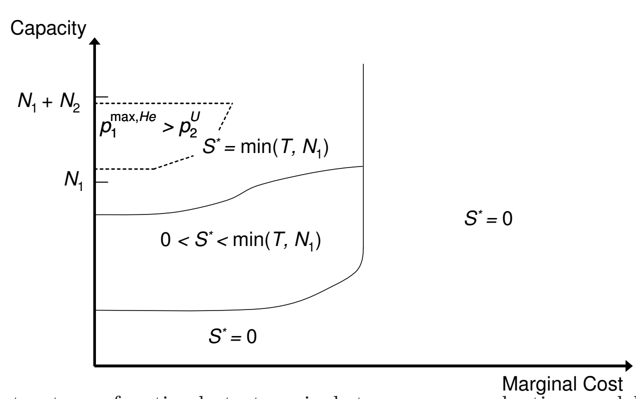

Figure 2.5. A typical structure of optimal strategy in heterogeneous valuation model ($N_1 = 9$, $N_2 = 7$, $\alpha \sim$ Uniform$[25, 35]$)

Interestingly, unlike the homogeneous case, a premium advance price is beneficial when capacity level is fairly high and marginal cost is low. The key driver in this case is consumer surplus: At a fairly high capacity level, the seller needs to cut the spot price to ensure the sales of all his capacity. As a result of this low spot price, if the customer is able to obtain the good, his/her surplus is likely to be fairly high. On the other hand, if the consumer cannot obtain the good (because of the shortage), he/she would receive zero surplus. Because of this big difference, they do not mind paying some premium in the advance period for the guaranteed availability. On the other hand, when the seller's capacity is tight, the tight capacity itself justifies a high spot price (even higher if some capacity is rationed to advance). A customer's expected surplus from buying in the spot period is low, as her realized valuation likely falls below the spot price, hence a discount is needed to persuade her to get the product in advance.

Consider the homogeneous-k model. In this case, valuations are homogeneous within each group, but each group's valuation is independent of another's. Clearly, as the number of subgroups increases, correlation among customers' valuations weakens and the variance of aggregate demand decreases. Note that homogeneous-k case includes as special cases both homogeneous valuation (when $k = 1$) and heterogeneous valuation (when $k = N_1 + N_2 - S$) models. In the homogeneous-1 case, the entire spot population forms one group while in the heterogeneous model, each remaining customer itself is a group. The impact of the number of subgroups k on the seller's optimal policy was numerically computed and their results are presented in Figure 2.6.

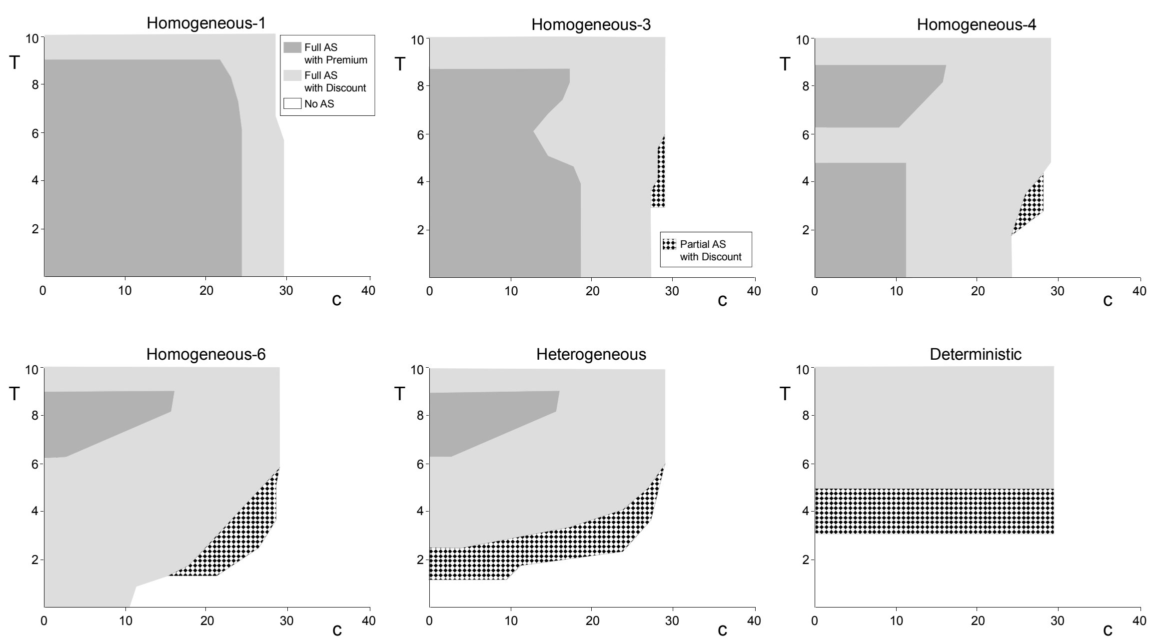

Figure 2.6. Seller's optimal policy for various valuation models when $N_1 = 5$, $N_2 = 5$, $\alpha \sim U[25, 35]$.

As valuation interdependence decreases (represented by increasing number of subgroups k), the region where advance selling is optimal shrinks and so does the subregion where a premium advance price is used. On the other hand, the region where the limited advance selling or no advance selling is used expands. This phenomenon can be intuitively justified: As consumer valuations become less interdependent, the aggregate demand in spot period is less variable. In such a case, the seller can charge a higher spot price without worrying about losing customers too much. This makes selling in advance less attractive. At the same time, from a customer's point of view, the chance of shortage is the highest in the homogeneous-1 case and gradually decreases as the number of subgroups increases. Thus, when moving from homogeneous-1 to heterogeneous and deterministic cases, a premium advance price is less likely deployed. While the total region of advance selling decreases, the region where the limited advance selling is used increases as rationing is especially beneficial when spot demand is not too variable.

While we do not provide separate graphs illustrating the effects of the size of the customer population, we report that, as N_1 or N_2 increase, the region where a premium advance price is used also shrinks. In addition, the seller's optimal strategy eventually converges to that in the deterministic model when the market size becomes very large.

2.6 Numerical Study

Our results in previous sections show that the seller can gain from offering advance selling. Advance selling can be optimal for a seller with unlimited and limited capacity. Exact conditions under which advance selling is beneficial depend on the valuation distribution and its interdependence among consumers. The obvious follow-up questions to be raised are how large the gain from advance selling is and what factors affect the gain. To answer these questions, we conduct a numerical study. In our numerical study, we measure the gain from advance selling under four valuation interdependence models - deterministic, heterogeneous, homogeneous-2, and homogeneous-1.

To evaluate the gain from advance selling, we use the percentage improvement in total profit over the spot-only strategy:

$$\Delta_t = \frac{\pi^t_{AS}(S^*) - \pi^t_S}{\pi^t_S} * 100\%, \quad t = D, He, H2, H$$

where S^* is the optimal capacity rationed in advance, and the spot-only profit is defined to be $\pi^t_S = \pi^t_{AS}(S = 0)$.

To measure the gain under various scenarios, we consider the following set of parameters: • Market size: $N = N_1 + N_2 \in \{5, 10, 20, 50, 100\}$;

- Proportion of customers arriving in advance: $\frac{N_1}{N} \in \{0.2, 0.4, 0.6, 0.8\}$;
- Ratio of capacity to market size: $\frac{T}{N} \in \{0.2, 0.4, 0.6, 0.8, 1\}$;
- Marginal cost: $c \in \{0, 8, 16, 40, 60\}$;
- Valuation distribution: $\alpha \sim \text{Uniform}[A - b, A + b]$, where $A = 75$, $b \in \{1, 5, 30, 50, 75\}$

We computed the gain for 2,500 instances under each of the three valuation interdependence models, resulting in 10,000 instances.

2.6.1 The Valuation Interdependence

Overall, the gain from advance selling is quite significant, with an average of 15.97%, many instances exceeding 50%, and some reaching 100%; see Table 4.1. Among the three valuation interdependence models, the average gain is the largest for the homogeneous-1 model and the smallest for the deterministic model. We also noticed that in many instances, the gain under the deterministic model is similar to the gain under the heterogeneous model.

	Mean	(Min, Max)
Δ_D	7.08%	(.00%, 61.98%)
Δ_{He}	7.48%	(.00%, 61.93%)
Δ_{H2}	15.93%	(.00%, 77.73%)
Δ_H	33.39%	(.00%, 100%)
Overall	15.97%	(.00%, 100%)

Table 2.1. Benefit of advance selling with respect to various customer behavior

2.6.2 The Seller's Capacity T

To measure the effect of capacity T relative to market size $N_1 + N_2$, we evaluate the gain as a function of the ratio $\frac{T}{N_1+N_2}$ in Figure 2.7. As $\frac{T}{N_1+N_2}$ approaches 1 (i.e., the unlimited capacity case), the gains under all four valuation models converge as shown in Theorem 1.

In general, the change of the gain under deterministic valuation is similar to that under heterogeneous valuation. For both cases, when capacity is small, the seller is better off selling only in spot at a high spot price instead of selling scarce capacity at a discount price in advance. As T increases, rationing some capacity becomes desirable since the seller can create scarcity in spot in order to charge a higher spot price. However, the gain eventually decreases when a significant portion of customers are offered a discount in advance.

On the other hand, the gain under homogeneous valuation always decreases in capacity. Notice that the optimal spot price is independent of the remaining capacity and is equal to p_2^U, and that the seller always uses full advance selling. With small capacity, the seller prefers selling all products in advance (sometimes at a premium price) to selling in uncertain spot market. As T increases and exceeds N_1, the seller cannot sell all units in advance and is forced to sell in spot, which decreases advance price and the gain.

At an intermediate level of valuation interdependence, i.e., under the homogeneous-2 valuation model, the trend is a mixture of the two extreme cases. The gain decreases in capacity when capacity is small, for a reason similar to that under homogeneous-1 model: regardless of advance rationing S, the remaining capacity in spot is always sold out whenever spot demand is positive and hence the optimal spot price is independent of the remaining capacity.[5] In the meanwhile, advance price decreases in T as the product availability in spot increases and hence the gain of advance selling decreases. On the other hand, when capacity is medium to large, the change of the gain and the underlying reason are similar to those under deterministic (or heterogeneous) model: the gain first increases since the seller can effectively raise prices in both periods by disposing some capacity in advance, but it eventually decreases as a large portion of the total capacity is sold in advance at a discount price.

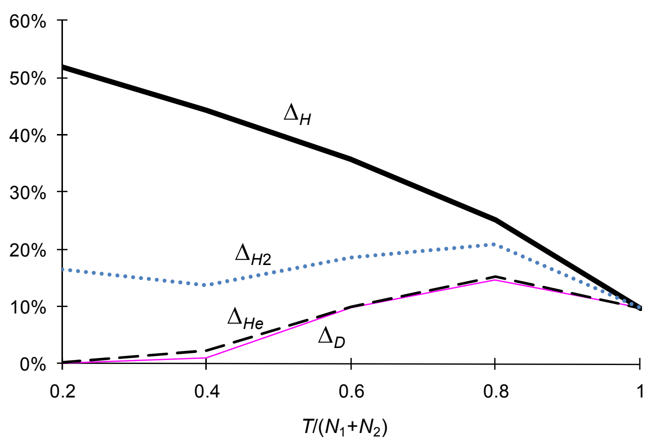

Figure 2.7. Average gains w.r.t various $T/(N_1 + N_2)$ levels

2.6.3 The Seller's Marginal Cost c

Average	$c = 0$	$c = 8$	$c = 16$	$c = 40$	$c = 60$
Δ_D	7.44%	8.00%	8.46%	8.22%	3.30%
Δ_{He}	8.24%	8.63%	8.93%	8.12%	3.50%
Δ_{H2}	18.32%	18.72%	18.85%	15.85%	7.93%
Δ_H	34.16%	35.90%	37.34%	37.08%	22.45%

Table 2.2. Average gains with respect to various marginal cost c

Table 2.2 implies that sellers with high production or service cost (e.g., when c is as high as

[5]Specifically, in such cases, optimal spot price maximizes the expected unit profit $(p - c)[1 - (G(p))^2]$, where $[1 - (G(p))^2]$ is the probability that spot demand is positive.

80% of the expected valuation) may still gain from advance selling. For all four behavioral models, the average gains first increase with higher marginal costs. The logic is the same as that for the unlimited-capacity model. Higher cost drives prices up and total sales down. But with advance selling, the total sales (number of customers) drop more slowly since all advance customers buy. Therefore, as c becomes very large, the average gains eventually diminish and in many cases advance selling is no longer beneficial, since the benefit in total sales can no longer compensate for the deep discount that the seller has to offer in advance.

2.6.4 Variability in Customers' Spot Valuations b

In our numerical study, variability of valuation is controlled by b, as individual valuation follows a uniform distribution on $[-b, b]$. Clearly, the variability of valuation influences the variability of spot demand. When customers are unsure about future valuations, the seller can take advantage of their information uncertainty, and the gain from advance selling is likely to increase. It can be shown that for given spot price and remaining capacity, both the spot profit and advance price discount decrease with more variable spot demand.

Average	$b = 1$	$b = 5$	$b = 30$	$b = 50$	$b = 75$
Δ_D	0.33%	1.77%	10.26%	11.87%	11.20%
Δ_{He}	0.74%	3.02%	11.09%	11.94%	10.64%
Δ_{H2}	1.66%	8.44%	25.04%	24.48%	20.04%
Δ_H	1.70%	10.56%	49.90%	54.97%	49.80%

Table 2.3. Average gains with respect to various uncertainty ratio b.

As shown in Table 2.3, the average gain increases when variability is small to medium, then eventually starts to decrease when variability is extremely high. Such behavior is observed in all three valuation models. To understand this behavior, note that in extreme case, where $b = 0$, there is no gain from advance selling. On the other hand, when b is very large, the seller charges a very high spot price to only serve customers with very high valuations. In general, a deep discount would be needed to sell in advance, and the benefit of advance selling disappears gradually.

2.6.5 Proportion of Customers Arriving In Advance $N_1/(N_1 + N_2)$

As the seller can sell up to $\min(N_1, T)$ in advance, a larger portion of customers arriving in advance, $N_1/(N_1 + N_2)$, provides the seller an opportunity to ration more capacity to advance, which raises prices in both periods and subsequently increases the gain from advance selling, as illustrated in Figure 2.8.

To summarize, the more homogeneous customers are, the more variable aggregate demand is,

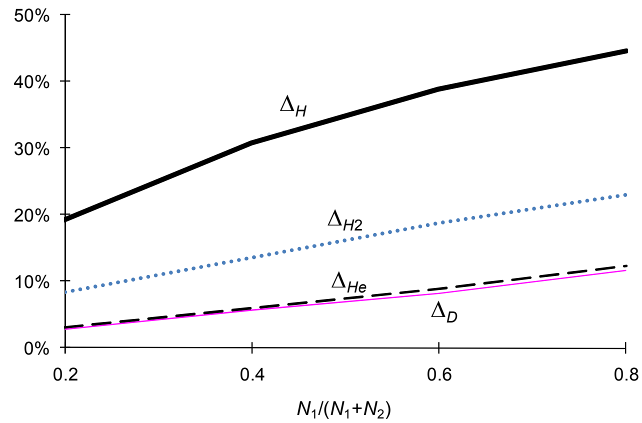

Figure 2.8. Average gains w.r.t. various $N_1/(N_1+N_2)$ levels

thus the benefit of advance selling tends to increase. An increase in the seller's total capacity may result in a lower benefit of advance selling and the underlying reason is dependent on the valuation model. The benefit of advance selling increases in the variability of individual valuation, but eventually decreases as the seller decides to serve few customers in spot instead of selling in both periods. Obviously, when more customers consider the product in advance, the benefit of advance selling increases. Our numerical study also shows that advance selling is an effective way to increase the seller's profit even when the marginal cost is high.

2.7 Extensions and Discussions

2.7.1 Seller's Capacity Choice

So far, we have analyzed the cases where the seller's initial capacity is given. While this setting fits several examples like sales of any particular flight where the capacity is fixed, in some situations, the seller may be able to adjust (choose) its capacity before pricing decisions. Conventional intuition might suggest that, if the firm can choose the price and is not committed to exhaust its capacity, the profit will always increase in capacity: If the seller can choose any capacity free of charge, the seller will choose the largest available capacity even if the seller may not use it. However, our analysis shows that this is not always the case. A larger capacity can lead to a lower profit. For example, consider a seller in a deterministic valuation model with $N_1 = N_2 = 20$, $\alpha \sim U[45, 105]$, and $c = 8$. We compare two initial capacity levels: $T = 32$ and $T = 40$. At $T = 32$, the seller will sell $S = 20$ units in the advance period at $p_1^* = 64.2$, and sell remaining 12 units in the spot period at $p_2^* = 69$. The seller's resultant profit is $\$1,856$. On the other hand, at $T = 40$, we still have $S = 20$ but the prices in both periods change: $p_1^* = 55.4$ and $p_2^* = 56.5$, resulting in the profit of $\$1,732$. This rather counter-intuitive phenomenon will not happen if the seller has only one chance to sell. To explicitly recognize the dependencies of optimal profit and rationing on T, let $\pi_{AS}^D(S^*(T)|T)$ be the seller's optimal profit when the capacity level is T. Figure 2.9 illustrates that the optimal profit $\pi_{AS}^D(S^*(T)|T)$ changes non-monotonically in the capacity.

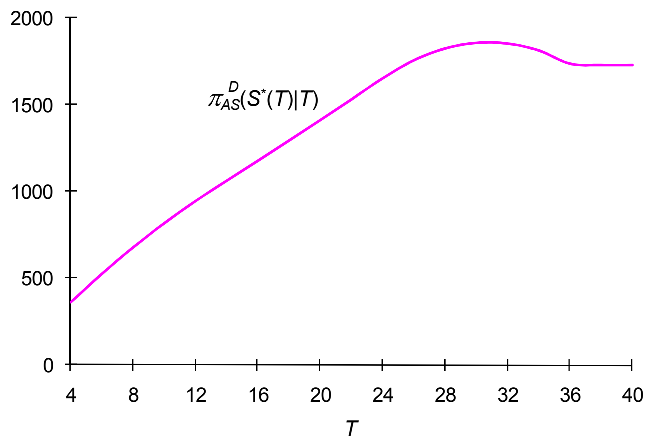

Figure 2.9. Seller's optimal profit as a function of capacity for deterministic valuation model with $N_1 = 20, N_2 = 20, c = 8, \alpha \sim U[45, 105]$

To understand how the seller's optimal capacity is determined, we examine the two valuation models: deterministic and homogeneous-1 models. In the deterministic valuation model, we show that the profit function is unimodal in capacity T. Utilizing the unimodality, the following theorem shows the existence of an optimal capacity T^*. In preparation, recall that T^D is a threshold capacity defined in Theorem 2.5.1 above which the full advance selling is optimal and below which the limited advance selling is optimal (assuming the seller finds it optimal to sell in advance).

Theorem 5 [(The seller's optimal capacity under the deterministic valuation model)]
Under the deterministic valuation model, $\pi_{AS}^D(S^*(T)|T)$ is unimodal in T. Furthermore, there exists an optimal capacity T^* between T^D and $N_1 + N_2\overline{G}(p_2^U)$.

We also characterize the seller's optimal capacity under the homogenous-1 valuation model. Once again, the seller with a larger capacity does not necessarily result in a higher profit. As in the deterministic valuation case, we define $\pi_{AS}^H(S^*(T)|T)$ to be the seller's optimal profit at the capacity level T.

Theorem 6 [The seller's optimal capacity under the homogeneous-1 valuation model]
Under the homogeneous-1 valuation model, $\pi_{AS}^H(S^*(T)|T)$ is nondecreasing when $T \leq N_1$ and convex when $T \geq N_1$. Hence, the optimal capacity, T^*, is either N_1 or $N_1 + N_2$, whichever leads to a higher profit.

An interesting observation here is that the seller's optimal capacity is equal to either the size of consumer population who arrive in the advance period, N_1 or the size of the total consumer population, $N_1 + N_2$. Once again, recall that, under the homogeneous-1 model, the realized valuations in the spot period are identical. If the seller only offers the product in the spot period, its expected profit function is simply linear in capacity T. As a result, The seller will always choose $N_1 + N_2$ as the optimal capacity. However, if the seller has an option to sell in the advance period, another possibility, $T^* = N_1$, emerges as an optimal capacity. The seller's total profit function (when selling in advance) is a piecewise linear function in T with a breakpoint at $T = N_1$. When the seller chooses to sell only in advance (i.e., $T^* = N_1$), the expected scarcity in spot enables the seller to charge a higher price in advance, which can be optimal to the seller.

As we have shown in Theorems 5 and 6, a large capacity does not always benefit the seller. One interesting question is whether the seller is more or less likely to offer advance selling if the seller decides to reduce a portion of its capacity. The next theorem precisely answers this question.

Theorem 7 [The optimal pricing and capacity rationing at the optimal capacity level]
Under both the deterministic and homogeneous-1 valuation model, suppose that the seller finds it optimal to decrease its capacity. Then, the following holds.
(i) Prices in both periods increase. (ii) If the seller will offer advance selling at the original capacity, he would continue to offer advance selling at the optimal capacity as well. (iii) Even if the seller will not offer advance selling at the original capacity, he may find it beneficial to offer advance selling at the optimal capacity.

Theorem 7 implies that if the seller can choose its capacity, advance selling becomes even more desirable. This is because disposing some capacity upfront enables the seller to raise prices in both periods.

2.7.2 Different Valuation Distributions in Spot and Advance Periods

So far, we have assume that valuations of all customers are drawn from the same distribution and shown that advance selling can result in a significant benefit to the seller. We now examine the case where there are multiple segments in the consumer population, each with different valuation distribution. Specifically, we assume that the valuation distribution for customers arriving in the advance period, denoted by $G_A(\cdot)$, is different from the valuation distribution for customers arriving in the spot period, denoted by $G_S(\cdot)$. We numerically evaluate the seller's optimal strategy.

This setting is similar to the settings of earlier work in the revenue management literature (c.f., Littlewood (1972), Robinson (1995), Talluri and van Ryzin (2004), and references therein). In these papers, each customer chooses a take-it-or-leave-it offer in the period they arrive without having an option to defer their decision. As a consequence, the price decision in one period does not affect the consumer behavior and demand in another period. However, in our models, consumers arriving in the advance period are strategic and can choose to wait until spot while the aforementioned papers. With strategic consumers, the price decision in one period has lingering effect on the consumer behavior and demand in another period. For example, the advance price not only affects the current period revenue, but also affects the number of customers who will buy in the spot price and their valuation distributions.

We examines two different scenarios under the deterministic valuation model. We assume that two equal populations ($N_1 = N_2 = 10$) but their valuations differ. In the first scenario, the valuation of customers arriving in advance (G_A) is stochastically larger (or smaller) than the valuation of customers arriving in the spot period (G_S). In the second scenario, the variance of G_A is different from the variance of G_S.

In the first scenario, we assume $G_A \sim U[0.5+\delta, 1.5+\delta]$ and $G_S \sim U[0.5, 1.5]$ and vary δ from -0.3 to 0.3. Notice that when δ is positive (negative), the valuation distribution of advance customers is stochastically larger (smaller) than that of spot customers. The results of the three representative cases: $\delta = -0.3, 0$, and 0.3 are presented in Figure 2.10.

We observe that, as the valuation distribution of advance customers becomes stochastically smaller (equivalently, δ decreases), the seller makes less capacity available in advance selling and the benefit of advance selling decreases. In particular, when the capacity is tight, the seller may abandon the advance selling and offer the product only in the spot-period. On the other hand, if G_A becomes stochastically larger (i.e., advance customers are likely to pay more), the benefit of advance selling increases and the seller is more likely to use advance selling. However, the examples in Figure 2.10 shows that the region under the partial advance selling expands considerably when $\delta = 0.3$. In particular, we notice that the seller prefers to sell only a portion of its capacity (partial advance selling) even when the seller has sufficient capacity. With partial advance selling, the seller force some advance customers to buy in spot. This additional mix of high paying customers will increase the valuation distribution of the remaining consumer population (i.e., a mixture of spot and advance valuation distributions), and the seller's spot price. This in turn allows the seller to charge a higher price in advance.

In the second scenario, we vary the variance of the distribution for advance customers, G_A while keeping the distribution for spot customers the same. Specifically, we consider the case where $G_A \sim U[0.5 - \delta, 1.5 + \delta]$ and $G_S \sim U[0.5, 1.5]$ for different values of δ. We present the results of three values of δ, -0.3, 0, 0.3 in Figure 2.11. Notice that if δ increases, the variance of the distribution for advance customers increases. Surprisingly, advance selling is less likely used when the variance of the distribution for advance customer is large. This is because advance customers make purchase decisions based on their expected value in advance, but realized value in spot. With higher dispersion, the seller finds it better off to have advance customers to buy in spot so as to charge a higher spot price. On the other hand, if the variance of advance customer's valuation is small, the seller prefers advance selling because the seller can extract most of the surplus through advance selling. Making them procrastinate will not necessarily increase the spot price.

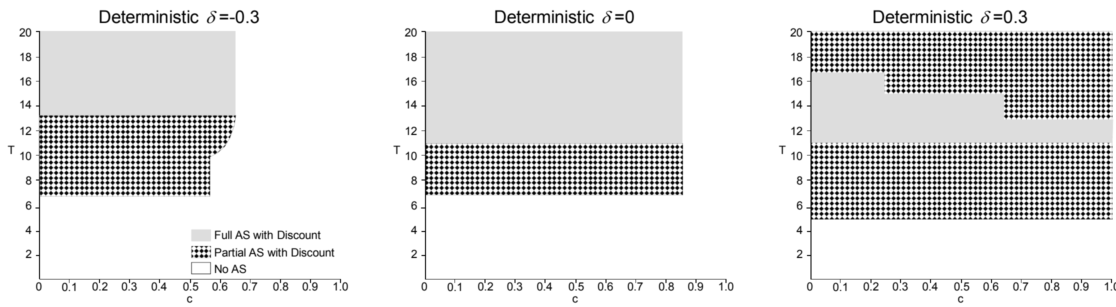

Figure 2.10. Seller's optimal policy with respect to the heterogeneity factor δ under deterministic valuation model, when $N_1 = 10$, $N_2 = 10$, and the valuation distributions in advance and spot are $U[0.5+\delta, 1.5+\delta]$ and $U[0.5, 1.5]$, respectively.

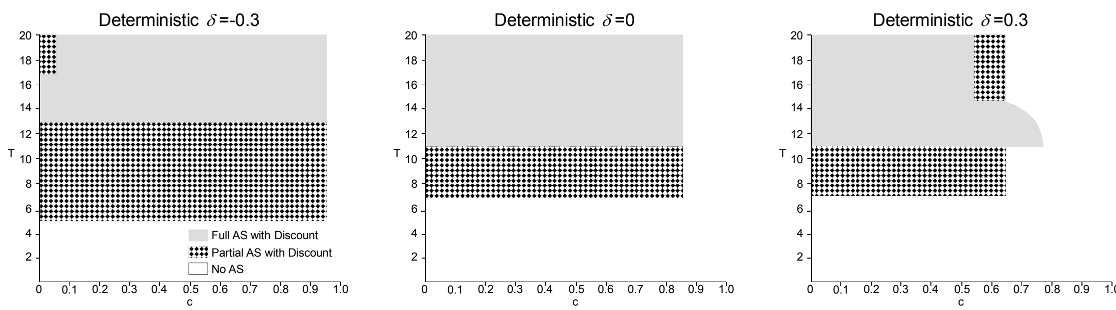

Figure 2.11. Seller's optimal policy with respect to the heterogeneity factor δ when $N_1 = 10$, $N_2 = 10$, and the valuation distributions in advance and spot are $U[0.5-\delta, 1.5+\delta]$ and $U[0.5, 1.5]$, respectively.

2.7.3 The Effect of Transaction Cost In Spot

In some cases, advance selling occurs in forms of mail intercepts, phone orders, or internet, all of which enable consumer to shop at relatively low transaction cost while consumers need to be physically present to purchase in spot. In such case, consumers incur additional transaction cost in the spot period. The next proposition characterizes how the seller's optimal policy will change with respect to the change in the transaction cost in the spot period, which we denote by θ.

Proposition 5 [The effect of transaction cost in spot on the seller's pricing and capacity rationing] *Suppose that consumers incur a transaction cost θ when buying in spot. Under both the deterministic and homogeneous valuation models,* *(i) the optimal spot price decreases in θ,*

(ii) the optimal advance price increases in θ, and

(iii) the quantity sold in the advance period increases in θ.

We illustrate the implication of Proposition 5 using Table 2.4. As the transaction cost increases, the seller will concede a portion of its margin to compensate for added inconvenience incurred by consumers, hence the spot price decreases. Since a compensation is only partial, the total cost of buying spot (i.e., spot price plus transaction cost) increases. This will increase the maximum price that the seller will charge in the advance period and increase the profit margin from advance selling.

One interesting observation is that an increased transaction cost in the spot period can actually increase the seller's overall profit in some cases. This happens when the seller's capacity is large and a majority of consumers arrive in the advance period. When this happens, the gain from the increased advance price outweighs the loss in the spot period (c.f., the leftmost panel of Table 2.4). This observation, together with Proposition 5 implies that the seller with large capacity could benefit by making consumer's spot purchase less convenient. For example, the seller could sell tickets for a sports event at multiple outlets in advance, but only at the event venue on the day of the event at which consumers may have to wait in a long line. Such added inconvenience will increase the customer's willingness to pay in advance and benefit the seller.

	$T=20, \frac{N_1}{N_1+N_2}=0.9$				$T=20, \frac{N_1}{N_1+N_2}=0.3$				$T=2, \frac{N_1}{N_1+N_2}=0.9$			
θ	S	p_1	p_2	π^D_{AS}	S	p_1	p_2	π^D_{AS}	S	p_1	p_2	π^D_{AS}
0	18	66.2	72.5	506.8	6	66.2	72.5	403.6	0	74.7	99.0	118.0
20	18	70.8	62.5	570.9	6	70.8	62.5	302.8	0	74.7	79.0	78.0
40	18	73.7	52.5	611.8	6	73.7	52.5	238.6	2	75.0	65.0	70.0
60	18	74.9	42.5	629.3	6	74.9	42.5	211.1	2	75.0	45.0	70.0
80	18	75.0	40.0	630.0	6	75	40.0	210.0	2	75.0	40.0	70.0

Table 2.4. Optimal strategy and profit with respect to the customers' spot transaction cost θ under deterministic valuation model, where $N_1 + N_2 = 20$, $c = 40$, and the valuation distribution is $U[45, 105]$.

2.8 Conclusion

Advance selling has been a standard marketing tool in many service and non-service industries (such as books and popular computer game consoles[6]). Due to the nature of different markets, consumers' valuations can be highly independent (when personal or idiosyncratic factors dominate) or closely correlated (when common factors prevail), resulting in qualitatively different aggregate demand in spot market. This paper is devoted to studying the seller's optimal pricing and capacity rationing strategy when consumer valuation interdependence varies.

[6] http://www.gamesindustry.biz/articles/ps3-is-most-pre-ordered-console-yet-says-playcom

This is the first paper, to the best of our knowledge, which studies the effect of consumer valuation interdependence on advance selling policy. We evaluate when and what type of advance selling (premium or discount advance price, the form of rationing) should be deployed in different environments. We show that the uncertainty in consumers' valuations and their interdependence play critical roles in advance selling. As opposed to many other stylized papers that replace the uncertainty in aggregate demand by certainty equivalent (or fluid model), we show that the benefits, as well as the policy, dramatically differ from one valuation model to the other. We also show that capacity plays a critical role in choice of the seller's policy.

Our analytical and numerical study shows that the seller's optimal policy (and profit) clearly depends on consumer valuation interdependence and the seller's capacity level. Valuation interdependence plays a weaker role at high capacity level, but plays a critical role at low capacity level. Our models and resultant insights are consistent with several examples we observe in practice and allow us to explain the phenomena we observe in various applications.

When consumers' valuation are fairly diverse (e.g., hotels), shortages are quite unlikely, and thus consumers are not willing to pay a premium price in advance (as typically observed for hotel reservations). In such cases, a seller with limited capacity may offer only a portion of capacity to advance selling (e.g., hotels may set booking limits for reservations made at lower rates) so as to raise spot price and reserve some capacity for spot to take advantage of the high price.

When consumers' valuation are highly correlated, advance selling can be very beneficial to the seller. In some cases, the sellers can exploit consumers' uncertainty about the product availability and charge premium price in advance. This explains why premium prices may be observed in situations like Broadway shows, or popular concerts or sports events. In a similar sprit, to take advantage of the premium advance price, partial advance selling becomes a less frequent choice (as observed in our analysis in the homogeneous-k model).

In general, advance selling is most advantageous for sellers with reasonably large capacity and moderate marginal cost. When the seller's capacity is tight or unit cost is high, selling only in spot market is a better choice as long as valuations are not highly interdependent. When valuations are highly interdependent as in our homogeneous-1 case, advance selling still benefits the seller even at tight capacity. Among all valuation models we have considered, the average gains from advance selling (compared to spot only) are biggest when valuations are homogeneous. Also, when a larger portion of the population considers buying in advance, the benefits are larger. The benefits do not necessarily increase in capacity, neither in variability of consumer valuations. In the same time, if the seller can choose its capacity or increase customers' inconvenience in spot purchase, advance selling can become more advantageous to the seller.

We also find that if advance and spot customers have different valuation distributions, benefit of advance selling decreases when the valuation distribution of advance customers becomes stochastically smaller or have a larger variance.

CHAPTER 3

Rationing Capacity in Advance to Signal Quality

3.1 Introduction

Nowadays many sellers offer customers opportunities to purchase a product or service considerably ahead of consumption (e.g., book a hotel room well in advance of the trip). This practice, usually known as advance selling, has become a standard marketing approach in service industries including travel, entertainment, professional sports, as well as many retailing industries, including toys, books, electronic gadgets, and media products (Gale and Holmes 1993, Dana 1998, Xie and Shugan 2001, Shugan and Xie 2004). Through buying in advance, customers can usually get guaranteed availability at discounted prices, but in some occasions have to commit early to products with uncertain quality. In such cases, advance selling may provide firms an opportunity to signal to customers the quality of their products.

For example, consider the advance selling of French wine, a centuries-long practice well-known as en primeur ("wine future" in French). Every spring, chateau estates offer customers opportunities to buy the new vintage which, at the time of primeur sales, is not yet finished and still in barrels. If customers accept the offer, payment is due up front but delivery occurs after the wine is finished and bottled, usually twelve to eighteen months later. Clearly, wine quality is unknown to customers when the wines are sold en primeur, as young wines taste and smell vastly different from the finished ones. In contrast, chateaus, as the producers of the wines, have better information about wine quality than customers, as they know exactly what has happened and will happen during the grape-growing and wine-making processes.[1] Aware of such asymmetry of information about wine quality (Hadj Ali and Nanges 2006, Dubois and Nauges 2006), smart customers may try deducing the quality from chateaus' actions in the primeur market.

The example of future wines does not stand alone. Asymmetry in quality information exists in advance when customers preorder a new product (e.g., Google G1 Phone) with uncertain attributes

[1] For example, Chateaus have first-hand information on all the important determinants of wine quality, such as climate, soil, viticultural and enological practices (Jackson, 2000). They also know better than customers on the final blending.

(e.g., design and functionality),[2] when customers prepay for an existing service with which they have no experience before - for example, a first-time Orlando visitor make a prepaid reservation for Cinderella's Royal Table character buffet in DisneyWorld,[3] or when a seller's credibility is uncertain and customers are unsure about whether the seller will deliver the product as promised - for example, investing in preconstruction condominiums marketed by an unknown real-estate developer.[4]

Such quality asymmetry in advance can be a double-edged sword to the sellers. On one hand, it creates opportunities for sellers with low-quality products to hide the inferior quality in advance and to boost their sales by locking many customers who would not have made the purchase if quality was known; On the other hand, the sellers may need to give a concession to induce customers, who are wary of quality uncertainty, to buy in advance.

Given these two opposite drivers, it is not clear whether and when the seller should offer advance selling. Many subsequent questions are interconnected: How does asymmetric quality information affect seller's profit and optimal advance-selling strategy? In particular, facing rational customers, can the seller of high-quality products use any signals in advance selling to credibly convey high quality and differentiate himself from the low-quality seller? Is it always or when is it desirable to do so? Which signals can he use?

We address these questions by analyzing a dynamic model of advance selling with asymmetric information. To our best knowledge, this paper is the first to examine the role of advance selling in signalling quality. Specifically, we focus on the signalling effect of a particular strategy in advance selling, *capacity rationing*.

Capacity rationing means limiting the sales for customers arriving early (Liu and van Ryzin 2008). That is, sellers choose to only satisfy a portion of demand in advance and reserve some of their capacity to spot. In en-primeur market, it is no secret that many wine merchants release only a "tranche" or proportion (usually about twenty percent) of their total production and intentionally limit the wine availability in advance market.[5] In electronic-device industries, limiting pre-orders of

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[2]The official launch of Google G1 phone was scheduled to be Oct 22, 2008, and the preorder of the then-new phone became available one month before, on Sep 23, 2008. At the time of pre-ordering, customers were uncertain about some details on the design and functionality including Bluetooth capability and the availability of third-party applications (http://voices.washingtonpost.com/fasterforward/2008/09/t-mobile_unveils_first_google.html).

[3]Cinderella's Royal Table character buffet is the most popular character dining event in DisneyWorld. Customers usually need to make reservation as early as 180 days before their visit and the payment is due at the time of reservation(ref: http://disneyworld.disney.go.com/parks/magic-kingdom/dining/cinderellas-royal-table, and http://hubpages.com/hub/Walt-Disney-World-Character-Dining). At the time as early as 180 days before their visits, customers can be unsure about the quality of the character buffet, for example, regarding the food quality, the atmosphere, the interaction with characters, etc.

[4]An example of buying preconstruction condominiums can be found at http://www.newcondorealty.com.

[5]The rationing behavior in en-primeur market has been noted in a report by the NY Times on March 22, 1989 (ref: http://www.nytimes.com/1989/03/22/garden/wine-talk-898389.html), and also addressed by many wine-merchant or wine-expert websites such as http://www.decanter.com/learning/basics/en_primeur.php and

new products is also a common practice. Two most well-known (and also controversial) examples are Microsoft Xbox 360 in 2005[6] and Sony PlayStation 3 in 2006[7], and a more recent example is Google G1 phone in 2008.[8] All of them are famous for nationwide shortage and sellout in advance markets.

Capacity rationing (and resultant supply shortage) is once perceived as a marketing tool to create hype and promote demand for new products (*Retailing Today* 2000, Dye 2000, Brown 2001). While acknowledging this, we ask whether rationing can be used as a valid signal of quality; if so, what is the economics implication behind it, and how efficient the rationing signal is compared to other forms of signal.

3.2 Literature Review

Our work is closely related to the literature on signalling quality. Several different forms of signals of quality have been examined in existing literature, including advertising (Kihlstrom and Riordan 1984, Milgrom and Roberts 1986), dynamic pricing (Bagwell and Riordan 1991), warranties (Lutz 1989), and scarcity (Stock and Balachander 2005). Our paper contributes to this stream of literature by showing that rationing in advance selling can signal quality. Furthermore, it shows that rationing is a more efficient signalling tool in comparison to pricing and advertising.

Among the aforementioned signalling literature, Stock and Balachander (2005) study scarcity as a signal of quality, which is similar to the rationing signal in our paper. They consider a seller who has ample capacity to meet all demand and is able to freely dispose some capacity to create scarcity. In a two-period model with fixed price and customers informed of quality ("innovators") arriving before those uninformed ("followers"), they show that high-quality seller can signal quality by making product scarce for followers and charging full-information price for all customers. They find that scarcity dominates pricing in terms of signalling efficiency since scarcity only affects the profit from followers, while any distortion in price influences the margin from all customers. They also acknowledge that the efficiency of scarcity signal relies on two important assumptions: fixed price and informed customers making purchase first, both of which ensure a big profit loss for the low-quality seller in any attempt to mimick the high-quality seller.

The rationing signal considered in our paper is different from the scarcity signal in Stock and Balachander (2005). Rationing means that the seller limits the sales for customers arriving early and

http://www.rarewineco.com/downloads/newsletter/archive/may801.pdf.

[6]Ref: http://www.gaming-age.com/news/2001/9/6-19

[7]Ref: http://www.joystiq.com/2006/10/09/ps3-pre-orders-open-tomorrow-at-eb-gamestop/

[8]Ref: http://androidcommunity.com/t-mobile-g1-pre-order-has-come-to-an-end-20080927/

reserves the rest of the capacity to sell later. As a concept originating from revenue management, it is specifically for a seller who is endowed with or has committed to a fixed amount of capacity (which may not be sufficient to satisfy all customers) and cannot freely change it. For example, at the time of primeur sales, a chateau's total wine production for the current year has been fixed. Furthermore, rationing and scarcity differ in the timing of the signalling. Consistent with the anecdotal evidence about advance selling, we assume that quality uncertainty exists in advance (e.g., when wines are sold en primeur) and resolves as consumption time approaches (e.g., when wines are finished and released to regular market). In other words, signalling by rationing is performed in advance, while signalling by scarcity in Stock and Balachander (2005) occurs close to consumption time. In addition, compared to the scarcity signal, the rationing signal in this paper is examined under a more practical setting, where the seller can dynamically change price over time and the advance customers can strategically choose when to buy, i.e., whether to buy in advance under imperfect quality information or wait till information is publicly revealed in spot. Such strategic customer behavior is supported by many empirical evidences (Liu and van Ryzin 2008, Su 2007) and is not captured in Stock and Balachander (2005).

Our paper is also related to the literature on advance selling, especially those considering customer uncertain valuations (e.g., Xie and Shugan, 2001; Gallego and Sahin, 2006; Zhao et al., 2006; Yu et al., 2007). This stream of literature confirms the profit advantage of advance selling in various situations, but all of the papers in this stream assume that all the information on sellers is publicly available and sellers do not have any private information. In contrast, our paper considers seller's private information on quality and examines the impact of asymmetric quality information on sellers' strategy and profit in advance selling.

Our paper is also linked to existing studies on capacity rationing within a broader context than advance selling. Among these studies, Liu and van Ryzin (2008) show that capacity rationing can induce risk-averse customers to buy early at the regular price instead of waiting for a clearance price. Zhang and Cooper (2006) evaluate the benefit of rationing with both fixed and flexible pricing. Gilbert and Klemperer (2000) find that rationing is preferred to market-clearing price when customers incur seller-specific sunk cost. These papers, however, all ignore the signalling effect of rationing, which is the focus of this paper.

The rest of the paper is organized as follows. We define the problem in section 3.3 and provide the analysis in section 3.4. Specifically, in section 3.4.1, we characterize the seller's optimal (equilibrium) strategy in a benchmark scenario where information asymmetry does not exist, i.e., a full-information setting (FI). In the following section 3.4.2, we then study the focus of the paper: the equilibrium strategy and outcome when quality information is asymmetric and the seller has the option of rationing capacity (OR). In section 3.4.3, we examine the value of rationing and characterize the conditions under which the seller prefers signalling through rationing. In section 3.5, we discuss two extensions of the basic models. The whole paper is then concluded in section 4.6. All proofs are presented in the appendix.

3.3 The Model

Consider a seller who has an option to sell his or her product in two periods, advance and spot. At the end of spot period, consumption takes place and there is no salvage value for any leftover capacity. The seller's product can be of either high or low quality.[9] The seller knows his own type, but customers in advance do not. Seller's total capacity is T and marginal cost is c, both of which are common knowledge and independent of the quality. In the basic model, we assume that seller's marginal cost is independent of quality, i.e., quality is costless and results from some non-monetary factors (e.g., the characteristics of a region's soil and a specific grape-growing site greatly affect grape quality[10]). We later relax this assumption and consider costly quality in Section 3.5.2.

Customers are forward-looking and make purchasing decision by evaluating both advance and spot buying opportunities. Customer i's preference is represented by a (net) utility function $U_i = A_t + \alpha_i - p$, where $A_t, t = H, L$ represents quality, $A_H > A_L$, and α_i is customer i's individual valuation, which reflects the heterogeneity in customers' willingness-to-pay and corresponds to the combined effect of all the idiosyncratic factors (for example, individual preferences on the taste and flavor of the wines or customers' mood at consumption time). N_1 customers arrive in advance and are uncertain about both individual valuation α_i and the product's quality A_t (for example, in en-primeur market customers are uncertain about their willingness to pay for the final wines, which are affected by both wine quality and customers' individual preferences). Customers share a common prior belief about product quality: it is high with probability q and low with $1 - q$, where q is a constant between 0 and 1. While customers' individual valuations α_i can be different from each other, in advance they follow the same prior distribution with cdf $G(\alpha)$. In spot, α_i is revealed to individual customer and product quality is revealed to public (for example, customers know exactly about their willingness-to-pay for wines after they taste samples of the finished wines). Another N_2

[9]We will refer to quality as the seller's type and use "a high-quality seller," "a high-type seller," and "high type" interchangeably

[10]A region's soil characteristics include depth, chemical composition, texture, etc., and the specific site characteristics include altitude, slope, exposure, prevailing winds, etc.
(ref: http://www.nysaes.cornell.edu/hort/faculty/pool/NYSite-Soils/terroir.html.)

customers arrive, who have perfect information on quality and their individual valuations. Following the commonly-used fluid model in advance-selling literature (e.g., Xie and Shugan 2002, Gallego and Sahin 2006, Yu et al. 2007), we assume that in spot, the *proportion* of customers having spot valuation less than or equal to α is $G(\alpha)$.

The sequence of events (Figure 3.1) is as follows: 1) N_1 advance customers arrive. 2) The seller decides whether to offer advance selling and, if so, announces the advance price p_1 and the capacity ration $S \in [0, \min(T, N_1)]$ available in advance. 3) Advance customers observe the seller's action, update their belief about the product quality, and decide whether to buy in advance or wait till spot. 4) In spot period, individual valuation and the product quality are revealed to individual customer. N_2 spot customers arrive. 5) The seller decides and announces spot price p_2. 6) Those customers who did not buy in advance and those who arrive in spot decide whether to buy in spot.

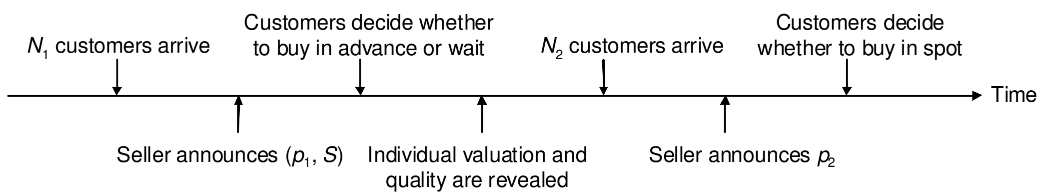


Figure 3.1. Sequence of events

Clearly, quality directly affects customers' utility and willingness-to-pay. While customers are uninformed about quality in advance, they anticipate that the seller have better information about quality and would try extracting the quality information from the seller's strategy. On the other hand, the seller with high-quality product naturally wants to use his strategy to convey the high quality to customers, such that customers would be willing to pay a high price in advance. In order words, the seller tries to signal the quality to customers. This is hence a sequential signalling game with incomplete information and the equilibrium concept we employ is perfect bayesian equilibrium (PBE).

Typically, two classes of Perfect bayesian equilibria, separating and pooling equilibria, exist for such a signalling game. In a separating equilibrium, high-quality seller can successfully distinguish himself from the low-quality seller in advance by executing strategies which low-quality seller does not have incentive to mimic. Consequently, customers can perfectly infer a seller's type by his strategy in advance. In contrast, in a pooling equilibrium, the high-quality seller cannot differentiate himself and both types of sellers adopt the same strategy in advance. Resultantly, customers cannot deduce any information about quality in equilibrium.

In a signaling game, multiple (sometimes a continuum of) equilibria may exist. To limit the number of equilibria, we impose the Intuitive Criterion (Cho and Kreps, 1987) on customers' beliefs on off-equilibrium paths. The Intuitive Criterion requires that if a customer observes an off-equilibrium strategy, which makes one and only one type of seller strictly better off than his equilibrium strategy, then the customer should assume that the strategy has been implemented by the type of seller who gets better off from doing so. In the meanwhile, to support the perfect Bayesian equilibrium, for the off-equilibrium strategy which does not satisfy the condition in the Intuitive Criterion, we require that a customer, upon observing such a strategy, should believe that it has been implemented by a L-type seller.

In addition to the Intuitive Criterion, we also impose the pareto-dominant criterion: if multiple equilibria exists, we choose the equilibrium that pareto dominates all the others from the seller's point of view, i.e., the equilibrium where both types of sellers obtain (weakly) higher profits than they do in any other equilibrium. Such a equilibrium constitutes a focal equilibrium, which is supported by evidences from behaviorial experiments (Schelling, 1960). Furthermore, since the seller moves first and customers can only choose accepting the offer or not, the seller is able to always pick the equilibrium most appealing to himself and expects customers to foresee his choice.

Throughout the paper, we impose the following technical conditions on the distribution $G(\cdot)$ of individual valuations:

(1) $G(\cdot)$ has a finite support $[\alpha_L, \alpha_H]$.

(2) $G(\cdot)$ is twice continuously differentiable.

(3) $g(\cdot) = G'(\cdot) > 0$ on (α_L, α_H).

(4) $g(\cdot)$ is log-concave.

(5) $\left(\frac{G(x)\bar{G}(x)}{g(x)}\right)' + \bar{G}(x) - k$ is positive-negative[11] for any $k \in [0, 1]$.

We use finite support for the ease of presentation, but all of the properties that we derive in the paper hold also when $\alpha_L = -\infty$ or $\alpha_H = \infty$. In case when α_L or α_H is infinite, however, we require that the mean and variance are both finite. A number of commonly-used log-concave distributions and their truncated versions [12], such as uniform, exponential, logistic, normal, extreme-value, Weibull (with shape parameter greater than or equal to 1), Beta (with two shape parameters greater than or equal to 1), Gamma (with shape parameter greater than or equal to 1), χ (with shape parameter greater than or equal to 1), χ^2 (with shape parameter greater than or equal to 2), and power function distributions, satisfy these conditions.

[11]A function $a(\cdot)$ is positive-negative, if $a(x_0) < 0$ implies $a(x) < 0$ for all $x > x_0$.

[12]Truncation of the distributions to a finite support is defined as a conditional distribution.

3.4 Analysis

3.4.1 Benchmark: Full Information (FI)

We first consider a benchmark case where asymmetry of quality information does not exist and all customers are fully informed about quality. This is the deterministic-valuation model studied in Yu et al.(2007). We briefly review the results in this subsection and examine how quality affects the seller's optimal strategy in the full-information setting.

We follow a backward induction and start from the spot period.

Spot Period

In spot, all customers are informed of both quality and their individual valuation. Given quality A_t, valuation α_i and spot price p_2, customers buy the product when their *ex-post* utility from a purchase exceeds the reservation utility (assumed to be zero), i.e.,

$$U_i = A_t + \alpha_i - p_2 \geq 0$$

In other words, any customer with valuation $\alpha_i \geq p_2 - A_t$ can afford the product. By fluid model, the proportion of customers who can afford the product equals to the probability that α_i exceeds $p_2 - A_t$, i.e., $\bar{G}(p_2 - A_t)$.

Given the capacity ration in advance S, the number of remaining customers in spot is N_1+N_2-S and remaining capacity is $T - S$. With spot price p_2, type-t seller's spot profit is given by

$$\pi_2^t(p_2, S) = (p_2 - c)\min\left(T - S, (N_1 + N_2 - S)\bar{G}(p_2 - A_t)\right)$$

Lemma 6 characterizes the optimal spot price as a function of S, which leads to an important result in Corollary 3: the seller always adjusts the spot price such that the remaining capacity in spot is sufficient to satisfy all the customers who can afford to buy in spot. The intuition is clear: since the spot demand is a deterministic function of price, the seller would never induce any demand over his remaining capacity.

Lemma 6 (Yu et al., 2007) *For given $S \in [0, \min(T, N_1)]$, $\pi_2^t(p_2, S)$ is quasi-concave in p_2 and has a unique maximizer $p_2^t(S) \in [\underline{p}_t, \bar{p}_t]$:*

$$p_2^t(S) = \max\left(p_{2t}^U, p_{2t}^B(S)\right) = \begin{cases} p_{2t}^U & \textit{if } \frac{T-S}{N_1+N_2-S} \geq \bar{G}(p_{2t}^U - A_t) \\ p_{2t}^B(S) & \textit{otherwise} \end{cases} \tag{3.1}$$

where

$$p_{2t}^U \begin{cases} = A_t + \alpha_H & \textit{if } c \geq \bar{p}_t \\ \in (A_t + \alpha_L, A_t + \alpha_H) \textit{ and is a solution to } p_2^U = c + \frac{\bar{G}(p_2^U - A_t)}{g(p_2^U - A_t)} & \textit{if } \underline{c}_t < c < \bar{p}_t \\ = A_t + \alpha_L & \textit{if } c \leq \underline{c}_t \end{cases} \quad (3.2)$$

$p_2^B(S)$ *is market-clearing price and is the solution of* $\bar{G}(p_2^B - A) = \frac{T-S}{N_1+N_2-S}$; $\underline{p}_t = A_t + \alpha_L$, $\bar{p}_t = A_t + \alpha_H$, *and* $\underline{c}_t = A_t + \alpha_L - \frac{1}{g(\alpha_L)}$.

Corollary 3 (Yu et al., 2007) *For* $S \in [0, \min(T, N_1)]$, $T - S \geq (N_1 + N_2 - S)\bar{G}(p_2^t(S) - A_t)$, *i.e., shortage in supply never occurs with the optimal spot price.*

Note that if $c > \bar{p}_H$, demand is trivially zero for any positive price charged by either type of seller. We exclude this extremely unrealistic (and trivial) case by assuming $c < \bar{p}_H$ for the rest of the paper. For ease of exposition, let $\pi_2^{*t}(S)$ denote the optimal spot profit as a function of S, i.e., $\pi_2^{*t}(S) = \pi_2^t(p_2^t(S), S)$.

Advance Period

When customers in advance have perfect information on product quality, they deduce the seller's optimal spot price and compare the expected utility from purchasing in advance with that from waiting. Denote by $\lambda_1(S)$ the probability that a customer obtains the product in advance. (If capacity rationed, S, is less than N_1, only a portion of advance customers are able to buy the product in advance.) The expected utility of a customer who attempts to buy the good in advance is

$$U_A(S) = \lambda_1(S)\mathrm{E}[A + \alpha - p_1] + (1 - \lambda_1(S))\mathrm{E}[\max(A + \alpha - p_2^t(S), 0)].$$

On the other hand, since supply shortage never occurs in spot by Corollary 3, the expected utility from waiting until spot is,

$$U_W(S) = \mathrm{E}[\max(A + \alpha - p_2^t(S), 0)].$$

Buying in advance is optimal for a customer if and only if $U_A(S) \geq U_W(S)$ and $U_A(S) \geq 0$. From $U_W(S) \geq 0$, the second inequality is redundant, hence buying in advance is optimal if and only if,

$$\lambda_1(S)\mathrm{E}[A + \alpha - p_1] + (1 - \lambda_1(S))\mathrm{E}[\max(A + \alpha - p_2^t(S), 0)] \geq \mathrm{E}[\max(A + \alpha - p_2^t(S), 0)]$$

Simplifying the inequality, we get

$$p_1 \leq A_t + \mathrm{E}[\alpha] - \mathrm{E}[\max(A_t + \alpha - p_2^t(S), 0)] = \mathrm{E}\left[\min(p_2^t(S), A_t + \alpha)\right] \quad (3.3)$$

Clearly, the right-hand side of Equation (3.3) represents the maximum price that, given spot price $p_2^t(S)$, a type-t seller can charge to induce advance purchase. For ease of exposition, define this maximum price as a function of spot price p_2:

$$p_1^{t,\max}(p_2) = \mathrm{E}\left[\min(p_2, A_t + \alpha)\right] \tag{3.4}$$

and define $p_1^t(S) = p_1^{t,\max}(p_2^t(S))$, $p_{1t}^U = p_1^{t,\max}(p_{2t}^U)$, and $p_{1t}^B(S) = p_1^{t,\max}(p_{2t}^B(S))$.

Clearly, the optimal advance price for type-t seller is $p_1^t(S)$. Lemma 7 shows that for given rationing, high-type seller charges higher prices that low-type seller does in both periods, since the informed customers are willing to pay more for a high-quality product.

Lemma 7 *For given S, $p_2^H(S) \geq p_2^L(S)$, $p_1^H(S) > p_1^L(S)$.*

Now we consider the seller's optimal rationing strategy. Given the optimal pricing strategy and the fact that the ration cannot exceed total capacity or market size in advance, the seller's rationing decision can be formulated as follows:

$$\max_S \ \pi_t^{FI}(S) = (p_1^t(S) - c)S + \pi_2^{*t}(S)$$
$$\text{subject to:} \quad 0 \leq S \leq \min(T, N_1)$$

The seller's optimal strategy as a function of total capacity T and marginal cost c is characterized in Theorem 8 and illustrated in Figure 3.2.

Theorem 8 (Yu et al., 2007) *When capacity rationing is optional, for given A_t, there exist two critical numbers, T_1 and T_D, $T_1 < T_D$, and a function $c^{OR}(T)$ for $T > T_1$, such that*

- *if $T \in (T_1, T_D)$ and $c \leq c^{OR}(T)$, then $0 < S^{FI,OR} < \min(T, N_1)$* **[limited advance selling]**,
- *if $T \geq T_D$ and $c \leq c^{OR}(T)$, then $S^{FI,OR} = \min(T, N_1)$* **[full advance selling]**, *and*
- *otherwise, $S^{FI,OR} = 0$* **[no advance selling]**.

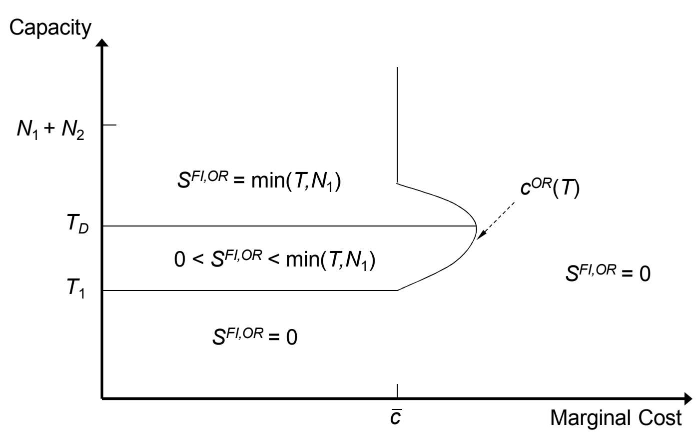


Figure 3.2. Seller's optimal strategy with full information and optional capacity rationing [Yu et al., 2007]

By Theorem 8, advance selling is profitable for a seller only if marginal cost is sufficiently low and total capacity is sufficiently large. Otherwise, it is not beneficial for the seller to offer the product at a discount price in advance. When rationing is optional, the seller rations capacity in advance only if capacity is medium, such that he can effectively raise prices in both periods and at the same time limit the capacity sold at a discounted advance price.

Furthermore, as shown in the following Proposition 6, we note that with full information, a high-type seller never rations less capacity in advance than a low-type seller does.

Proposition 6 $S_H^{FI,OR} \geq S_L^{FI,OR}$.

The intuition behind Proposition 6 is as follows. When quality information are symmetric, for given rationing, the price discount needed in advance is always (weakly) smaller for the H-type seller. That is because, when quality is high, the probability of a customer not consuming in spot is low and thus a small discount is sufficient to persuade the customer to buy in advance. Since the H-type seller can secure the sales early at a smaller price discount, he has a stronger incentive to offer advance selling and would ration more in advance. As we shall see, the comparative result in Proposition 6 is reversed when quality information is asymmetric in advance.

For ease of exposition, we denote type-t seller's full-information optimal strategy by ($p_{1t}^{FI,OR}$, $S_t^{FI,OR}$), where $S_t^{FI,OR}$ is the optimal rationing characterized in Theorem 8 and $p_{1t}^{FI,OR} = p_1^t\left(S_t^{FI,OR}\right)$ is the optimal price in advance. Similarly, denote type-t seller's asymmetric-information equilibrium strategy by $\left(p_{1t}^{AI,OR}, S_t^{AI,OR}\right)$.

3.4.2 Asymmetric Information with Optional Capacity Rationing (OR)

When customers in advance are uninformed of quality and the seller can ration capacity, quality can potentially be conveyed through two signals: price and capacity ration in advance. In this subsection we formally prove that rationing is a valid signal of quality. Specifically, we show that, whereas pricing may also communicate quality, customers can perfectly deduce quality *solely* from capacity rationing in advance period.

Our analysis, again, follows a backward induction and we first note that since quality information is fully revealed in spot period, the seller's optimal strategy and profit in spot period are identical with those characterized in section 3.4.1. In advance period, the seller announces advance price and ration (p_1, S). If the advance offer is accepted by customers, given the seller's type t, his (or her) total profit is

$$\pi_t^{AI}(p_1, S) = (p_1 - c)S + \pi_2^{*t}(S) \qquad (3.5)$$

We prove an important lemma regarding $\pi_t^{AI}(p_1, S)$, as follows.

Lemma 8 *For given p_1,*

$$\frac{\partial \pi_H^{AI}(p_1, S)}{\partial S} < \frac{\partial \pi_L^{AI}(p_1, S)}{\partial S}$$

Lemma 8 indicates that for given advance price, the marginal benefit of selling one additional unit of capacity in advance for a high-type seller is always less than or equal to that for a low-type seller. That is because high type always has an incentive to reserve more capacity to spot, when quality is publicly revealed and his high-quality product can get fully appreciated.

Graphically, if we plot different types' iso-profit curves in the rationing-price strategy space (Figure 3.3), Lemma 8 implies that at the same rationing level, the slopes of high-quality seller's iso-profit curves are always greater than those of low-quality. As a result, different types' iso-profit curves intersect with each other at most once. We shall refer to this result as the single-crossing property, which is critical for all the subsequent signaling outcome.

Furthermore, we note that in Figure 3.3, the band area between the two curves $\{S, p_1^L(S)\}$ and $\{S, p_1^H(S)\}$ defines the region containing all the strategy points that can be possibly sustained in a perfect bayesian equilibrium, since for given S, any price higher than $p_1^H(S)$ is definitely rejected by customers and any lower than $p_1^L(S)$ is strictly dominated by $p_1^L(S)$.

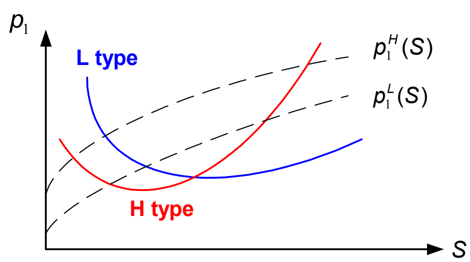


Figure 3.3. Typical iso-profit curves for two types of sellers

Separating Equilibrium

We define a separating equilibrium as an equilibrium in which either only one of the two types of sellers offer advance selling, or both types sell in advance but differ in their advance price *or* advance ration. In a separating equilibrium, customers can perfectly infer the seller's type from the observed advance price (p_1) and advance capacity ration (S), based on which they make their choice on whether to buy in advance or wait till spot.

For given advance ration S, it is clear that, regardless of their belief, customers will accept any price lower than or equal to $p_2^L(S)$ and reject any price higher than $p_2^H(S)$. Based on this knowledge, neither types of sellers will charge advance price lower than $p_2^L(S)$ or higher than $p_2^H(S)$. Meanwhile, in any separating equilibrium, customers can perfectly deduce the seller's type and clearly they won't accept any price higher than $p_2^L(S)$ when they are facing a L-type seller. Combining these facts, we prove that in any separating equilibrium, L type always adopts his full-information strategy.

Lemma 9 *In any separating equilibrium, L-type seller's equilibrium strategy is the same as his full-information strategy, i.e., $p_{1L}^{AI,OR} = p_{1L}^{FI,OR}$ and $S_L^{AI,OR} = S_L^{FI,OR}$.*

Suppose a separating equilibrium exists, then H-type seller's equilibrium strategy must be a solution to the following optimization problem, which maximizes H-type seller's total profit subject to the constraints that (i) his price is acceptable to advance customers (equation (3.6)), (ii) his strategy won't be mimicked by L-type seller (equation (3.7)), (iii) likewise he won't have an incentive to mimic L-type seller's strategy (equation (3.8)), (iv) rationing is within the feasible domain (equation (3.9)), and (v) his strategy is not identical to L-type seller's (equation (3.10)).

$$\max_{p_1,S} \pi_H^{AI}(p_1,S) = (p_1 - c)S + \pi_2^{*H}(S)$$

$$\text{subject to:} \qquad p_1 \leq p_1^H(S) \tag{3.6}$$

$$\pi_L^{AI}(p_1,S) \leq \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) \tag{3.7}$$

$$\pi_H^{AI}(p_1,S) \geq \pi_H^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) \tag{3.8}$$

$$0 \leq S \leq \min(T, N_1) \tag{3.9}$$

$$(p_1,S) \neq \left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) \tag{3.10}$$

Applying the single-crossing property, we characterize a separating equilibrium in Theorem 9, the main theorem of this subsection.

Theorem 9

(i)A separating equilibrium exists if and only if both types offer advance selling in full-information setting, i.e., $S_H^{FI,OR} > 0$ and $S_L^{FI,OR} > 0$.

(ii)In a separating equilibrium, L type executes his full-information strategy, while H type rations strictly less capacity and charges lower price in advance than what he would in full-information setting, i.e., $S_H^{AI,OR} < S_H^{FI,OR}$ and $p_{1H}^{AI,OR} = p_1^H(S_H^{AI,OR}) \leq p_{1H}^{FI,OR}$.

(iii) In a separating equilibrium, H type rations strictly less capacity in advance than L type does, i.e., $S_H^{AI,OR} < S_L^{AI,OR}$.

Theorem 9 (i) specifies the condition under which a high-quality seller can use advance selling to signal quality. If either type finds advance selling not desirable under full-information setting, then with asymmetric information, he would not voluntarily engage in a self-revealing equilibrium involving advance selling either, due to the cost associated with information revelation.

Theorem 9 (ii) highlights the impact of asymmetric quality information on sellers' advance-selling strategy. Information asymmetry only affects the behavior of the high-type seller. The low-type seller will always act as if full information about quality was released to customers. For the high-type seller, to differentiate himself from the low-type, he distorts his strategy and essentially burns money to the extent which the low-type seller cannot afford.

Although distortion in strategy is typical in a signaling equilibrium, the most intriguing result is usually the direction of such a distortion. Reducing, instead of raising, capacity ration in advance can effectively distinguish H type from L type because it costs H type less to reduce rationing than it does L type, while for raising the advance ration, the conclusion is the opposite.[13] That is because, by reducing the advance ration, a high-type seller can make a higher profit from selling those units of capacity in spot than a low-type seller, due to the advantage of fully-revealed high quality.

In the meantime, note that by Theorem 9 (ii), H type's advance price is also distorted and can be lower than the full-information price. Interestingly, such a price distortion is not for the purpose of signalling quality, since the equilibrium price equals to the full-information advance price at the equilibrium rationing level. In other words, the seller would have charged the same advance price if he rationed $S_H^{AI,OR}$ in a full-information setting and hence no advance margin is lost because of information asymmetry. Instead, the price is distorted downwards from the full-information level solely because of the distortion in capacity rationing and customers' strategic response to it. Essentially, as the capacity ration decreases in advance, customers expect that more capacity is available in spot and resultantly, spot price will drop. Hence customers lower their reservation price in advance accordingly, which drives the advance price down.

The discussion above leads us to the conclusion that, although pricing may convey quality in other models (e.g., Bagwell and Riordan, 1991; Stock and Balachander, 2005), it does not serve as a quality signal in our model and the high-type seller signals quality solely by capacity rationing. Intuitively, this is because customers are homogeneous in advance and for given rationing, changing advance price within the acceptable range (i.e., $[p_1^L(S), p_1^H(S)]$) only influences the seller's advance profit. In such a case, distorting advance price affects two types of sellers' payoff in a same rate and hence pricing does not have any differentiating power.

The last part of Theorem 9, point (iii), is the key result of the paper. It formally proves the role

[13] By Lemma 8, for given p_1, S, and Δ_s,

$$\pi_H^{AI}(p_1, S) - \pi_H^{AI}(p_1, S - \Delta_S) \leq \pi_L^{AI}(p_1, S) - \pi_L^{AI}(p_1, S - \Delta_S),$$

and

$$\pi_H^{AI}(p_1, S) - \pi_H^{AI}(p_1, S + \Delta_S) \geq \pi_L^{AI}(p_1, S) - \pi_L^{AI}(p_1, S + \Delta_S).$$

of capacity rationing as a signal of quality. A high-quality seller imposes smaller ration for capacity in advance than what a low-quality seller does, to signal the high quality. Such a strategy is both cost-efficient and consistent with customers' rational expectation. Customers, from observing a low ration in advance, infer that the seller is very confident in his product's quality and reserves a lot of capacity to sell in spot, and hence are convinced of high quality.

Recalling the rationing examples (premium French wines, popular electronic devices) that we discussed at the beginning of the paper, Theorem 9 (iii) indicates that, instead of purely promoting hype about the new products, signalling high quality can be the true economic motivation behind the sellers' practice of limiting supply in advance market.

Pooling Equilibrium

We define a pooling equilibrium as an equilibrium in which either both types of sellers only sell in spot, or both types sell in advance and use the same advance price and ration. We show that Intuitive Criterion eliminates all the pooling equilibria where both types offer advance selling.

Theorem 10 *By Intuitive Criterion, a pooling equilibrium in which both types offer advance selling can never be sustained.*

Figure 3.4 illustrates the proof of Theorem 10. Suppose a pooling equilibrium (p_1^E, S^E) exists where $S^E > 0$. By the single-crossing property, there always exists a point $(p_1^H(S'), S')$ which satisfies $S' < S^E$ and lies in the region between the two indifference curves crossing (p_1^E, S^E). Compared to the equilibrium point (p_1^E, S^E), $(p_1^H(S'), S')$ makes H type strictly better off and L strictly worse off. By Intuitive Criterion, customers would believe that the seller was H type if a deviation from (p_1^E, S^E) to $(p_1^H(S'), S')$ was observed and thus would accept $(p_1^H(S'), S')$, which in turn supports H type's unilateral deviation and breaks down the hypothetical pooling equilibrium.

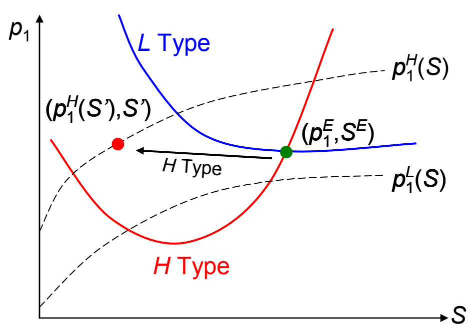


Figure 3.4. Collapse of a pooling equilibrium

Theorem 10 further strengthens the efficacy of rationing signal proved in Theorem 9. If rationing is feasible and advance selling is desirable, H type would never voluntarily pool with L type in advance period since he can instantaneously gain from revealing himself to customers by unilaterally lowering the rationing in advance.

Immediately from Theorem 9 and 10, only a separating equilibrium can be sustained when $S_L^{FI,OR} > 0$. To complete the picture, in Theorem 11, we characterize the equilibrium outcome when $S_L^{FI,OR} = 0$. Not surprising, in such a case, neither type sells in advance.

Theorem 11 *A pooling equilibrium in which neither type offers advance selling can be sustained if and only if* $S_L^{FI,OR} = 0$.

Collecting the results from Theorem 9 to Theorem 11, Theorem 12 (illustrated in Figure 3.5) summarizes the equilibrium strategies as a function of the seller's capacity and marginal cost.

Theorem 12 *When capacity rationing is optional, the equilibrium strategies are as follows:*

- *If* $c \leq c_L^{OR}(T)$ *and* $T > T_1$*, only a separating equilibrium can be sustained in which* H *type's and* L *type's pricing and rationing strategies are* $(p_1^H(S_H^{AI,OR}), S_H^{AI,OR})$ *and* $(p_{1L}^{FI,OR}, S_L^{FI,OR})$*, respectively, and* $0 \leq S_H^{AI,OR} < S_L^{FI,OR} \leq \min(T, N_1)$.
- *Otherwise, only a pooling equilibrium can be sustained in which neither types of sellers offers advance selling.*

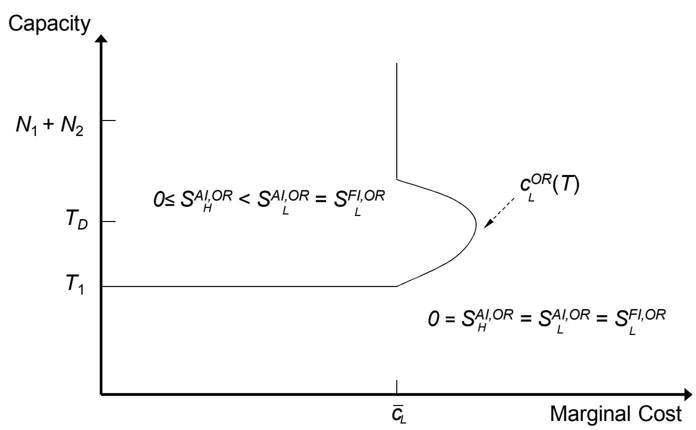


Figure 3.5. Summary of equilibrium with optional capacity rationing

Similar to the full-information scenario, advance selling is advantageous to a seller only if the seller's marginal cost is not too high and total capacity is not too small. Asymmetric quality information does not affect the strategy or profit of those sellers offering low-quality products. Nevertheless, for sellers offering high-quality products, asymmetric information on quality adversely affects the profit that the sellers can gain from advance period and makes advance selling less desirable. Specifically, comparing Figure 3.2 with Figure 3.5, the region under which H type offers advance selling shrinks. That is because, with customers' uncertainty about quality, a high-quality seller needs to sacrifice a portion of his profit to signal himself in advance, which sometimes makes advance selling less favorable than selling only in spot. Note that, due to the asymmetric information on quality, a high-quality seller's advance ration is always less than or equal to a low-quality seller's, which is in stark contrast with the result in full-information scenario.

3.4.3 Value of Rationing Flexibility

Section 3.4.2 proves the efficacy of capacity rationing as a signal of quality. Several natural subsequent questions are: how much can the seller benefit from capacity rationing? In particular, if the seller follows a naive no-rationing strategy and at any point of time, always serves as many customers as his total capacity allows, how much does he lose compared to the optional-rationing case? Can he even gain from no rationing? In other words, can the seller ever get worse off by the rationing flexibility? We study these questions in this subsection.

Under full information, the question about value of rationing flexibility is trivial. An option to ration capacity never hurts the seller since it enlarges the seller's decision space and hence never lowers his total profit. Furthermore, rationing has the potential to increase seller's total profit by effectively raising prices in both periods, as well as limiting the sales in advance to avoid huge loss of revenue resulting from the discounted advance price.

The benefit of capacity rationing, however, becomes less evident when quality information is asymmetric. As we have seen in section 3.4.2, whenever rationing is feasible, a high-type seller always has an incentive to self-reveal and by doing so, inevitably incurs some signaling cost. The signaling cost could make the seller worse off compared to the case when he can pool with a low-type seller, which may occur only if rationing is not feasible.

To confirm this intuition, we examine the equilibrium outcome for the case when capacity rationing is not feasible. Since the seller does not ration capacity, customers anticipate that as long as the seller offers advance selling, he accepts pre-orders up to his total capacity (in other words, capacity available in advance is always $\min(T, N_1)$). Therefore, two types of sellers never differ in the level of capacity available in advance and can no longer signal quality through capacity decision.

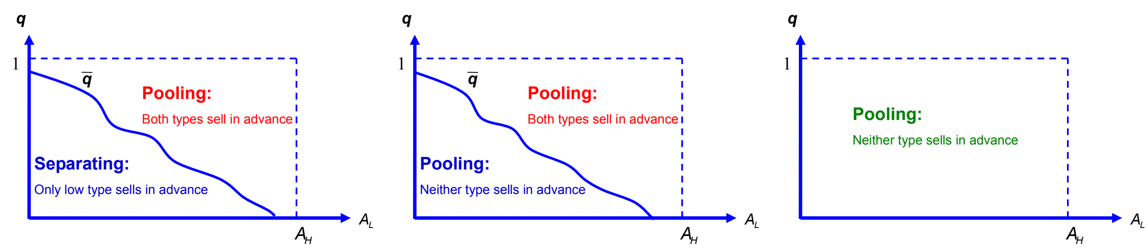
In such a case, the only possible signal of quality is advance price. As we discussed in the last section, pricing cannot effectively signal quality, since distorting advance price affects both types' total profit at a same rate and a low-type seller always has an incentive to mimic any high price. As a result, a high-type seller can never fully differentiate himself by pricing alone. Theorem 13 (illustrated in Figure 3.6) characterizes a unique focal equilibrium (i.e., pareto dominant from the seller's point of view).

In preparation, define

$$p_1^E = qp_1^H(\min(T, N_1)) + (1-q)p_1^L(\min(T, N_1)),$$
$$\bar{q} = 1 - \frac{\pi_H^{FI}(\min(T, N_1)) - \pi_H^{FI}(0)}{\min(T, N_1)\left[p_1^H(\min(T, N_1)) - p_1^L(\min(T, N_1))\right]}.$$

Theorem 13 *When rationing is not feasible, there exists a unique focal equilibrium (i.e., pareto dominant from viewpoint of the seller). Specifically, there exists two functions of T, $c_L^{NR}(T)$ and $c_H^{NR}(T)$, such that $c_L^{NR}(T) \leq c_H^{NR}(T)$ and the focal equilibrium is as follows:*

- *For $c \leq c_L^{NR}(T)$, if $q \geq \bar{q}$, a pooling equilibrium in which both types offer full advance selling at p_1^E; otherwise, a separating equilibrium in which only L type offers full advance selling at $p_1^L(\min(T, N_1))$.*
- *For $c_L^{NR}(T) < c \leq c_H^{NR}(T)$, if $q \geq \bar{q}$, a pooling equilibrium in which both types offer full advance selling at p_1^E; otherwise, a pooling equilibrium in which neither type sells in advance.*
- *For $c \geq c_H^{NR}(T)$, a pooling equilibrium in which neither type sells in advance.*



if $c \leq c_L^{NR}(T)$ if $c_L^{NR}(T) < c \leq c_H^{NR}(T)$; if $c > c_H^{NR}(T)$.

Figure 3.6. Illustration of equilibrium outcome for no-rationing model

By Theorem 13, if rationing is not feasible, a low-type seller offers advance selling as long as the marginal cost is not too high. Because his low quality will be fully revealed in spot, she has a strong incentive to get rid of a lot of capacity in advance. On the other hand, a high-type seller chooses to sell in advance only if customers' prior belief for high quality is sufficiently high. In such cases, the prior belief supports a high pooling price in advance, which makes pooling in advance more profitable than selling only in spot for both types of sellers.

Theorem 13 further implies that, both types of sellers may benefit from no-rationing. Owing to H type's inability to signal by advance price, when the prior belief is sufficiently high and the marginal cost is sufficiently low, L type can successfully hide her inferior quality by pooling with H type and enjoy a higher advance margin than what she can under optional rationing. At the same time, H type may also prefer no rationing, since when customers are sufficiently optimistic about the product's quality (i.e., when q is large), pooling with L type can secure him a reasonably high advance price, which makes burning money to fully differentiate himself less profitable.

In fact, there indeed exist scenarios where both types of sellers get strictly worse off by the option of rationing. Figure 3.7 illustrates such an example. Each point in the figure corresponds to a strategy pair (p_1, S) and numbers in the brackets represent two types of sellers' total profits if they apply the strategy pair and customers buy in advance., i.e., $\left(\pi_L^{AI}(p_1, S), \pi_H^{AI}(p_1, S)\right)$. The curves in different colors are iso-profit curves for two types.

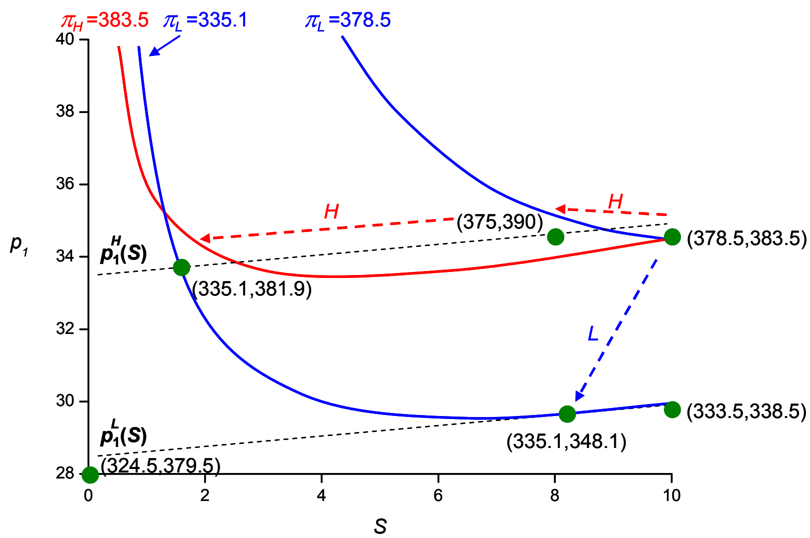


Figure 3.7. An example where both types are strictly worse off by optional capacity rationing: $q = 0.9, N_1 = 10, N_2 = 10, \alpha \sim \text{Uniform}[-5, 5], T = 11, c = 0, A_H = 35, A_L = 30$.

In the example illustrated in Figure 3.7, when capacity rationing is not allowed, the unique focal equilibrium is $(p_1 = 34.45, S = 10)$, i.e., both types of sellers offer full advance selling at the advance price 34.45. It is focal as it achieves a higher profit, (378.5,383.5), for both types of sellers than in any other equilibrium: (333.5,379.5) in the separating equilibrium where only L type offers full advance selling, and (324.5,379.5) in the pooling equilibrium where neither type offers advance selling. Without rationing flexibility, neither type has incentive to deviate from the focal equilibrium, and the cooperative outcome is attained.

Nevertheless, $(p_1 = 34.45, S = 10)$ cannot be sustained as an equilibrium when the seller has the option to ration capacity. As the strategy space for seller's advance rationing expands, H type always has an incentive to deviate to smaller ration and higher price, e.g., $(p_1 = 34.6875, S = 8)$ resulting in profit pair (375,390), which, compared to $(p_1 = 34.45, S = 10)$, makes H type better off and L type worse off. By Intuitive Criterion, such a unilateral deviation is supported by customers' posterior belief and distinguishes a high-type seller. Consequently, the pooling equilibrium $(p_1 = 34.45, S = 10)$ is overturned.

In such a case, where is the new equilibrium under optional rationing? $(p_1 = 34.6875, S = 8)$ itself cannot be sustained as H type's strategy in a *separating* equilibrium either, since L type is supposed to play her full-information strategy ($p_1 = 29.75, S = 8.39$) in any separating equilibrium, but she can get strictly better off by mimicking H type and deviating to ($p_1 = 34.6875, S = 8$). To eliminate any mimicking incentive for L type, H type keeps cutting price and lowering advance ration, until L type is indifferent between her full-information strategy and H type's equilibrium strategy. The evolution hence eventually stops at ($p_1 = 33.69777, S = 1.62$), where both types end up with strictly lower profits than those in the no-rationing focal equilibrium ($p_1 = 34.45, S = 10$).

In short, both types of sellers get worse off by rationing flexibility, since for the high type, the incentive to fully reveal himself leads to an increase in the signalling cost, which outweighs the benefit of rationing, and for the low type, rationing eliminates any possibility of pooling with high type in advance and the resultant lost in profit dominates the value of rationing.

It may seem quite counterintuitive that rationing flexibility can hurt a seller. One may ask, if both types of sellers foresee the final lose-lose outcome, why wouldn't they stick to the no-rationing equilibrium, even if they have the flexibility to ration? The answer is that, similar to the situation in the prisoners' dilemma, the no-rationing equilibrium, while achieving the pareto-dominant outcome, is not self-enforcing once rationing becomes feasible. H type, induced by the short-term increase of profit, can never resist the temptation to unilaterally reduce the advance ration. This irresistible incentive of rationing, however, just like a curse, activates a series of strategic interactions and leads to the inevitable lose-lose outcome.

The curse of rationing illustrated in Figure 3.7 usually occurs when the prior belief of high quality is large enough. It is worthy noting that the rationing flexibility does not always hurt the seller and when the prior belief is small enough, we can find examples under which at least one or both types of sellers can get better off by rationing. In fact, we can show that keeping everything else the same, the value of rationing flexibility decreases in the prior belief q. Intuitively, that is because as q increases, customers are more optimistic about high quality and are willing to pay a higher price at the pooling no-rationing equilibrium. On the other hand, by Theorem 12, the optional-rationing equilibrium is independent of the prior belief q. Hence, as q increases, the increase in the pooling price makes the no-rationing equilibrium more appealing and the rationing option less desirable.

As a brief summary, while capacity rationing can be an effective signal of quality, it can also be very costly and make both types of sellers worse off compared to the no-rationing case. As the prior belief of high quality decreases, optional rationing becomes more desirable for the seller and both types of sellers can get better off from the rationing flexibility.

3.5 Extensions and Discussions

3.5.1 Signalling: Rationing Versus Advertising

In marketing literature and practice, advertising has been widely recognized and used as a signal of quality. So far we have shown that capacity rationing, an operational decision, also conveys quality. In this subsection we extend our basic model to compare the efficiency of these two signals of quality. Specifically, if both signals are available, which signal(s), rationing or advertising or both, are used by the high-quality seller in equilibrium.

To focus on the signaling effect of advertising, we consider uninformative advertising, i.e., advertising does not affect customers' awareness of the product or market size; it is a pure dissipative cost for the seller. Such an assumption has been made in many marketing literature studying the signal of advertising (e.g., Kihlstrom and Riordan 1984, Milgrom and Roberts 1986). Should customers be perfectly informed about quality in advance, such uninformative advertising clearly would not be used by any seller, as it is a pure cost without any gain. However, with asymmetric quality information, uninformative advertising may be used as a signal of quality and may benefit the seller: if advertising can convince customers of the high quality, customers are willing to pay a high price or/and more customers are willing to buy.

In the extended model, the sequence of events are exactly the same as in the basic model, except that besides advance price and ration, the seller also needs to decide on advertising expenditure in advance, and that customers update their belief of quality based all of the three decisions made by the seller.

We also re-define the seller's profit function. Denote the amount of advertising expenditure by Q. If a type-t seller chooses (p_1, S, Q) and customers buy in advance, the seller's total profit is

$$\pi_t^{AI,AD}(p_1, S, Q) = (p_1 - c)S + \pi_2^{*t}(S) - Q$$

The equilibrium definitions are similar to those in the basic model: a separating equilibrium is one in which either only one type offers advance selling, or both types sell in advance but choose different advance price, or advance ration, or advertising expenditure, and in contrast, a pooling equilibrium is one in which either neither type sells in advance, or both types sell in advance and choose exactly the same price, rationing, and advertising expenditure.

We first note that similarly to Theorem 10, any pooling equilibrium in which both types offer advance selling can be eliminated by Intuitive Criterion. Hence, we focus on the characterization of a separating equilibrium. In any separating equilibrium, it is easy to show that L-type stays at her full-information strategy $\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right)$ since she is always perfectly identified in a separating equilibrium and hence cannot benefit from any derivation. For H-type, similar to the basic model and per definition of a separating equilibrium, his equilibrium strategy profile (p_1, S, Q) is a solution to the following problem:

$$\max_{p_1,S,Q} \pi_H^{AI,AD}(p_1, S, Q) = (p_1 - c)S + \pi_2^{*H}(S) - Q$$

$$\text{subject to} \qquad p_1 \leq p_1^H(S) \tag{3.11}$$

$$\pi_L^{AI,AD}(p_1, S, Q) \leq \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \tag{3.12}$$

$$\pi_H^{AI,AD}(p_1, S, Q) \geq \pi_H^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \tag{3.13}$$

$$0 \leq S \leq \min(T, N_1),\ Q \geq 0 \tag{3.14}$$

$$(p_1, S, Q) \neq \left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \tag{3.15}$$

A separating equilibrium is sustained if and only if there exists a solution (p_1, S, Q) to H-type's problem defined above.

Although advertising has been shown in existing literature to serve as a signal of quality (e.g., Kihlstrom and Riordan 1984, Milgrom and Roberts 1986), we find that in our model, when capacity rationing is also available as a signal, the seller never uses the advertising signal. We formally prove this result in Theorem 14.

Theorem 14 *In any separating equilibrium, neither type of seller invests on advertising, i.e.* $Q_H = Q_L = 0$. *Quality is conveyed through capacity rationing.*

Theorem 14 implies that in our model, signaling by rationing is more efficient than by advertising. Compared to uninformative advertising, which is a pure cost for the seller, rationing can positively contribute to the seller's profit by raising prices in both periods. Consequently, sending a signal by rationing costs a high-type seller less than doing so by advertising and is a better choice for the seller.

3.5.2 Costly Quality

So far we assume that sellers with different quality incur the same marginal cost. In this subsection, we generalize the basic model by considering costly quality, i.e., high-quality seller incurs a higher marginal cost than low-quality seller does. Denote the high-quality and low-quality sellers' marginal cost by c_H and c_L, respectively. The cost of quality is thus equal to $c_H - c_L > 0$. We further assume that the increase in customers' utility from higher quality is sufficient to cover the cost of quality, i.e., $A_H - A_L > c_H - c_L$. That is, if both priced at marginal cost, high-quality product is preferred by customers to low-quality. Same as in the basic model, we restrict $c_H < \bar{p}_H$.

As shown in the following Theorem 15, the cost of quality does not affect the efficacy of rationing as a signal of quality. In fact, the high type's incentive to separate by rationing is reinforced by the costly quality, since at a given advance price, high type's advance margin is less than low type's due to the higher marginal cost, which makes pooling in advance even less preferable for the high type. Hence, the single-crossing property still holds, which implies the same characteristics of the separating and pooling equilibria as those in the basic model.

Furthermore, advance price again fails to convey quality, even with costly quality. While two types of sellers no long share the same advance profit function due to the difference in marginal cost, distorting price still affects two types' advance profits at exactly the same rate, since two types of sellers face the same advance demand: at a given price, either all or none of the advance customers buy.

Theorem 15 *When two types' marginal costs satisfy* $0 < c_H - c_L < A_H - A_L$*, Lemma 7, Lemma 8, Theorem 9, and Theorem 10 still hold.*

3.6 Conclusion

Advance selling has been widely applied in business practice and recently received much attention in academic research. To the best of our knowledge, this paper is the first one to study asymmetric information on quality in advance selling.

We study a two-period model of advance selling where quality information is asymmetric in advance period and publicly revealed in spot. A seller has the option of selling in both periods at different prices and using advance selling to signal his quality to forward-looking customers.

We show that rationing capacity in advance, if feasible, is an effective signal of quality. A high-quality seller always has an incentive to differentiate himself by rationing less to advance than a low-quality seller does. The efficacy of rationing as a quality signal originates from the fact that at a given advance price, a high-quality seller prefers reserving more capacity to spot, when his high quality is fully revealed, while a low-quality seller would rather get rid of more capacity in advance, when she may hide her low quality.

Our finding on rationing capacity to signal quality is consistent with several examples in practice. One of such examples is observed in the premium French wine's advance (en primeur) market, where chateaux intentionally limit the availability of wine sold in en primeur to convey high quality.

In our model, capacity rationing, as a signal of quality, is found to be more cost-efficient than other marketing signals like pricing or advertising. The efficacy of conveying quality through capacity rationing is proved for both costless and costly quality. Nevertheless, compared to the case when rationing is not feasible, rationing capacity may hurt both high-quality and low-quality sellers since both of these two types of sellers may sometimes prefer pooling in advance, which is not a self-enforcing equilibrium with rationing flexibility.

Regarding the impact of asymmetric quality information, a seller offering high-quality product always gets worse off, since his product cannot get fully appreciated in advance due to customer uncertainty about quality. If a high-quality seller wants to differentiate himself, he has to distort his selling strategy from the full-information strategy and incur some signalling cost. As a result, asymmetric information on quality makes selling in advance less desirable for a high-quality seller. In contrast, a seller offering low-quality product is never worse off, and can even get benefit from the asymmetric quality information when she can pool with the high-quality seller in advance and enjoy a higher margin than what she deserves.

CHAPTER 4

Demand Shaping and Product Overselling to Better Match Supply and Demand for Assemble-to-Order Firms

4.1 Introduction

In Assemble-to-Order (ATO) processes, components are acquired (or produced) to stock, while the assembly of final products is delayed until detailed product specifications are available (Wemmerlov 1984). ATO is particularly desirable when the time to assemble products is negligible compared to the substantial lead-time needed to replenish component inventory (Song and Zipkin 2004). By postponing final assembly and pooling component inventory, ATO is used to mass customize products. In recent years, it has been widely adopted by many manufacturers in computers (e.g., Dell Computer), automobiles (e.g., Toyota "Build Your Toyota"), jewelry (e.g., "Build Your Own Ring" at amazon.com or bluenile.com), bags (e.g., "Build Your Own Bag" at timbuk2.com), and shoes (e.g., "Design Your Shoes" at nikeid.nike.com or converse.com) industries.

Despite its increasing popularity in practice, ATO processes are "notoriously difficult to analyze and manage" (Benjaafar and ElHafsi, 2006). This is mainly due to the analytical difficulty rooted in the nature of an ATO system: it is essentially a special case of multi-product multi-resource system in which products' demand can be inter-correlated and common components are shared among products. Thus, the products and components need to be managed jointly. Moreover, due to the relatively long lead time in replenishing component inventory, ATO systems are often operated under hard constraints on component supply and the major challenge is to match limited inventory of components with stochastic demand of final products.

To address such a challenge, the existing literature on ATO systems has been largely focusing on supply management (particularly, inventory replenishment policies) with exogenous demand profiles (ref. a comprehensive review of this stream of literature in Song and Zipkin (2004) and the references within). Our work takes a different perspective and focuses on demand management in ATO systems. Specifically, we consider *the joint pricing and order fulfillment decisions* for an ATO firm offering multiple substitutable products with a given pool of component inventory.

To optimize the joint decisions, we focus on both demand shaping and product overselling strategies. Specifically, due to the price-dependent nature of product demand, we assume that the firm can shape demand by adjusting product prices according to component availability. Meanwhile, for order management policy, we consider the firm with the flexibility to dynamically control order acceptance and fulfillment as product orders sequentially arrive. In particular, we assume that the firm can, at its discretion, cancel (or delay the fulfillment of) some accepted lower-margin orders to satisfy higher-margin ones which arrive at a later time. That is, the firm can initially oversell products (i.e., to accept more orders than what it can fulfill) and then decide on the actual order fulfillment as the entire demand process unfolds.

Product overselling has the potential to benefit the firm by reducing the losses from both high yield spill (losing high-margin sales due to fulfilling low-margin orders earlier) and inventory spoilage (waiting in vain for high-margin orders and losing the opportunity to fulfill low-margin ones). Various forms of product overselling (aka. callable products) have been used in a number of different industries including airlines and media advertising, or even in a business-to-business setting (Gallego et al. 2008). In the recent years, product overselling has also received attention from the academic research (e.g., Biyalogorsky et al., 1999 and Gallego et al. 2008). However, the existing work has been limited to a single product with exogenous prices. Very little insight has been drawn for either multiple products or joint decisions of overselling and pricing.

When the overselling and pricing decisions are jointly considered for multiple products, many questions naturally arise: when to oversell? how much to oversell? with endogenous pricing, how to make these decisions based on products' demand characteristics? how is the optimal pricing decision different when overselling is involved? how much value can demand shaping and overselling contribute to the firm, either individually or jointly? what is the impact of overselling strategy on consumer and social surplus?

We study these questions in this paper. Our work contributes to the literature by incorporating the overselling strategy into firms' joint pricing and order-acceptance decisions on substitutable products. In addition to ATO firms, our model can also be applied to service industries such as airlines (selling itineraries) and media (selling advertising time slots). The common features of these problems are first, the firms offer multiple substitutable products which share limited resources, and second, the order fulfillment (production or allocation) can be decided or adjusted after demand is revealed.

We find that, when combined with demand shaping, product overselling can effectively enhance the firm's profitability, especially when demand variability is high. In terms of the optimal selling policy, although the dynamic order acceptance/fulfillment problem can be very hard to solve, we prove that a special class of policies, partial postponement in order fulfillment, is asymptotically optimal. Under such policy, we further characterize the optimal overselling and pricing strategies for firms with different flexibilities in pricing and overselling. We also show how the optimal decisions and values of the flexibilities depend on the key operational parameters, including component inventory level and cancellation compensation to customer. Finally, we find that, in addition to enhancing the firm's profit, overselling may also improve the consumer and social surpluses.

4.2 Literature Review

Our paper considers the joint pricing and overselling decisions for assemble-to-order firms offering multiple substitutable products. Hence, it is closely related to five streams of literature: overselling with opportunistic cancellations, multi-product optimization, assemble-to-order systems, production postponement, and demand substitution.

Overselling with opportunistic cancellations

The concept of overselling with opportunistic cancellations was first discussed by Biyalogorsky et.al (1999, 2000) in the context of airlines. They characterized a firm's optimal order-acceptance policy in a two-period model where prices are exogenous and the sale to some low-paying customers may be cancelled by the firm to satisfy high-paying customers' demand. They show that, different from the traditional overbooking concept which is driven by uncertainty in customer no-shows, such overselling can be profitable even if all customers show up. Modarres and Bolandifar (2008) extend the model in Biyalogorsky et.al (1999, 2000) to a multi-period setting with dynamic pricing, but as opposed to Biyalogorsky et.al, Modarres and Bolandifar (2008) consider exogenous order-acceptance policy. Biyalogorsky and Gerstner (2004) and Gallego et al. (2008) further developed the idea by endogenizing firms' decision on the cancellation compensation and optimizing the design of contingent pricing.

Different from the above papers, we focus on joint optimization of pricing and order-acceptance decisions for given cancellation compensation. Instead of considering a single product with two pre-prioritized demand classes, we examine arbitrary number of partially-substitutable products with endogenized ordering.

Multi-product optimization

Compared to single-product literature, relatively limited research exists on joint management of multiple products, with correlation in product demand or joint resource constraints. Early multi-product research focuses on resource constraints. The optimized decisions are resource allocation (e.g., newsstand problems in Lau and Lau 1995, 1996) or resource investment (e.g., Harrison and Van Mieghem 1999). Pricing and demand are assumed to be exogenous.

Until recently, the multi-product pricing has received little attention in the literature. Recently, under various assumptions on product demands, optimal pricing strategies are characterized. For example, Birge et al. (1998) examine how to determine the optimal prices or capacity levels for two products. Aydin and Porteus (2008), Zhu and Thonemann (2009), Wang and Kapuscinski (2009), Song and Xue (2007) consider joint optimization of pricing and inventory decisions for multiple products. Krausa and Yano (2003) and Hopp and Xu (2005) address the problem of choosing both pricing and product line (assortment). Maddah and Bish (2007) study joint pricing, inventory, and assortment. Most of the work in this stream, however, do not consider resource constraints across products.

Our paper examines multi-product pricing under resource constraints. The existing work in this area, including Gallego and van Ryzin (1997) and Cooper (2001), focus on dynamic pricing strategy and prove the asymptotic optimality of some heuristics. Maglaras and Meissner (2006) show that the dimensionality of the dynamic-pricing problem can be greatly reduced if the firm's control is on resource-consumption rate rather than pricing or demand rate. Tang and Yin (2007) characterize the optimal pricing and quantity decisions for two products which share a common resource and have deterministic demand. Within the context of assemble-to order systems, Bertsimas and de Boer (2002) consider joint decisions on pricing and quantity for multiple products in a stochastic environment, but ignored the cross-price elasticity of product demand. Song and Xue (2008) analyze a multi-period joint pricing, quantity, and bundling problem for multiple products, and assume all unmet demands are backlogged.

Our paper contributes to this stream of literature by being the first to analyze the multi-product joint pricing and quantity decision problem in which resources are constrained and product demands are stochastic, price-based substitutable, and lost in case of stock-out. More importantly, we incorporate the overselling strategy into firms' decisions and show that overselling has the potential to effectively enhance the firm's profitability via improving the match between products' demand and component inventory.

Assemble-to-order systems

As we noted in the introduction section, a large portion of the existing work on ATO systems is dedicated to decisions on ex-ante inventory replenishment and/or ex-post component allocation (e.g., Gerchak and Henig 1986, 1989, Zhang 1997, Hausman et al. 1998, Agrawal and Cohen 2001, Akcay and Xu 2004, see a comprehensive survey in Song and Zipkin 2004). The two papers reviewed earlier, Bertsimas and de Boer (2002) and Song and Xue (2008), consider joint pricing and component allocation decisions, but they both assume that the component allocation is decided ex ante, i.e., before demand arrives. All of these papers, by assuming either ex-ante or ex-post allocation of components, take a simplified view on firms' order-acceptance process. Specifically, ex-ante allocation assigns dedicated component supply to each product and precludes the possibility of inventory pooling, while ex-post allocation presumes that regardless of the component availability, the firm never rejects any orders during the selling process.

A few papers allow ATO firms to dynamically control order acceptance/rejection (e.g., Balakrishnan et al. 1996, Defregger and Kuhn 2007, Benjaafar and ElHafsi 2006). In particular, firms can block some orders of low-margin products to reserve components for orders of high-margin products. However, all of the papers in this stream of literature assume that the firm never accepts more orders than what can be fulfilled with its on-hand inventory and precludes the possibility of overselling. Besides, none of them examined joint decisions on rationing and pricing, which is the focus of our paper. Specifically, we show that the joint optimization of demand shaping and product overselling strategies can bring substantial benefit to assemble-to-order firms.

Production postponement

ATO systems allow for production postponement (aka. delayed differentiation). The strategy and value of postponement have been analyzed in both monopolistic and competitive models (e.g., Anand and Mendelson 1998, Van Mieghem and Dada 1999, Anupindi and Jiang 2008). Our paper adds to this stream of literature by considering cross-price demand elasticity and product overselling strategy in a multi-product setting.

Demand substitution

The demand substitution modelled in our paper is customer-driven price-based (static) substitution. In literature, many other kinds of demand substitution have been studied, including firm-driven substitution (e.g., Bassok et al. 1999, Hale et al. 2001), customer-driven stockout-based (dynamic) substitution on final products (e.g., Parlar 1988, Lippman and McCardle 1997, Mahajan and Van Ryzin 2001, Netessine and Rudi 2003), and customer-driven stockout-based substitution on components (Iravani et al., 2003). Among all these, price-based substitution best fits the purposes of our paper.

The remainder of this paper is organized as follows. In section 4.3, we present the model formulation, define the partial-postponed order fulfillment policy, and prove its asymptotical optimality. To further characterize the policy and evaluate the benefits, we define four strategies for firms with different flexibilities in pricing and overselling. In section 4.4, we focus on the optimal pricing and order-acceptance policies under each of the four strategies. Subsequently, in section 4.5, we evaluate the firm's gain from pricing and operational flexibilities both individually and jointly. We also numerically test the impact of overselling strategy on consumer surplus and social surplus. We then conclude the paper in section 4.6 with a summary . All the proofs are provided in the appendix.

4.3 The Model

4.3.1 The Dynamics

We consider a continuous-time model and the formulation is similar to Cooper (2001). Consider a firm having a given inventory of components which can be used to produce multiple horizontally-differentiated products. Let the sets of components and products be $I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$, respectively. $[A_{ij}]_{m\times n}$ is the bill-of-material matrix, where A_{ij} is the number of component i required to produce one unit of product j, i.e., . Denote the vector of unit costs of components by $\mathbf{c} = (c_i)_{m\times 1}$ and the on-hand component inventories by $\mathbf{y} = (y_i)_{m\times 1}$.

We assume that the selling horizon is a finite time interval $(0, \tau]$, during which there is no replenishment of component inventory and customers' requests for the products arrive according to a stochastic process. Denote the process of customer arrival (aka., demand process) by $\{\mathbf{d}(t, \mathbf{p}) = (d_j(t, \mathbf{p}))_{n\times 1} : t \in (0, \tau]\}$, where $d_j(t, \mathbf{p})$ is the number of orders for product j in $(0, t]$ and $\mathbf{p} = (p_j)_{n\times 1}$ is the price vector. Note that one product's demand process depends on not only its own price, but also all the other products' prices.

We assume that each customer requests for a single product and that given any price vector, almost surely at most one customer arrives at any time point and the expected total number of orders for each product arriving during $(0, \tau]$ is finite. Note that we do not impose any additional assumption on the demand process: it does not have to be a Poisson process or a Markov process. This highlights the generality of our asymptotic results. The sequence of events is illustrated in Figure 4.1.

The firm announces the prices $\mathbf{p}$ upfront and keeps the prices fixed during the selling horizon. In practice the firm may dynamically adjust the prices over a long period of time, but it is reasonable to assume that the prices remain unchanged for a short planning horizon. Besides, the fixed-price assumption enhances the tractability of the dynamic order-management problem, which is the focus of our model. Such a setting has also been adopted by numerous authors, including Lee and Hersh (1993), Cooper (2001), and Maglaras and Meissner (2006), when studying similar problems.

During the selling horizon $(0, \tau]$, whenever an order arrives, the firm decides on whether to accept or reject the order. In the meanwhile, at any time t, the firm can choose to cancel some of the previously-accepted orders. It is profitable to do so if, say, based on the knowledge acquired from the realized demand process, the firm expects a large amount of higher-margin orders arriving soon, which cannot be satisfied with the remaining unallocated components, and hence wants to release some of the components which are tied up with the lower-margin orders accepted earlier. For the orders which are first accepted and then cancelled, the firm pays the customers some monetary compensation in proportion to the corresponding profit margin. Denote the vector of compensation ratio by $\boldsymbol{\gamma} = (\gamma_j)_{n\times 1}$. We assume that $\boldsymbol{\gamma}$ is exogenously given and independent of how long an order stays in the system before it is cancelled. Such a cancellation compensation structure has been observed in practice (see, e.g., Constantin et al. (2009) for its application in television advertisement slot allocation and cancellation). The cancelled order cannot be accepted again at later time of the selling horizon.

In practice, the firm, instead of cancelling the orders, may choose to postpone the delivery of the orders or to request expedited supply of components to meet all orders. For the purpose of our model, we do not differentiate these scenarios. For each accepted but unfulfilled order, we assume that the firm loses the revenue and incurs the compensation cost.

After the selling horizon ends, the firm assembles components to fulfill those orders which have been accepted and have not been cancelled by time τ. For simplicity, we assume that any unsold component inventory has zero salvage value.

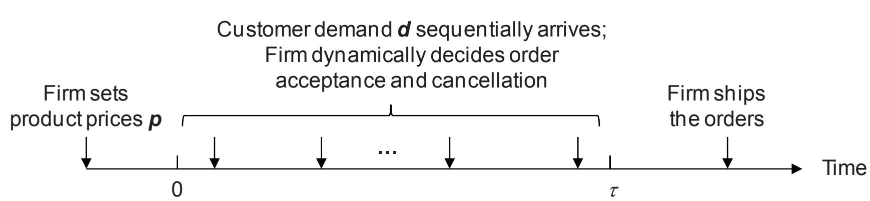


Figure 4.1. Sequence of events

The firm seeks a price vector and an order-management policy to maximize the expected total net profit, which equals to the total profit earned from the fulfilled orders subtracted by the order-cancellation compensation. Formally, let $\mathcal{P}$ and Φ represent the sets of all feasible price vectors and order-management policies, respectively. Clearly any feasible price vector needs to ensure that for each product, profit margin is nonnegative, i.e., $\mathcal{P} = \{\mathbf{p} : \mathbf{p} \geq A^T \mathbf{c}\}$, where A^T is the transpose of matrix A.

On the other hand, an order-management policy specifies the acceptance/rejection decision when an order arrives and the cancellation decision at any time point. Hence, an element $\{\phi(t), t \in (0,\tau]\}$ of the set Φ is composed of two parts, $\{\phi^1(t), t \in (0,\tau]\}$ and $\{\phi^2(t), t \in (0,\tau]\}$, where the first component dictates acceptance decision and the second cancellation decision. For each t, $\phi^1(t)$ maps the history up to t to $\{0,1\}^n$: $\phi^1_j(t)$ equals to one if an order of product j arriving at time t will be accepted, and zero otherwise. Similarly, $\phi^2(t)$ maps the history up to t to $(\mathbb{Z}_{\geq 0})^n$: the value of $\phi^2_j(t)$ represents the number of order j to be cancelled at time t.

An order-management policy $\{\phi(t), t \in (0,\tau]\}$ is feasible if it satisfies the following three conditions almost surely: first, the component requirement of the orders to be fulfilled at the end of the selling season cannot exceed the on-hand component inventory; second, up to any time t, the total number of accepted orders cannot exceed the total demand; and third, up to any time t, the total number of orders which are accepted and then cancelled cannot exceed that of the accepted orders. Formally, let $N^\phi(t)$ and $R^\phi(t)$ be two n-dimensional vectors, representing the number of accepted and cancelled orders up to time t under policy $\{\phi(t), t \in (0,\tau]\}$, respectively. By definition and for given $j \in J$,

$$N^\phi_j(t) = \int_{(0,t]} \phi^1_j(s) d_j(ds, \mathbf{p}), \quad R^\phi_j(t) = \int_{(0,t]} \phi^2_j(ds).$$

where $d_j(ds, \mathbf{p})$ equals to one if there is a product-j order arriving at time s and zero otherwise; and $\phi^2_j(ds)$ is the number of order j to be cancelled at time s. By the feasibility conditions, we have

$$\Phi = \{\{\phi(t), t \in (0,\tau]\} : A(N^\phi(\tau) - R^\phi(\tau)) \leq \mathbf{y} a.s.;\ \forall t \in (0,\tau], 0 \leq N^\phi(t) \leq \mathbf{d}(t,\mathbf{p}), 0 \leq R^\phi(t) \leq N^\phi(t)\ , a.s.\}$$

The firm's decision problem can then be formulated as follows:

$$\Pi^* = \sup_{\mathbf{p} \in \mathcal{P}, \phi \in \Phi} \{\mathrm{E}[(\mathbf{p} - A^T \mathbf{c})^T (N^\phi(\tau) - (1 + \boldsymbol{\gamma})^T R^\phi(\tau))]\} \tag{4.1}$$

Given the general demand process and the complicated nature of the problem, it is very difficult to characterize an optimal joint pricing and order-management policy: it could be non-Markovian and depend on the sample path of the demand process. In this paper, we do not try to find the optimal policy. Instead, we focus on a heuristic policy which is easily implementable and achieves asymptotically optimal performance.

4.3.2 The Policy: Partial Postponement of Order Fulfillment

For the problem described in the last subsection, we propose the following heuristic policy, which we shall refer to as *partial postponement of order fulfillment.* The policy is composed of two parts:

order acceptance and order fulfillment. The order acceptance decision is made at the beginning of the selling season. For each product, the firm determines the maximum number of orders acceptable (aka. up-to level, or protection level, or rationing level). These up-to levels are set at time zero and fixed thereafter. During the selling horizon, the firm accepts orders first come, first serve until all the up-to levels are reached or the period ends, whichever comes first. This type of order acceptance policy is known as discrete allocation policy (Cooper 2001) and also similar in spirit to the make-to-stock policy considered in Gallego and van Ryzin (1997). Cooper (2001) pointed out that this type of policy is well-known to practitioners and is appealing for its simplicity.

While the order-acceptance decision is made upfront, the order-fulfillment decision is postponed till the end of the horizon. At the time, the firm chooses which orders to fulfill (or to cancel) based on the component inventory and the accepted orders, so as to maximize its *ex-post* net profit. The sequence of events under such policy is illustrated in Figure 4.2.

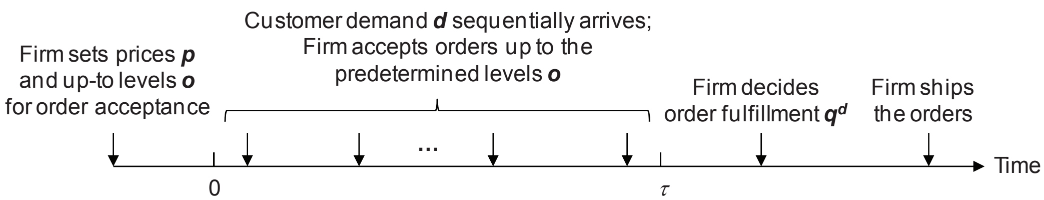


Figure 4.2. Sequence of events with partially postponed decision on order fulfillment

From the name of the proposed policy, partial postponement of order fulfillment, one can easily think of the two extreme cases of the policy. On one extreme, when there is no postponement of order fulfillment, all the accepted orders are guaranteed to be fulfilled. In such a case the firm does not need to pay any order cancellation penalty, but bears the risk of committing components to lower-margin orders too early and losing the higher-margin sales which arrives later in the selling horizon (i.e., high-yield spill). On the other extreme, when the firm fully postpones the fulfillment decision to ex post and accepts all the incoming orders, it can take advantage of the fully-revealed demand information, but may end up having to cancel a lot of orders which cannot be fulfilled with the on-hand inventory. From this point of view, the partial postponement of order fulfillment decision can potentially achieve a balance between the lost from high-yield spill and the cost of fully observing demand. Clearly, it dominates the two extreme policies, no postponement or full postponement of order fulfillment.

Next we introduce the formal formulation of the partial-postponed fulfillment policy. Let $\mathbf{o} = (o_j)_{n\times 1}$ denote the up-to level vector for order acceptance and $\mathbf{q^d} = (q_j^{\mathbf{d}})_{n\times 1}$ be the number of orders to be fulfilled at the end of horizon. Note that $\mathbf{q^d}$ depends on the realization of demand profile and hence bears a superscript $\mathbf{d}$. The proposed policy is then defined by:

$$\phi_j^1(t) = 1_{(d_j(t,\mathbf{p}) \le o_j)}, \forall t \in (0, \tau], \tag{4.2}$$

$$\phi_j^2(t) = 0, \forall t \in (0, \tau),\ \phi_j^2(\tau) = \min(o_j, d_j(\tau, \mathbf{p})) - q_j^{\mathbf{d}}. \tag{4.3}$$

where $1_{(\cdot)}$ is the indicator function. Correspondingly, we have

$$N_j^{\phi}(t) = \min(o_j, d_j(t, \mathbf{p})), \forall t \in (0, \tau],$$
$$R_j^{\phi}(t) = 0, \forall t \in (0, \tau),\ R_j^{\phi}(\tau) = \min(o_j, d_j(\tau, \mathbf{p})) - q_j^{\mathbf{d}}.$$

So far we have not specified how to determine the values of $\mathbf{o}$, $\mathbf{q^d}$, or $\mathbf{p}$. By backward induction, they can be solved by the following steps:

Ex-post: Order Fulfillment

The firm chooses $\mathbf{q^d}$ to maximize the ex-post net profit $\pi(\mathbf{p}, \mathbf{o}, \mathbf{d})$, subject to the bill-of-material constraint and the constraint that only the accepted orders can be fulfilled.

$$\begin{aligned}\pi(\mathbf{p}, \mathbf{o}, \mathbf{d}) &= \max_{\mathbf{q^d}} [\mathbf{p} - A^T\mathbf{c}]^T [N^{\phi}(\tau) - (1 + \boldsymbol{\gamma})^T R^{\phi}(\tau)] \\ &= \max_{\mathbf{q^d}} [\mathbf{p} - A^T\mathbf{c}]^T \left\{\mathbf{q^d} - \boldsymbol{\gamma}^T [\min(\mathbf{o}, \mathbf{d}(\tau, \mathbf{p})) - \mathbf{q^d}]\right\} \end{aligned} \tag{4.4}$$

$$\text{subject to} \quad A\mathbf{q^d} \le \mathbf{y},\ \ 0 \le \mathbf{q^d} \le \min(\mathbf{o}, \mathbf{d}(\tau, \mathbf{p})) \tag{4.5}$$

Ex-ante: Order Acceptance and Pricing

The firm chooses prices $\mathbf{p}$ and order-acceptance up-to levels $\mathbf{o}$ to maximize the expected net profit $\pi^{OO}(\mathbf{p}, \mathbf{o})$, subject to the constraints that the product margins and the up-to levels are nonnegative. Denote the optimal net profit by $\Pi^{ID,OO}$, where ID represents inventory-dependent pricing and OO stands for optional overselling.[1]

$$\Pi^{ID,OO} = \max_{\mathbf{p},\mathbf{o}} \pi^{OO}(\mathbf{p}, \mathbf{o}) = \mathrm{E}_{\mathbf{d}}[\pi(\mathbf{p}, \mathbf{o}, \mathbf{d})] \tag{4.6}$$

$$\text{subject to} \quad \mathbf{p} \ge A^T\mathbf{c},\ \ \mathbf{o} \ge 0 \tag{4.7}$$

It is easy to check that the policy $(\mathbf{p}, \{\phi(t), t \in (0, \tau]\})$ defined by equation (4.2) through (4.7) is feasible. Next we show that the policy as defined is asymptotically optimal.

4.3.3 Asymptotic Optimality of the Partial-Postponement Policy

In this subsection we show that the policy defined by equation (4.2) through (4.7) is asymptotically optimal when the potential demand and component inventory become proportionally large. Following Gallego and van Ryzin (1997), Cooper (2001), and Maglaras and Meissner (2006), we consider a

[1] The superscripts are chosen in preparation for the comparison later with the profits under inventory-independent pricing or no overselling.

sequence of problem indexed by integers k. For the kth problem, the initial component inventory is $k\mathbf{y}$ and the demand arrival process is $\{\mathbf{d}^{(k)}(t,\mathbf{p})\}$ with the total expected number of orders equal to $k\mathrm{E}[\mathbf{d}^{(1)}(\tau,\mathbf{p})]$. Similar to Cooper (2001), we assume that, for given price vector $\mathbf{p}$, the normalized number of total arrivals converges in distribution[2] to the original expected number or arrivals. That is,

$$\frac{\mathbf{d}^{(k)}(\tau,\mathbf{p})}{k} \xrightarrow{\mathcal{D}} \mathrm{E}[\mathbf{d}^{(1)}(\tau,\mathbf{p})] \tag{4.8}$$

Cooper (2001) shows that condition (4.8) is satisfied by a few classes of arrival processes including the commonly-used (nonhomogeneous) Poisson process. In addition to those processes, we note that, since condition (4.8) is only imposed on the total number of demand arrivals, it is also satisfied when the total demand for product j takes the following form:

$$d_j^{(k)}(\tau,\mathbf{p}) = k\lambda(\tau)z_j(\mathbf{p}) + [k\lambda(\tau)z_j(\mathbf{p})]^{\beta}\epsilon_j, j = 1, ..., n \tag{4.9}$$

In demand function (4.9), $\lambda(\tau)$ is a function of τ and represents the total potential market size for the selling horizon $(0,\tau]$; β is a constant in $[0,1)$ and indicates that demand's coefficient of variation is nonincreasing in volume; $z_j(\mathbf{p})$ is the demand rate function and reflects the ratio of the expected demand for product j to the total market size; ϵ_j is a mean-zero random variable and represents the random shock on the demand.

The demand function in equation (4.9) is known as mixed "multiplicative-additive"(Maddah and Bish 2007) and unifies several commonly used models in the literature: when $\beta = 0$, it is "additive" demand (e.g., Zhu and Thonemann 2009); when $\beta = 1/2$ and ϵ is normally distributed, it is a normal approximation of Poisson demand (e.g., van Ryzin and Mahajan 1999); $\mathbf{z}(\mathbf{p})$ is linear (e.g., Wang and Kapuscinski 2009) or multinomial logit (e.g., Maddah and Bish 2007).

The examples of demand processes and functions listed above highlight the generality of condition (4.8). Next we show that condition (4.8) alone is sufficient for the asymptotic result. In preparation, for the kth problem, let $\Pi^*_{(k)}$ and $\Pi^{ID,OO}_{(k)}$ denote the expected profits under the optimal policy and the proposed heuristic policy, respectively. The asymptotical optimality is presented in the following theorem.

Theorem 16 *Under condition (4.8),* $\lim_{k\to\infty} \Pi^{ID,OO}_{(k)}/\Pi^*_{(k)} = 1$.

Theorem 16 shows that, when the potential demand and component inventory are both very large, the dynamic control of order fulfillment can be substituted, with little lost in the total expected

[2] Recall that for a sequence of random variables $\{X_k\}$, X_k converges in distribution to a random variable X if $\lim_{k\to\infty} P_k(X_k \in S) = P(X \in S)$ for every X-continuity set S, where P_k and P are distribution of X_k and X, respectively. (Cooper, 2001)

profit, by the combination of ex-ante order acceptance and ex-post order fulfillment. This is clearly good news for the managers, as the latter is much easier to understand and implement.

Following Theorem 16, for the remainder of the paper, we will focus on characterizing the firm's optimal pricing and order-fulfillment strategy under the partially-postponed order fulfillment policy.

4.3.4 A Special Case with Two Products and One Common Component

For the sake of tractability, our subsequent analysis will focus on a special case of the problem: two products sharing one common component (i.e., $m = 1$ and $n = 2$); two products' unit component consumptions are $A_{11} = A$ and $A_{12} = 1$, respectively; the unit cost of the component is C and on-hand inventory is y; the cancellation compensation ratio is the same for two products: $\gamma_1 = \gamma_2 = \gamma$. Recall that, by the problem formulation under the partially-postponed order fulfillment policy, the optimal strategy is independent of the demand arrival process; instead, it is only a function of the total demand arriving in $(0, \tau]$. We adopt the demand function in equation (4.9) and further assume that the total demand is linear in both prices and has additive uncertainty (i.e., $\beta = 0$); total market size $\lambda(\tau)$ is normalized to unity. For notational convenience, we write the total demand only as functions of prices and the random shocks:

$$d_1(p_1, p_2, \epsilon_1) = z_1(p_1, p_2) + \epsilon_1 = a_1 - b_1 p_1 + c_1 p_2 + \epsilon_1,$$
$$d_2(p_1, p_2, \epsilon_2) = z_2(p_1, p_2) + \epsilon_2 = a_2 - b_2 p_2 + c_2 p_1 + \epsilon_2,$$

where $a_1, a_2 > 0$ represent base demands, $b_1, b_2 > 0$ denote self-price sensitivities, and $c_1, c_2 \geq 0$ stand for cross-price sensitivities. We also impose the standard diagonal-dominance assumption: $\min(b_1, b_2) > \max(c_1, c_2)$. The assumption has very intuitive interpretation and implies that the total demand of all products becomes stochastically smaller when price of any product increases and that one product's demand is more responsive to its own price than to all the other products' prices, such that if prices of all products increases by the same amount, demand of any product decreases.

The diagonal-dominance assumption ensures that the linear demand function has an inverse:

$$\begin{pmatrix} p_1(z_1, z_2) \\ p_2(z_1, z_2) \end{pmatrix} = \frac{1}{b_1 b_2 - c_1 c_2} \left[- \begin{pmatrix} b_2 & c_1 \\ c_2 & b_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} a_1 b_2 + c_1 a_2 \\ b_1 a_2 + c_2 a_1 \end{pmatrix} \right] \tag{4.10}$$

Based on equation (4.10), we can and shall use the demand rate $\mathbf{z}$, instead of $\mathbf{p}$, as the firm's decision variables on pricing. Such a change in variables facilitates the analysis and is used in many multi-product revenue management papers, e.g., Gallego and van Ryzin (1997) and Song and Xie (2007).

The linear demand function guarantees that when the on-hand component inventory is unlimited, the expected profit $[\mathbf{p}(\mathbf{z}) - A^T\mathbf{c}]^T\mathbf{z}$ is bounded, continuous and strictly concave in $\mathbf{z}$ (Song and Xie 2007). This implies that there exists a unique maximizer to the unconstrained profit function. Denote the unconstrained optimal demand rates and the corresponding prices by $\mathbf{z}^U$ and $\mathbf{p}^U$, respectively. It is natural to expect that the unconstrained optimal prices contribute positive margins, i.e., with unlimited component availability, it is profitable for the firm to offer all products. Furthermore, notice that theoretically the demand model defined in equation (4.9) allows negative demand, which is practically meaningless. Hence, we assume that at the unconstrained optimal prices, all products' demands are nonnegative almost surely, i.e., $\mathbf{p}^U > A^T\mathbf{c}$ and $\mathbf{d}(\mathbf{p}^U, \boldsymbol{\epsilon}) \geq 0$ a.s.

In addition to the assumptions on the demand rate function, we make the following regularity assumptions on the random shocks ϵ_1, ϵ_2. Assume that ϵ_1 and ϵ_2 are independently distributed. For $j = 1, 2$, let $[-L_j, H_j]$ be the support of ϵ_j, where $L_j > 0$ and $H_j > 0$. Denote the c.d.f. of ϵ_j by $G_j(\cdot)$, the p.d.f. by $g_j(\cdot)$, and the tail probability function by $\bar{G}_j(\cdot) = 1 - G_j(\cdot)$. Assume that $G_j(x)$ is twice continuously differentiable for $x \in [-L_j, H_j]$ with $g_j(x) > 0$ for $x \in (-L_j, H_j)$. Moreover, assume that $G_j(x)$ satisfies the Increasing-Failure-Rate property, i.e., $\frac{g_j(x)}{1-G_j(x)}$ is nondecreasing in x for $x \in [-L_j, H_j]$. Note that from the twice continuously differentiability of $G_j(\cdot)$, $\bar{G}_j(\cdot)$ has an inverse, denoted by $(\bar{G}_j)^{-1}(\cdot)$.

All of these assumptions are standard in literature and are satisfied by many commonly-used distributions and their truncated versions: for example, uniform, exponential, logistic, normal, extreme-value, power function with shape parameter greater than or equal to 1, Weibull with shape parameter greater than or equal to 1, beta with both shape parameters greater than or equal to 1, gamma with shape parameter greater than or equal to 1, χ with shape parameter greater than or equal to 1, and χ^2 with shape parameter greater than or equal to 2 (Bergstrom and Bagnoli 2005).

Furthermore, we impose a boundary condition requiring that when both products are priced as low as their corresponding costs, demand for each product shall always be positive. That is, $z_1(AC, C) > L_1$ and $z_2(AC, C) > L_2$. Note that these two inequalities jointly imply $p_1(L_1, L_2) > AC$ and $p_2(L_1, L_2) > C$. The same assumption has been adopted by Wang and Kapuscinski (2009).

Lastly, we set upper and lower bounds on product 1's unit component usage: $\max(c_1, c_2)/b_1 < A < b_2/\max(c_1, c_2)$. This requires that product 1's unit component usage shall not deviate too far from unity[3] and essentially limits the asymmetry in the two products regarding their component usage. This is consistent with the assumption that two products are horizontally differentiated. In practice, if one product uses a lot more components than the other (e.g., one desktop equipped with

[3]Note that by the diagonal-dominance assumption, $A = 1$ is always in the desired range.

four 256M RAM sticks versus another with similar configuration but only one 256M RAM stick), the two products are more likely to be vertically differentiated.

To evaluate the impact of demand shaping (i.e., inventory-dependent pricing) and product overselling, we shall examine and compare four types of firm's strategies:

- Inventory-Independent Pricing, No Overselling (II,NO)

  The firm ignores component availability and adopts a naive pricing strategy: always quotes the unconstrained optimal prices $\mathbf{p}^U$, regardless of the on-hand component inventory. In the meanwhile, the firm does not oversell products, due to either lack of such operational capability, or a very high cost for order cancellation (e.g., $\gamma = \infty$). In such a case, the firm only accepts those orders which can be fulfilled with the on-hand component inventory. That is, the order-acceptance up-to levels always satisfy the bill-of-material constraints.

- Inventory-Independent Pricing, Optional Overselling (II,OO)

  Similar to the (II,NO) strategy, the firm always uses the inventory-independent prices $\mathbf{p}^U$. Nevertheless, the firm has the option of overselling products and optimizes the order-fulfillment decision according to the on-hand component inventory.

- Inventory-Dependent Pricing, No Overselling (ID,NO)

  Similar to the (II,NO) strategy, the firm has limited flexibility in operations and does not oversell products. However, the firm can adjust prices based on initial component availability.

- Inventory-Dependent Pricing, Optional Overselling (ID,OO)

  This is the case when the firm has full flexibility in both pricing and overselling.

Evidently, these four strategies represent the decision problems faced by firms with different limitation in operational and pricing flexibilities. To be consistent with the net profit $\Pi^{ID,OO}$ defined earlier, denote the optimal net profit under the other three strategies by $\Pi^{II,NO}$, $\Pi^{II,OO}$, and $\Pi^{ID,NO}$, respectively.

4.4 Optimal Policy

In this section, we characterize the joint optimal pricing and order-fulfillment decisions under the four strategies defined earlier. We first rule out a trivial case: when component inventory is large enough that the firm can always serve as many orders as what it would with unlimited capacity, the four strategies converge and result in the same total profit for the firm. Formally, let $\bar{y} = A(z_1^U + H_1) + (z_2^U + H_2)$, then for $y \geq \bar{y}$, under any of the four strategies, the optimal prices and order-acceptance up-to levels are $\mathbf{p}^U$ and $\mathbf{z}^U + \mathbf{H}$, respectively. Consequently, we shall assume $y < \bar{y}$

in the following analysis.

4.4.1 Inventory-Independent Pricing: (II,NO) and (II,OO)

We first examine the optimal order-fulfillment decision under the inventory-independent pricing, i.e., without demand shaping. The results provide baselines for analyzing the demand shaping strategy.

From the symmetry of the problem, we shall focus on the case when product 1's *relative margin*, defined as margin per unit of component usage, is greater than or equal to product 2's. That is, $(p_1^U - AC)/A \geq p_2^U - C$, or $p_1^U \geq Ap_2^U$. Given that product 1 has a higher relative margin, the priority in order fulfillment is pre-determined: orders for product 1 are always granted higher priority and the firm would never cancel any product 1's order to satisfy product 2's. In other words, if overselling is optimal, the firm always oversells product 2.

Inventory-Independent Pricing, No Overselling (II,NO)

When overselling is not feasible, the firm pre-allocates the component inventory to the two products, subject to the total availability of components. The firm's problem under (II,NO) strategy is hence a typical resource-allocation problem under demand uncertainty:

$$\Pi^{II,NO} = \max_{o_1,o_2} \pi^{NO}(z_1^U, z_2^U, o_1, o_2) = [p_1^U - AC]\mathrm{E}[\min(o_1, z_1^U + \epsilon_1)] + [p_2^U - C]\mathrm{E}[\min(o_2, z_2^U + \epsilon_2)]$$

$$\text{subject to} \quad o_1, o_2 \geq 0, \quad Ao_1 + o_2 \leq y$$

where the function $\pi^{NO}(z_1, z_2, o_1, o_2)$ is the expected no-overselling profit function when the demand rates are z_1, z_2 and the order-acceptance up-to levels are o_1, o_2, respectively:

$$\pi^{NO}(z_1, z_2, o_1, o_2) = (p_1(z_1, z_2) - AC)\mathrm{E}[\min(o_1, z_1 + \epsilon_1)] + (p_2(z_1, z_2) - C)\mathrm{E}[\min(o_2, z_2 + \epsilon_2)]$$

Naturally, it is never optimal to leave any component unallocated and Lemma 10 shows that under the no-overselling strategies (either ID or II), the bill-of-material constraint is always binding at optimum.

Lemma 10 *Under either the (II,NO) or the (ID,NO) strategy,* $Ao_1^* + o_2^* = y$*.*

By Lemma 10, the firm's problem is simplified to

$$\Pi^{II,NO} = \max_{o_1} \pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1) \quad \text{subject to} \quad 0 \leq o_1 \leq y/A.$$

Proposition 7 characterizes the optimal order-acceptance rationing levels for the (II,NO) strategy. Note that in some cases, the optimal solution may not be unique.[4] In such cases, we follow convention

[4]For example, when two products are symmetric (i.e., $p_1^U = p_2^U$, $A = 1$, $G_1 \equiv G_2$) and $y < z_1^U - L_1$, the two products contribute the same margin and the firm can guarantee to sell out all the components. Hence the component allocation does not matter and any o_1 between 0 and y is optimal.

(e.g., Federgruen and Heching 1999) to select the lexicographically largest. Such a criterion is also used in all the subsequent analysis whenever multiple optimal solutions exist.

Proposition 7 *(Optimal Order Acceptance Policy with Inventory-Independent Pricing)* *Under the (II,NO) strategy and given* $p_1^U \geq Ap_2^U$, $o_1^* < z_1^U + H_1$ *and there exists a critical number* $\hat{y}_1 \in [0, \bar{y}]$ *such that if* $y \in (0, \hat{y}_1]$, $o_1^* = y/A$, $o_2^* = 0$*; if* $y \in (\hat{y}_1, \bar{y})$, $o_1^* \in (0, y/A)$ *and satisfies* $(p_1^U - AC)\bar{G}_1(o_1 - z_1^U) = A(p_2^U - C)\bar{G}_2(y - Ao_1 - z_2^U)$, $o_2^* = y - Ao_1^*$. *Specifically,* $\hat{y}_1 = Az_1^U + A(\bar{G}_1)^{-1}\left(\frac{A(p_2^U - C)}{p_1^U - AC}\right)$.

By Proposition 7, when component inventory is low, the firm only offers product 1 and accepts product-1 orders until the component inventory is exhausted. On the other hand, when component inventory is relatively high, the firm offers both products and yet also rations orders of both products. In such a case, pre-blocking some product-1 orders is desirable, because if all components were pre-committed to product 1, the chance of not receiving enough product-1 orders to consume all capacity (i.e., spoilage risk of capacity) would be very high. In other words, since all decisions are made ex-ante under the no-overselling strategy, demand uncertainty plays an important role and the firm pre-allocates capacity according to, instead of the relative margin, *the expected relative return*, which equals to the relative margin times the probability that demand exceeds the pre-allocated amount (i.e., the tail probability). As product-1's pre-allocation increases, the relative expected return from product 1 decreases and that from product 2 increases. Hence, starting from zero, the firm continues to increase the pre-allocation for product 1 until product 1's expected relative return is not higher than product 2's.

Inventory-Independent Pricing, Optional Overselling (II,OO)

The firm's problem under (II,OO) strategy can be obtained by simply replacing the pricing decision in the (ID,OO) formulation with the unconstrained prices $\mathbf{p}^U$ (or equivalently, substituting the endorgenized demand rates by the unconstrained optimal $\mathbf{z}^U$):

<u>Ex-post:</u>

$$\pi(z_1^U, z_2^U, o_1, o_2, \epsilon_1, \epsilon_2) = \max_{q_1, q_2}\{(p_1^U - AC)[q_1 - \gamma(\min(o_1, z_1^U + \epsilon_1) - q_1)] + (p_2^U - C)[q_2 - \gamma(\min(o_2, z_2^U + \epsilon_2) - q_2)]\}$$

$$\text{subject to:} \quad 0 \leq q_1 \leq \min(o_1, z_1^U + \epsilon_1), 0 \leq q_2 \leq \min(o_2, z_2^U + \epsilon_2), 0 \leq Aq_1 + q_2 \leq y$$

Ex-ante:

$$\begin{aligned} \Pi^{II,OO} &= \max_{o_1,o_2} \pi^{OO}(z_1^U, z_2^U, o_1, o_2) = \mathrm{E}_{\epsilon_1,\epsilon_2}\pi(z_1^U, z_2^U, o_1, o_2, \epsilon_1, \epsilon_2) \\ \text{subject to:} &\qquad 0 \le o_1 \le \min(y/A, z_1^U + H_1), 0 \le o_2 \le \min(y, z_2^U + H_2) \end{aligned}$$

Note that compared to the formulation in equation (4.7), here we add two upper bounds for the up-to levels: $o_1 \le \min(y/A, z_1^U + H_1)$ and $o_2 \le \min(y, z_2^U + H_2)$. This addition is without loss of optimality, as the firm obviously would neither accept any orders which cannot be satisfied for sure, nor set up-to levels beyond the maximal demands. Obviously, $o_1^* = \min(y/A, z_1^U + H_1)$ indicates that it is optimal to accept product-1 orders up to the total inventory, while $o_1^* < \min(y/A, z_1^U + H_1)$ implies that blocking some product-1 orders is optimal.

Since $p_1^U \ge Ap_2^U$, clearly the optimal solution to the ex-post order-fulfillment problem is:

$$\begin{aligned} q_1^*(z_1^U, z_2^U, o_1, o_2, \epsilon_1, \epsilon_2) &= \min(y/A, \min(o_1, z_1^U + \epsilon_1)) = \min(o_1, z_1^U + \epsilon_1), \\ q_2^*(z_1^U, z_2^U, o_1, o_2, \epsilon_1, \epsilon_2) &= \min(\min(o_2, z_2^U + \epsilon_2), y - Aq_1^*) \\ &= \min(\min(o_2, z_2^U + \epsilon_2), y - A\min(o_1, z_1^U + \epsilon_1)) \end{aligned}$$

Consequently, for given overselling quantities, the expected profit is

$$\begin{aligned} &\pi^{OO}(z_1^U, z_2^U, o_1, o_2) \\ &= (p_1^U - AC)\mathrm{E}[\min(o_1, z_1^U + \epsilon_1)] + (p_2^U - C)\mathrm{E}[\min(o_2, z_2^U + \epsilon_2)] \\ &\quad - (p_2^U - C)(1+\gamma)\mathrm{E}[\max(A\min(o_1, z_1^U + \epsilon_1) + \min(o_2, z_2^U + \epsilon_2) - y, 0)] \end{aligned} \tag{4.11}$$

Note that from equation (4.11), o_1 is the *guaranteed* component supply for product 1, meaning that the firm can guarantee fulfilling up to o_1 product-1 orders. Of these, $(y - o_2)/A$ represents *reserved* allocation, i.e., the number of product-1 orders which can be fulfilled without having to cancel any product-2 order. In contrast, o_2 is composed of two parts: $y - Ao_1$ is the *guaranteed* allocation for product 2 and $Ao_1 + o_2 - y$, if positive, represents the "callable" allocation (Gallego et al. 2008) which may be cancelled in the need of satisfying product-1 demand.

Proposition 8 characterizes the optimal order-acceptance policy with inventory-independent pricing.

Proposition 8 *(Optimal Order-Acceptance Policy with Inventory-Independent Pricing)*
Under the (II,OO) strategy and given $p_1^U \ge Ap_2^U$, there exists a critical number $\hat{y}_2 \in (0, \bar{y})$ such that if $y \in (0, \hat{y}_2]$, overselling is not optimal (i.e., $Ao_1^* + o_2^* \le y$), and if $y \in (\hat{y}_2, \bar{y})$, overselling is optimal (i.e., $Ao_1^* + o_2^* > y$). Specifically, the value of $\hat{y}_2$ and the optimal overselling policy are as follows:

- *If* $\gamma = 0$, $\hat{y}_2 = 0$.

  For all $y > 0$, $o_1^* = \min(y/A, z_1^U + H_1)$ *and* $o_2^* = \min(y, z_2^U + H_2)$;

- *If* $0 < \gamma \le \gamma_0 = \frac{p_1^U - Ap_2^U}{A(p_2^U - C)}$, $\hat{y}_2 = Az_1^U + A(\bar{G}_1)^{-1}\left(\frac{1}{1+\gamma}\right)$.

  If $y \le \hat{y}_2$, $o_1^* = y/A$ *and* $o_2^* = 0$; *otherwise,* $o_1^* = \min(y/A, z_1^U + H_1)$ *and* $o_2^* = \min(\hat{o}_2, z_2^U + H_2)$.

- *If* $\gamma > \gamma_0$, $\hat{y}_2 = Az_1^U + z_2^U + A(\bar{G}_1)^{-1}\left(\frac{1}{1+\gamma}\right) + (\bar{G}_2)^{-1}\left(\frac{p_1^U - AC}{A(p_2^U - C)(1+\gamma)}\right)$.

  If $y \in (0, \hat{y}_2]$, o_1^* *and* o_2^* *are as given in Proposition 7; otherwise,* $o_1^* = \min(\hat{o}_1, z_1^U + H_1)$ *and* $o_2^* = \min(\hat{o}_2, z_2^U + H_2)$.

  More specifically, there exists a $\hat{y}_3 \in (\hat{y}_2, \bar{y})$ *such that if* $y \in (0, \hat{y}_1]$, $o_1^* = y/A$ *and* $o_2^* = 0$; *if* $y \in (\hat{y}_1, \hat{y}_2]$, $o_1^* \in (0, \min(y/A, z_1^U + H_1))$ *and* $o_2^* = y - Ao_1^*$; *if* $y \in (\hat{y}_2, \hat{y}_3)$, $o_1^* \in (0, \min(y/A, z_1^U + H_1))$ *and* $o_2^* > y - Ao_1^*$; *and if* $y \in [\hat{y}_3, \bar{y})$, $o_1^* = z_1^U + H_1$ *and* $o_2^* > y - Ao_1^*$,

  where $\hat{o}_1 = \frac{1}{A}\left[y - z_2^U - (\bar{G}_2)^{-1}\left(\frac{p_1^U - AC}{A(p_2^U - C)(1+\gamma)}\right)\right]$, $\hat{o}_2 = y - Az_1^U - A(\bar{G}_1)^{-1}\left(\frac{1}{1+\gamma}\right)$, $\hat{y}_1$ *is defined in Proposition 7, and* $\hat{y}_3 = Az_1^U + AH_1 + z_2^U + (\bar{G}_2)^{-1}\left(\frac{p_1^U - AC}{A(p_2^U - C)(1+\gamma)}\right)$.

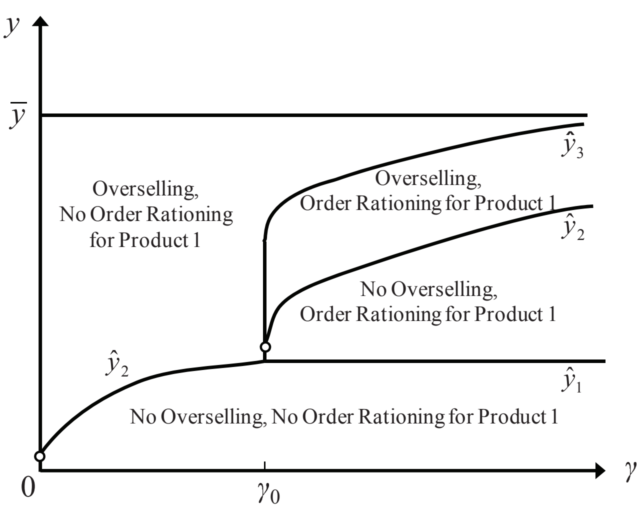


Figure 4.3. Structure of optimal order-acceptance policy under inventory-independent pricing.

Proposition 8 is illustrated in Figure 4.3. Note that due to existence of multiple optimal solutions, $\hat{y}_2$ is discontinuous at $\gamma = 0$ and $\gamma = \gamma_0$. To see why the discontinuity exists, take $y \in (0, Az_1^U - AL_1)$ for example. In such a case, the firm can guarantee to sell out all the components by only offering product 1. When cancellation compensation $\gamma = 0$, the firm is indifferent accepting any product 2's order, since product-2 orders cannot never be fulfilled and can be cancelled for free. Hence, overselling and no-overselling are both optimal when $\gamma = 0$. In contrast, for any positive compensation $\gamma > 0$, the firm strictly prefers not accepting any product-2 orders, i.e., no overselling.

From Proposition 8 and Figure 4.3, overselling is optimal only when component inventory level is high and cancellation compensation is low. This is because, with low inventory level, the chance of exhausting all components is high and the firm does not need to accept more orders than what he can fulfill; and with a high cancellation compensation, overselling is simply not economical.

Rationing product-1 orders is never optimal when cancellation compensation γ is low. This is exactly the situation studied in the aforementioned airline-overselling literature (esp. Biyalogorsky et al. 1999 and 2000). With a low compensation (specifically, when $p_1^U - AC \geq (1+\gamma)A(p_2^U - C)$), cancelling A units of product-2 orders to fulfill a product-1 order is always profitable. Hence, the firm is better off accepting as many product-1 orders as possible and cancelling product-2 orders whenever needed.

Nevertheless, limiting product-1 orders can be optimal when cancellation compensation is high. This case has been overlooked by the existing overselling literature. In particular, it is optimal when inventory level is medium (i.e., $y \in (\hat{y}_1, \hat{y}_3)$) and can be accompanied with either overselling or no-overselling policy. Recall that with no-overselling, rationing product-1 orders is aimed to balance the expected relative returns from the two products. With overselling, it exists for a different reason: when cancellation compensation is high (specifically, when $p_1^U - AC < (1+\gamma)A(p_2^U - C)$), accepting an additional order of product 1 is not necessarily profitable from the firm's viewpoint ex-ante. Essentially, the tradeoff is between the gain from satisfying a product-1 order and the potential loss from having to cancel A units of product-2 orders - which occurs only if the product-2 demand exceeds the *guaranteed* component allocation for product 2 (i.e., when $d_2(z_2^U, \epsilon_2) \geq y - Ao_1$). Hence, the firm continues to accept product-1 orders as long as product-1 margin outweighs the *expected* loss of cancelling product-2 orders, i.e., $p_1^U - AC \geq (1+\gamma)A(p_2^U - C)\bar{G}_2(y - Ao_1 - z_2^U)$.[5] Similarly, the rationing level for product 2 is determined by equating the gain from fulfilling a product-2 order (i.e., product 2's margin $p_2^U - C$) and the expected loss of ending up cancelling the order to fulfill product-1 demand - which occurs only if demand for product 1 exceeds its *reserved* allocation (i.e., $(p_2^U - C)(1+\gamma)\Pr[d_1(z_1^U, \epsilon_1) \geq (y - o_2)/A]$).[6] Interestingly, one product's rationing level depends on the other product's demand distribution, instead of its own.

When component inventory is very large ($y > \hat{y}_3$), rationing product-1 orders becomes undesirable, since with a lot of components in stock, the firm has a good chance to satisfy all product-1 orders without having to cancel any product-2 order. In such a case, blocking any product-1 order would very likely result in leftover components.

A direct corollary of Proposition 8 is that keeping everything else the same and for arbitrarily given prices $p_1 \geq Ap_2$, when product 1's price increases, it is more likely for the firm to oversell

[5]Ref. expression of $\hat{o}_1$ in Proposition 7

[6]Ref. expression of $\hat{o}_2$ in Proposition 7

(mathematically, $\hat{y}_2$ decreases in p_1). Two factors contribute to this result. On one hand, when price 1 increases, selling product 1 becomes more profitable and the value of overselling is enhanced as product 2's orders can be used as capacity buffer for product 1. On the other hand, an increase in price 1 results in a (stochastic) decrease in the aggregate demand of the two products. Hence, for given initial inventory, the likelihood of having leftover inventory increases and overselling can benefit the firm by hedging against the risk of inventory spillage.

In the meantime, the effect of product 2's price on overselling is not as clear. That is because, even though increasing price 2 has a similar effect on the aggregate product demand as increasing price 1 (which drives the value of overselling up), an increase in price 2 makes it less profitable to cancel a product-2 order to fulfill a product 1 order, which drives the value of overselling down. Hence, depending on the parameter values, increasing p_2 may or may not make overselling more desirable for the firm.

To see how demand uncertainty affects overselling strategy, assume that both ϵ_1 and ϵ_2 follow symmetric distributions[7] and that if the variance of either ϵ_1 or ϵ_2 increases, both distributions remain symmetric. Note that uniform or (truncated) normal distribution satisfy these conditions. In such a case, we can show that if $\gamma < 1$, then the region of overselling expands when the variance of either ϵ_1 or ϵ_2 increases.[8] That is, when the cancellation ratio is sufficiently low, overselling becomes more desirable when demand of either product becomes more variable. This is clearly consistent with the role of overselling in hedging the demand risk.

4.4.2 Inventory-Dependent Pricing, No Overselling (ID,NO)

So far we have assumed that prices are exogenously given and independent of initial inventory. Starting from this subsection, we endorgenize the pricing decision and consider the joint optimization of pricing and order-acceptation strategies.

We start with the firm's optimal policy without overselling flexibility (i.e. the (ID,NO) strategy). As we noted earlier, this strategy can also be regarded as a special case of (ID,OO) strategy with infinite cancellation compensation.

Under the (ID-NO) strategy, the firm's problem is to maximize the expected profit $\pi^{NO}(z_1, z_2, o_1, o_2)$ subject to the bill-of-material constraint and the nonnegativity constraint on product margins. Furthermore, to preclude the possibility of negative product demand, we impose an additional constraint

[7] That is to require that for $i = 1, 2$, $L_i + H_i = 0$ and $g_i(x) = g_i(-x)$ for $x \in [-L_i, H_i]$.

[8] The proof is straightforward by noting that if $\gamma < 1$, then $\frac{p_1^U - AC}{A(p_2^U - C)(1+\gamma)} > \frac{1}{1+\gamma} > 1/2$. Hence, by the assumptions on demand distribution, $\hat{y}_2$ decreases in the variance of either ϵ_1 or ϵ_2.

on demand rates $\mathbf{z}$, requiring that under any feasible $\mathbf{z}$, all products' demand are nonnegative almost surely. This constraint is equivalent to setting an lower bound for demand rates $\mathbf{z}$ (i.e., an upper bound for prices $\mathbf{p}$). Hence, the firm's optimization problem under (ID-NO) strategy is as follows:

$$\max_{z_1,z_2,o_1,o_2} \pi^{NO}(z_1,z_2,o_1,o_2) = (p_1(z_1,z_2) - AC)\mathrm{E}[\min(o_1, z_1+\epsilon_1)] + (p_2(z_1,z_2) - C)\mathrm{E}[\min(o_2, z_2+\epsilon_2)],$$
$$s.t. \quad z_1 \geq L_1, z_2 \geq L_2, p_1(z_1,z_2) \geq AC, p_2(z_1,z_2) \geq C, o_1, o_2 \geq 0, Ao_1 + o_2 \leq y.$$

The problem formulated above is a typical multi-product joint pricing and inventory decision problem with lost-sale and price-based substitution. Even without any feasibility constraint, the problem has been notorious for its analytical difficulty arising from a non-concave profit function. It was not until recently that the unconstrained version of the problem was shown to have a unique optimal solution (Aydin and Porteus 2008, Wang and Kapuscinski 2009). Here we extend this stream of literature by incorporating multiple feasibility constraints including bill of material, nonnegativity on margins and up-to levels, and almost-surely nonnegativity on demands. In the following we prove that the uniqueness of the optimal solution still holds for the constrained decision problem. Further, we show some structural properties of the optimal policy.

We first note that by Lemma 10, the firm's problem can be simplified to

$$\max_{z_1,z_2,o_1} \pi^{NO}(z_1,z_2,o_1) = (p_1(z_1,z_2) - AC)\mathrm{E}[\min(o_1, z_1+\epsilon_1)] + (p_2(z_1,z_2) - C)\mathrm{E}[\min(y - Ao_1, z_2+\epsilon_2)],$$
$$s.t. \quad z_1 \geq L_1, z_2 \geq L_2, p_1(z_1,z_2) \geq AC, p_2(z_1,z_2) \geq C, 0 \leq o_1 \leq y/A.$$

We show that the optimal "safety stock" (i.e., order-acceptance up-to level minus expected demand) for each product is always within the support of the corresponding random shock. That is, the firm neither over- nor under- protects itself against demand uncertainty.

Lemma 11 *(Optimal "Safety Stock") When $0 < y < \bar{y}$, the optimal policy (z_1^*, z_2^*, o_1^*) satisfies $-L_1 \leq o_1^* - z_1^* \leq H_1$ and $-L_2 \leq y - Ao_1^* - z_2^* \leq H_2$. Further, at least one of the two inequalities $o_1^* - z_1^* \leq H_1$ and $y - Ao_1^* - z_2^* \leq H_2$ is strict.*

Lemma 11 refines the problem's feasible region: without loss of optimality, we shall impose the constraints stated in Lemma 11 on any feasible policies. Within the refined feasible region, we prove the uniqueness of the optimal policy for the case when the two cross-price sensitivity parameters are both positive, i.e., $c_1 > 0$ and $c_2 > 0$. Theorem 17 follows.

Theorem 17 *(Uniqueness of Solution) When* $c_1 > 0$ *and* $c_2 > 0$*,*

(i) for given $o_1 \in [0, y/A]$*, there exists a unique pair of* (z_1, z_2)*, denoted by* $(z_1^*(o_1), z_2^*(o_1))$*, which maximizes the profit function* $\pi^{NO}(z_1, z_2, o_1)$ *subject to* $z_1 \geq L_1, z_2 \geq L_2, p_1(z_1, z_2) \geq AC, p_2(z_1, z_2) \geq C$*;*

(ii) $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ *is continuously differentiable and strictly quasi-concave in* o_1 *for* $o_1 \in [0, y/A]$*;*

(iii) there exists a unique solution (z_1^*, z_2^*, o_1^*) *maximizing the profit function* $\pi^{NO}(z_1, z_2, o_1)$ *subject to* $z_1 \geq L_1, z_2 \geq L_2, 0 \leq o_1 \leq y/A, p_1(z_1, z_2) \geq AC, p_2(z_1, z_2) \geq C$*.*

The idea underlying the proof of Theorem 17 is similar to that used in Petruzzi and Dada (1999) for their study of a single-product unconstrained joint pricing and inventory decision problem. It was extended to an N-product unconstrained problem in Aydin and Porteus (2008). In its core, the original idea was to show that first, the global maximum is always attained at a stationary point; second, the Hessian of the profit function is negative-definite in the neighborhood of any stationary point. Hence, any stationary point is a strict local maximum and there exists a unique global maximum.

The feasibility constraints bring considerable complexity such that a direct application of the aforementioned logic to the constrained problem is impossible. Specifically, the global and local maxima may not always be attained at stationary points - they can exist at boundaries for all variables, or alteratively, at boundaries for some variables and in the interior for others. In the proof, we first show that the second-order conditions are satisfied in all these scenarios and then formally extend the analysis to the constrained problem. The detailed proof is provided in the appendix.

When one of the cross-price sensitivity parameter is zero ($c_1 = 0$ or $c_2 = 0$), multiple optimal solutions may exist. Specifically, imagining that $c_1 = 0$ and in an optimal solution, the firm only offers product 1 (i.e., $o_1^* = y/A$). In such a case, the firm's profit is independent of product 2's price (or demand rate), which implies that any price for product 2 is optimal. To break the tie and also to be consistent with the optimal policy characterized in Theorem 17, we set $z_2^* = L$ when $c_1 = 0$ and $o_1 = 0$, and $z_1^* = L$ when $c_2 = 0$ and $o_2 = 0$. The uniqueness of optimal solution can then be extended to the case when $c_1 = 0$ or $c_2 = 0$. The proof follows the same logic as that for Theorem 17 and is thus omitted for brevity.

The significance of the unique solution is in two folds. First, it ensures that any gradient-based searching algorithm can find the optimal solution, which is important for practical application. Second, it forms the basis for further characterization of the optimal policy.

Building on the uniqueness of optimal solution, Theorem 18 (illustrated in Figure 4.4) identifies the firm's optimal assortment as a function of demand characteristics, component usage, and on-hand component inventory.

Theorem 18 *(Optimal Assortment)*

(i) If $p_1(L_1, L_2) > Ap_2(L_1, L_2)$, i.e., if $\frac{a_1 - L_1}{Ab_1 - c_1} \geq \frac{a_2 - L_2}{b_2 - Ac_2}$, there exists a critical number $\acute{y} > 0$ such that $o_1^* = y/A$ if $0 < y \leq \acute{y}$ and $0 < o_1^* < y/A$ if $y > \acute{y}$; If $p_1(L_1, L_2) < Ap_2(L_1, L_2)$, there exists another critical number $\grave{y} > 0$, such that $o_1^* = 0$ if $0 < y \leq \grave{y}$ and $0 < o_1^* < y/A$ if $y > \grave{y}$; If $p_1(L_1, L_2) = Ap_2(L_1, L_2)$, $0 < o_1^* < y/A$ for all $y > 0$.

(ii) $\grave{y}$ is non-decreasing in A.

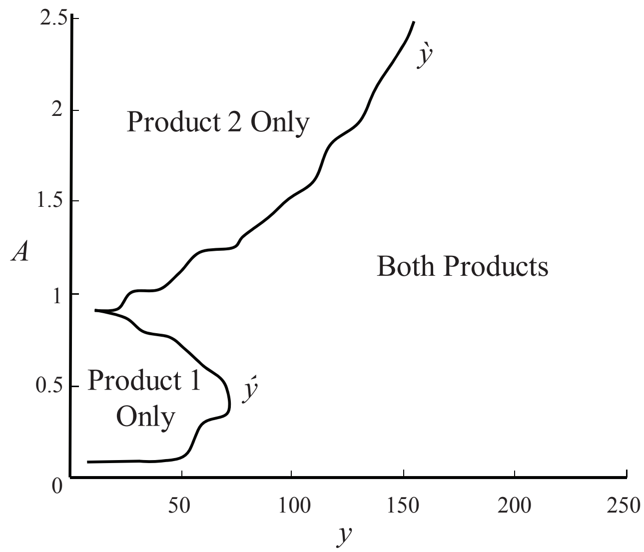


Figure 4.4. Optimal assortment as a function of component inventory y and product 1's unit component consumption A: $a_1 = 500, a_2 = 400, b_1 = 15, b_2 = 10, c_1 = 1, c_2 = 0.5, c = 0, \epsilon_1, \epsilon_2 \sim \text{Uniform}[-50, 50]$.

The structure illustrated in Figure 4.4 is quite intuitive: when the firm has a lot of component inventory, it is better off offering both products; while when the inventory level is very low, completely blocking one product becomes necessary.

The more intriguing question is which product to offer with very limited component inventory. Theorem 18 provides an product index based on which the firm can prioritize among products. The index is the maximal price that the firm can charge for a product (attained when both products' demand rates are at their corresponding lower bound) divided by the product's component usage. It aggregates the effects of the product's demand characteristics and component usage. Specifically, a product tends to be given high priority when it has a large worst-case base demand ($a_j - L_j$), a low self-price sensitivity (b_j), a high cross-price sensitivity (c_j), and a low component usage (A). It is interesting to notice that Tang and Yin (2007) use a similar index, $a_j/(b_j - c_j)$, for a two-product problem with deterministic demand and identical component usages. It is easy to see that their index is a special case of the one identified in Theorem 18.

Under some mild conditions on the demand characteristics, we further show that the optimal order-acceptance up-to levels are non-decreasing in the component inventory. Theorem 19 follows.

Theorem 19 *(Monotonicity of Optimal Order-Acceptance Up-to Levels)*

Let $o_2^* = y - Ao_1^*$.

(i) If $Ab_1b_2 + Ac_1^2 - Ac_2^2 - b_2c_1 \geq 0$, o_1^* is non-decreasing in y.

(ii) If $b_1b_2 + c_2^2 - c_1^2 - Ab_1c_2 \geq 0$, o_2^* is non-decreasing in y.

Note that the conditions in Theorem 19 (i) and (ii) jointly require that the absolute difference between c_1 and c_2 be sufficiently small, i.e., it limits the extent to which the two products are asymmetric in terms of cross-price sensitivity. When $c_1 = c_2 = 0$, we further prove the monotonicity of both pricing and order-acceptance policies in component inventory and product component usage. Proposition 9 follows.

Proposition 9 *When $c_1 = c_2 = 0$, let $(o_1^*, o_2^*, z_1^*, z_2^*, p_1^*, p_2^*)$ denote the optimal policy.*

(i) Both o_1^* and o_2^* are non-decreasing in y; o_1^* is non-increasing in A.

(ii) Both z_1^* and z_2^* are non-decreasing in y; z_1^* is non-increasing in A.

(iii) Both p_1^* and p_2^* are non-increasing in y; p_1^* is non-decreasing in A.

The monotonicities characterized in Theorem 19 and Proposition 9 all make intuitive sense: with more inventory of components, the firm tend to quote lower prices and accept more orders; when a product uses more components, its price tends to increase and order ration decrease.

4.4.3 Inventory-Dependent Pricing, Optional Overselling (ID,OO)

In this subsection, we consider the (ID,OO) strategy, where the firm has full flexibility in pricing and order management. Recall the firm's problem:

Ex-post:

$$\begin{aligned}
&\pi(z_1, z_2, o_1, o_2, \epsilon_1, \epsilon_2) \\
&= \max_{q_1, q_2} \{(p_1(z_1, z_2) - AC)[q_1 - \gamma(\min(o_1, z_1 + \epsilon_1) - q_1)] + (p_2(z_1, z_2) - C)[q_2 - \gamma(\min(o_2, z_2 + \epsilon_2) - q_2)]\} \\
&\text{subject to: } 0 \leq q_1 \leq \min(o_1, z_1 + \epsilon_1), 0 \leq q_2 \leq \min(o_2, z_2 + \epsilon_2), 0 \leq Aq_1 + q_2 \leq y
\end{aligned}$$

Ex-ante:

$$\Pi^{ID,OO} = \max_{z_1,z_2,o_1,o_2} \pi^{OO}(z_1,z_2,o_1,o_2) = \mathrm{E}_{\epsilon_1,\epsilon_2}\pi(z_1,z_2,o_1,o_2,\epsilon_1,\epsilon_2)$$

$$\text{subject to: } z_1 \geq L_1, z_2 \geq L_2, p_1(z_1,z_2) \geq AC, p_2(z_1,z_2) \geq C, 0 \leq o_1 \leq \min(y/A, z_1+H_1),$$
$$0 \leq o_2 \leq \min(y, z_2+H_2)$$

Note that similarly to the (ID,NO) strategy, we impose the constraints that both products' demand should be nonnegative with probability one. Similarly to Lemma 11, Lemma 12 refines the feasible region for the decision problem under (ID,OO) strategy.

Lemma 12 *(Refining Feasible Region for (ID,OO) Strategy) The optimal policy satisfies* $A(z_1^*+H_1)+z_2^*+H_2 > y$, $o_1^* \geq (y-(z_2^*+H_2))/A$, *and* $o_2^* \geq y - A(z_1^*+H_1)$.

By Lemma 12, without loss of optimality, we can and shall impose the following constraints on feasible policies: $A(z_1+H_1)+z_2+H_2 > y$, $o_1 \geq (y-(z_2+H_2))/A$, and $o_2 \geq y - A(z_1+H_1)$.

Within the refined feasible region, we first characterize the optimal order-acceptance policy for given prices. Note that this problem is actually the same with the one under the (II,OO) strategy and we can directly borrow the results from there.

Proposition 10 *Under the (ID,OO) strategy, the optimal order-acceptance policy for given prices, denoted by* $\boldsymbol{o}^*(\boldsymbol{z})$, *is the same with that characterized in Proposition 8 with* z_1^U *substituted by* z_1, z_2^U *by* z_2, p_1^U *by* $p_1(z_1,z_2)$, *and* p_2^U *by* $p_2(z_1,z_2)$.

By Proposition 10, the firm's problem can be simplified to

$$\Pi^{ID,OO} = \max_{z_1,z_2} \pi^{OO}(z_1,z_2,o_1^*(z_1,z_2),o_2^*(z_1,z_2))$$
$$\text{subject to: } z_1 \geq L_1, z_2 \geq L_2, p_1(z_1,z_2) \geq AC, p_2(z_1,z_2) \geq C$$

Not surprisingly, due to the complicated interactions between the pricing and overselling policies (ref. Proposition 8), compared with the unimodal profit function under (ID,NO) strategy, the profit function in the (ID,OO) model tends to be ill-behaved. Specifically, we find that the profit function $\pi^{OO}(z_1,z_2,o_1^*(z_1,z_2),o_2^*(z_1,z_2))$ may have multiple local maxima. An numerical example is illustrated in Figure 4.5.

The existence of multiple local maxima implies that the optimal solution may not be unique. For some special case, we can actually show that the profit function is bimodal and can have two optimal solutions (ref: section C.10). Without a unique optimal solution, any further characterization of the optimal policy is very difficult. To obtain some insight on this strategy, next we consider a special case of the general problem: when the two products are symmetric, i.e., $a_1=a_2=a$, $b_1=b_2=b$, $c_1=c_2=c$, and $A=1$.

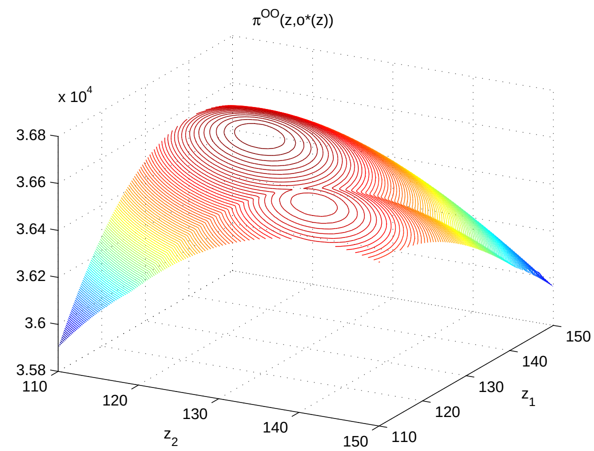


Figure 4.5. An example where $\pi^{OO}(\mathbf{z}, \mathbf{o}^*(\mathbf{z}))$ has two local maxima: $a_1 = a_2 = 600$, $b_1 = 3$, $b_2 = 2.95$, $c_1 = c_2 = 0.3$, $c = 0$, $y = 216$, $\gamma = 0.1$, $A = 1$, $\epsilon_1, \epsilon_2 \sim$ Uniform$[-60, 60]$.

When the two products are symmetric (e.g., red and black Lenovo S10 Ideapad) and share one critical component (e.g., motherboard for the S10 Ideapad), we shall focus on the symmetric optimal policies. In such a case, clearly the optimal policy can be found by maximizing

$$\pi^{OO}(z,o) = 2(p(z) - C)\left\{\mathrm{E}[\min(o, z+\epsilon_1)] - (1+\gamma)\mathrm{E}[\max(\min(o, z+\epsilon_1) + \min(o, z+\epsilon_2) - y, 0)]\right\}$$

subject to $z \geq L, p(z) \geq C, 0 \leq o \leq \min(y, z+H)$, where $p(z) = \frac{a-z}{b-c}$ and ϵ_1, ϵ_2 are i.i.d with c.d.f $G(\cdot)$, p.d.f $g(\cdot)$, and support $[-L, H]$. Also when $y \leq \bar{y} = 2(z^U + H)$, by Lemma 12, any feasible (z, o) should also satisfy $2z + H > y$ and $o + z + H \geq y$.

We denote the optimal policy by $(z^{ID,OO}, o^{ID,OO})$. Proposition 11 (illustrated in Figure 4.6) establishes the uniqueness of the optimal policy and provides some structural properties.

Proposition 11 *(Symmetric Products with Inventory-Dependent Pricing and Optional Overselling)*

(i) For given z, there exists a unique symmetric solution, denoted by $o^*(z)$, maximizing $\pi^{OO}(z,o)$ subject to $0 \leq o \leq y$;

(ii) $\pi^{OO}(z, o^*(z))$ is strictly quasi-concave in z in the feasible domain of z;

(iii) $(z^{ID,OO}, o^{ID,OO})$ uniquely exists;

(iv) There exists a critical number $\tilde{y}_1 \in [0, \bar{y})$ such that $o^{ID,OO} = y/2$ if $y \leq \tilde{y}_1$ and $o^{ID,OO} > y/2$ if $y \in (\tilde{y}_1, \bar{y})$. That is, the firm oversells if and only if $y \in (\tilde{y}_1, \bar{y})$;

(v) There exists another critical number $\tilde{y}_2 \in [\tilde{y}_1, \bar{y})$ such that $o^{ID,OO} < \min(y, z^{ID,OO} + H)$ if $y < \tilde{y}_2$ and $o^{ID,OO} = \min(y, z^{ID,OO} + H)$ if $y \geq \tilde{y}_2$. That is, the firm rations orders if and only if $y < \tilde{y}_2$;

(vi) When $\gamma = 0$, $\tilde{y}_1 = \tilde{y}_2 = 0$. When $\gamma > 0$, both $\tilde{y}_1$ and $\tilde{y}_2$ are non-decreasing in γ.

The structure illustrated in Figure 4.6 is very similar to the one in Figure 4.3 (with $p_1^U = Ap_2^U$).

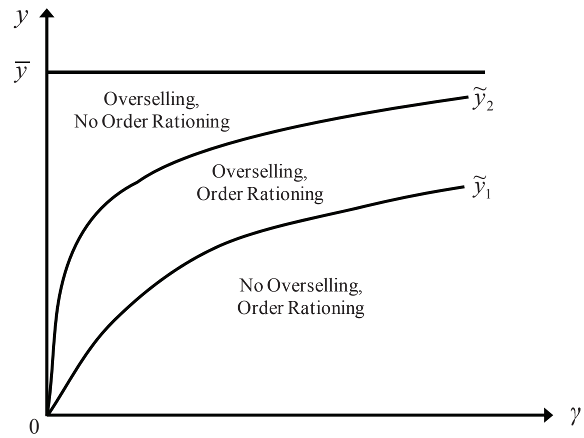


Figure 4.6. Structure of optimal order-acceptance policy with inventory-dependent pricing and two symmetric products

Specifically, overselling is optimal with large component inventory and low compensation ratio, while order rationing is desirable only if component inventory is small and compensation ratio is high. Such a difference originates from the different tradeoffs faced by the firm while making decision on overselling or order rationing. With overselling, the firm reduces both yield and spoilage losses from ex-post order management, but risks paying substantial cancellation compensation. In contrast, with order rationing, the firm can potentially gain from reduction on yield loss and saving on cancellation compensation, but can suffer huge spoilage loss from left-over components. Notice that when two products are symmetric, yield loss does not exist. Hence, the main tradeoff is between spoilage loss and compensation cost, on which the effects of overselling and order rationing are in two opposite directions.

Figure 4.7 and Figure 4.8 illustrate the effects of compensation ratio γ, component inventory y, and demand variability L on the optimal policies for the case with two symmetric products.

As shown in Figure 4.7, the compensation ratio γ only affects the policy under optional overselling. When the compensation ratio increases, naturally the firm accepts fewer orders and $o^{ID,OO}$ decreases. In the meanwhile, two forces pull the optimal price in opposite directions. On one hand, as $o^{ID,OO}$ decreases, the risk of inventory spoilage increases and the firm tends to lower the price to invite more orders. On the other hand, as $o^{ID,OO}$ decreases, the expected number of accepted orders decreases and the firm has the incentive to raise the price to make up for the loss in demand. When γ increases from zero, $o^{ID,OO}$ starts at a very high level and hence the force of increasing margin dominates that of inducing demand, which results in an increase of optimal price (equivalently, a decrease in optimal demand rate). After γ reaches a certain level, the loss from increased inventory spoilage outweighs that of potential margin and the optimal price eventually decreases (i.e., demand rate increases). Compared to the optimal price under (ID,NO), the (ID,OO) price is lower only if γ is very small, since in such cases, the firm tends to accept far more orders in (ID,OO) model than what he does in (ID,NO) model and hence needs to cut price to achieve a better balance between demand and margin.

Different from the cancellation ratio γ, component inventory level y influences the optimal policies under either (ID,NO) or (ID,OO) strategy, as shown in Figure 4.8. Quite intuitively, in either case, both the optimal demand rate and order-acceptance up-to level increase in y. Meanwhile, the (ID,OO) price is lower than the (ID,NO) price only when y is high and the reason is similar to what we have discussed earlier regarding γ.

Figure 4.9 illustrates that with a reasonably low cancellation cost, as demand variability increases, the firm raises the order-acceptance thresholds to reduce the risk of having inventory leftover. In the meanwhile, the optimal demand rates under either (ID,NO) or (ID,OO) strategy (as well as their relative magnitude) may not be monotonic in demand variability.

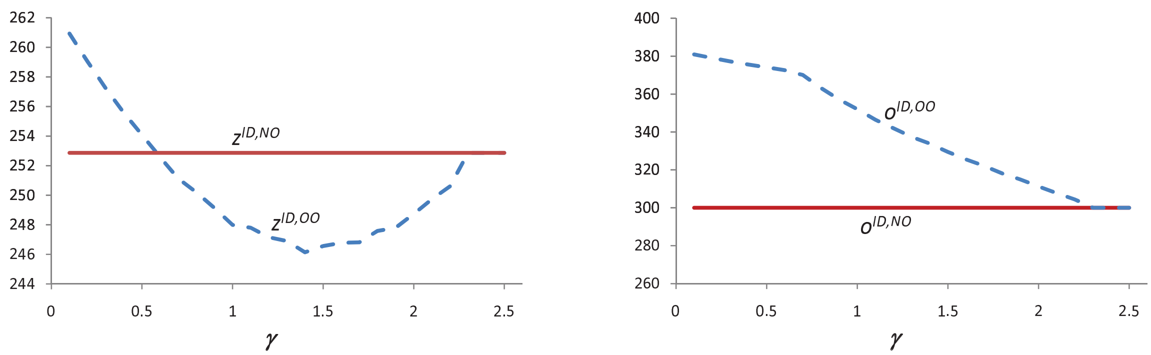


Figure 4.7. Optimal policy as a function of cancellation compensation ratio γ: $a_1 = a_2 = 600$, $b_1 = b_2 = 6$, $c_1 = c_2 = 3$, $C = 0$, $\epsilon_1, \epsilon_2 \sim$ Uniform$[-120, 120]$, $y = 600$.

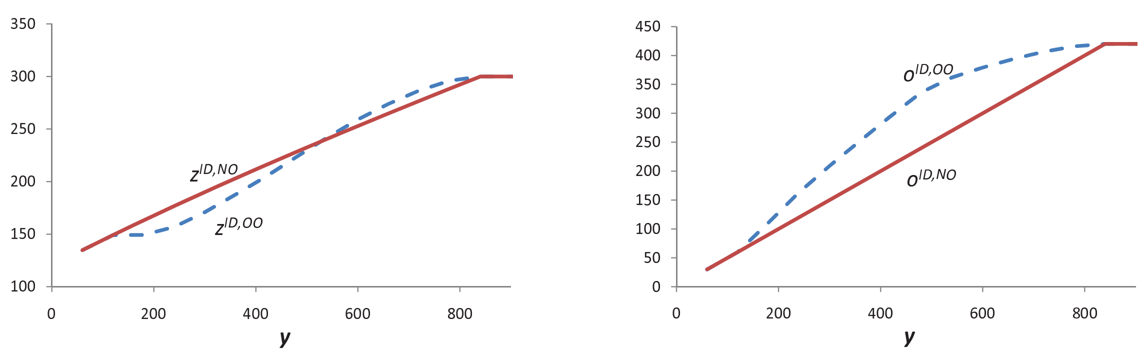


Figure 4.8. Optimal policy as a function of component inventory y: $a_1 = a_2 = 600$, $b_1 = b_2 = 6$, $c_1 = c_2 = 3$, $C = 0$, $\epsilon_1, \epsilon_2 \sim$ Uniform$[-120, 120]$, $\gamma = 0.2$.

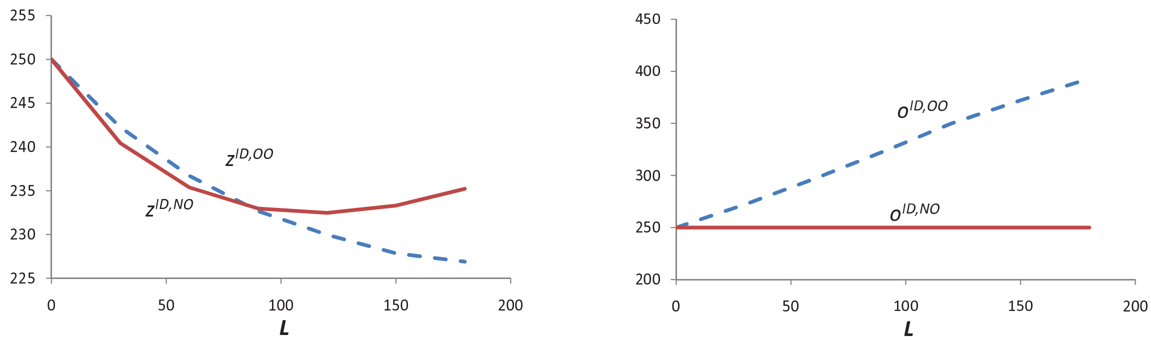


Figure 4.9. Optimal policy as a function of demand variability L: $a_1 = a_2 = 600$, $b_1 = b_2 = 6$, $c_1 = c_2 = 3$, $C = 0$, $\epsilon_1, \epsilon_2 \sim$ Uniform$[-L, L]$, $y = 500$, $\gamma = 0.2$.

4.5 Value of Pricing and Operational Flexibilities

So far we examine the optimal policies under the four strategies which differ in the combination of pricing and operational flexibilities. The natural follow-up question is how much the firm can gain by having these two flexibilities. In this section we evaluate the value of pricing and operational flexibilities. For this purpose, define the value of flexibility by the percentage improvement in total profit with a given flexibility:

$$\text{(Individual Value):} \quad \Delta^{OO} = \frac{\Pi^{II,OO} - \Pi^{II,NO}}{\Pi^{II,NO}} * 100\%, \quad \Delta^{ID} = \frac{\Pi^{ID,NO} - \Pi^{II,NO}}{\Pi^{II,NO}} * 100\%,$$

$$\text{(Joint Value):} \quad \Delta^{ID,OO} = \frac{\Pi^{ID,OO} - \Pi^{II,NO}}{\Pi^{II,NO}} * 100\%,$$

To isolate the effects of price sensitivity and price-based substitution in the numerical study, we follow the literature (e.g., McGuire and Staelin 1983, Wang and Kapuscinski 2009) to assume the following linear demand function:

$$d_1 = a_1 - \frac{\beta_1}{1-\theta_{12}} p_1 + \frac{\beta_1 \theta_{12}}{1-\theta_{12}} p_2 + \epsilon_1, \quad d_2 = a_2 - \frac{\beta_2}{1-\theta_{21}} p_2 + \frac{\beta_2 \theta_{21}}{1-\theta_{21}} p_1 + \epsilon_2.$$

where, for $i, j \in \{1, 2\}$ and $i + j = 3$, β_j represents demand j's own-price sensitivity and θ_{ij} denotes the substitution ratio - the ratio of cross-price sensitivity to own-price sensitivity (McGuire and Staelin 1983).

The following parameters are used:

- Base demand: $a_1 = a_2 = 60$;
- Price sensitivity: $\beta_1 \in \{1, 3, 5, 8, 10\}$, $\beta_2 = 5$;
- Substitution ratio: $\theta_{12} \in \{0.05, 0.1, 0.2, 0.3\}$, $\theta_{21} = 0.2$;
- Demand uncertainty: ϵ_1, ϵ_2 are i.i.d and follow uniform distribution on $[-L, L]$, where $L \in \{0, 3, 6, 9, 12\}$;
- Ratio of component inventory to sum of base demands: $\frac{y}{a_1+a_2} \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$;

• Cancellation compensation ratio: $\gamma \in \{0, 0.05, 0.1, 0.15, 0.2\}$;

• Component unit cost and component usage: $C = 0$ and $A = 1$;

resulting in 2,500 instances.

Table 4.1 reports the overall statistics on the value of flexibilities. The value of pricing flexibility, either by itself or combined with overselling, is quite significant: with averages of 13.16% individually and 14.69% jointly, and some instances reaching 80%. In contrast, the firm's gain from pure overselling is relatively smaller, with an average of 1.01% and maximum of 10.72%. Hence, in general, pricing flexibility is more valuable to the firm than overselling flexibility. This observation is consistent with the asymptotic performance in the previous subsection. Further, it implies that for a firm with neither flexibility initially, the first step of improvement is to invest on the pricing flexibilty.

	Mean	(Min, Max)
Δ^{OO}	1.01%	(.00%, 10.72%)
Δ^{ID}	13.16%	(.00%, 80.00%)
$\Delta^{ID,OO}$	14.69%	(.00%, 80.00%)

Table 4.1. Value of flexibilities: summary

Figure 4.10 presents the average values of flexibilities as a function of each key parameter.

Price sensitivity β_1: For a fixed $\beta_2 = 5$, increasing β_1 means varying the extent to which the two products are asymmetric. As shown in Figure 4.10, when the two products are symmetric (i.e., when $\beta_1 = 5$), the joint value and the value of pricing both reach their corresponding peaks, while the value of overselling hits its bottom. With symmetric products, failing to adjust price according to component availability results in very low fill ratios for both products since component inventory is equally allocated to the two products. Inventory-dependent pricing effectively improves the fill ratios and profits for both products, and thus creates a maximal value. On the other hand, with symmetric products, two prices are the same and the yield loss does not exist. Hence, the benefit from overselling is minimal.

Substitution ratio θ_{12}: For a fixed θ_{21}, increasing θ_{12} also results in a change of the asymmetry in the two products and hence the effect is similar to what we have discussed for β_1.

Ratio of inventory to total base demand $\frac{y}{a_1+a_2}$: The value of overselling is very small when the level of component inventory is at the two extremes: with very low inventory, the firm can guarantee to sell all the inventory and overselling is unnecessary; while with very high inventory, the firm can fulfill all the orders and overselling is infeasible. The value of pricing, either by itself or

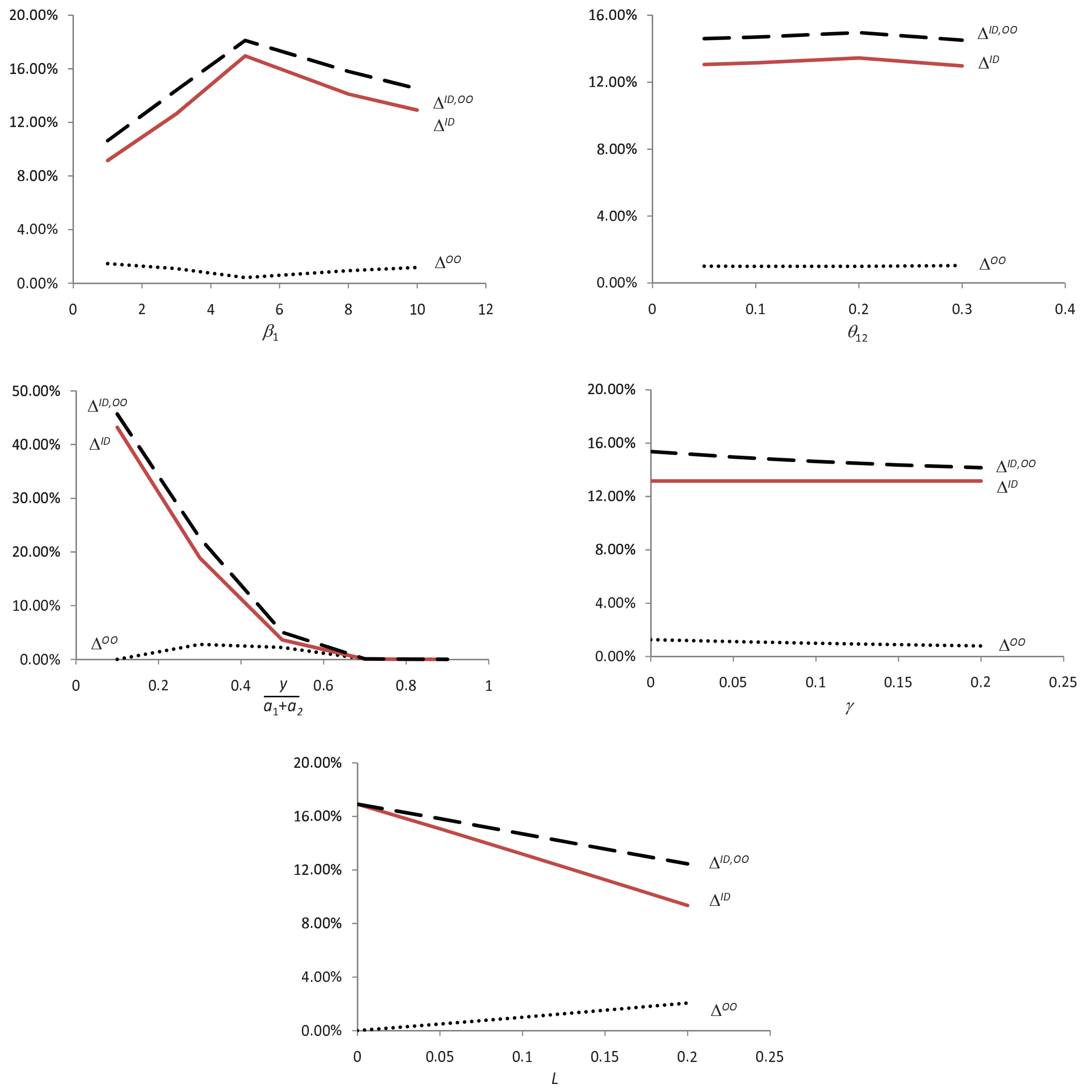


Figure 4.10. Value of flexibilities: effects of key parameters

combined with overselling, decreases in component inventory, as the optimal inventory-dependent prices eventually converge with the inventory-independent ones (i.e., unconstrained optimal prices).

Cancellation compensation ratio γ: When the compensation ratio increases, overselling becomes less desirable, and thus both the joint value and the value of overselling decrease. In the meantime, the individual value of pricing is independent of the compensation ratio.

Demand uncertainty L: With increased demand uncertainty, the firm expects high yield loss and spoilage loss, which enhances the value of ex-post component re-allocation, as well as the effectiveness of overselling strategy. In contrast, the ex-ante control of pricing becomes less efficient as demand variance increases and the value of pricing flexibility declines.

To summarize, the joint application of pricing and overselling flexibilities creates significant benefit to the firm, but individually, pricing flexibility is more valuable than overselling flexibility and should be given higher priority. In the meanwhile, the gain from inventory-dependent pricing tends to be larger with symmetric products, low component inventory, and low demand variability. In contrast, overselling brings maximal benefit when products are asymmetric, component inventory is medium, cancellation compensation is small, and demand variance is large.

4.5.1 Overselling May Benefit All

So far we have focused on the value of overselling flexibility to the firm, i.e., whether the overselling strategy is economically justifiable. Nevertheless, another important issue in implementing the overselling strategy is the "legal and public image aspects" associated with cancelling orders (Biyalogorsky et al. 1999). To help address this issue, Biyalogorsky et al. (1999) show that overselling can actually benefit the customers as well. Their analysis and conclusion, however, are limited within the framework they considered, which assumes a fixed pricing independent of inventory or whether the firm adopts the overselling strategy. In this subsection we extend their analysis to include the effect of inventory-dependent pricing or demand shaping.

We find that with demand shaping, overselling can also benefit all the entities: the firm, the customers, and the entire social surplus. Table 4.2 presents such an example. The detailed calculations of consumer surplus and social surplus are provided in the Appendix.

	z_1	z_2	o_1	o_2	p_1	p_2	Π	CS	SS
No Overselling (ID,NO)	27.2	23.9	35.4	24.6	28.1	9.3	953.6	368.8	1322.4
Optional Overselling (ID,OO)	27.7	27.6	39.7	39.6	27.6	8.6	981.6	397.6	1379.2

Table 4.2. An example where overselling increases firm's profit, consumer surplus, and social surplus: $a_1 = a_2 = 60$, $\beta_1 = 1$, $\beta_2 = 5$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $L = 12$, $y = 60$, $\gamma = 0.05$.

In the example shown in Table 4.2, the firm quotes lower prices and accepts more orders for both products. Customer surplus is increased, since fewer orders are blocked upon arrival and among the admitted orders, those getting fulfilled enjoy lower prices and those being cancelled obtain the compensation. Such win-win outcome is usually observed when total inventory is high or compensation ratio is low, in which cases the firm tends to lower prices after adopting overselling.

4.6 Conclusion

Matching stochastic product demand with limited component supply has always been a challenge for assemble-to-order firms. Two common risks faced by these firms are high-yield loss and inventory spoilage loss. In this paper, overselling strategy has been shown to have the potential to mitigate both sources of risks. Specifically, we find that with a reasonably low cancellation ratio, overselling is most valuable to the firm when demand uncertainty is high.

In addition, we consider the firm's joint optimization of overselling and demand shaping strategies and evaluate the benefits of these strategies. Specifically, our findings are as follows:

When the potential demand and component inventory are both very large, the partial postponement of order fulfillment policy can be asymptotically optimal for the continuous-time problem with dynamic control. That is, the dynamic control of order fulfillment can be substituted, with little lost in the total expected profit, by an optimal combination of ex-ante order acceptance and ex-post order fulfillment.

When the firm does not have flexibility to oversell, the decision problem with demand shaping has a well-behaved profit function and a unique optimal solution on pricing and order-acceptance. Nevertheless, when overselling is feasible, multiple optimal solutions may exist.

Overselling products is optimal only when component inventory is sufficiently large and cancellation compensation is sufficiently low. In those cases, the optimal prices under overselling tend to be lower than the no-overselling prices. In the meantime, with medium inventory level and relatively high cancellation compensation, the firm is better off rationing orders even if overselling is deployed.

The firm's gain from joint adopting demand shaping and overselling is significant. In the meanwhile, the benefit of overselling is relatively small compared to that of demand shaping. The value of pricing flexibility tends to be larger with symmetric products, low component inventory, and low demand variability. In contrast, overselling flexibility brings maximal benefit when products are asymmetric, component inventory is medium, cancellation compensation is small, and demand variance is large.

In addition to enhancing firms' profitability, product overselling can also improve consumer and social surpluses.

CHAPTER 5

Summary

This dissertation focuses on joint optimization of pricing and capacity decisions with explicit modelling of customer behavior. The three essays contained in the dissertation provide theoretical results and numerical evaluations of the joint decision problem within two different business frameworks: advance selling and assemble to order.

The first essay concentrates on a firm's optimal dynamic pricing and capacity allocation decisions when offering advance selling to customers with inter-dependent valuations. I show that customer behavior, particularly the degree of inter-dependence in valuations, dramatically influences the firms decisions on pricing (e.g., discount or premium pricing in advance) and capacity (e.g., whether to ration capacity in advance).

The second essay concentrates on a firm's signalling strategy in advance selling when information about quality is asymmetric. Specifically, I assess and compare the effectiveness and efficiency of signalling through pricing, capacity rationing, and advertising. I conclude that capacity rationing can be a very effective signal of quality, but can also be very costly for the firm to use. In addition, I find that asymmetric information about quality can make advance selling not beneficial at all.

The third essay concentrates on an assemble-to-order firm's joint pricing, order-acceptance, and order-fulfillment decisions when products' demands are price-based substitutable. I characterize the optimal joint decisions for firms with different flexibilities in pricing and overselling, and show that the value of jointly applying demand shaping and overselling is significant. In particular, I find that overselling is desirable when component inventory is sufficiently large and cancellation compensation is sufficiently low. Furthermore, I illustrate with an numerical example that, in addition to enhancing firms' profitability, product overselling can also improve consumer and social surpluses.

APPENDICES

APPENDIX A

Proofs in Chapter 2

A.1 Proof of Theorem 1

Proof: We first show that the spot profit function takes the same form for all four customer valuation models. Note that with unlimited capacity, expected sales equals to expected demand. Hence, it suffices to show that given D_2 customers in spot and spot price p_2, the expected spot demand is the same for all four customer valuation models.

• In deterministic valuation model, the portion of customers who buy in spot is $\overline{G}(p_2) = \Pr(\alpha \geq p_2)$. That is, exactly $D_2\overline{G}(p_2)$ customers have nonnegative utility and spot demand is $D_2\overline{G}(p_2)$.

• In heterogeneous valuation model, the valuation of customer i, α_i, is independently drawn from $G(\cdot)$ and the probability of customer i buys in spot is $\overline{G}(p_2)$. Hence, the spot demand follows a binomial distribution with mean $D_2\overline{G}(p_2)$.

• In homogeneous-1 valuation model, all customers share a same valuation α: if $\alpha \geq p_2$, all of the D_2 customers buy; otherwise, no one buys. Thus, the expected demand in spot is, again, $D_2\overline{G}(p_2)$.

• In homogeneous-k valuation model, the valuation for group i is independently drawn from $G(\cdot)$ and the probability of group i with valuation p_2 or higher is $\overline{G}(p_2)$. Hence, the number of subgroups with valuation p_2 or higher is binomially distributed with mean $k\overline{G}(p_2)$. Clearly, the expected spot demand is the size of the population in a subgroup times the expected number of subgroups with valuation p_2 or higher, i.e., $\frac{D_2}{k} \cdot k\overline{G}(p_2) = D_2\overline{G}(p_2)$.

Thus, it is clear that all four models share the expected spot profit function, as in equation (2.1). Consequently, the optimal spot price must be the same for the four valuation models, resulting in an identical expected spot profit. ■

A.2 Proof of Lemma 1

Proof: Since $\overline{G}(L) = 1$ and $\overline{G}(H) = 0$, any $p_2 > H$ or $p_2 < L$ cannot be optimal and it suffices to consider $p_2 \in [L, H]$. We divide the proof into three cases.

• ($c \geq \bar{p}$) No price yields a positive profit, thus $p_2^U = H$, the seller will not sell in spot, and

$\pi_2^*(p_1) = 0$.

• ($\underline{c} < c < \bar{p}$) Taking the derivative with respect to p_2, we have

$$\frac{\partial \pi_2(p_2)}{\partial p_2} = D_2[\overline{G}(p_2) - (p_2 - c)g(p_2)] = D_2 g(p_2)\left[\frac{\overline{G}(p_2)}{g(p_2)} - (p_2 - c)\right]$$

Evaluating at two boundary points $p_2 = L$ and $p_2 = H$, we have

$$\left.\frac{\partial \pi_2}{\partial p_2}\right|_{p_2=L} = D_2\{1 - (L-c)g(L)\} > 0, \quad \left.\frac{\partial \pi_2}{\partial p_2}\right|_{p_2=H} = D_2\{0 - (H-c)g(H)\} < 0.$$

Hence, the optimal solution is interior. Since $\frac{\overline{G}(x)}{g(x)} - x$ is strictly decreasing, $\pi(p_2)$ is strictly quasi-concave in p_2 and solving the first-order condition, $p_2^* = c + \frac{\overline{G}(p_2^*)}{g(p_2^*)}$, results in a unique optimal solution in (L, H).

• ($c \leq \underline{c}$) It is easy to show that $\frac{\partial \pi_2}{\partial p_2}|_{p_2=L} \leq 0$ and $\frac{\partial \pi_2}{\partial p_2}|_{p_2=H} < 0$. Furthermore, $G(\cdot)$ satisfies IFR, thus $\frac{\partial \pi_2}{\partial p_2} \leq 0$ for all $p_2 \in [L, H]$. Thus, p_2^U must be the boundary point, L. ■

A.3 Proof of Proposition 1 and Proposition 2

Proof: The result trivially follows as $\min\{x, \alpha\} \leq x$. ■

A.4 Proof of Lemma 2

Proof: a) From Lemma 1, $p_2^U = L$ for all $c \leq \underline{c}$. At this price, all customers will buy in spot independent of α since $\alpha - p_2^U \geq 0$ for any α. Thus,

$$\pi_S^U(c) = (N_1 + N_2)(p_2^U - c)$$

If the seller sells in advance, he will charge the same price in both advance and spot since

$$p_1^{\max,U} = \mathrm{E}[\alpha] - \int \max(\alpha - p_2^U, 0)\, dG(\alpha) = p_2^U$$

Thus, all advance customers will buy and

$$\pi_A^U(c) = (N_1 + N_2)(p_2^U - c)$$

Clearly, $\pi_A^U(c) = \pi_S^U(c)$. All customers buy regardless of whether the seller offers advance selling.

b) From Lemma 1, $p_2^U = H$ for all $c \geq H$. Since $\alpha - p_2^U \leq 0$ for any α, no customers buy in spot. If the seller chooses to sell in advance, he would charge

$$p_1^{\max,U} = \mathrm{E}[\alpha] - \int \max(\alpha - p_2^U, 0)\, dG(\alpha) = \mathrm{E}[\alpha] < H \leq c$$

But doing so results in loss, as for all $c \geq H$,

$$\pi_A^U(c) - \pi_S^U(c) = N_1\left[p_1^{\max,U} - c - (p_2^U - c)\overline{G}(p_2^U)\right] = N_1\left[\mathrm{E}[\alpha] - c\right] < 0 \tag{A.1}$$

Note that $\pi_S^U(c)$ is continuous in c due to the continuity of $g(\cdot)$ and $\overline{G}(\cdot)$, as well as the IFR property. In the meanwhile, p_2^U is a continuous function of c, so is $p_1^{\max,U}$ and, therefore, also $\pi_A^U(c)$. As $\pi_A^U(c) - \pi_S^U(c)$ is continuous in c and $\pi_A^U(\underline{c}) - \pi_S^U(\underline{c}) = 0$, there must exist a $\bar{c} \in [\underline{c}, \bar{p})$, such that $\pi_A^U(c) = \pi_S^U(c)$ at $c = \bar{c}$ and $\pi_A^U(c) < \pi_S^U(c)$ for all $c > \bar{c}$. ■

A.5 Proof of Theorem 2

Proof: We first note two boundary conditions: by Lemma 1, $\pi_A^U(c) = \pi_S^U(c)$ for $c \leq \underline{c}$; for $c \geq H$, from equation (A.1), $\pi_A^U(c) - \pi_S^U(c)$ is negative and strictly decreasing in c. Meanwhile, it is shown in the proof of Lemma 2 that $\pi_A^U(c) - \pi_S^U(c)$ is continuous in c. Hence, to prove the theorem, it suffices to show that $\pi_A^U(c) - \pi_S^U(c)$ is quasi-concave for $\underline{c} < c < H$. Below we prove the quasi-concavity by showing that the derivative of the function is positive-negative. For the ease of exposition, we suppress c in $\pi_A^U(c)$ and $\pi_S^U(c)$.

For $\underline{c} < c < H$,

$$\begin{aligned}\pi_A^U - \pi_S^U &= N_1 \left\{ p_1^{\max,U} - c - (p_2^U - c)\overline{G}(p_2^U) \right\} \\ &= N_1 \left\{ \mathrm{E}[\alpha] - \mathrm{E}[\max(\alpha - p_2^U, 0)] - c - (p_2^U - c)\overline{G}(p_2^U) \right\}\end{aligned}$$

Taking derivative with respect to c,

$$\frac{d\{\pi_A^U - \pi_S^U\}}{dc} = N_1 \left\{ -\frac{d\mathrm{E}(\max(\alpha - p_2^U, 0))}{dc} - 1 - \frac{d(p_2^U - c)\overline{G}(p_2^U)}{dc} \right\}$$

where

$$\frac{d\mathrm{E}(\max(\alpha - p_2^U, 0))}{dc} = \frac{d\mathrm{E}(\max(\alpha - p_2^U, 0))}{dp_2^U} \cdot \frac{dp_2^U}{dc} = -\overline{G}(p_2^U) \cdot \frac{dp_2^U}{dc} \tag{A.2}$$

and since p_2^U is the solution to equation (1) (see Lemma 1),

$$\begin{aligned}\frac{d(p_2^U - c)\overline{G}(p_2^U)}{dc} &= \frac{\partial(p_2^U - c)\overline{G}(p_2^U)}{\partial p_2^U} \cdot \frac{dp_2^U}{dc} + \frac{\partial(p_2^U - c)\overline{G}(p_2^U)}{\partial c} \\ &= \left[\overline{G}(p_2^U) - (p_2^U - c)g(p_2^U)\right] \cdot \frac{dp_2^U}{dc} - \overline{G}(p_2^U) = -\overline{G}(p_2^U), \qquad \text{(A.3)} \\ \frac{dp_2^U}{dc} &= \frac{1}{1 - \left(\frac{\overline{G}(x)}{g(x)}\right)'\Big|_{x=p_2^U}}. \qquad \text{(A.4)}\end{aligned}$$

Applying equation (C.10) through (A.4) to the expression of $\frac{d\{\pi_A^U - \pi_S^U\}}{dc}$,

$$\begin{aligned}
\frac{d\{\pi_A^U - \pi_S^U\}}{dc} &= N_1 \left\{ \frac{\overline{G}(x)}{1 - \left(\frac{\overline{G}(x)}{g(x)}\right)'} - G(x) \right\}\Bigg|_{x=p_2^U} \\
&= N_1 \left\{ \frac{1}{1 - \left(\frac{\overline{G}(x)}{g(x)}\right)'} \left[\overline{G}(x) + G(x)\left(\frac{\overline{G}(x)}{g(x)}\right)' + \overline{G}(x) - 1 \right] \right\}\Bigg|_{x=p_2^U} \\
&= N_1 \left\{ \frac{1}{1 - \left(\frac{\overline{G}(x)}{g(x)}\right)'} \left[\left(\frac{G(x)\overline{G}(x)}{g(x)}\right)' + \overline{G}(x) - 1 \right] \right\}\Bigg|_{x=p_2^U}
\end{aligned}$$

Note that by equation (A.4) and the IFR property, p_2^U strictly increases in c.[1] In the meantime, $\left(\frac{\overline{G}(x)}{g(x)}\right)' \leq 0$ by the IFR property and $\left(\frac{G(x)\overline{G}(x)}{g(x)}\right)' + \overline{G}(x) - 1$ is positive-negative from Assumption (A). Hence the derivative above is positive-negative in c and the proof is complete. ■

A.6 Proof of Lemma 3

Proof: It suffices to consider the case with $T - S > 0$. If $T - S \geq (N_1 + N_2 - S)\overline{G}(p_2^U)$, then clearly the unlimited-capacity optimal price p_2^U is optimal. Now suppose $0 < T - S < (N_1 + N_2 - S)\overline{G}(p_2^U)$. The continuity of $G(\cdot)$ guarantees that $p_2^B(S)$ always exists. Since $\overline{G}(p_2)$ decreases in p_2, we have $p_2^B(S) > p_2^U$. Recall, from Lemma 1, that $(p_2 - c)\overline{G}(p_2)$ is quasi-concave and is maximized at p_2^U, thus $(N_1 + N_2 - S)(p_2 - c)\overline{G}(p_2)$ decreases in p_2 for $p_2 > p_2^B(S)$. Therefore, $\pi_2(p_2, S)$ increases in p_2 for $p_2 \leq p_2^B(S)$ and decreases in p_2 for $p_2 > p_2^B(S)$. Clearly, the market-clearing price $p_2^B(S)$ is optimal. ■

A.7 Proof of Theorem 3

Recall that the seller's total profit function is

$$\pi_{AS}^D(S) = \left(p_1^{\max,D}(S) - c\right) S + (p_2^*(S) - c)\min(T - S, (N_1 + N_2 - S)\overline{G}(p_2^*(S)))$$

For easy of exposition, we define $p_1^{\max}(p_2) = \mathrm{E}[\min(p_2, \alpha)]$, which is the maximum price the seller can charge in advance if the spot price is p_2 and no shortage occurs in spot. Clearly, $p_1^{\max,D}(S) = p_1^{\max}(p_2^*(S))$ and $p_1^{\max,U} = p_1^{\max}(p_2^U)$. Hence,

$$\pi_{AS}^D(S) = (p_1^{\max}(p_2^*(S)) - c)\, S + (p_2^*(S) - c)\min(T - S, (N_1 + N_2 - S)\overline{G}(p_2^*(S)))$$

[1]To see why the monotonicity is strict, note that for $\underline{c} < c < H$, $p_2^U \in (L, H)$. Since $g(x)$ is a probability density function and continuous differentiable, $|g'(x)| < +\infty$ for $x \in (L, H)$. Also note that $g(x) > 0$ for $x \in (L, H)$. Hence, $\left(\frac{\overline{G}(x)}{g(x)}\right)' = -1 - \frac{\overline{G}(x)g'(x)}{g(x)} > -\infty$ for $x \in (L, H)$ and by equation (A.4) and the IFR property, $\frac{dp_2^U}{dc} > 0$.

Note that as shown in Lemma 3, the remaining capacity influences the seller's spot pricing strategy. When sufficiently large capacity remains in spot (this happens when S is small), the seller charges the unlimited-capacity spot price p_2^U. On the other hand, when only small capacity remains, the seller charges the maximum price, $p_2^B(S)$, that clears the remaining capacity $T - S$.

Let $S^=$ be the *lowest* amount of capacity used in advance above which the remaining capacity will be binding in the spot period. Clearly, $S^=$ is the solution to $\frac{T-S}{N_1+N_2-S} = \overline{G}(p_2^U)$ and $S^= = \frac{T-(N_1+N_2)\overline{G}(p_2^U)}{G(p_2^U)}$. From Lemma 3, $p_2^* = p_2^U$ for $0 \leq S \leq S^=$, and $p_2^* = p_2^B(S)$ for $S \geq S^=$.

We define two functions, $f^U(S)$ and $f^B(S)$, corresponding to the seller's total profits when remaining capacity, $T-S$, is not binding and when it is binding, respectively. For $S \in [0, \min(T, N_1)]$,

$$\begin{aligned} f^U(S) &:= \left(p_1^{\max}(p_2^U) - c\right) S + (N_1 + N_2 - S)\overline{G}(p_2^U)(p_2^U - c) \\ &= \frac{1}{N_1}\left[\pi_A^U - \pi_S^U\right] S + (N_1 + N_2)\overline{G}(p_2^U)(p_2^U - c) \\ f^B(S) &:= \left(p_1^{\max}\left(p_2^B(S)\right) - c\right) S + (T - S)\left(p_2^B(S) - c\right) \end{aligned}$$

Hence,

$$\pi_{AS}^D(S) = \begin{cases} f^U(S) & \text{for } S \leq S^= \text{ and } S \in [0, \min(T, N_1)] \\ f^B(S) & \text{for } S > S^= \text{ and } S \in [0, \min(T, N_1)] \end{cases}$$

Note that $f^U(S)$ is a linear function of S and its slope is determined by c. Lemma A.7.1 below characterizes $f^B(S)$ and is used in the proof of the theorem. We defer the proof of Lemma A.7.1 to the end of this section.

Lemma A.7.1

i) $f^B(S)$ *is strictly quasi-concave in* S *and has a unique maximizer* S^B *on* $[0, \min(T, N_1)]$, *which is independent of* c.

ii) S^B *is nondecreasing in* T.

iii) $S^B = 0$ *for* $0 < T \leq T_1$.

iv) There exists a critical number $T^D \in (T_1, T_2)$ *such that* $0 < S^B < \min(T, N_1)$ *for* $T_1 < T < T^D$, *and* $S^B = \min(T, N_1)$ *for* $T^D \leq T < N_1 + N_2$.

To find S^* which maximizes $\pi_{AS}^D(S)$ in $[0, \min(T, N_1)]$, we first note the following two facts: first, depending on the combination of T and c, $S^=$ may be out of the feasible domain, i.e., it can be negative or greater than $\min(T, N_1)$; second, for some values of c, $f^U(S)$ is linearly decreasing in S, which makes $\pi_{AS}^D(S)$ a bimodal function (as illustrated in Figure A.1). These two facts, together with the properties of $f^B(S)$ shown in Lemma A.7.1, naturally divide the proof into five cases for different combination of T and c, as illustrated in Figure A.2.

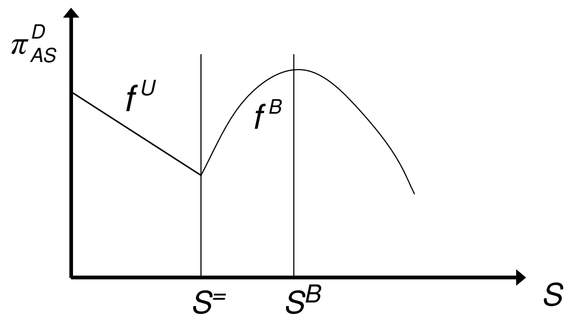


Figure A.1. π^D_{AS} may be bimodal when $c > \bar{c}$

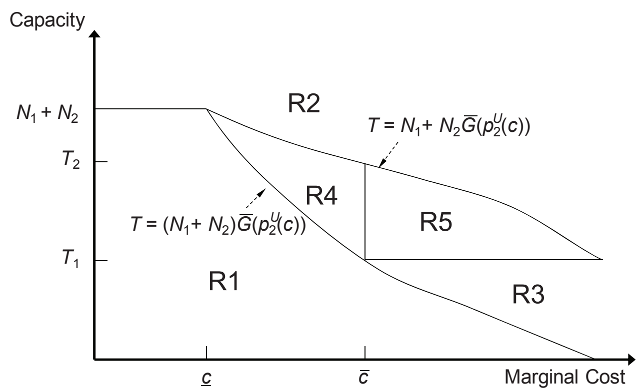


Figure A.2. Five regions in the capacity-cost space

(R1): $T \leq (N_1 + N_2)\overline{G}(p_2^U(c))$.

In this case, $S^= \leq 0$, implying that the total capacity is small enough that independent of S, the remaining capacity is always binding in spot. Clearly $\pi^D_{AS}(S) = f^B(S)$ for all feasible S and $S^* = S^B$. Therefore, the result follows from Lemma A.7.1 (iii)-(iv).

(R2): $T \geq N_1 + N_2\overline{G}(p_2^U(c))$.

In this case, $S^= \geq N_1$, implying that the total capacity is large enough that independent of S, the remaining capacity is never binding in spot. Thus, $\pi^D_{AS}(S) = f^U(S)$ for all feasible S. As in the unlimited-capacity case, for $c \leq \bar{c}$, we have $\pi^U_A - \pi^U_S \geq 0$, implying $S^* = N_1$, and for $c > \bar{c}$, $S^* = 0$.

Now we consider the remaining region:

$$\mathcal{R} = \{(c, T) | (N_1 + N_2)\overline{G}(p_2^U(c)) < T < N_1 + N_2\overline{G}(p_2^U(c))\}$$

Note that $0 < S^= < \min(T, N_1)$ in this region. As a result, the amount of capacity allocated in advance (i.e., S) directly affects the spot price: $p_2^* = p_2^U$ for $S < S^=$, $p_2^* = p_2^B(S)$, otherwise. We further divide this region into three cases (R3), (R4), and (R5).

(R3): $\mathcal{R} \cap \{(c,T) | T \leq T_1 = (N_1+N_2)\overline{G}(p_2^U(\bar{c})), c \geq \bar{c}\}$.

For $c \geq \bar{c}$, $\pi_A^U - \pi_S^U \leq 0$ and hence $f^U(S)$ is nonincreasing in S. In addition, $f^B(S)$ is decreasing in S from Lemma A.7.1 (i) and (iii). Therefore, $\pi_{AS}^D(S)$ (which is either f^U or f^B, where feasible) is nonincreasing in S and maximized at $S^* = 0$.

(R4): $\mathcal{R} \cap \{(c,T) | T > T_1, c \leq \bar{c}\}$.

Unlike the previous case, $f^U(S)$ is nondecreasing in S for $c \leq \bar{c}$. Below we prove that $S^* = S^B$ by showing that the derivative of $f^B(S)$ is nonnegative at $S = S^=$. Recall that

$$f^B(S) = T\left(p_2^B(S) - c\right) - S\left(p_2^B(S) - p_1^{\max}\left(p_2^B(S)\right)\right)$$

where $p_1^{\max}(p_2)$ is a function of p_2. Taking derivative with respect to S, we get

$$\frac{df^B(S)}{dS} = T \cdot \frac{dp_2^B(S)}{dS} - S\left(1 - \frac{dp_1^{\max}(p_2^B(S))}{dp_2^B(S)}\right) \cdot \frac{dp_2^B(S)}{dS} - \left(p_2^B(S) - p_1^{\max}\left(p_2^B(S)\right)\right) \tag{A.5}$$

Recall that $p_1^{\max}(p_2^B(S))$ is defined by

$$p_1^{\max}(p_2^B(S)) = \mathrm{E}[\alpha] - \mathrm{E}[\max(\alpha - p_2^B(S), 0)] = \mathrm{E}[\alpha] - \int_{y \geq p_2^B(S)} y - p_2^B(S)\, dG(y)$$

Using Leibnitz's rule,

$$\frac{dp_1^{\max}(p_2^B(S))}{dp_2^B(S)} = \overline{G}(p_2^B(S)) \tag{A.6}$$

Meanwhile, by definition of $p_2^B(S)$, $\overline{G}(p_2^B) = \frac{T-S}{N_1+N_2-S}$. Taking derivative with respect to S on both sides and rearranging:

$$\frac{dp_2^B}{dS} = \frac{N_1+N_2-T}{g(p_2^B)(N_1+N_2-S)^2} = \frac{G(p_2^B)}{g(p_2^B)(N_1+N_2-S)} > 0 \tag{A.7}$$

Replacing $\frac{dp_1^{\max}(p_2^B(S))}{dp_2^B(S)}$ and $\frac{dp_2^B(S)}{dS}$ with equations (A.6) and (A.7) in equation (A.5), and applying some algebra, we get

$$\frac{df^B(S)}{dS} = \frac{N_1+N_2}{N_1+N_2-S} \cdot \frac{\overline{G}(p_2^B(S))G(p_2^B(S))}{g(p_2^B(S))} - \left(p_2^B(S) - p_1^{\max}\left(p_2^B(S)\right)\right) \tag{A.8}$$

Now, applying the fact that $p_2^B(S^=) = p_2^U$, together with equation (2.2) and definition of $S^=$, to equation (A.8), we get

$$\left.\frac{df^B(S)}{dS}\right|_{S=S^=} = \frac{(N_1+N_2)\left[G(p_2^U)\right]^2}{N_1+N_2-T}(p_2^U - c) - \left(p_2^U - p_1^{\max}\left(p_2^U\right)\right)$$

Rearranging and noting that $G(p_2^U) = \frac{N_1+N_2-T}{N_1+N_2-S^=}$ and $\frac{N_1+N_2}{N_1+N_2-S^=} > 1$,

$$\begin{aligned}\left.\frac{df^B(S)}{dS}\right|_{S=S^=} &= (p_1^{\max,U} - c) - (p_2^U - c)\left\{1 - \frac{(N_1+N_2)\left[G(p_2^U)\right]^2}{N_1+N_2-T}\right\} \\ &= (p_1^{\max,U} - c) - (p_2^U - c)\left\{1 - \frac{(N_1+N_2)G(p_2^U)}{N_1+N_2-S^=}\right\} \\ &> (p_1^{\max,U} - c) - (p_2^U - c)\left\{1 - G(p_2^U)\right\} \\ &= \frac{1}{N_1}\left[\pi_A^U - \pi_S^U\right] \geq 0.\end{aligned}$$

Hence, $S^* = S^B$ and S^* follows from Lemma A.7.1 (i) and (iv).

(R5): $\mathcal{R} \cap \{(c,T)|T > T_1, c \geq \bar{c}\}$.

As in case (R3), $f^U(S)$ is nonincreasing in S. Therefore, $S^* = S^B$ if $S^= \leq S^B$ and $f^B(S^B) \geq f^U(0)$, and $S^* = 0$ otherwise. To show the existence of a switching curve $c^D(T)$ such that $S^* = S^B$ if and only if $\bar{c} \leq c \leq c^D(T)$, we show that (R5-i) $f^B(S^B) \geq f^U(0)$ implies $S^= \leq S^B$. This fact immediately implies that $S^* = S^B$ if and only if $f^B(S^B) \geq f^U(0)$; and (R5-ii) $f^B(S^B) - f^U(0)$ is nonincreasing in c (i.e., if no advance selling is better than selling S^B units in advance at some c_0, it is still better at all $c > c_0$). Based on (R5-i) and (R5-ii), we define $c^D(T)$ in (R5-iii) and show that it is well-defined for each $T \in [T_1, T_2]$. In the end we show $c^D(T_1) = c^D(T_2) = \bar{c}$ in (R5-iv) to complete the proof of case (R5).

(R5-i) It suffices to show that if $S^= > S^B$, then $f^B(S^B) < f^U(0)$. To this end, first note that by the definition of $S^=$, when $S^= > S^B$, $p_2^B(S^B) < p_2^U$. This fact, together with the facts that $p_1^{\max}(p_2)$ is nondecreasing in p_2 and that p_2^U is the unique maximizer of $(p_2 - c)\overline{G}(p_2)$ (proved in Lemma 1), implies $p_1^{\max}(p_2^B(S^B)) \leq p_1^{\max}(p_2^U)$ and $(p_2^B(S^B) - c)\overline{G}(p_2^B(S^B)) < (p_2^U - c)\overline{G}(p_2^U)$. Applying these results to the expression of $f^B(S^B)$:

$$\begin{aligned}f^B(S^B) &= \left(p_1^{\max,D}\left(S^B\right) - c\right)S^B + \left(T - S^B\right)\left(p_2^B(S^B) - c\right) \\ &= \left(p_1^{\max}(p_2^B(S^B)) - c\right)S^B + \left(N_1 + N_2 - S^B\right)\left(p_2^B(S^B) - c\right)\overline{G}(p_2^B(S^B)) \\ &< \left(p_1^{\max}(p_2^U) - c\right)S^B + \left(N_1 + N_2 - S^B\right)\left(p_2^U - c\right)\overline{G}(p_2^U) \\ &= f^U(S^B) \leq f^U(0)\end{aligned}$$

where the last inequality is by the fact that $f^U(S)$ is nonincreasing in S.

(R5-ii) Consider now the difference between $f^B(S^B)$ and $f^U(0)$. Define the profit difference as a function of c:

$$\begin{aligned}h(c) &= f^B(S^B) - f^U(0) \\ &= \left\{\left(p_1^{\max,D}\left(S^B\right) - c\right)S^B + \left(T - S^B\right)\left(p_2^B(S^B) - c\right)\right\} - (N_1 + N_2)\overline{G}\left(p_2^U(c)\right)\left(p_2^U(c) - c\right).\end{aligned}$$

Note that neither $p_2^B(S^B)$ nor $p_1^{\max,D}(S^B)$ depends on c, since S^B is independent of c by Lemma A.7.1(i). Thus

$$\frac{dh(c)}{dc} = -T + (N_1 + N_2)\overline{G}\left(p_2^U(c)\right) < 0$$

Hence, $h(c)$ strictly decreases in c.

(R5-iii) Based on (R5-i) and (R5-ii), for given $T \in [T_1, T_2]$, we define

$$c^D(T) = \max\{c \geq \bar{c} : T \leq N_1 + N_2\overline{G}(p_2^U(c)), h(c) \geq 0\}.$$

Note that $c^D(T)$ exists for each $T \in [T_1, T_2]$. To see this, it suffices to show that all conditions are met at $c = \bar{c}$. The first condition is easily satisfied since $T \leq N_1 + N_2\overline{G}(p_2^U(\bar{c})) = T_2$ for any $T \in [T_1, T_2]$. It is also easy to verify the second condition, since $\frac{df^B(S)}{dS}\Big|_{S=S^=} \geq 0$ at $c = \bar{c}$ from the proof of case (R4) above. For the last condition, we first note that the definition of $\bar{c}$ implies $\pi_A^U(\bar{c}) = \pi_S^U(\bar{c})$. Thus, for all $S \in [0, S^=]$, at $c = \bar{c}$,

$$\pi_{AS}^D(S) = f^U(S) = \left[\pi_A^U - \pi_S^U\right] S + (N_1 + N_2)\overline{G}(p_2^U)(p_2^U - c) = (N_1 + N_2)\overline{G}(p_2^U)(p_2^U - c).$$

Then, from the definition of S^B, $h(\bar{c}) = \pi_{AS}^D(S^B) - \pi_{AS}^D(0) \geq \pi_{AS}^D(S^=) - \pi_{AS}^D(0) = 0$. Thus, $c^D(T)$ is well defined: $S^* = S^B$ if $\bar{c} \leq c \leq c^D(T)$ and $S^* = 0$ for $c > c^D(T)$.

(R5-iv) Finally, we show that $c^D(T) = \bar{c}$ at the two boundary points, $T_1 = (N_1 + N_2)\overline{G}(p_2^U(\bar{c}))$ and $T_2 = N_1 + N_2\overline{G}(p_2^U(\bar{c}))$. To show $c^D(T_1) = \bar{c}$, it suffices to show that when $T = T_1$, $S^= = S^B$ at $c = \bar{c}$ and $S^= > S^B$ for all $c > \bar{c}$. When $T = T_1$, $S^B = 0$ by Lemma A.7.1(iii). In the meanwhile, by definition of $S^=$ and T_1, $S^= = 0$ when $T = T_1$ and $c = \bar{c}$, which immediately implies $S^= = S^B$ at $c = \bar{c}$. Furthermore, by (R5-i) and the fact $\frac{dp_2^U}{dc} > 0$ at $c = \bar{c}$ (see proof of Theorem 2), $S^= > S^B$ for $c > \bar{c}$. Hence, $c^D(T_1) = \bar{c}$. Similarly, we can show $c^D(T_2) = \bar{c}$.

Figure A.3 summarizes the optimal solution S^* for all five regions, where S^B is characterized by Lemma A.7.1. Note that due to the nature of (R2), we can expand the domain of $c^D(T)$ to $T > T_2$ by defining $c^D(T) = \bar{c}$ for all $T > T_2$. Theorem 3 immediately follows.

Proof of Lemma A.7.1

Proof: i) To show the strict quasi-concavity of $f^B(S)$, it suffices to show that (i-1) the derivative of $f^B(S)$ with respect to S is positive-negative; (i-2) there exists at most one S in $[0, \min(T, N_1)]$ satisfying the first-order condition $\frac{df^B(S)}{dS} = 0$.

(i-1) Recall equation (A.8):

$$\frac{df^B(S)}{dS} = \frac{N_1 + N_2}{N_1 + N_2 - S} \cdot \frac{\overline{G}(p_2^B(S))G(p_2^B(S))}{g(p_2^B(S))} - \left(p_2^B(S) - p_1^{\max}\left(p_2^B(S)\right)\right)$$

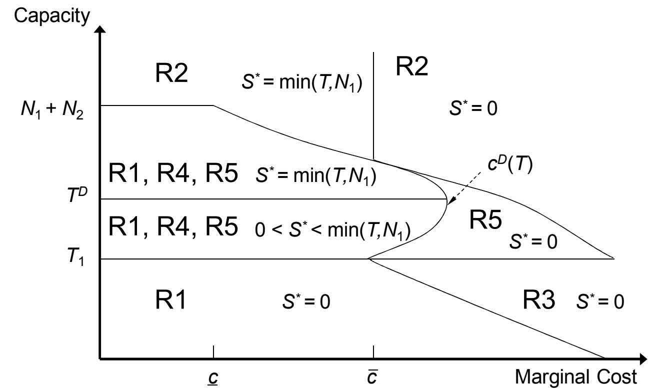


Figure A.3. Summary of optimal S^* for the five regions

By definition of $p_1^{\max}\left(p_2^B(S)\right)$,

$$p_2^B(S) - p_1^{\max}(p_2^B(S)) = \int_{y \le p_2^B(S)} p_2^B(S) - y \, dG(y) \tag{A.9}$$

Rewriting the right-hand-side of equation (A.9) as a double integral and changing the order of integration,

$$p_2^B(S) - p_1^{\max}(p_2^B(S)) = \int_{y \le p_2^B(S)} \int_y^{p_2^B(S)} g(y) \, dt \, dy = \int_{t \le p_2^B(S)} G(t) \, dt \tag{A.10}$$

Replacing $p_2^B(S) - p_1^{\max}(p_2^B(S))$ with the above identity in equation (A.8), we get

$$\frac{df^B(S)}{dS} = \frac{N_1+N_2}{N_1+N_2-S} \cdot \frac{\overline{G}(p_2^B(S))G(p_2^B(S))}{g(p_2^B(S))} - \int_{y \le p_2^B(S)} G(y) \, dy \tag{A.11}$$

$$= \left\{ \frac{N_1+N_2}{N_1+N_2-T} \cdot \frac{\overline{G}(x)\left(G(x)\right)^2}{g(x)} - \int_{y \le x} G(y) \, dy \right\} \Bigg|_{x=p_2^B(S)} \tag{A.12}$$

Note that $p_2^B(S)$ increases in S and $p_2^B(S) = \overline{G}^{-1}(\frac{T-S}{N_1+N_2-S}) \in (L, H]$ for $T < N_1 + N_2$ and $S \in [0, \min(T, N_1)]$. Hence, to show $\frac{df^B(S)}{dS}$ is positive-negative in S, it suffices to show that the terms inside the braces in equation (A.12) is positive-negative in $x \in (L, H]$. Define the terms inside the braces as a function of x, $M(x)$:

$$M(x) = \frac{N_1+N_2}{N_1+N_2-T} \cdot \frac{\overline{G}(x)\left(G(x)\right)^2}{g(x)} - \int_{y \le x} G(y) \, dy. \tag{A.13}$$

Taking derivative with respect to x and applying Assumption (A), we observe that the derivative is positive-negative:

$$M'(x) = \frac{N_1+N_2}{N_1+N_2-T} \left[G(x) \left(\frac{\overline{G}(x)G(x)}{g(x)} \right)' + \overline{G}(x)G(x) \right] - G(x)$$

$$= G(x) \frac{N_1+N_2}{N_1+N_2-T} \left[\left(\frac{\overline{G}(x)G(x)}{g(x)} \right)' + \overline{G}(x) - \frac{N_1+N_2-T}{N_1+N_2} \right]. \tag{A.14}$$

Furthermore, $M(L) = \frac{N_1+N_2}{N_1+N_2-T} \cdot \frac{(G(L))^2}{g(L)} \geq 0$. Hence, $M(x)$ must be positive-negative. Thus, $\frac{df^B(S)}{dS}$ itself must be positive-negative.

(i-2) Notice that $p_2^B(S) > L$ for all feasible S and $p_2^B(S)$ is a strictly monotone function in S. Thus, it suffices to show that there can be no more than one solution to $M(x) = 0$ in $x \in (L, H]$.

To this end, we consider two cases, $M(L) > 0$ and $M(L) = 0$. If $M(L) > 0$, the uniqueness of solution easily follows from the facts that $M(x)$ is continuous and that both $M(x)$ and $M'(x)$ are positive-negative in x.

Now we prove the case $M(L) = 0$ by contradiction. Suppose there exist two distinct solutions to $M(x) = 0$ on $(L, H]$ and denote them by x_0 and x_1 ($L < x_0 < x_1 \leq H$). Such a case is possible only if $M(x) = 0$ and $M'(x) = 0$ for all x that $L < x < x_0$. To see this, suppose there exists a point $x_2 \in (L, x_0)$ with $M(x_2) > 0$. Since $M(x_0) = 0$, there must exist a point $x_3 \in (x_2, x_0)$ such that $M'(x_3) < 0$. From the fact that $M'(x)$ is positive-negative, $M'(x) < 0$ for all $x > x_3$, implying that $M(x) < 0$ for all $x > x_0$. This, however, contradicts the fact $M(x_1) = 0$ and $x_1 > x_0$.

Next we show a contradiction from the fact that $M(x) = 0$ and $M'(x) = 0$ for all x that $L < x < x_0$. By $M(x) = 0$, rearranging equation (A.13), we have

$$\frac{\overline{G}(x)G(x)}{g(x)} = \frac{N_1 + N_2}{N_1 + N_2 - T} \cdot \frac{\int_{y \leq x} G(y)\, dy}{G(x)} \tag{A.15}$$

Since $g(\cdot)$ is log-concave, $\int_{y \leq x} G(y)\, dy$ is also log-concave (Bergstrom and Bagnoli, 2005). From this, it follows immediately that $\frac{\int_{y \leq x} G(y)\, dy}{G(x)}$ and $\frac{\overline{G}(x)G(x)}{g(x)}$ are both (weakly) increasing in $x \in (L, x_0)$. As a result, $\left(\frac{\overline{G}(x)G(x)}{g(x)}\right)'$ must be nonnegative. Combining this with $M'(x) = 0$, we have $\overline{G}(x) \leq \frac{N_1+N_2-T}{N_1+N_2}$ for $x \in (L, x_0)$. However, this cannot occur as $\overline{G}(x)$ is a strictly decreasing function in x on support $[L, H]$ and converges to 1 at L. Therefore, there can be no more than one $x \in (L, H]$ satisfying $M(x) = 0$. This completes the proof of the strict quasi-concavity of $f^B(S)$.

Define

$$S^B = \begin{cases} 0 & \text{if } \left.\frac{df^B(S)}{dS}\right|_{S=0} < 0 \\ \text{the solution to} \frac{df^B(S)}{dS} = 0 & \text{if } \left.\frac{df^B(S)}{dS}\right|_{S=0} \geq 0 \text{ and } \left.\frac{df^B(S)}{dS}\right|_{S=\min(T,N_1)} < 0 \\ \min(T, N_1) & \text{otherwise} \end{cases} \tag{A.16}$$

By the strict quasi-concavity of $f^B(S)$, S^B is the unique maximizer of $f^B(S)$. From (A.11), $df^B(S)/dS$ is independent of c. Hence S^B is also independent of c.

ii) Let $S^B(T)$ be the maximizer of $f^B(S)$ for given T. To explicitly recognize the dependence of $f^B(S)$ on parameter T, within this proof we explicitly write $f^B(S|T)$. Since $S^B(T) \geq 0$ for all T, the monotonicity follows if we prove **(a)** if $0 < S^B(T) < \min(T, N_1)$ for some T, $S^B(T') \geq S^B(T)$ for all $T' > T$, and **(b)** if $S^B(T) = \min(T, N_1)$, then $S^B(T') = \min(T', N_1)$ for all $T' > T$.

We prove **(a)** by contradiction. Suppose $S^B(T') < S^B(T)$ for some $T' > T$. From part (i), $\frac{df^B}{dS}(S = S^B(T) \; |T) = 0$ and $\frac{df^B}{dS}(S = S^B(T) \; |T') < 0$. Define $x_1 = \overline{G}^{-1}\left(\frac{T - S^B(T)}{N_1 + N_2 - S^B(T)}\right)$ and $x_2 = \overline{G}^{-1}\left(\frac{T' - S^B(T)}{N_1 + N_2 - S^B(T)}\right)$. Clearly $x_1 > x_2$ as $\overline{G}(x)$ strictly decreases in x. From the definitions of $M(x|T)$, x_1 and x_2, $M(x_1|T) = 0$ and $M(x_2|T') < 0$. On the other hand, $M(x|T)$ is increasing in T for given x as $\frac{\partial M(x|T)}{\partial T} = \frac{N_1+N_2}{(N_1+N_2-T)^2} \cdot \frac{\overline{G}(x)(G(x))^2}{g(x)} \geq 0$, hence $M(x_1|T') \geq M(x_1|T) = 0$. Thus, we must have $M(x_1|T') \geq 0 > M(x_2|T')$ and $x_1 > x_2$, which cannot occur since $M(x|T')$ is positive-negative in x (i.e., if $M(x_1|T')$ is nonnegative, so is $M(x_2|T')$).

For **(b)**, we first consider the case with $T \geq N_1$. In such a case, if $S^B(T) = \min(T, N_1) = N_1$, by part (i), $\frac{df^B}{dS}(S = N_1 \; |T) \geq 0$. As in **(a)**, it can be shown that $\frac{df^B}{dS}(S = N_1 \; |T') \geq 0$ for all $T' > T$, hence $S^B(T') = N_1$.

Now consider the case with $T < N_1$ and $S^B(T) = T$ (i.e., $\frac{df^B}{dS}(S = T \; |T) \geq 0$). First let $T' \in (T, N_1]$. We show that $\frac{df^B}{dS}(S = T'|T') \geq 0$. Since $p_2^B(T) = H$, equation $(A.11)$ simplifies to

$$\frac{df^B}{dS}(S = T \; |T) = \frac{N_1 + N_2}{N_1 + N_2 - T} \cdot \frac{\overline{G}(H)}{g(H)} - \int_L^H G(y)\, dy,$$

implying that $\frac{df^B}{dS}(S = T \; |T)$ is nondecreasing in T for $T \leq N_1$. As a result, $\frac{df^B}{dS}(S = T' \; |T') \geq \frac{df^B}{dS}(S = T \; |T) \geq 0$ and $S^B = T'$ for all $T' \in (T, N_1]$. Since we already proved monotonicity for $T \geq N_1$, the result for $T' \geq N_1$ is immediate.

iii) $S^B = 0$ *for* $0 < T \leq T_1$.

Since S^B is monotone in T, it suffices to show that $\frac{df^B}{dS}(S = 0 \; |T = T_1) = 0$. When $S = 0$ and $T = T_1 = (N_1 + N_2)\overline{G}(p_2^U(\bar{c}))$, $p_2^B(S) = p_2^U(\bar{c})$ and $\frac{\overline{G}(p_2^B(S))}{g(p_2^B(S))} = \frac{\overline{G}(p_2^U(\bar{c}))}{g(p_2^U(\bar{c}))} = p_2^U(\bar{c}) - \bar{c}$ (by Lemma 1). Apply these to equation (A.11) and note that $\int_{y \leq p_2} G(y)\, dy = p_2 - p_1^{\max}(p_2)$,

$$\begin{aligned} \frac{df^B}{dS}(S = 0 \; |T = T_1) &= G\left(p_2^U(\bar{c})\right)\left(p_2^U(\bar{c}) - \bar{c}\right) - \left(p_2^U(\bar{c}) - p_1^{\max}\left(p_2^U(\bar{c})\right)\right) \\ &= \left(p_1^{\max,U} - \bar{c}\right) - \left(p_2^U(\bar{c}) - \bar{c}\right)\overline{G}(p_2^U(\bar{c}) \\ &= \frac{1}{N_1}\left[\pi_A^U(\bar{c}) - \pi_S^U(\bar{c})\right] = 0 \text{ (from the definition of } \bar{c}). \qquad \text{(A.17)} \end{aligned}$$

iv) We first show that there exists a $T^D \in (T_1, T_2)$ such that $S^B < \min(T, N_1)$ for $T_1 < T < T^D$ and $S^B = \min(T, N_1)$ for $T > T^D$. We then show that $S^B > 0$ for all $T > T_1$ to complete the claim.

To show the existence of T^D in (T_1, T_2), from part (ii.b), it suffices to show $\frac{df^B(S)}{dS}(S = \min(T, N_1)|T) < 0$ at $T = T_1$ and $\frac{df^B(S)}{dS}(S = \min(T, N_1)|T) > 0$ at $T = T_2 = N_1 + N_2\overline{G}(p_2^U(\bar{c}))$.

The fact $\frac{df^B(S)}{dS}(S = \min(T, N_1)|T = T_1) < 0$ trivially follows from (i) and (iii). Now consider the case when $T = T_2$. When $T = T_2 = N_1 + N_2\bar{G}(p_2^U(\bar{c}))$ and $S = \min(T, N_1) = N_1$, $p_2^B(S) = p_2^U(\bar{c})$ and $\frac{\overline{G}(p_2^B(S))}{g(p_2^B(S))} = \frac{\overline{G}(p_2^U(\bar{c}))}{g(p_2^U(\bar{c}))} = p_2^U(\bar{c}) - \bar{c}$ (by Lemma 1). Apply these facts to (A.11) and note $\int_{y \le p_2} G(y)\, dy = p_2 - p_1^{\max}(p_2)$,

$$\begin{aligned}
\frac{df^B}{dS}(S = N_1 \,|T = T_2) &= \frac{N_1 + N_2}{N_2} G\left(p_2^U(\bar{c})\right)\left(p_2^U(\bar{c}) - \bar{c}\right) - \left(p_2^U(\bar{c}) - p_1^{\max}\left(p_2^U(\bar{c})\right)\right) \\
&= \left(p_1^{\max,U} - \bar{c}\right) - \left(p_2^U(\bar{c}) - \bar{c}\right)\left(1 - \frac{N_1 + N_2}{N_2} G(p_2^U(\bar{c}))\right) \\
&> \left(p_1^{\max,U} - \bar{c}\right) - \left(p_2^U(\bar{c}) - \bar{c}\right)\overline{G}(p_2^U(\bar{c})) \\
&= \frac{1}{N_1}\left[\pi_A^U(\bar{c}) - \pi_S^U(\bar{c})\right] = 0 \text{ (from the definition of } \bar{c}\text{).}
\end{aligned} \tag{A.18}$$

To show $S^B > 0$ for $T > T_1$, it suffices to show $\frac{d}{dT}\frac{df^B}{dS}(S = 0\,|T) > 0$ at $T = T_1$. Then, from equation (A.17) and part (i), the result will follow. To this end, we first take derivative of equation (A.11) at $S = 0$ with respect to T:

$$\frac{d}{dT}\frac{df^B}{dS}(S = 0\,|T) = \left\{\left(\frac{\overline{G}(x)G(x)}{g(x)}\right)' + \overline{G}(x) - 1\right\}\Bigg|_{x = p_2^B(0)} \cdot \frac{dp_2^B(0)}{dT}$$

where $p_2^B(0) = \overline{G}^{-1}\left(\frac{T}{N_1 + N_2}\right)$ strictly decreases in T. By Assumption (A), to show $\frac{d}{dT}\frac{df^B}{dS}(S = 0\,|T) > 0$ at $T = T_1$, it suffices to find a $T' > T_1$ such that $\frac{d}{dT}\frac{df^B}{dS}(S = 0\,|T) > 0$ at $T = T'$. To see the existence of such T', note that by part (i) and equation (A.18), $\frac{df^B}{dS}(S = 0\,|T = T_2) > 0$. This fact, together with equation (A.17) and the fact that $\frac{df^B}{dS}(S = 0\,|T)$ is differentiable in T, implies the existence of a $T' \in (T_1, T_2)$ such that $\frac{d}{dT}\frac{df^B}{dS}(S = 0\,|T) > 0$ at $T = T'$. ■

A.8 Proof of Lemma 4

Proof: (i) Since $T - S < N_1 + N_2 - S$,

$$\pi_2(S) = (T - S)\mathrm{E}[\mathbf{1}_{\{\alpha \ge p_2\}}](p_2 - c) = (T - S)\overline{G}(p_2)(p_2 - c)$$

Note that $\pi_2(S)$ is the same function of price p_2 as in the unlimited-capacity case except that potential sales D_2 are replaced by $T - S$. Thus, the optimal spot price is equal to the optimal spot price of the unlimited-capacity model, i.e., $p_2^*(S) = p_2^U$.

(ii) By definition of the homogeneous-1 valuation model, supply shortage occurs in spot whenever $\alpha > p_2^*$. That is, probability of shortage is $\overline{G}(p_2^*) = \overline{G}(p_2^U)$. Meanwhile, recall that by Lemma 1, $c < H$ implies $\overline{G}(p_2^U) > 0$. ■

A.9 Proof of Lemma 5

Proof: The second order derivative of $\pi_{AS}^H(S)$, with respect to S, is:

$$\frac{d^2\pi_{AS}^H(S)}{dS^2} = \frac{dp_1^{\max,H}(S)}{dS} + \mathrm{E}[\max(\alpha - p_2^U, 0)] \cdot \left[\frac{N_1+N_2-T}{(N_1+N_2-S)^2} + S \cdot \frac{2(N_1+N_2-T)}{(N_1+N_2-S)^3}\right] \geq 0$$

since
$$\frac{dp_1^{\max,H}(S)}{dS} = \frac{N_1+N_2-T}{(N_1+N_2-S)^2}\mathrm{E}[\max(\alpha - p_2^U, 0)] \geq 0$$

Thus, $\pi_{AS}^H(S)$ is convex in S and optimal S is either 0 or $\min(T, N_1)]$. ■

A.10 Proof of Theorem 4

Recall that from Lemma 5, for any $T \in (0, N_1+N_2)$, we have $S^* = \min(T, N_1)$ (full advance selling) or 0 (spot only). Thus, the difference in profits between full advance selling and spot only, as a function of marginal cost c, is simply

$$\begin{aligned}
\Delta(c) &= \pi_{AS}^H(\min(T,N_1)) - \pi_{AS}^H(0) \\
&= \min(T,N_1)\left(p_1^{\max,H}(\min(T,N_1)) - c\right) + (T - \min(T,N_1))\overline{G}(p_2^U)(p_2^U(c) - c) - T\overline{G}(p_2^U(c))(p_2^U(c) - c) \\
&= \min(T,N_1)\left\{(p_1^{\max,H}(\min(T,N_1)) - c) - \overline{G}(p_2^U(c))(p_2^U(c) - c)\right\}
\end{aligned}$$

where $p_1^{\max,H}(\min(T,N_1))$ is defined in equation (2.8).

We divide the proof into two cases: $c \leq \bar{c}$ and $c > \bar{c}$.

• $c \leq \bar{c}$:

For any $T \in (0, N_1+N_2)$, $\frac{T-\min(T,N_1)}{N_1+N_2-\min(T,N_1)} < 1$. Thus, by equation (2.8),

$$p_1^{\max,H}(\min(T,N_1)) > \mathrm{E}[\alpha] - \mathrm{E}[\max(\alpha - p_2^U(c), 0)],$$

which then leads to

$$\begin{aligned}
\Delta(c) &> \min(T,N_1)\left[\mathrm{E}[\alpha] - \mathrm{E}[\max(\alpha - p_2^U(c),0)] - c - \overline{G}(p_2^U(c))(p_2^U(c) - c)\right] \\
&= \frac{\min(T,N_1)}{N_1}(\pi_A^U - \pi_S^U) \geq 0.
\end{aligned}$$

Thus, full advance selling is always optimal when $c \leq \bar{c}$.

• $c > \bar{c}$:

- Consider first the case when $N_1 \leq T < N_1+N_2$. We have

$$\Delta(c) = N_1\left\{\mathrm{E}[\alpha] - \frac{T-N_1}{N_2}\mathrm{E}[\max(\alpha - p_2^U(c),0)] - c - \overline{G}(p_2^U(c))(p_2^U(c) - c)\right\} \quad \text{(A.19)}$$

Noting that $\mathrm{E}[\max(\alpha - p_2^U(c), 0)]$ decreases in c, we have

$$\begin{aligned}
\frac{d\Delta(c)}{dc} &< N_1 \cdot d\left\{\mathrm{E}[\alpha] - \mathrm{E}\left[\max(\alpha - p_2^U(c),0)\right] - c - \overline{G}(p_2^U(c))(p_2^U(c) - c)\right\}/dc \\
&= \frac{d(\pi_A^U - \pi_S^U)}{dc} < 0.
\end{aligned}$$

The last inequality comes from the fact that $\pi_A^U - \pi_S^U$ is quasi-concave and decreasing in $c \geq \bar{c}$ (shown in the proof of Theorem 2). Thus, $\Delta(c)$ is monotonically decreasing in c for $c \geq \bar{c}$. Furthermore, $\Delta(\bar{c}) > 0$, and $\Delta(\bar{p}) = N_1 (\mathrm{E}[\alpha] - \bar{p}) < 0$. Therefore, there must exist a $c^H(T) \in (\bar{c}, \bar{p})$ such that $S^* = S^B$ for $c \leq c^H(T)$ and $S^* = 0$ otherwise.
- If $T < N_1$, then $p_1^{\max,H}(\min(T, N_1))$ is reduced to $p_1^{\max,H}(T) = \mathrm{E}[\alpha]$ and

$$\Delta(c) = T \left\{ \mathrm{E}[\alpha] - c - \overline{G}(p_2^U(c))(p_2^U(c) - c) \right\} \tag{A.20}$$

Note that this is a special case of $\Delta(c)$ for the case above, with $N_1 = T$. Thus $c^H(T)$ exists and is constant for all $T \leq N_1$, $c^H(T) = c^H(N_1)$.

To show that $c^H(T)$ is nonincreasing in $T \in (0, N_1 + N_2)$, it suffices to consider the case $N_1 < T < N_1 + N_2$ (note that $c^H(T) = c^H(N_1)$ for $T \leq N_1$). Since $\mathrm{E}[\max(\alpha - p_2^U(c), 0)] \geq 0$, $\Delta(c)$ is nonincreasing in T, thus $c^H(T)$ is nonincreasing in T for $T \in [N_1, N_1 + N_2)$.

A.11 Proof of Proposition 3

Proof: (⇒) Since $S^* > 0$, $S^* = \min(T, N_1)$ by Theorem 4. Substituting $S^* = \min(T, N_1)$ into $p_1^{\max,H}(S)$, the price premium is

$$p_1^{\max,H}(S^*) - p_2^U = \mathrm{E}[\alpha] - p_2^U - \tfrac{T - \min(T,N_1)}{N_1 + N_2 - \min(T,N_1)} \mathrm{E}\left[\max(\alpha - p_2^U, 0)\right] \tag{A.21}$$

In equation (A.21), note that $\tfrac{T - \min(T,N_1)}{N_1 + N_2 - \min(T,N_1)} \mathrm{E}\left[\max(\alpha - p_2^U, 0)\right]$ is always nonnegative. Hence we immediately have $p_1^{\max,H}(S^*) - p_2^U > 0$ only if $\mathrm{E}[\alpha] - p_2^U(c) > 0$. To see the necessity of the other condition, by contradiction, suppose $p_1^{\max,H}(S^*) - p_2^U > 0$ and $T \geq N_1 + \frac{\mathrm{E}[\alpha] - p_2^U(c)}{\mathrm{E}[\max(\alpha - p_2^U(c), 0)]} N_2$, then $\min(T, N_1) = N_1$ and (after easy re-arrangement) the right-hand-side of equation (A.21) is non-positive, which contradicts the assumption $p_1^{\max,H}(S^*) - p_2^U > 0$.
(⇐) We divide the proof into two cases: $T < N_1$ and $T \geq N_1$. For each case, we first show $S^* > 0$ by contradiction and then prove $p_1^{\max,H}(S^*) > p_2^U$.
- $T < N_1$: Suppose $S^* = 0$, then $\pi_{AS}^H(0) \geq \pi_{AS}^H(\min(T, N_1)) = \pi_{AS}^H(T)$. By the expressions of $\pi_{AS}^H(S)$ and $p_1^{\max,H}(S)$, we have $(p_2^U - c)\overline{G}(p_2^U)T \geq (\mathrm{E}[\alpha] - c)T$, which cannot occur for any positive T since $\mathrm{E}[\alpha] - c > p_2^U \geq (p_2^U - c)\overline{G}(p_2^U)$. This proves $S^* > 0$. This fact, together with Theorem 4, further implies $S^* = \min(T, N_1) = T$. Hence, $p_1^{\max,H}(T) = \mathrm{E}[\alpha]$, which is greater than p_2^U by the assumption.
- $T \geq N_1$: Suppose $S^* = 0$, then $\pi_{AS}^H(0) \geq \pi_{AS}^H(\min(T, N_1)) = \pi_{AS}^H(N_1)$. By the expressions of $\pi_{AS}^H(S)$ and $p_1^{\max,H}(S)$, we have

$$(p_2^U - c)\overline{G}(p_2^U)T \geq \left\{ \mathrm{E}[\alpha] - \frac{T - N_1}{N_2} \mathrm{E}\left[\max(\alpha - p_2^U, 0)\right] - c \right\} N_1 + (T - N_1)(p_2^U - c)\overline{G}(p_2^U)$$

which can be simplified to

$$T \geq N_1 + N_2 \frac{\mathrm{E}[\alpha] - c - (p_2^U - c)\overline{G}(p_2^U)}{\mathrm{E}\left[\max(\alpha - p_2^U, 0)\right]} \tag{A.22}$$

Since $p_2^U(c) \geq c$ and $\overline{G}(p_2^U) \in [0,1]$, equation (C.11) holds only if $T \geq N_1 + N_2 \frac{\mathrm{E}[\alpha]-p_2^U}{\mathrm{E}\left[\max(\alpha-p_2^U,0)\right]}$. This condition, however, directly contradicts our assumption. This proves $S^* > 0$ and further implies $S^* = \min(T, N_1) = N_1$. By equation (A.21), the second condition is exactly what is needed for $p_1^{\max,H}(S^*) - p_2^U > 0$. ■

A.12 Proof of Proposition 4

Proof: Assume that customer valuation follows a uniform distribution on $[A-b, A+b]$, where A and b are both positive. To prove that the region of premium price shrinks in b, by Proposition 3, it suffices to show that (a) p_2^U is nondecreasing in b; (b) for given c satisfying $\mathrm{E}[\alpha] - p_2^U(c) > 0$, $\frac{\mathrm{E}[\alpha]-p_2^U(c)}{\mathrm{E}[\max(\alpha-p_2^U(c),0)]}$ is nonincreasing in b.

To prove (a), first note that by Lemma 1, $p_2^U = \min(\max(p_2^I, L), H)$, where p_2^I is the solution to the first-order condition $p_2 - c = \frac{\overline{G}(p_2)}{g(p_2)}$. Hence, to show the monotonicity of p_2^U in b, it suffices to show the monotonicity of p_2^I in b. For uniform distribution on $[A-b, A+b]$, it is easy to get p_2^I equal to $(A+b+c)/2$, which is clearly nondecreasing in b.

To prove (b), first note that if $c \leq \underline{c} = A-3b$, the result is trivial: $p_2^U(c) = L$ and $\frac{\mathrm{E}[\alpha]-p_2^U(c)}{\mathrm{E}[\max(\alpha-p_2^U(c),0)]} = 1$. Now, if $c > \underline{c} = A - 3b$ and $p_2^U(c) < \mathrm{E}[\alpha]$, by Lemma 1 and part (a) of the proof, $p_2^U = p_2^I = (A+b+c)/2$ and

$$\frac{\mathrm{E}[\alpha] - p_2^U(c)}{\mathrm{E}[\max(\alpha - p_2^U(c), 0)]} = \frac{A - (A+b+c)/2}{\int_{(A+b+c)/2}^{A+b} \frac{x-(A+b+c)/2}{2b}\, dx} = \frac{8b(A-b-c)}{(A+b-c)^2} \tag{A.23}$$

Take derivative of equation (A.23) with respect to b,

$$\left(\frac{\mathrm{E}[\alpha] - p_2^U(c)}{\mathrm{E}[\max(\alpha - p_2^U(c), 0)]}\right)' = \frac{8(A-c)(A-3b-c)}{(A+b-c)^4} \tag{A.24}$$

Since $c > A - 3b$ and $p_2^U(c) = (A+b+c)/2 < \mathrm{E}[\alpha] = A$, $A - 3b - c > 0$ and $A - c > b > 0$. Hence, the derivative in equation (A.24) is positive and $\frac{\mathrm{E}[\alpha]-p_2^U(c)}{\mathrm{E}[\max(\alpha-p_2^U(c),0)]}$ is nondecreasing in b. ■

A.13 Proof of Theorem 5

In preparation, recall the function $f^B(S)$ defined in the proof of Theorem 5: $f^B(S) = (p_1^{\max,D}(S) - c)S + (p_2^B(S) - c)(T - S)$. Also recall that by Lemma A.7.1, S^B defined in equation (A.16) is the unique maximizer of $f^B(S)$ on $[0, \min(T, N_1)]$. The following lemma is useful and the proof of it is deferred to the end of this section.

Lemma A.13.1 *$f^B(S^B)$ is continuous in T and has at most two modes: it is concave for $0 < T \le T_1$, convex for $T_1 < T < T^D$, and concave for $T^D \le T < N_1 + N_2$. Furthermore, $f^B(S^B)$ is differentiable in T at $T = T_1$ and $T = T^D$.*

We divide the proof of Theorem 5 in two cases: (i) $c \le \bar{c}$, (ii) $c > \bar{c}$. In each case, we first prove the continuity and unimodality of $\pi^D_{AS}(S^*(T)|T)$ in T, and then show that there exists an optimal T^* in $[T^D, N_1 + N_2\overline{G}(p^U_2)]$.

(i) $c \le \bar{c}$.

In this case, by proof of Theorem 3,

$$\pi^D_{AS}(S^*(T)|T) = \begin{cases} f^B(S^B) & \text{if } T < N_1 + N_2\overline{G}(p^U_2(c)) \\ f^U(N_1) & \text{if } T \ge N_1 + N_2\overline{G}(p^U_2(c)) \end{cases} \tag{A.25}$$

To show the continuity of $\pi^D_{AS}(S^*(T)|T)$, note that $f^B(S^B)$ is continuous in T (by the continuity of $f^B(S)$ in (S,T) and Lemma A.13.1) and that $f^U(N_1)$ is independent of T. Hence, it suffices to show that when $T = N_1 + N_2\overline{G}(p^U_2(c))$, $f^B(S^B) = f^U(N_1)$. Recall that by the proof of Theorem 3, when $T = N_1 + N_2\overline{G}(p^U_2(c))$ and $c \le \bar{c}$, $S^B = N_1$ and $p^B_2(S^B) = p^U_2(c)$. Applying these facts to the expression of $f^B(S^B)$:

$$\begin{aligned} & f^B(S^B)|_{T=N_1+N_2\overline{G}(p^U_2(c))} \\ =& (p^{\max}_1(p^B_2(N_1)) - c)N_1 + (T - N_1)(p^B_2(N_1) - c) \\ =& (p^{\max}_1(p^U_2(c)) - c)N_1 + N_2\overline{G}(p^U_2(c))(p^U_2(c) - c) \\ =& f^U(N_1) \end{aligned}$$

To show the unimodality of $\pi^D_{AS}(S^*(T)|T)$, by its continuity and the fact that $f^U(N_1)$ is independent of T, it suffices to show that $f^B(S^B)$ is unimodal. By Lemma A.13.1, it suffices to show that $\frac{df^B(S^B)}{dT}$ is nonnegative at $T = T_1$, which then implies that $f^B(S^B)$ is nondecreasing in T when $T < T^D$ and concave when $T \ge T^D$ and resultantly, unimodal (see Figure A.4).

To show $\frac{df^B(S^B)}{dT}$ is nonnegative at $T = T_1$, recall from the proof of Theorem 3 that $S^B = 0$ and $p^B_2(0) = p^U_2(\bar{c})$ when $T = T_1$. By equation (A.32) and Lemma 1,

$$\left.\frac{df^B(S^B)}{dT}\right|_{T=T_1} = p^B_2(0) - c - \frac{\overline{G}(p^B_2(0))}{g(p^B_2(0))} = p^U_2(\bar{c}) - c - \frac{\overline{G}(p^U_2(\bar{c}))}{g(p^U_2(\bar{c}))} = \bar{c} - c, \tag{A.26}$$

which is nonnegative for $c \le \bar{c}$.

Furthermore, the existence of T^* directly follows from the fact that $\pi^D_{AS}(S^*(T)|T)$ is nondecreasing for $T < T^D$ and constant for $T \ge N_1 + N_2\overline{G}(p^U_2(c))$.

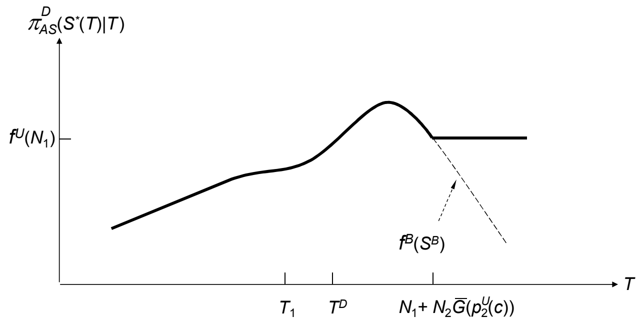


Figure A.4. $\pi_{AS}^{D}(S^*(T)|T)$ when $c \le \bar{c}$

(ii) $c > \bar{c}$

In this case, by the proof of Theorem 3,

$$\pi_{AS}^{D}(S^*|T) = \begin{cases} f^B(S^B) = f^B(0) & \text{if } T < (N_1+N_2)\overline{G}(p_2^U(c)) \\ f^U(0) = \max(f^B(S^B), f^U(0)) & \text{if } (N_1+N_2)\overline{G}(p_2^U(c)) \le T < T_1 \\ \max(f^B(S^B), f^U(0)) & \text{if } T_1 \le T < N_1 + N_2\overline{G}(p_2^U(c)) \\ f^U(0) = \max(f^B(S^B), f^U(0)) & \text{if } T \ge N_1 + N_2\overline{G}(p_2^U(c)) \end{cases}$$

$$= \begin{cases} f^B(S^B) = f^B(0) & \text{if } T < (N_1+N_2)\overline{G}(p_2^U(c)) \\ \max(f^B(S^B), f^U(0)) & \text{if } T \ge (N_1+N_2)\overline{G}(p_2^U(c)) \end{cases} \quad \text{(A.27)}$$

Note that $f^U(0) = (N_1+N_2)(p_2^U(c) - c)\overline{G}(p_2^U(c))$, which is independent of T. Also by Lemma A.13.1 and equation (A.26), when $c > \bar{c}$, $f^B(S^B)$ is concave for $T < T_1$, strictly decreasing at $T = T_1$, convex for $T \in (T_1, T^D)$, and concave for $T \in [T^D, N_1 + N_2)$ (see Figure A.5).

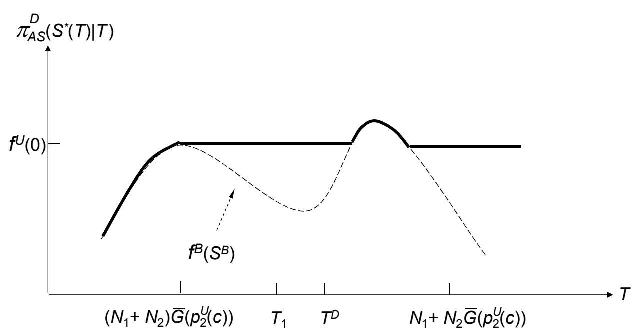


Figure A.5. $\pi_{AS}^{D}(S^*(T)|T)$ when $c > \bar{c}$

In preparation, we first prove a technical result: at the switching point $T = (N_1+N_2)\overline{G}(p_2^U(c))$, $f^B(0) = f^U(0)$ and $\frac{df^B(S^B)}{dT} = 0$. By the proof of Theorem 3, when $T = (N_1+N_2)\overline{G}(p_2^U(c))$ and $c > \bar{c}$, $S^B = 0$ and $p_2^B(0) = p_2^U(c)$. Applying these to the expressions of $f^B(0)$ and $\frac{df^B(0)}{dT}$ (equation

(A.32)):

$$f^B(0)|_{T=(N_1+N_2)\overline{G}(p_2^U(c))} = T(p_2^B(0)-c) = (N_1+N_2)\overline{G}(p_2^U(c))(p_2^B(0)-c)$$
$$=(N_1+N_2)\overline{G}(p_2^U(c))(p_2^U(c)-c) = f^U(0). \tag{A.28}$$

and

$$\left.\frac{df^B(0)}{dT}\right|_{T=(N_1+N_2)\overline{G}(p_2^U(c))} = p_2^B(0)-c+T\frac{dp_2^B(0)}{dT} = p_2^U(c)-c-\frac{\overline{G}(p_2^U(c))}{g(p_2^U(c))} = 0, \tag{A.29}$$

where the last equality is by Lemma 1 and the fact that $c \in (\bar{c}, H)$.

To prove the continuity of $\pi_{AS}^D(S^*(T)|T)$, note that both $f^B(S^B)$ and $f^U(0)$ are continuous in T. Meanwhile, by equation (A.28), $\pi_{AS}^D(S^*(T)|T)$ is continuous at $T = (N_1+N_2)\overline{G}(p_2^U(c))$. These facts jointly imply that $\pi_{AS}^D(S^*(T)|T)$ is continuous.

To prove the unimodality of $\pi_{AS}^D(S^*(T)|T)$, first note that equation (A.29) and the concavity of $f^B(S^B)$ in T for $T \in (0, T_1)$ imply that $f^B(S^B)$ is nondecreasing in T for $T < (N_1+N_2)\overline{G}(p_2^U(c))$ and nonincreasing in T for $T \in [(N_1+N_2)\overline{G}(p_2^U(c)), T_1)$. This fact, together with Lemma A.13.1, implies that there must exist a point $\bar{T} \in (T_1, T^D]$ such that $f^B(S^B)$ is nondecreasing in T for $T < (N_1 + N_2)\overline{G}(p_2^U(c))$, nonincreasing for $T \in ((N_1+N_2)\overline{G}(p_2^U(c)), \bar{T})$, and unimodal for $T \in [\bar{T}, N_1+N_2)$.[2] Hence, by equation (A.27), $\pi_{AS}^D(S^*|T)$ is nondecreasing when $T \leq (N_1+N_2)\overline{G}(p_2^U(c))$, a constant (i.e., $f^U(0)$) when $T \in ((N_1+N_2)\overline{G}(p_2^U(c)), \bar{T})$, and the maximum of a unimodal function and a constant when $T \in [\bar{T}, N_1+N_2)$, which implies that $\pi_{AS}^D(S^*(T)|T)$ is unimodal for $T \in (0, N_1+N_2)$.

Finally, to prove the existence of T^*, consider two cases: if $f^B(S^B) \leq f^U(0)$ for all $T \geq T^D$, $\pi_{AS}^D(S^*|T)$ is maximized at any T greater than $(N_1+N_2)\overline{G}(p_2^U(c))$. In such a case, we can let T^* equal to $N_1 + N_2\overline{G}(p_2^U(c))$. If, however, $f^B(S^B) > f^U(0)$ for some $T \geq T^D$, the existence of T^* follows from the fact that $\pi_{AS}^D(S^*(T)|T)$ is nondecreasing for $T \leq T^D$ and constant for $T \geq N_1 + N_2\overline{G}(p_2^U(c))$. This completes the proof of Theorem 5 for case (ii).

Proof of Lemma A.13.1 *Proof:* To explicitly recognize the dependence of S^B, $f^B(S)$, and $p_2^B(S)$ in T, within this proof we explicitly write $S^B(T)$, $f^B(S,T)$, and $p_2^B(S,T)$. Recall that for $S \in [0, \min(T, N_1)]$,

$$f^B(S,T) := \left(p_1^{\max}\left(p_2^B(S,T)\right) - c\right)S + (T-S)\left(p_2^B(S,T)-c\right)$$

and that by Lemma A.7.1, $S^B(T) = 0$ for $0 < T \leq T_1$, $S^B(T) \in (0, \min(T, N_1))$ for $T_1 < T < T^D$ and satisfies $\frac{\partial f^B(S,T)}{\partial S} = 0$, and $S^B(T) = \min(T, N_1)$ for $T^D \leq T < N_1 + N_2$.

[2]Specifically, if $\frac{df^B(S^B)}{dT}$ is negative at $T = T^D$, $\bar{T} = T^D$, and otherwise, $\bar{T}$ is the largest T in $(T_1, T^D]$ above which $\frac{df^B(S^B)}{dT}$ is nonnegative.

Hence, we have

$$f^B(S^B(T),T) = \begin{cases} T\left(p_2^B(0,T)-c\right) & \text{if } 0<T\le T_1 \\ \left(p_1^{\max}\left(p_2^B(S^B(T),T)\right)-c\right)S^B + (T-S^B(T))\left(p_2^B(S^B(T),T)-c\right) & \text{if } T_1<T<T^D \\ \left(p_1^{\max}\left(p_2^B(\min(T,N_1),T)\right)-c\right)N_1 + (T-\min(T,N_1))\left(p_2^B(\min(T,N_1),T)-c\right) & \text{if } T^D\le T<N_1+N_2 \end{cases}$$

where $S^B(T)$ satisfies $\frac{\partial f^B(S,T)}{\partial S}=0$ for $T_1<T<T^D$.

We further divide the case $T^D\le T<N_1+N_2$ into two subcases: when $T^D<T<\max(T^D,N_1)$, $S^B(T)=\min(T,N_1)=T$ which implies $p_2^B(S^B(T),T)=H$ and $p_1^{\max}(p_2^B(S^B(T),T))=\mathrm{E}[\alpha]$; when $\max(T^D,N_1)\le T<N_1+N_2$, $S^B(T)=\min(T,N_1)=N_1$ which implies $p_2^B(S^B(T),T)=p_2^B(N_1,T)$ and $p_1^{\max}(p_2^B(S^B(T),T))=p_1^{\max}(p_2^B(N_1,T))$. That is,

$$f^B(S^B(T),T) = \begin{cases} f^B(0,T)=T\left(p_2^B(0,T)-c\right) & \text{if } 0<T\le T_1 \\ \left(p_1^{\max}\left(p_2^B(S^B(T),T)\right)-c\right)S^B + (T-S^B(T))\left(p_2^B(S^B(T),T)-c\right) & \text{if } T_1<T<T^D \\ (\mathrm{E}[\alpha]-c)\,T & \text{if } T^D\le T<\max(T^D,N_1) \\ f^B(N_1,T)=\left(p_1^{\max}\left(p_2^B(N_1,T)\right)-c\right)N_1+(T-N_1)\left(p_2^B(N_1,T)-c\right) & \text{if } \max(T^D,N_1)\le T<N_1+N_2 \end{cases}$$

First note that since both $f^B(S,T)$ and $\frac{\partial f^B(S,T)}{\partial S}$ are continuous in (S,T), clearly both $S^B(T)$ and $f^B(S^B(T),T)$ are continuous in T. In preparation for further analysis, we derive the total derivative of $f^B(S^B(T),T)$ with respect to T:

$$\frac{df^B(S^B(T),T)}{dT} = \left.\frac{\partial f^B(S,T)}{\partial S}\right|_{S=S^B(T)} \cdot \frac{dS^B(T)}{dT} + \left.\frac{\partial f^B(S,T)}{\partial T}\right|_{S=S^B(T)}$$

Notice that if $0<T\le T_1$ or $\max(T^D,N_1)\le T<N_1+N_2$, $\frac{dS^B(T)}{dT}=0$, and that if $T_1<T<T^D$, $\left.\frac{\partial f^B(S,T)}{\partial S}\right|_{S=S^B(T)}=0$. Therefore, the total derivative of $f^B(S^B(T),T)$ with respect to T in the interior of each case is

$$\frac{df^B(S^B(T),T)}{dT} = \begin{cases} \frac{\partial f^B(0,T)}{\partial T} & \text{if } 0<T<T_1 \\ \left.\frac{\partial f^B(S,T)}{\partial T}\right|_{S=S^B(T)} & \text{if } T_1<T<T^D \\ \mathrm{E}[\alpha]-c & \text{if } T^D<T<\max(T^D,N_1) \\ \frac{\partial f^B(N_1,T)}{\partial T} & \text{if } \max(T^D,N_1)<T<N_1+N_2 \end{cases} \tag{A.30}$$

We further derive the partial derivative of $f^B(S,T)$ in T as follows. Note that by definitions of $p_2^B(S,T)$ and $p_1^{\max}(p_2^B(S,T))$,

$$\frac{\partial p_2^B(S,T)}{\partial T} = -\frac{1}{(N_1+N_2-S)g(p_2^B(S,T))}, \quad \frac{\partial p_1^{\max}(p_2^B(S,T))}{\partial T} = \overline{G}(p_2^B(S,T))\frac{\partial p_2^B(S,T)}{\partial T}.$$

These facts imply

$$\frac{\partial f^B(S,T)}{\partial T} = \frac{\partial p_1^{\max}(p_2^B(S,T))}{\partial T} S + (T-S)\frac{\partial p_2^B(S,T)}{\partial T} + p_2^B(S,T) - c$$
$$= p_2^B(S,T) - c - \frac{(N_1+N_2)\overline{G}(p_2^B(S,T))}{(N_1+N_2-S)g(p_2^B(S,T))} \tag{A.31}$$

Applying equation (A.31) to equation (A.30), we get

$$\frac{df^B(S^B(T),T)}{dT}$$
$$= \begin{cases} p_2^B(0,T) - c - \frac{\overline{G}(p_2^B(0,T))}{g(p_2^B(0,T))} & \text{if } 0 < T < T_1 \\ p_2^B(S^B(T),T) - c - \frac{(N_1+N_2)\overline{G}(p_2^B(S^B(T),T))}{(N_1+N_2-S^B(T))g(p_2^B(S^B(T),T))} & \text{if } T_1 < T < T^D \\ \mathrm{E}[\alpha] - c & \text{if } T^D < T < \max(T^D, N_1) \\ p_2^B(N_1,T) - c - \frac{(N_1+N_2)\overline{G}(p_2^B(N_1,T))}{N_2 g(p_2^B(N_1,T))} & \text{if } \max(T^D,N_1) < T < N_1+N_2 \end{cases} \tag{A.32}$$

Using equation (A.32), next we show that (i) $f^B(S^B(T),T)$ is differentiable at $T = T_1$ and $T = T^D$; (ii) $\frac{df^B(S^B(T),T)}{dT}$ is nonincreasing for $0 < T \le T_1$ and $\max(T^D, N_1) \le T < N_1 + N_2$; (iii) $\frac{df^B(S^B(T),T)}{dT}$ is nondecreasing for $T \in (T_1, T^D)$; (iv) when $T^D \le N_1$, $\lim_{T\to N_1^-} \frac{df^B(S^B(T),T)}{dT} \ge \lim_{T\to N_1^+} \frac{df^B(S^B(T),T)}{dT}$. Clearly, (i) through (iv) jointly imply Lemma A.13.1.

(i) Recall that by Lemma A.7.1, $S^B(T_1) = 0$. By equation (A.32), it is easy to see that $\lim_{T\to T_1^+} \frac{df^B(S^B(T),T)}{dT} = \lim_{T\to T_1^-} \frac{df^B(S^B(T),T)}{dT}$ and hence $\frac{df^B(S^B(T),T)}{dT}$ exists at $T = T_1$. Similarly, we can show that $\frac{df^B(S^B(T),T)}{dT}$ exists at $T = T^D$ if $T^D \ge N_1$ since in such a case, $S^B(T^D) = N_1$.

Now consider the case when $T^D < N_1$. In such a case, $S^B(T^D) = T^D$, $p_2^B(S^B(T^D), T^D) = H$, and $p_1^{\max}(p_2^B(S^B(T^D), T^D)) = \mathrm{E}[\alpha]$. Furthermore, by definition of T^D (ref. the proof of Lemma A.7.1), $\left.\frac{\partial f^B(S,T)}{\partial S}\right|_{S=\min(T,N_1),T=T^D} = 0$. From equation (A.8), that is

$$\frac{N_1+N_2}{N_1+N_2-T^D} \cdot \frac{\overline{G}(p_2^B(T^D,T^D))}{g(p_2^B(T^D,T^D))} - \left(p_2^B(T^D,T^D) - p_1^{\max}(p_2^B(T^D,T^D))\right) = 0 \tag{A.33}$$

Applying $p_2^B(T^D,T^D) = H$ and $p_1^{\max}(p_2^B(T^D,T^D)) = \mathrm{E}[\alpha]$, equation (A.33) implies

$$\frac{N_1+N_2}{N_1+N_2-T^D} \cdot \frac{\overline{G}(H)}{g(H)} = H - \mathrm{E}[\alpha] \tag{A.34}$$

By equation (A.32) and (A.34),

$$\lim_{T\to T^{D-}} \frac{df^B(S^B(T),T)}{dT} = p_2^B(T^D,T^D) - c - \frac{(N_1+N_2)\overline{G}(p_2^B(T^D,T^D))}{(N_1+N_2-T^D)g(p_2^B(T^D,T^D))}$$
$$= H - c - \frac{(N_1+N_2)\overline{G}(H)}{(N_1+N_2-T^D)g(H)}$$
$$= \mathrm{E}[\alpha] - c$$
$$= \lim_{T\to T^{D+}} \frac{df^B(S^B(T),T)}{dT}.$$

(ii) By equation (A.32), it suffices to show that for given S, $p_2^B(S,T) - c - \frac{(N_1+N_2)\overline{G}(p_2^B(S,T))}{(N_1+N_2-S)g(p_2^B(S,T))}$ is nonincreasing in T. By definition of $p_2^B(S,T)$, it is nonincreasing in T for given S. Meanwhile, since $G(\cdot)$ satisfies IFR property, $\frac{\overline{G}(x)}{g(x)}$ is nonincreasing in x. The desired result immediately follows.

(iii) To show the monotonicity of $\frac{df^B(S^B(T),T)}{dT}$ in T for $T_1 < T < T^D$, we apply a change of variable and let $\lambda = \frac{T-S}{N_1+N_2-S}$. Substitute $S = N_1 + N_2 - \frac{N_1+N_2-T}{1-\lambda}$ into $f^B(S)$ and define the resultant function $f_\lambda^B(\lambda, T)$ by

$$f_\lambda^B(\lambda,T) = \left(p_2^B(\lambda) - c\right)T - \left(N_1 + N_2 - \frac{N_1+N_2-T}{1-\lambda}\right)\left(p_2^B(\lambda) - p_1^{\max}(p_2^B(\lambda))\right),$$

where $\overline{G}(p_2^B(\lambda)) = \lambda$, implying that $p_2^B(\lambda)$ is independent of T.

Note that for given T, N_1 and N_2, $\lambda \in \left[\frac{T-\min(T,N_1)}{N_1+N_2-\min(T,N_1)}, \frac{T}{N_1+N_2}\right]$ and $S \in [0, \min(T, N_1)]$ are in one-to-one correspondence. Hence, by Lemma A.7.1, for given T, $f_\lambda^B(\lambda, T)$ is strictly quasi-concave in λ and has an unique maximizer $\lambda^B(T)$ defined by $\lambda^B(T) = \frac{T-S^B(T)}{N_1+N_2-S^B(T)}$. Specifically, when $T_1 < T < T^D$, $\lambda^B(T)$ satisfies $\frac{\partial f_\lambda^B(\lambda,T)}{\partial\lambda} = 0$. Since $f^B(S^B(T),T) = f_\lambda^B(\lambda^B(T),T)$, to show the monotonicity of $\frac{df^B(S^B(T),T)}{dT}$ in T, it suffices to show it for $\frac{df_\lambda^B(\lambda^B(T),T)}{dT}$. Note that the total derivative of $f_\lambda^B(\lambda^B(T),T)$ in T is

$$\begin{aligned}\frac{df_\lambda^B(\lambda^B(T),T)}{dT} &= \left.\frac{\partial f_\lambda^B(\lambda,T)}{\partial\lambda}\right|_{\lambda=\lambda^B(T)} \cdot \frac{d\lambda^B(T)}{dT} + \left.\frac{\partial f_\lambda^B(\lambda,T)}{\partial T}\right|_{\lambda=\lambda^B(T)} \\ &= \left.\frac{\partial f_\lambda^B(\lambda,T)}{\partial T}\right|_{\lambda=\lambda^B(T)} \\ &= p_2^B(\lambda^B(T)) - c - \frac{p_2^B(\lambda^B(T)) - p_1^{\max}(p_2^B(\lambda^B(T)))}{1-\lambda^B(T)} \qquad \text{(A.35)}\end{aligned}$$

where the second equality is from the fact that $\lambda^B(T)$ satisfies $\frac{\partial f_\lambda^B(\lambda,T)}{\partial\lambda} = 0$. Note that by equation (A.35), the total derivative of $f_\lambda^B(\lambda^B(T),T)$ depends on T only through $\lambda^B(T)$. Hence,

$$\frac{d^2 f_\lambda^B(\lambda^B(T),T)}{dT^2} = \left(\frac{df_\lambda^B(\lambda^B(T),T)}{dT}\right)'_{\lambda^B} \cdot \frac{d\lambda^B(T)}{dT} \qquad \text{(A.36)}$$

Note that in equation (A.36),

$$\begin{aligned}\left(\frac{df_\lambda^B(\lambda^B(T),T)}{dT}\right)'_{\lambda^B} &= \frac{dp_2^B(\lambda^B)}{d\lambda^B} - \frac{G(p_2^B(\lambda^B))}{1-\lambda^B}\frac{dp_2^B(\lambda^B)}{d\lambda^B} - \frac{p_2^B(\lambda^B) - p_1^{\max}(p_2^B(\lambda^B))}{(1-\lambda^B)^2} \\ &= -\frac{p_2^B(\lambda^B) - p_1^{\max}(p_2^B(\lambda^B))}{(1-\lambda^B)^2} \\ &\leq 0 \qquad \text{(A.37)}\end{aligned}$$

where the second equality follows from $\overline{G}(p_2^B(\lambda^B)) = \lambda^B$ and the last inequality is by the definition of $p_1^{\max}(p_2)$, $p_2 \geq p_1^{\max}(p_2)$ for any p_2.

In the meanwhile, since $\lambda^B(T)$ satisfies $\frac{\partial f_\lambda^B(\lambda,T)}{\partial \lambda} = 0$,

$$\frac{d\lambda^B(T)}{dT} = -\left.\frac{\partial^2 f_\lambda^B(\lambda,T)/\partial\lambda\partial T}{\partial^2 f_\lambda^B(\lambda,T)/\partial\lambda^2}\right|_{\lambda=\lambda^B(T)} \tag{A.38}$$

where

$$\left.\frac{\partial^2 f_\lambda^B(\lambda,T)}{\partial\lambda\partial T}\right|_{\lambda=\lambda^B(T)} = -\frac{p_2^B(\lambda^B(T)) - p_1^{\max}(p_2^B(\lambda^B(T)))}{(1-\lambda^B(T))^2} \leq 0 \tag{A.39}$$

and noting that $f_\lambda^B(\lambda,T) = f^B(S(\lambda,T),T)$ with $S(\lambda,T) = N_1 + N_2 - \frac{N_1+N_2-T}{1-\lambda}$ and $S(\lambda^B,T) = S^B(T)$,

$$\begin{aligned}
\frac{\partial f_\lambda^B(\lambda,T)}{\partial\lambda} &= \frac{\partial f^B(S,T)}{\partial S}\cdot\frac{\partial S(\lambda,T)}{\partial\lambda} \\
\frac{\partial^2 f_\lambda^B(\lambda,T)}{\partial\lambda^2} &= \frac{\partial^2 f^B(S,T)}{\partial S^2}\cdot\left(\frac{\partial S(\lambda,T)}{\partial\lambda}\right)^2 + \frac{\partial f^B(S,T)}{\partial S}\cdot\frac{\partial^2 S(\lambda,T)}{\partial\lambda^2} \\
\left.\frac{\partial^2 f_\lambda^B(\lambda,T)}{\partial\lambda^2}\right|_{\lambda=\lambda^B(T)} &= \left.\frac{\partial^2 f^B(S,T)}{\partial S^2}\right|_{S=S^B(T)}\cdot\left.\left(\frac{\partial S(\lambda,T)}{\partial\lambda}\right)^2\right|_{\lambda=\lambda^B(T)} + \left.\frac{\partial f^B(S,T)}{\partial S}\right|_{S=S^B(T)}\cdot\left.\frac{\partial^2 S(\lambda,T)}{\partial\lambda^2}\right|_{\lambda=\lambda^B(T)} \\
&= \left.\frac{\partial^2 f^B(S,T)}{\partial S^2}\right|_{S=S^B(T)}\cdot\left.\left(\frac{\partial S(\lambda,T)}{\partial\lambda}\right)^2\right|_{\lambda=\lambda^B(T)} \leq 0 \qquad \text{(A.40)}
\end{aligned}$$

where the last equation is due to the fact that $S^B(T)$ satisfies $\frac{\partial f^B(S,T)}{\partial S} = 0$ and $\frac{\partial^2 f^B(S,T)}{\partial S^2} \leq 0$ (implied by the strict quasi-concavity of $f^B(S,T)$ in S).

Collecting all the facts from equation (A.36) through (A.40), we have $\frac{d^2 f_\lambda^B(\lambda^B(T),T)}{dT^2} \geq 0$, which implies that both $\frac{df_\lambda^B(\lambda^B(T),T)}{dT}$ and $\frac{df^B(S^B(T),T)}{dT}$ are nondecreasing in T.

(iv) If $T^D \leq N_1$, when $T = N_1 \geq T^D$, $S^B(N_1) = N_1$ (by Lemma A.7.1), which implies $p_2^B(S_B(N_1),N_1) = p_2^B(N_1,N_1) = H$ and $p_1^{\max}(p_2^B(S_B(N_1),N_1)) = \mathrm{E}[\alpha]$. By equation (A.34) and the fact $N_1 \geq T^D$,

$$\begin{aligned}
\lim_{T\to N_1^+}\frac{df^B(S^B(T),T)}{dT} &= p_2^B(S_B(N_1),N_1) - c - \frac{(N_1+N_2)\overline{G}(p_2^B(S_B(N_1),N_1))}{(N_1+N_2-N_1)g(p_2^B(S_B(N_1),N_1))} \\
&\leq p_2^B(S_B(N_1),N_1) - c - \frac{(N_1+N_2)\overline{G}(p_2^B(S_B(N_1),N_1))}{(N_1+N_2-T^D)g(p_2^B(S_B(N_1),N_1))} \\
&= H - c - \frac{(N_1+N_2)\overline{G}(H)}{(N_1+N_2-T^D)g(H)} = \mathrm{E}[\alpha] - c = \lim_{T\to N_1^-}\frac{df^B(S^B(T),T)}{dT} \qquad \blacksquare
\end{aligned}$$

A.14 Proof of Theorem 6

Proof: It suffices to show that $\pi_{AS}^H(S^*|T)$ is (i) continuous in T, (ii) nondecreasing in T for $0 < T \leq N_1$, and (iii) convex for $T > N_1$.

Recall that by Lemma 5, S^* is either 0 or $\min(T, N_1)$, whichever leads to the higher total profit. Hence, the optimal profit function $\pi_{AS}^H(S^*|T)$ equals to $\max(\pi_{AS}^H(0), \pi_{AS}^H(\min(T,N_1)))$. Also recall that

$$\begin{aligned}\pi_{AS}^{H}(S) &= (p_1^{\max,H}(S) - c)S + (T-S)(p_2^U - c)\\ &= \left\{\mathrm{E}[\alpha] - \frac{T-S}{N_1+N_2-S}\mathrm{E}[\max(\alpha - p_2^U, 0)] - c\right\} S + (T-S)(p_2^U - c)\overline{G}(p_2^U(c))\end{aligned}$$

and

$$\pi_{AS}^{H}(0) = T(p_2^U - c)\overline{G}(p_2^U)$$

$$\pi_{AS}^{H}(\min(T,N_1)) = \begin{cases} (\mathrm{E}[\alpha]-c)\,T & \text{if } 0 < T \le N_1, \\ \left(\mathrm{E}[\alpha] - \frac{T-N_1}{N_2}\mathrm{E}\left[\max(\alpha - p_2^U, 0)\right] - c\right) N_1 + (T-N_1)(p_2^U(c) - c)\overline{G}(p_2^U(c)) & \text{if } T > N_1. \end{cases}$$

(i) By the continuity of $\pi_{AS}^{H}(S|T)$ in (S,T), both $\pi_{AS}^{H}(0)$ and $\pi_{AS}^{H}(\min(T,N_1))$ are continuous in T. This immediately implies the continuity of $\pi_{AS}^{H}(S^*|T)$ in T.

(ii) To show that $\pi_{AS}^{H}(S^*|T)$ is nondecreasing in T for $0 < T \le N_1$, first note that $\pi_{AS}^{H}(0)$ is always nondecreasing in T. We then consider two cases: if $\mathrm{E}[\alpha] \ge c$, $\pi_{AS}^{H}(\min(T,N_1))$ is also nondecreasing in T for $0 < T \le N_1$, thus so is $\pi_{AS}^{H}(S^*|T)$; if, however, $\mathrm{E}[\alpha] < c$, $p_1^{\max,H}(\min(T,N_1)) < c$ and obviously $\pi_{AS}^{H}(\min(T,N_1)) < \pi_{AS}^{H}(0)$. This implies that $\pi_{AS}^{H}(S^*|T)$ equals to $\pi_{AS}^{H}(0)$ and thus is nondecreasing in T.

(iii) Notice that for $T > N_1$, both $\pi_{AS}^{H}(0)$ and $\pi_{AS}^{H}(\min(T,N_1))$ are linear in T. Since $\pi_{AS}^{H}(S^*|T)$ equals to the maximum of $\pi_{AS}^{H}(0)$ and $\pi_{AS}^{H}(\min(T,N_1))$, $\pi_{AS}^{H}(S^*|T)$ is clearly convex for $T > N_1$. ■

A.15 Proof of Theorem 7

Proof: In preparation, let T_0 and T^* denote the capacity levels before and after the reduction, respectively. Assume $T_0 > T^*$. Let (p_1^0, p_1^0, S^0) denote the seller's optimal pricing and rationing policy with original capacity level T_0, and (p_1^r, p_2^r, S^r) be their counterparts after reducing capacity to T^*. It then suffices to show that (i) $p_1^0 \le p_1^r$, $p_2^0 \le p_2^r$; (ii) If $S^0 > 0$, then $S^r > 0$; (iii) if $S^0 = 0$, then S^r may be positive sometimes.

We provide separate proofs for deterministic and homogeneous-1 valuation models and start with **deterministic valuation model**.

(i) We first note that by equation (2.6), to show $p_1^0 \le p_1^r$, it suffices to prove $p_2^0 \le p_2^r$. We prove $p_2^0 \le p_2^r$ by contradiction. Suppose $p_2^0 > p_2^r$. By equation (2.4), Theorem 3, and the fact $T_0 > T^* \ge T^D$ (by Theorem 5), $p_2^0 > p_2^r$ can only occur if $p_2^0 = p_2^B(S^0|T^0)$, $p_2^r = p_2^B(S^r|T^*)$, $S^0 = \min(T^0, N_1)$, and $S^r = \min(T^*, N_1)$. However,

$$p_2^B(S^0|T^0) = \overline{G}^{\,-1}\left(\frac{T_0 - \min(T_0,N_1)}{N_1+N_2-\min(T_0,N_1)}\right) \le \overline{G}^{\,-1}\left(\frac{T^* - \min(T^*,N_1)}{N_1+N_2-\min(T^*,N_1)}\right) = p_2^B(S^r|T^*),$$

where the second equality follows from the facts that $\frac{T-\min(T,N_1)}{N_1+N_2-\min(T,N_1)}$ is nondecreasing in T, $\overline{G}(\cdot)$ is a nonincreasing function, and $T_0 > T^*$. This implies $p_2^0 \le p_2^r$ and contradicts with the hypothesis.
(ii) When $S^0 > 0$, by Theorem 3, Theorem 5, and the fact $T_0 > T^* \ge T^D$, $S^r = \min(T^*, N_1) > 0$.
(iii) It suffices to provide an example where $S^0 = 0$ and $S^r > 0$. Consider the following case: $N_1 = N_2 = 7$, $\alpha \sim$ U$[20, 40]$, $c = 27$, and $T_0 = 10$. It is easy to verify that in such a case, $S^0 = 0$, $p_2^0 = 33.5$, $\pi_{AS}^D(S^0|T_0) = 29.6$; $T^* = 8.5$, $S^r = 7$, $p_1^r = 29.5$, $p_2^r = 35.7$, $\pi_{AS}^D(S^r|T^*) = 30.8$.

Now, we proceed to prove the theorem for the **homogeneous-1 valuation model**.
(i) First recall that by Lemma 4, optimal price always equals to p_2^U. Hence, $p_2^0 \le p_2^r$ trivially holds. To show $p_1^0 \le p_1^r$, By equation (2.8), it suffices to show $\frac{T_0-S^0}{N_1+N_2-S^0} \ge \frac{T^*-S^r}{N_1+N_2-S^r}$. Prove by contradiction. Suppose $\frac{T_0-S^0}{N_1+N_2-S^0} < \frac{T^*-S^r}{N_1+N_2-S^r}$. By Theorem 4 and the fact $T_0 > T^*$, it can occur only if $S^r = \min(T^*, N_1)$. Since $S^0 \le \min(T_0, N_1)$ and $T_0 > T^*$, we have

$$\frac{T_0 - S^0}{N_1 + N_2 - S^0} \ge \frac{T_0 - \min(T_0, N_1)}{N_1 + N_2 - \min(T_0, N_1)} \ge \frac{T^* - \min(T^*, N_1)}{N_1 + N_2 - \min(T^*, N_1)} = \frac{T^* - S^r}{N_1 + N_2 - S^r},$$

where the first equation follows from the fact that $\frac{T_0-S}{N_1+N_2-S}$ is nonincreasing in S and the second is due to the fact that $\frac{T-\min(T,N_1)}{N_1+N_2-\min(T,N_1)}$ is nondecreasing in T.
(ii) When $S^0 > 0$, by Theorem 4 and the fact $T_0 > T^*$, $S^r = \min(T^*, N_1) > 0$.
(iii) It suffices to provide an example where $S^0 = 0$ and $S^r > 0$. Consider the following case: $N_1 = 30$, $N_2 = 7$, $\alpha \sim$ U$[20, 40]$, $c = 27$, $T_0 = 36$. It is easy to verify that in such a case, $S^0 = 0$, $p_2^0 = p_2^r = 33.5$, $\pi_{AS}^H(S^0|T_0) = 76.05$, $T^* = 30$, $S^r = 30$, $p_1^r = 30$, $\pi_{AS}^H(S^r|T^*) = 90$. ■

A.16 Proof of Proposition 5

Proof: (i) By Lemma 3 and 4, it suffices to show that both p_2^U and $p_2^B(S)$ are nonincreasing in θ. Note that with the fixed purchasing cost θ in spot, a customer with valuation α buys in spot if and only if $\alpha \ge p_2 + \theta$. Hence, the expected spot demand is $(N_1 + N_2 - S)\overline{G}(p_2 + \theta)$. Therefore, p_2^U maximizes $(p_2 - c)\overline{G}(p_2 + \theta)$ and $p_2^B(S) = (\overline{G})^{-1}\left(\frac{T-S}{N_1+N_2-S}\right) - \theta$. This immediately implies that $p_2^B(S)$ is nonincreasing in θ. Further, by Lemma 1 and the IFR property, p_2^U is nonincreasing in θ.

(ii) With the fixed spot purchasing cost θ, the expected utilities of an advance customer buying in advance and waiting to spot are $\mathrm{E}[\alpha] - p_1$ and $\mathrm{E}[\lambda_2 \max(\alpha - p_2 - \theta, 0)]$, respectively, where λ_2 is the probability of obtaining the product in spot. Therefore,

$$p_1^{\max,D}(S) = \mathrm{E}[\min(p_2^*(S) + \theta, \alpha)], \quad p_1^{\max,H}(S) = \mathrm{E}[\alpha] - \frac{T - S}{N_1 + N_2 - S}\mathrm{E}[\max(\alpha - p_2^U - \theta, 0)]$$

To show that both $p_1^{\max,D}(S)$ and $p_1^{\max,H}(S)$ are nondecreasing in θ, it then suffices to show that both $p_2^U + \theta$ and $p_2^B(S) + \theta$ are nondecreasing in θ. By part (i), $p_2^B(S) + \theta = (\overline{G})^{-1}\left(\frac{T-S}{N_1+N_2-S}\right)$, which is independent of θ. In the meanwhile, $p_2^U + \theta$ maximizes $(p_2 - c - \theta)\overline{G}(p_2)$, where $c + \theta$ can be equivalently treated as the aggregate marginal cost. Since the optimal price is nondecreasing in the marginal cost, so is $p_2^U + \theta$ in θ. This completes the proof of part (ii).

(iii) Under the deterministic valuation model, to show that the optimal advance rationing is nondecreasing in θ, by the proof of Theorem 3, it suffices to show that both $\frac{df^U(S)}{dS}$ and $\frac{df^B(S)}{dS}$ are nondecreasing in θ. Incorporated with the fixed purchasing cost θ, the derivatives of $f^U(S)$ and $f^B(S)$ are as follows (refer: their corresponding definitions and also equation (A.8)):

$$\begin{aligned}
\frac{df^U(S)}{dS} &= \frac{1}{N_1}\left[p_1^{\max}(p_2^U) - c - \overline{G}(p_2^U + \theta)(p_2^U - c)\right] \\
&= \frac{1}{N_1}\left[\mathrm{E}[\min(p_2^U + \theta, \alpha)] - c - \overline{G}(p_2^U + \theta)(p_2^U - c)\right] \\
\frac{df^B(S)}{dS} &= \frac{N_1 + N_2}{N_1 + N_2 - S} \cdot \frac{\overline{G}(p_2^B(S) + \theta)G(p_2^B(S) + \theta)}{g(p_2^B(S) + \theta)} - \left(p_2^B(S) - p_1^{\max}\left(p_2^B(S)\right)\right) \\
&= \frac{N_1 + N_2}{N_1 + N_2 - S} \cdot \frac{\overline{G}(p_2^B(S) + \theta)G(p_2^B(S) + \theta)}{g(p_2^B(S) + \theta)} - \left(p_2^B(S) - \mathrm{E}[\min(p_2^B(S) + \theta, \alpha)]\right)
\end{aligned}$$

Recall that part (i) and (ii) prove the facts that $p_2^U + \theta$ is nondecreasing in θ, $p_2^B(S) + \theta$ is independent of θ, and $p_2^B(S)$ is nonincreasing in θ. Meanwhile, it is easy to show that $\overline{G}(p_2^U + \theta)(p_2^U - c)$ is nonincreasing in θ. Therefore, both $\frac{df^U(S)}{dS}$ and $\frac{df^B(S)}{dS}$ are nondecreasing in θ.

Under the homogeneous-1 valuation model, to show that the optimal advance rationing is nondecreasing in θ, first it is easy to prove that with the extra fixed cost θ, Lemma 5 still holds, i.e., either full advance selling or selling only in spot is optimal. Hence, it suffices to show that $\pi_{AS}^H(\min(T, N_1)) - \pi_{AS}^H(0)$ is nondecreasing in θ. Incorporating the fixed purchasing cost θ into equation (A.19), we have

$$\pi_{AS}^H(\min(T, N_1)) - \pi_{AS}^H(0) = \min(T, N_1)\left\{p_1^{\max,H}(\min(T, N_1)) - c - \overline{G}(p_2^U + \theta)(p_2^U - c)\right\}$$

By part (ii) and the fact that $\overline{G}(p_2^U + \theta)(p_2^U - c)$ is nonincreasing in θ, $\pi_{AS}^H(\min(T, N_1)) - \pi_{AS}^H(0)$ is nondecreasing in θ. ■

APPENDIX B

Proofs in Chapter 3

B.1 Proof of Lemma 7

Proof: **To show** $p_2^H(S) \geq p_2^L(S)$, it is equivalent to show that $p_2^t(S) = \max(p_{2t}^U, p_{2t}^B(S))$ is weakly increasing in A_t. Hence, it suffices to show that both p_{2t}^U and $p_{2t}^B(S)$ are weakly increasing in A_t. The monotonicity of $p_{2t}^B(S)$ is straightforward since by definition, $p_{2t}^B(S) = (\bar{G})^{-1}\left(\frac{T-S}{N_1+N_2-S}\right) + A_t$. To show the monotonicity of $p_{2t}^U(S)$, note that by equation (3.2),

$$p_{2t}^U = \max\left(\min\left(A_t + \alpha_H, p_{2t}^M\right), A_t + \alpha_L\right) \tag{B.1}$$

where p_{2t}^M satisfies $p_{2t}^M = c + \frac{\bar{G}(p_{2t}^M - A_t)}{g(p_{2t}^M - A_t)}$.

It is easy to check that

$$\frac{dp_{2t}^M}{dA_t} = \frac{-\left(\frac{\bar{G}(\cdot)}{g(\cdot)}\right)'}{1-\left(\frac{\bar{G}(\cdot)}{g(\cdot)}\right)'} \geq 0 \text{ (by IFR of } G)$$

which immediately implies $\frac{dp_{2t}^U}{dA_t} \geq 0$.

To show $p_1^H(S) > p_1^L(S)$, it suffices to show that $p_1^t(S) = \mathrm{E}[\min(p_2^t(S), A_t+\alpha)]$ is strictly increasing in A_t. Since $p_2^t(S) \in [A_t + \alpha_L, A_t + \alpha_H]$, we consider the following three cases.

- $p_2^t(S) = A_t + \alpha_L$: $p_1^t(S) = A_t + \alpha_L$, which strictly increases in A_t.
- $p_2^t(S) = A_t + \alpha_H$: $p_1^t(S) = A_t + \mathrm{E}[\alpha]$, which strictly increases in A_t.
- $p_2^t(S) \in (A_t + \alpha_L, A_t + \alpha_H)$: it is easy to check that the derivative of $p_1^t(S)$ with respect to A_t is positive:

$$\frac{dp_1^t(S)}{dA_t} = \frac{\partial p_1^t(S)}{\partial p_2^t(S)} \cdot \frac{dp_2^t(S)}{dA_t} + \frac{\partial p_1^t(S)}{\partial A_t} = \bar{G}(p_2^t(S) - A_t) \cdot \frac{dp_2^t(S)}{dA_t} + G(p_2^t(S) - A_t) > 0$$

  where the inequality follows from the facts that $p_2^H(S) \geq p_2^L(S)$ (implying $\frac{dp_2^t(S)}{dA_t} \geq 0$) and that $G(p_2^t(S) - A_t) > 0$ for $p_2^t(S) \in (A_t + \alpha_L, A_t + \alpha_H)$. ■

B.2 Proof of Proposition 6

Proof: By Theorem 8, it suffices to show that T_1 and T_D are independent of A_t and that $c^{OR}(T)$ increases in A_t.

(i) To show the independence of T_1 in A_t, first recall the definition of T_1 from Yu et al. (2007): $T_1 = (N_1 + N_2)\bar{G}(p_{2t}^U(\bar{c}) - A_t)$, where $\bar{c} \in (\underline{c}, \bar{p})$ is the unique solution to $p_{1t}^U(c) - c - (p_{2t}^U(c) - c)\bar{G}(p_{2t}^U(c) - A_t) = 0$ and $p_{2t}^U(c)$ satisfies $p_{2t}^U(c) = c + \frac{\bar{G}(p_{2t}^U(c)-A_t)}{g(p_{2t}^U(c)-A_t)}$. It is easy to show that when A_t increases by δ, all of $\bar{c}$, $p_{2t}^U(\bar{c})$, and $p_{1t}^U(\bar{c})$ increase by δ. Therefore, T_1 remains the same when A_t changes.

(ii) To show the independence of T_D in A, first recall the definition of T_D from Yu et al. (2007): let $f^B(S, A_t) = (p_{1t}^B(S) - c)S + (p_{2t}^B(S) - c)(T - S)$, then T_D is the unique solution to $\frac{\partial f^B(S,A_t)}{\partial S}\Big|_{S=\min(T,N_1)} = 0$. It then suffices to show that $\frac{\partial f^B(S,A_t)}{\partial S}$ is independent of A_t, i.e., $\frac{\partial^2 f^B(S,A_t)}{\partial S \partial A_t} = 0$. To this end, first note that By definitions of $p_{2t}^B(S)$ and $p_{1t}^B(S)$, it is easy to show that $\frac{dp_{2t}^B(S)}{dA_t} = \frac{dp_{1t}^B(S)}{dA_t} = 1$. This implies $\frac{\partial f^B(S,A_t)}{\partial A_t} = T$ and $\frac{\partial^2 f^B(S,A_t)}{\partial S \partial A_t} = 0$.

(iii) To show the monotonicity of $c^{OR}(T)$ in A, recall that from Yu et al. (2007), $c^{OR}(T) = \bar{c}$ for $T \geq T_2$, and for $T_1 < T < T_2$, $c^{OR}(T) \geq \bar{c}$ and is the solution to

$$\Delta(A_t) = f^B(S^M, A_t) - (N_1 + N_2)\bar{G}(p_{2t}^U - A_t)(p_{2t}^U - c) = 0$$

where $T_2 = N_1 + N_2\bar{G}(p_{2t}^U(\bar{c}) - A_t)$ and S^M is the unique maximizer of $f^B(S, A_t)$ for $S \in [0, \min(T, N_1)]$. By Yu et al. (2007), S^M is independent of c. In the meanwhile, by part (ii), S^M is also independent of A_t. Furthermore, by definitions of $p_{2t}^B(S)$ and $p_{1t}^B(S)$, it is easy to show that $\frac{dp_{2t}^B(S^M)}{dA_t} = \frac{dp_{1t}^B(S^M)}{dA_t} = 1$ and $\frac{dp_{2t}^B(S^M)}{dc} = \frac{dp_{1t}^B(S^M)}{dc} = 0$.

To show that $c^{OR}(T)$ increases in A_t, first note that similarly to the proof of part (i), we can show that T_2 is also independent of A_t. For $T \geq T_2$, $c^{OR}(T) = \bar{c}$, which increases in A_t as shown in part (i). For $T_1 < T < T_2$, by implicit differentiation and equation (3.2),

$$\frac{dc^{OR}(T)}{dA_t} = -\frac{\partial \Delta(A_t)/\partial A_t}{\partial \Delta(A_t)/\partial c} = -\frac{T - (N_1 + N_2)\bar{G}(p_{2t}^U - A_t)}{-T + (N_1 + N_2)\bar{G}(p_{2t}^U - A_t)} = 1 > 0.$$

■

B.3 Proof of Lemma 8

To prove Lemma 8, we first state and prove two lemmas. Lemma 14 directly implies Lemma 8 and is repeatedly used in other proofs in the appendix. Lemma 13 is used in the proof of Lemma 14.

Lemma 13

$$\bar{G}\left(p_{2H}^U - A_H\right) \geq \bar{G}\left(p_{2L}^U - A_L\right),\ (p_{2H}^U - c)\bar{G}\left(p_{2H}^U - A_H\right) > (p_{2L}^U - c)\bar{G}\left(p_{2L}^U - A_L\right).$$

Proof: Define $\delta = \bar{G}\left(p_{2H}^U - A_H\right) - \bar{G}\left(p_{2L}^U - A_L\right)$, $\Delta = (p_{2H}^U - c)\bar{G}\left(p_{2H}^U - A_H\right) - (p_{2L}^U - c)\bar{G}\left(p_{2L}^U - A_L\right)$. By equation (3.2) and the facts that $c < \bar{p}_H$ and $\underline{c}_L < \underline{c}_H$, we show $\delta \geq 0$ and $\Delta > 0$ in all of the following five cases:

- $p_{2H}^U = A_H + \alpha_L$ and $p_{2L}^U = A_L + \alpha_L$: $\delta = 1 - 1 = 0$ and $\Delta = A_H - A_L > 0$.
- $p_{2H}^U = A_H + \alpha_L$ and $p_{2L}^U \in (A_L + \alpha_L, A_L + \alpha_H)$: $\delta = 1 - \bar{G}\left(p_{2L}^U - A_L\right) > 0$, and

$$\begin{aligned}\Delta &= p_{2H}^U - c - (p_{2L}^U - c)\bar{G}\left(p_{2L}^U - A_L\right)\\ &\geq p_{2L}^U - c - (p_{2L}^U - c)\bar{G}\left(p_{2L}^U - A_L\right) \quad \text{(by Lemma 7)}\\ &= (p_{2L}^U - c)G\left(p_{2L}^U - A_L\right)\\ &= \frac{\bar{G}\left(p_{2L}^U - A_L\right)G\left(p_{2L}^U - A_L\right)}{g(p_{2L}^U - A_L)} \quad \text{(by equation (3.2))}\\ &> 0\end{aligned}$$

- $p_{2H}^U = A_H + \alpha_L$, $p_{2L}^U = A_L + \alpha_H$: $\delta = 1 - 0 = 1 > 0$ and $\Delta = p_{2H}^U - c > 0$.
- $p_{2H}^U \in (A_H + \alpha_L, A_H + \alpha_H)$ and $p_{2L}^U = A_L + \alpha_H$: $\delta = \bar{G}\left(p_{2H}^U - A_H\right) > 0$ and $\Delta = (p_{2H}^U - c)\bar{G}\left(p_{2H}^U - A_H\right) > 0$.
- $p_{2t}^U = p_{2t}^M \in (A_t + \alpha_L, A_t + \alpha_H)$ for $t = H$ and $t = L$: It suffices to show that $\bar{G}(p_{2t}^M - A_t)$ weakly increases in A_t and that $(p_{2t}^M - c)\bar{G}(p_{2t}^M - A_t)$ strictly increases in A_t:

$$\frac{d\left\{\bar{G}(p_{2t}^M - A_t)\right\}}{dA_t} = -g(p_{2t}^M - A_t)\left(\frac{dp_{2t}^M}{dA_t} - 1\right) = \left.\left\{\frac{g(x)}{1 - \left(\frac{\bar{G}(x)}{g(x)}\right)'}\right\}\right|_{x = p_{2t}^M - A_t} > 0$$

$$\begin{aligned}\frac{d\left\{(p_{2t}^M - c)\bar{G}(p_{2t}^M - A_t)\right\}}{dA_t} &= \frac{dp_{2t}^M}{dA_t}\bar{G}(p_{2t}^M - A_t) - (p_{2t}^M - c)g(p_{2t}^M - A_t)\left(\frac{dp_{2t}^M}{dA_t} - 1\right)\\ &= \bar{G}(p_{2t}^M - A_t) > 0 \quad \text{(by equation (3.2))}\end{aligned}$$

■

Lemma 14

$$\pi_2^{*H}(S) - \pi_2^{*L}(S) \quad \textit{strictly decreases in } S.$$

Proof: For $S \in [0, \min(T, N_1)]$, define $Q(S) = \pi_2^{*H}(S) - \pi_2^{*L}(S)$. Clearly, $Q(S)$ is continuous in S. To show $Q(S)$ strictly decreases in S, by definition of $\pi_2^{*t}(S)$, equation (3.2), and Lemma 13, it suffices to consider the following three cases:

- $\frac{T-S}{N_1+N_2-S} \geq \bar{G}(p_{2H}^U - A_H)$

In such case, $p_2^H(S) = p_{2H}^U$, $p_2^L(S) = p_{2L}^U$, and

$$\begin{aligned}Q(S) &= (N_1 + N_2 - S)\left(p_{2H}^U - c\right)\bar{G}\left(p_{2H}^U - A_H\right) - (N_1 + N_2 - S)\left(p_{2L}^U - c\right)\bar{G}(p_{2L}^U - A_L)\\ &= (N_1 + N_2)\left[(p_{2H}^U - c)\bar{G}(p_{2H}^U - A_H) - (p_{2L}^U - c)\bar{G}(p_{2L}^U - A_L)\right]\\ &\quad + \left[(p_{2L}^U - c)\bar{G}(p_{2L}^U - A_L) - (p_{2H}^U - c)\bar{G}(p_{2H}^U - A_H)\right]S\end{aligned}$$

By Lemma 13, $Q(S)$ strictly decreases in S.

- $\bar{G}(p_{2L}^U - A_L) < \frac{T-S}{N_1+N_2-S} \leq \bar{G}(p_{2H}^U - A_H)$

In such case, $p_2^H(S) = p_{2H}^B$, $p_2^L(S) = p_{2L}^U$, and

$$
\begin{aligned}
Q(S) &= (p_{2H}^B - c)(T - S) - (N_1 + N_2 - S)(p_{2L}^U - c)\bar{G}(p_{2L}^U - A_L) \\
&= (p_{2H}^B - c)(T - S) + (p_{2L}^U - c)\bar{G}(p_{2L}^U - A_L)S - (N_1 + N_2)(p_{2L}^U - c)\bar{G}(p_{2L}^U - A_L)
\end{aligned}
$$

To show the monotonicity of $Q(S)$, it suffices to show that $Q(S)$ is strictly concave and strictly decreases in S at the lower bound of the domain. Taking first-order and second-order derivative w.r.t S:

$$
\begin{aligned}
\frac{dQ(S)}{dS} =& (T - S)\frac{dp_{2H}^B}{dS} - (p_{2H}^B - c) + (p_{2L}^U - c)\bar{G}(p_{2L}^U - A_L) \\
& \left(\text{recall } \bar{G}(p_{2H}^B - A_H) = \frac{T - S}{N_1 + N_2 - S} \text{ and } \frac{dp_{2H}^B}{dS} = \frac{N_1 + N_2 - T}{(N_1 + N_2 - S)^2 \cdot g(p_{2H}^B - A_H)}\right) \\
=& \frac{\bar{G}(p_{2H}^B - A_H)}{g(p_{2H}^B - A_H)} \cdot \frac{N_1 + N_2 - T}{N_1 + N_2 - S} - (p_{2H}^B - c) + (p_{2L}^U - c)\bar{G}(p_{2L}^U - A_L)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2Q(S)}{dS^2} &= \left(\frac{\bar{G}(x)}{g(x)}\right)'\Bigg|_{x=p_{2H}^B - A_H} \cdot \frac{dp_{2H}^B}{dS} \cdot \frac{N_1 + N_2 - T}{N_1 + N_2 - S} + \frac{\bar{G}(p_{2H}^B - A_H)}{g(p_{2H}^B - A_H)} \cdot \frac{N_1 + N_2 - T}{(N_1 + N_2 - S)^2} - \frac{dp_{2H}^B}{dS} \\
&= \left(\frac{\bar{G}(x)}{g(x)}\right)'\Bigg|_{x=p_{2H}^B - A_H} \cdot \frac{dp_{2H}^B}{dS} \cdot \frac{N_1 + N_2 - T}{N_1 + N_2 - S} + \frac{\bar{G}(p_{2H}^B - A_H)}{g(p_{2H}^B - A_H)} \cdot \frac{N_1 + N_2 - T}{(N_1 + N_2 - S)^2} \\
&\quad - \frac{N_1 + N_2 - T}{(N_1 + N_2 - S)^2 \cdot g(p_{2H}^B - A_H)} \\
&= \left(\frac{\bar{G}(x)}{g(x)}\right)'\Bigg|_{x=p_{2H}^B - A_H} \cdot \frac{dp_{2H}^B}{dS} \cdot \frac{N_1 + N_2 - T}{N_1 + N_2 - S} - \frac{G(p_{2H}^B - A_H)}{g(p_{2H}^B - A_H)} \cdot \frac{N_1 + N_2 - T}{(N_1 + N_2 - S)^2} \\
&= \left[\left(\frac{\bar{G}(x)}{g(x)}\right)'\Bigg|_{x=p_{2H}^B - A_H} - 1\right] \cdot \frac{dp_{2H}^B}{dS} \cdot G(p_{2H}^B - A_H) \\
&< 0 \quad \left(\text{since } G \text{ satisfies IFR, } \frac{dp_{2H}^B}{dS} > 0 \text{ and } G(p_{2H}^B - A_H) = \frac{N_1 + N_2 - T}{N_1 + N_2 - S} > 0\right)
\end{aligned}
$$

Hence, $Q(S)$ is strictly concave in S.

Meanwhile, at the lower bound of S's domain in this case, i.e., when S satisfies $\frac{T-S}{N_1+N_2-S} = \bar{G}(p_{2H}^U - A_H)$, $p_{2H}^B = p_{2H}^U$ and

$$
\begin{aligned}
\frac{dQ(S)}{dS} &= (p_{2H}^U - c) \cdot G(p_{2H}^U - A_H) - (p_{2H}^U - c) + (p_{2L}^U - c)\bar{G}(p_{2L}^U - A_L) \\
&\qquad (\text{recall that } p_{2H}^U \text{ satisfies } p_{2H}^U - c = \frac{\bar{G}(p_{2H}^U - A_H)}{g(p_{2H}^U - A_H)}) \\
&= -(p_{2H}^U - c)\bar{G}(p_{2H}^U - A_H) + (p_{2L}^U - c)\bar{G}(p_{2L}^U - A_L) \\
&< 0 \quad (\text{by Lemma 13})
\end{aligned}
$$

Therefore, by strict concavity of $Q(S)$, for S satisfying $\frac{T-S}{N_1+N_2-S} \leq \bar{G}(p_{2H}^U - A_H)$, $\frac{dQ(S)}{dS} < 0$ and $Q(S)$ strictly decreases in S.

- $\frac{T-S}{N_1+N_2-S} < \bar{G}(p_{2L}^U - A_L)$

In such case, $p_2^H(S) = p_{2H}^B$, $p_2^L(S) = p_{2L}^B$, and

$$Q(S) = \left(p_{2H}^B(S) - c\right)(T-S) - \left(p_{2L}^B(S) - c\right)(T-S)$$

Taking derivative w.r.t S,

$$\begin{aligned}\frac{dQ(S)}{dS} =& [p_{2L}^B(S) - p_{2H}^B(S)] + (T-S)\left(\frac{dp_{2H}^B(S)}{dS} - \frac{dp_{2L}^B(S)}{dS}\right)\\ =& [p_{2L}^B(S) - p_{2H}^B(S)] + \frac{N_1+N_2-T}{B(N_1+N_2-S)} \cdot \left[\frac{\bar{G}(p_{2H}^B(S)-A_H)}{g(p_{2H}^B(S)-A_H)} - \frac{\bar{G}(p_{2L}^B(S)-A_L)}{g(p_{2L}^B(S)-A_L)}\right]\end{aligned}$$

By definition of $p_{2t}^B(S)$, $p_{2H}^B(S) - A_H = p_{2L}^B(S) - A_L = (\bar{G})^{-1}\left(\frac{T-S}{N_1+N_2-S}\right)$. Hence, $\frac{dQ(S)}{dS} = p_{2L}^B(S) - p_{2H}^B(S) = A_L - A_H < 0$. ■

Proof of Lemma 8

Proof: Directly from the definition of $\pi_t^{AI}(p_1, S)$ and Lemma 14. ■

B.4 Proof of Lemma 9

Proof: Assume L type's equilibrium strategy is (p_1^*, S^*) in a separating equilibrium. We prove the lemma by considering two cases: a) $S_L^{FI,OR} = 0$; b) $S_L^{FI,OR} > 0$.

a) When $S_L^{FI,OR} = 0$, L type sells only in spot under full-information setting, hence it suffices to show $S^* = 0$. Suppose $S^* > 0$, then p_1^* must equal to $p_1^L(S^*)$, since as we noted, L-type seller will not charge any price lower than $p_1^L(S^*)$ and in a separating equilibrium, customers will not pay any price higher than $p_1^L(S^*)$ to a L-type seller. However, since $S_L^{FI,OR} = 0$ maximizes $\pi_L^{FI}(S) = \pi_L^{AI}(p_1^L(S), S)$, this contradicts with the hypothesis $S^* > 0$.

b) When $S_L^{FI,OR} > 0$, we show $(p_1^*, S^*) = (p_{1L}^{FI,OR}, S_L^{FI,OR})$ in two steps: b.1) For given S^*, $p_1^* = p_1^L(S^*)$; b.2) $S^* = S_L^{FI,OR}$.

b.1) As we noted, for given S^*, advance customers will accept $p_1^L(S^*)$ regardless of their belief. If $p_1^* < p_1^L(S^*)$, L-type seller always has an incentive to deviate to $p_{1L}(S^*)$, which is higher than p_1^* and guaranteed to be accepted. Meanwhile, if $p_1^* > p_1^L(S^*)$, in equilibrium customers know it is offered by a L-type seller and would reject it, which makes (p_1^*, S^*) equivalent to $\left(p_1^L(0), 0\right)$. Recall that $S_L^{FI,OR}$ is positive and hence L-type seller always has an incentive to deviate to $(p_{1L}^{FI,OR}, S_L^{FI,OR})$, which is guaranteed to be accepted and makes him strictly better off. We thus conclude that $p_1^* = p_1^L(S^*)$.

b.2) From b.1), we know L type plays $\left(p_1^L(S^*), S^*\right)$ in equilibrium. As a result, S^* equals to $S_L^{FI,OR}$, which maximizes $\pi_L^{AI}(p_1^L(S), S)$. ■

B.5 Proof of Theorem 9

We first prove a useful lemma.

Lemma 15 $S_H^{AI,OR} \le S_L^{FI,OR}$.

Proof: By equation (3.7) and (3.8),

$$(p_{1H}^{AI,OR} - c)S_H^{AI,OR} + \pi_2^{*L}(S_H^{AI,OR}) \le (p_{1L}^{FI,OR} - c)S_L^{FI,OR} + \pi_2^{*L}(S_L^{FI,OR}) \tag{B.2}$$

$$(p_{1H}^{AI,OR} - c)S_H^{AI,OR} + \pi_2^{*H}(S_H^{AI,OR}) \ge (p_{1L}^{FI,OR} - c)S_L^{FI,OR} + \pi_2^{*H}(S_L^{FI,OR}) \tag{B.3}$$

Multiplying (B.2) by -1 and adding it to (B.3), we get

$$\pi_2^{*H}(S_H^{AI,OR}) - \pi_2^{*L}(S_H^{AI,OR}) \ge \pi_2^{*H}(S_L^{FI,OR}) - \pi_2^{*L}(S_L^{FI,OR})$$

By Lemma 14, $S_H^{AI,OR} \le S_L^{FI,OR}$. ■

Proof of Theorem 9

Proof: (i) By Proposition 6, if $S_L^{FI,OR} > 0$, then $S_H^{FI,OR} > 0$. Therefore, it suffices to show that a separating equilibrium exists if and only if $S_L^{FI,OR} > 0$.

($\Rightarrow$) Suppose $S_L^{FI,OR} = 0$ and a separating equilibrium exists. By Lemma 9 and the definition of a separating equilibrium, $S_H^{AI,OR} > 0$. However, by equation (3.9) and Lemma 15, $S_H^{AI,OR} = 0$. We then reach a contradiction and complete the proof.

($\Leftarrow$) It suffices to show that when $S_L^{FI,OR} > 0$, there exists a feasible solution satisfying constraint (3.6) through (3.10). In the following we construct a feasible solution (p_1, S) which satisfies $p_1 = p_1^H(S)$. Note that by $S_L^{FI,OR} > 0$,

$$\begin{aligned} \pi_L^{AI}(p_1^H(0), 0) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) &= \pi_L^{AI}(p_1^L(0), 0) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) \le 0 \\ \pi_L^{AI}\left(p_1^H(S_L^{FI,OR}), S_L^{FI,OR}\right) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) &= \left(p_1^H(S_L^{FI,OR}) - p_{1L}^{FI,OR}\right) S_L^{FI,OR} > 0 \end{aligned} \tag{B.4}$$

Hence, there exists at least a $S \in [0, S_L^{FI,OR})$ such that $\pi_L^{AI}(p_1^H(S), S) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) = 0$. Let

$$S_0 = \max\left\{S \in \left[0, S_L^{FI,OR}\right) : \pi_L^{AI}(p_1^H(S), S) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) = 0\right\}$$

Clearly $(p_1^H(S_0), S_0)$ satisfies (3.6), (3.7), (3.9) and (3.10). It is easy to show that (3.8) is also satisfied:

$$\pi_H^{AI}(p_1^H(S_0), S_0) - \pi_H^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) > \pi_L^{AI}(p_1^H(S_0), S_0) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) = 0$$

where the inequality follows from Lemma 8 and the fact that $S_0 < S_L^{FI,OR}$. Therefore, $(p_1^H(S_0), S_0)$ is a feasible solution to H-type's optimization problem and hence there must exist a separating equilibrium.

(ii) We show that H-type's equilibrium strategy satisfies (ii-a) $p_{1H}^{AI,OR} = p_1^H(S_H^{AI,OR})$ and (ii-b) $S_H^{AI,OR} < S_H^{FI,OR}$ and $p_{1H}^{AI,OR} \leq p_{1H}^{FI,OR}$.

(ii-a) First note that for given S, both $\pi_H^{AI}(p_1, S)$ and $\pi_L^{AI}(p_1, S)$ increase in p_1. Hence at least one of the constraint (3.6) and constraint (3.7) must be binding when the maximum of the objective is attained. Prove by contradiction. Suppose constraint (3.6) is not binding at the attained maximum, then constraint (3.7) must be binding. That is, $\pi_L^{AI}\left(p_{1H}^{AI,OR}, S_H^{AI,OR}\right) = \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right)$, which implies

$$\left(p_{1H}^{AI,OR} - c\right) S_H^{AI,OR} = \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) - \pi_2^{*L}\left(S_H^{AI,OR}\right) \tag{B.5}$$

Apply equation (B.5) to H-type seller's objective function:

$$\pi_H^{AI}\left(p_{1H}^{AI,OR}, S_H^{AI,OR}\right) = \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) - \pi_2^{*L}\left(S_H^{AI,OR}\right) + \pi_2^{*H}\left(S_H^{AI,OR}\right) \tag{B.6}$$

To reach a contradiction, it suffices to find another feasible strategy pair which, compared to $(p_{1H}^{AI,OR}, S_H^{AI,OR})$, strictly improves H-type's total profit. Similar to (i), let

$$S_1 = \min\left\{S \in \left[0, S_L^{FI,OR}\right) : \pi_L^{AI}(p_1^H(S), S) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) = 0\right\}$$

It is easy to prove that $(p_1^H(S_1), S_1)$ is feasible. In the meanwhile, $S_1 < S_H^{AI,OR}$ by the definition of S_1 and the result $\pi_L^{AI}(p_1^H(S_H^{AI,OR}), S_H^{AI,OR}) > \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right)$, as shown below:

$$\begin{aligned}
&\pi_L^{AI}(p_1^H(S_H^{AI,OR}), S_H^{AI,OR}) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) \\
=&\left(p_1^H(S_H^{AI,OR}) - c\right) S_H^{AI,OR} + \pi_2^{*L}\left(S_H^{AI,OR}\right) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) \\
>&\left(p_{1H}^{AI,OR} - c\right) S_H^{AI,OR} + \pi_2^{*L}\left(S_H^{AI,OR}\right) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) \\
&\quad \text{(since constraint (3.6) is not binding)} \\
=&\,0 \quad \text{(by equation (B.5))}
\end{aligned}$$

We now prove $\pi_H^{AI}(p_1^H(S_1), S_1) > \pi_H^{AI}(p_{1H}^{AI,OR}, S_H^{AI,OR})$. By the definition of S_1, Equation (B.6), Lemma 14, and the fact that $S_H^{AI,OR} > S_1$,

$$\begin{aligned}
&\pi_H^{AI}(p_1^H(S_1), S_1) - \pi_H^{AI}\left(p_{1H}^{AI,OR}, S_H^{AI,OR}\right) \\
=&\ \left(p_1^H(S_1) - c\right) S_1 + \pi_2^{*H}(S_1) - \pi_H^{AI}\left(p_{1H}^{AI,OR}, S_H^{AI,OR}\right) \\
=&\,\pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) - \pi_2^{*L}(S_1) + \pi_2^{*H}(S_1) \\
&- \left[\pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) - \pi_2^{*L}\left(S_H^{AI,OR}\right) + \pi_2^{*H}\left(S_H^{AI,OR}\right)\right] \\
=&\,\pi_2^{*H}(S_1) - \pi_2^{*L}(S_1) - \left[\pi_2^{*H}\left(S_H^{AI,OR}\right) - \pi_2^{*L}\left(S_H^{AI,OR}\right)\right] > 0
\end{aligned}$$

(ii-b) We first prove $S_H^{AI,OR} < S_H^{FI,OR}$ by contradiction. Suppose $S_H^{AI,OR} \geq S_H^{FI,OR}$. By Proposition 6 and Lemma 15, $S_H^{AI,OR} = S_L^{FI,OR}$. However, by equation (B.4), $(p_1^H(S_L^{FI,OR}), S_L^{FI,OR})$ is not feasible, as it violates constraint (3.7). This contradicts with (ii-a) and completes the proof.

We then note that $p_{1H}^{AI,OR} \leq p_{1H}^{FI,OR}$ follows from $S_H^{AI,OR} < S_H^{FI,OR}$, (ii-a), definition of $p_{1H}^{FI,OR}$, and the fact that $p_1^H(S)$ is non-decreasing in S.

(iii) To show $S_H^{AI,OR} < S_L^{FI,OR}$, note that by the definition of S_0, for any $S \in (S_0, S_L^{FI,OR}]$, $(p_1^H(S), S)$ is not feasible since $\pi_L^{AI}(p_1^H(S), S) - \pi_L^{AI}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}\right) > 0$, violating constraint (3.7). This result, together with (ii-a) and Lemma 15, implies that $S_H^{AI,OR} \leq S_0 < S_L^{FI,OR}$.

■

B.6 Proof of Theorem 10

Proof: Suppose there exists a pooling equilibrium in which both types offer advance selling at (p_1^E, S^E). Without loss of generality, assume p_1^E is accepted by advance customers and $S^E > 0$. We prove this equilibrium can not be sustained in three steps: **(a)** Since p_1^E is accepted by advance customers, p_1^E is strictly less than $p_1^H(S^E)$; **(b)** there always exists a $S' \in (0, S^E)$ such that $(p_1^H(S'), S')$ (if accepted by customers) is strictly preferred by H type to (p_1^E, S^E), while (p_1^E, S^E) is strictly preferred by L type to $(p_1^H(S'), S')$; **(c)** by intuitive criterion, customers should believe the seller is H type with probability one if $(p_1^H(S'), S')$ is observed. Hence $(p_1^H(S'), S')$ will always be accepted and H type always has incentive to deviate from (p_1^E, S^E) to $(p_1^H(S'), S')$, which breaks the pooling equilibrium.

(a) Clearly, in order for p_1^E to be accepted by advance customers, it must be true that $p_1^E \leq qp_1^H(S^E) + (1-q)p_1^L(S^E) < p_1^H(S^E)$.

(b) We first define two functions of S:

$$D_H(S) = \pi_H^{AI}(p_1^H(S), S) - \pi_H^{AI}(p_1^E, S^E)$$

$$D_L(S) = \pi_L^{AI}(p_1^H(S), S) - \pi_L^{AI}(p_1^E, S^E)$$

Since $p_1^E < p_1^H(S^E)$ and $S^E > 0$, it is clear that $D_H(S^E) > 0$ and $D_L(S^E) > 0$. Furthermore, $D_H(0) \leq 0$ and $D_L(0) \leq 0$, since otherwise at least one of the two types will have incentive to deviate to not offer advance selling. Therefore, the sets $\{S \in [0, S^E) : D_H(S) = 0\}$ and $\{S \in [0, S^E) : D_L(S) = 0\}$ are both non-empty. Let $\bar{S}_H = \max\{S : S \in [0, S^E), D_H(S) = 0\}$ and

$\bar{S}_L = \max\{S : S \in [0, S^E), D_L(S) = 0\}$. Note that for $S < S^E$,

$$\begin{aligned}
&D_H(S) - D_L(S) \\
&= (p_1^H(S) - c)S + \pi_2^{*H}(S) - \left[(p_1^E - c)S^E + \pi_2^{*H}(S^E)\right] \\
&\quad - \left\{(p_1^H(S) - c)S + \pi_2^{*L}(S) - \left[(p_1^E - c)S^E + \pi_2^{*L}(S^E)\right]\right\} \\
&= \pi_2^{*H}(S) - \pi_2^{*L}(S) - \left(\pi_2^{*H}(S^E) - \pi_2^{*L}(S^E)\right) > 0 \quad \text{(by Lemma 14)}
\end{aligned}$$

Hence, $\bar{S}_H < \bar{S}_L$, $D_L(\bar{S}_H) < 0$, and $D_H(S) > 0$ for all $S > \bar{S}_H$. As a result, in the right neighborhood of $\bar{S}_H$, there exists a S' such that $D_H(S') > 0$ and $D_L(S') < 0$. That is, $(p_1^H(S'), S')$ (if accepted by customers) is strictly preferred by H type to $(p_1^E,\ S^E)$, while (p_1^E, S^E) is strictly preferred by L type to $(p_1^H(S'), S')$.

(c) From (b), if customers observe deviation from (p_1^E, S^E) to $(p_1^H(S'), S')$, such deviation has to be made by H-type seller. Therefore, by intuitive criterion, customers will believe the seller is H type with probability one if $(p_1^H(S'), S')$ is observed. An important implication of this belief is that $(p_1^H(S'), S')$ will always be accepted and constitute a beneficial deviation for H type. Therefore, H type is never willing to stay at (p_1^E, S^E) and the pooling equilibrium can not be sustained. ■

B.7 Proof of Theorem 11

Proof: (⇒) Prove by contradiction. Suppose $S_L^{FI,OR} > 0$ and a pooling equilibrium exists in which neither type offers advance selling. In such an equilibrium, L type always has an incentive to deviate to $(p_{1L}^{FI,OR}, S_L^{FI,OR})$, which is guaranteed to be accepted and makes him better off, since $S_L^{FI,OR} > 0$ and $\pi_L^{AI}(p_{1L}^{FI,OR}, S_L^{FI,OR}) > \pi_L^{AI}(p_1^L(0), 0)$. Therefore, the pooling equilibrium cannot be sustained.

(⇐) It suffices to support the pooling equilibrium by constructing a customer-belief system as follows: customers believe that the seller is L type w.p.1 if any $S > 0$ is observed. In such a case, L type clearly does not have incentive to deviate from selling only in spot, since by allocating any $S > 0$ to advance, he can at most collect a total profit $\pi_L^{AI}(p_1^L(S), S)$, which is less than or equal to the total profit from selling only in spot $\pi_L^{AI}(p_1^L(0), 0)$. As for H type, he will not get better off by offering advance selling, either, since for any $S > 0$, by Lemma 14,

$$\begin{aligned}
\pi_H^{AI}(p_1^L(S), S) - \pi_2^{*H}(0) &= (p_1^L(S) - c)S + \pi_2^{*H}(S) - \pi_2^{*H}(0) \\
&< (p_1^L(S) - c)S + \pi_2^{*L}(S) - \pi_2^{*L}(0) \\
&= \pi_L^{AI}(p_1^L(S), S) - \pi_L^{AI}(p_1^L(0), 0) \le 0
\end{aligned}$$

■

B.8 Proof of Theorem 13

We prove Theorem 13 in three steps: first, Lemma 16 shows the seller's optimal no-rationing strategy under full-information scenario, which establishes the existence of two functions, $c_H^{NR}(T)$ and $c_L^{NR}(T)$, and $c_L^{NR}(T) \le c_H^{NR}(T)$ for all T; second, Lemma 17 and 18 characterize the separating equilibrium and pooling equilibrium for the no-rationing model, respectively; third, we collect all the results and prove Theorem 13.

Lemma 16 *When capacity rationing is not allowed and all customers in advance are informed of quality, for given A_t, there exists a function $c_t^{NR}(T)$ such that $S_t^{FI,NR} = \min(T, N_1)$ if $c \le c_t^{NR}(T)$ and $S_t^{FI,NR} = 0$ otherwise. Meanwhile, $c_t^{NR}(T)$ increases in A_t.*

Proof: Since the results are shown for every given A_t, we can drop the subscript t in all variables for now.

When rationing is not feasible, the optimal S is either 0 or $\min(T, N_1)$, whichever results in a higher total profit. To show the existence and monotonicity of $c^{NR}(T)$, it then suffices to show that for $T < N_1 + N_2\bar{G}(p_2^U - A)$, the profit difference $\pi^{FI}(\min(T, N_1)) - \pi^{FI}(0)$ both decreases in c and increases in A; and that for $T \ge N_1 + N_2\bar{G}(p_2^U - A)$, the profit difference $\pi^{FI}(\min(T, N_1)) - \pi^{FI}(0)$ is nonnegative if and only if $c \le \bar{c}$ and $\bar{c}$ increases in A, where $\bar{c}$ is as defined in the proof of Proposition 6.

To facilitate the proof, define a function of c, $p_2^U(c)$, which satisfies $p_2^U(c) = c + \frac{\bar{G}(p_2^U(c)-A)}{g(p_2^U(c)-A)}$ and divide the capacity-cost space into the following five regions, as illustrated in Figure B.1:

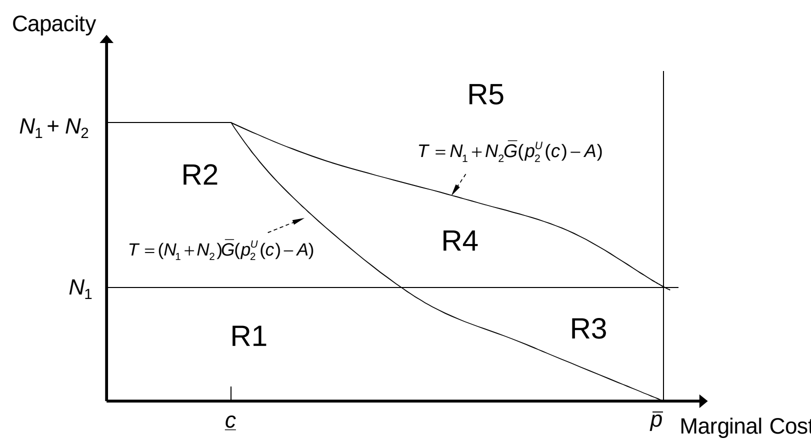


Figure B.1. Five regions for full-Information no-rationing setting

(R1): $T \le N_1$ and $T \le (N_1 + N_2)\bar{G}(p_2^U - A)$

In such a case, $p_2(\min(T,N_1)) = p_2^B(T) = \bar{p}$, $p_1(\min(T,N_1)) = A + \mathrm{E}[\alpha]$, $p_2(0) = p_2^B(0) = A + \bar{G}^{-1}\left(\frac{T}{N_1+N_2}\right)$, and

$$\pi^{FI}(\min(T,N_1)) - \pi^{FI}(0) = (A + \mathrm{E}[\alpha] - c)\,T - (p_2^B(0) - c)T = \left(\mathrm{E}[\alpha] - \bar{G}^{-1}\left(\frac{T}{N_1+N_2}\right)\right)T$$

Clearly, the profit difference is independent of both c and A.

(R2): $T > N_1$ and $T \leq (N_1+N_2)\bar{G}(p_2^U - A)$

In such a case, $p_2(\min(T,N_1)) = p_2^B(N_1)$, $p_1(\min(T,N_1)) = p_1^B(N_1)$, $p_2(0) = p_2^B(0) = A + \bar{G}^{-1}\left(\frac{T}{N_1+N_2}\right)$, and

$$\begin{aligned}\pi^{FI}(\min(T,N_1)) - \pi^{FI}(0) &= (p_1^B(N_1) - c)N_1 + (p_2^B(N_1) - c)(T - N_1) - (p_2^B(0) - c)T \\ &= [p_1^B(N_1) - p_2^B(N_1)]N_1 + [p_2^B(N_1) - p_2^B(0)]T\end{aligned}$$

Clearly, the difference in profit is independent of c. It is easy to show that for given S, $dp_2^B(S)/dA = dp_1^B(S)/dA = 1$, and hence the profit difference is also independent of A.

(R3): $T \leq N_1$ and $T > (N_1+N_2)\bar{G}(p_2^U - A)$

In such a case, $p_2(\min(T,N_1)) = p_2^B(T) = \bar{p}$, $p_1(\min(T,N_1)) = A + \mathrm{E}[\alpha]$, $p_2(0) = p_2^U$, and

$$\pi^{FI}(\min(T,N_1)) - \pi^{FI}(0) = (A + \mathrm{E}[\alpha] - c)\,T - (p_2^U - c)(N_1+N_2)\bar{G}(p_2^U - A)$$

Take first-order derivative of the profit difference with respect to c and A, respectively:

$$\begin{aligned}\frac{d\{\pi^{FI}(\min(T,N_1)) - \pi^{FI}(0)\}}{dc} &= -T + (N_1+N_2)\bar{G}(p_2^U - A) < 0 \\ \frac{d\{\pi^{FI}(\min(T,N_1)) - \pi^{FI}(0)\}}{dA} &= T - (p_2^U - c)(N_1+N_2)g(p_2^U - A) \\ &= T - (N_1+N_2)\bar{G}(p_2^U - A) > 0 \text{ (by equation (3.2))}\end{aligned}$$

(R4): $T > N_1$ and $(N_1+N_2)\bar{G}(p_2^U - A) < T < N_1 + N_2\bar{G}(p_2^U - A)$

In such a case, $p_2(\min(T,N_1)) = p_2^B(N_1)$, $p_1(\min(T,N_1)) = p_1^B(N_1)$, $p_2(0) = p_2^U$, and

$$\pi^{FI}(\min(T,N_1)) - \pi^{FI}(0) = (p_1^B(N_1) - c)N_1 + (p_2^B(N_1) - c)(T - N_1) - (p_2^U - c)(N_1+N_2)\bar{G}(p_2^U - A)$$

Take first-order derivative of the profit difference with respect to c and A, respectively, and note that for given S, $dp_2^B(S)/dA = dp_1^B(S)/dA = 1$,

$$\begin{aligned}\frac{d\{\pi^{FI}(\min(T,N_1)) - \pi^{FI}(0)\}}{dc} &= -T + (N_1+N_2)\bar{G}(p_2^U - A) < 0 \\ \frac{d\{\pi^{FI}(\min(T,N_1)) - \pi^{FI}(0)\}}{dA} &= N_1 + T - N_1 - (p_2^U - c)(N_1+N_2)g(p_2^U - A) \\ &= T - (N_1+N_2)\bar{G}(p_2^U - A) > 0\end{aligned}$$

(R5): $T \geq N_1 + N_2\bar{G}(p_2^U - A)$

In such a case, $p_2(\min(T, N_1)) = p_2^U$, $p_1(\min(T, N_1)) = p_1^U$, $p_2(0) = p_2^U$, and

$$\begin{aligned}\pi^{FI}(\min(T, N_1)) - \pi^{FI}(0) &= (p_1^U - c)N_1 + (p_2^U - c)N_2\bar{G}(p_2^U - A) - (p_2^U - c)(N_1 + N_2)\bar{G}(p_2^U - A) \\ &= \left(p_1^U - c - (p_2^U - c)\bar{G}(p_2^U - A)\right) N_1\end{aligned}$$

By the definition of $\bar{c}$, $\pi^{FI}(\min(T, N_1)) - \pi^{FI}(0)$ is nonnegative for $c \leq \bar{c}$ and negative otherwise. In the meantime, by the proof of Proposition 6, $\bar{c}$ increases in A. ∎

Lemma 17 *In any separating equilibrium, L type offers (full) advance selling at price* $p_1^L(\min(T, N_1))$*, while H type sells only in spot.*

Proof: We prove the lemma by showing that (i) no separating equilibrium exists such that both types of sellers offer (full) advance selling; (ii) no separating equilibrium exists such that H type offers (full) advance selling and L type does not; (iii) L type's equilibrium price is $p_1^L(\min(T, N_1))$.

(i) Suppose there exists a separating equilibrium such that both types of sellers offer (full) advance selling and charge advance prices $p_1^H \neq p_1^L$. All customers buy in advance in such a separating equilibrium because seller has full information about customer behavior; if he decides to offer advance selling, the price quoted will stimulate advance purchase from all customers. Without loss of generality, suppose $p_1^H > p_1^L$. Such separating equilibrium, however, cannot be sustained since L type always has incentive to take advantage of customers' belief and to mimic H type by charging p_1^H in advance. The reason is as follows: in a separating equilibrium, customers always believe seller is H type after observing p_1^H and buy in advance. In advance, L type increases profit by selling to all advance customers at a higher price. In spot, all information is revealed and L type's spot profit is not influenced. Thus, his overall profit is increased by deviating to offer p_1^H.

(ii) Suppose there exists a separating equilibrium such that H type offers advance selling at p_1^H and L type does not offer advance selling. By Lemma 14,

$$\begin{aligned}\pi_L^{AI}(p_1^H, \min(T, N_1)) - \pi_L^{AI}(p_1, 0) &= (p_1^H - c)\min(T, N_1) + \pi_2^{*L}(\min(T, N_1)) - \pi_2^{*L}(0) \\ &> (p_1^H - c)\min(T, N_1) + \pi_2^{*H}(\min(T, N_1)) - \pi_2^{*H}(0) \\ &= \pi_H^{AI}(p_1^H, \min(T, N_1)) - \pi_H^{AI}(p_1, 0) \qquad \text{(B.7)}\end{aligned}$$

That is, L type can always guarantee a benefit no less than that of H type, simply from mimicking H type to offer advance selling at p_1^H. If advance selling at p_1^H is beneficial for H type, it is also beneficial for L type. Hence L type always has incentive to make such unilateral deviation and the separating equilibrium cannot be sustained.

(iii) Let L type's equilibrium price be p_1^*. p_1^* must always equal to $p_1^L(\min(T, N_1))$ because first, p_1^* can not be higher than $p_1^L(\min(T, N_1))$ since otherwise it will be rejected and strictly dominated by $p_1^L(\min(T, N_1))$; second, p_1^* can not be lower than $p_1^L(\min(T, N_1))$ since otherwise L type can deviate to $p_1^L(\min(T, N_1))$ which is guaranteed to be accepted and generates higher margin. ■

Lemma 18 *In any focal pooling equilibrium where both types of sellers sell in advance, the followings are true:*

(i) The equilibrium price is p_1^E*;*

(ii) Such equilibrium is sustained if and only if $q \geq \bar{q}$*;*

(iii) It is the unique focal equilibrium if sustained.

Proof: (i) Assume such equilibrium is sustained at p_1^*. By the definition of pooling equilibrium, customers' belief after observing p_1^* should be consistent with seller's equilibrium strategy, i.e., the belief is the same as prior belief. Meanwhile, by similar arguments as in the proof of Lemma 17(i), all customers buy in advance in such equilibrium. These imply $p_1^* \leq p_1^E$ since it is easy to see that p_1^E is the maximum price that customers with the prior belief are willing to pay in advance.

Clearly all pooling equilibrium with $p_1^* < p_1^E$ is pareto dominated from the seller's point of view. Therefore, $p_1^* = p_1^E$ in any focal pooling equilibrium.

(ii). The equilibrium can be sustained if and only if advance selling at p_1^E provides both types of sellers nonnegative benefit, compared to selling only in spot. By equation (B.7), the latter is equivalent to requiring $\pi_H^{AI}(p_1^E, \min(T, N_1)) - \pi_H^{AI}(p_1, 0) \geq 0$. By the definition of p_1^E, $\bar{q}$, and $\pi_H^{FI}(\min(T, N_1))$, as well as the fact $\pi_H^{AI}(p_1, 0) = \pi_H^{FI}(0)$, it is equivalent to requiring $q \geq \bar{q}$.

(iii) To see the uniqueness, suppose a separating equilibrium also exists when $q \geq \bar{q}$. By Lemma 17, in such equilibrium L type sells in advance at $p_1^L(\min(T, N_1))$ and H type does not sell in advance. This equilibrium, however, is pareto dominated by the pooling equilibrium where both sell in advance at p_1^E. To see why, note that $p_1^E > p_1^L(\min(T, N_1))$ and L-type's total profit is strictly increased by charging a higher price in advance. For H-type, as shown in (ii), advance selling at p_1^E dominates selling only in spot when $q \geq \bar{q}$. Therefore, the separating equilibrium is not a focal equilibrium. ■

Proof of Theorem 13

Proof: The existence of the focal equilibrium is trivial. To see the uniqueness of the focal equilibrium, note that by Lemma 18 (ii) and (iii), pooling equilibrium at p_1^E is the unique focal equilibrium if and only if $q \geq \bar{q}$. Therefore, we focus on the scenario where $q < \bar{q}$ (implying $\bar{q} > 0$) and consider the following three cases:

• $c \leq c_L^{NR}(T)$: To see why the separating equilibrium is sustained when $q < \bar{q}$, it suffices to verify that neither type has incentive to mimic the other type's strategy. For L type, she does not have incentive to mimic high type since for $c \leq c_L^{NR}(T)$, by Lemma 16, advance selling at price $p_1^L(\min(T, N_1))$ makes her better off than selling only in spot. For H type, he does not have incentive to mimic L type as long as his benefit from doing that is nonpositive, i.e., $\pi_H^{AI}(p_1^L(\min(T, N_1)), \min(T, N_1)) - \pi_H^{AI}(p_1, 0) \leq 0$. This is, however, guaranteed by $\bar{q} > 0$. The uniqueness is obvious since the only equilibrium that has not been ruled out is a pooling equilibrium in which neither type offers advance selling; this is clearly not sustainable since L type has an incentive to deviate to advance selling alone.

• $c_L^{NR}(T) \leq c < c_H^{NR}(T)$: In such case, by Lemma 16, selling in advance at price $p_1^L(\min(T, N_1))$ makes L type worse off compared to selling only in spot. Consequently, a separating equilibrium cannot be sustained and the pooling equilibrium where neither type sells in advance is the only equilibrium, in which any deviation to sell in advance is perceived by customers as a sign of L type. In such an equilibrium, H type does not have incentive to offer advance selling because the highest price he can charge is $p_1^L(\min(T, N_1))$, which makes him worse off when $\bar{q} > 0$.

• $c \geq c_H^{NR}(T)$: Same as the proof for the second bullet except that in this case $\bar{q} \geq 1$, since by Lemma 16, $\pi_H^{FI}(\min(T, N_1)) \leq \pi_H^{FI}(0)$ when $c \geq c_H^{NR}(T)$. Therefore, pooling equilibrium in which neither offers advance selling is the unique equilibrium for any q between 0 and 1. ■

B.9 Proof of Theorem 14

We first claim and prove three useful lemmas. Lemma 19 and 20 are used to prove Corollary 4. Lemma 19 and Corollary 4 are then used to prove Lemma 21. In the end, Lemma 19 and Lemma 21 are used in the proof of Theorem 14.

Lemma 19 *If (p_1^*, S^*, Q^*) is an optimal solution to H-type's problem with $S^* > 0$ and $Q^* > 0$, then (p_1^*, S^*, Q^*) satisfies the following three conditions:*

$$\pi_L^{AI,AD}(p_1^*, S^*, Q^*) = \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right); \tag{B.8}$$

$$\pi_H^{AI,AD}(p_1^*, S^*, Q^*) = \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) - \pi_2^{*L}(S^*) + \pi_2^{*H}(S^*); \tag{B.9}$$

$$\pi_2^{*L}(0) - Q^* < \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right); \tag{B.10}$$

$$S^* \leq S_L^{FI,OR} \tag{B.11}$$

Proof: Since $Q^* > 0$ is part of an optimal solution, (3.12) is held as equality at (p_1^*, S^*, Q^*), otherwise Q^* can be decreased by ϵ such that all the constraints are still satisfied and H type's total profit $\pi_H^{AI,AD}(p_1^*, S^*, Q)$ is strictly improved. The equality version of (3.12) is exactly equation (B.8). Apply equation (B.8) to the objective function:

$$\pi_H^{AI,AD}(p_1^*, S^*, Q^*) = (p_1^* - c)S^* + \pi_2^{*H}(S^*) - Q^* = \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) - \pi_2^{*L}(S^*) + \pi_2^{*H}(S^*)$$

This proves equation (B.9).

Suppose condition (B.10) does not hold, i.e., $\pi_2^{*L}(0) - Q^* \geq \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right)$. Since $Q^* > 0$, $\pi_2^{*L}(0) > \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right)$, which contradicts with the fact that $(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0)$ is L-type's full-information optimal strategy.

To show (B.11), note that from (3.13) and (B.9):

$$\pi_H^{AI,AD}(p_1^*, S^*, Q^*) = \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) - \pi_2^{*L}(S^*) + \pi_2^{*H}(S^*) \geq \pi_H^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right)$$

This implies

$$\pi_2^{*L}(S_L^{FI,OR}) - \pi_2^{*H}(S_L^{FI,OR}) - \pi_2^{*L}(S^*) + \pi_2^{*H}(S^*) \geq 0$$

By Lemma 14, $S^* \leq S_L^{FI,OR}$. ∎

Lemma 20 *For given p_1 and Q, $\pi_t^{AI,AD}(p_1, S, Q)$ is concave in S.*

Proof: For given p_1 and Q, recall that

$$\begin{aligned}\pi_t^{AI,AD}(p_1, S, Q) &= (p_1 - c)S + \pi_2^{*t}(S) - Q \\ &= (p_1 - c)S + (p_2^t(S) - c)\min(T - S, (N_1 + N_2 - S)\bar{G}(p_2^t(S) - A_t)) - Q\end{aligned}$$

Let $S_t^{=}$ be the solution to $\frac{T-S}{N_1+N_2-S} = \bar{G}(p_{2t}^U - A_t)$, $S_t^{=} = \frac{T-(N_1+N_2)\bar{G}(p_2^U - A)}{G(p_2^U - A)}$. By definition of $p_2^t(S)$, for $S \in [0, \min(T, N_1)]$,

$$\pi_t^{AI,AD}(p_1, S, Q) = \begin{cases} (p_1 - c)S + (p_{2t}^U - c)(N_1 + N_2 - S)\bar{G}(p_{2t}^U - A_t) - Q, & \text{if } S \leq S_t^{=}, \\ (p_1 - c)S + (p_{2t}^B(S) - c)(T - S) - Q, & \text{if } S > S_t^{=}. \end{cases}$$

The proof is naturally divided into two cases: (1) $S \leq S_t^{=}$ and (2) $S > S_t^{=}$. Since $\pi_t^{AI,AD}(p_1, S, Q)$ is continuous in S, to show the concavity, it suffices to show that in either case $\pi_t^{AI,AD}(p_1, S, Q)$ is concave in S and $d\pi_t^{AI,AD}(p_1, S, Q)/dS$ is continuous at $S = S_t^{=}$.

(1) When $S \leq S_t^{=}$,

$$\pi_t^{AI,AD}(p_1, S, Q) = (N_1 + N_2)(p_{2t}^U - c)\bar{G}(p_{2t}^U - A_t) + \left(p_1 - c - (p_{2t}^U - c)\bar{G}(p_{2t}^U - A_t)\right) S - Q$$

which is linear in S.

(2) When $S > S_t^{=}$, check the first and second order derivatives of $\pi_t^{AI,AD}(p_1, S, Q)$ w.r.t. S:

$$\begin{aligned}
\frac{d\pi_t^{AI,AD}(p_1,S,Q)}{dS} &= p_1 - c + (T-S)\frac{dp_{2t}^B}{dS} - (p_{2t}^B - c) \\
&\qquad \left(\text{recall } \bar{G}(p_{2t}^B - A_t) = \frac{T-S}{N_1+N_2-S} \text{ and } \frac{dp_{2t}^B}{dS} = \frac{N_1+N_2-T}{(N_1+N_2-S)^2 \cdot g(p_{2t}^B - A_t)}\right) \\
&= p_1 - c + \frac{\bar{G}(p_{2t}^B - A_t)}{g(p_{2t}^B - A_t)} \cdot \frac{N_1+N_2-T}{N_1+N_2-S} - (p_{2t}^B - c) \qquad \text{(B.12)}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2\pi_t^{AI,AD}(p_1,S,Q)}{dS^2} &= \left(\frac{\bar{G}(x)}{g(x)}\right)'\Bigg|_{x=p_{2t}^B - A_t} \cdot \frac{dp_{2t}^B}{dS} \cdot \frac{N_1+N_2-T}{N_1+N_2-S} + \frac{\bar{G}(p_{2t}^B - A_t)}{g(p_{2t}^B - A_t)} \cdot \frac{N_1+N_2-T}{(N_1+N_2-S)^2} - \frac{dp_{2t}^B}{dS} \\
&= \left(\frac{\bar{G}(x)}{g(x)}\right)'\Bigg|_{x=p_{2t}^B - A_t} \cdot \frac{dp_{2t}^B}{dS} \cdot \frac{N_1+N_2-T}{N_1+N_2-S} + \frac{\bar{G}(p_{2t}^B - A_t)}{g(p_{2t}^B - A_t)} \cdot \frac{N_1+N_2-T}{(N_1+N_2-S)^2} \\
&\quad - \frac{N_1+N_2-T}{(N_1+N_2-S)^2 \cdot g(p_{2t}^B - A_t)} \\
&= \left(\frac{\bar{G}(x)}{g(x)}\right)'\Bigg|_{x=p_{2t}^B - A_t} \cdot \frac{dp_{2t}^B}{dS} \cdot \frac{N_1+N_2-T}{N_1+N_2-S} - \frac{G(p_{2t}^B - A_t)}{g(p_{2t}^B - A_t)} \cdot \frac{N_1+N_2-T}{(N_1+N_2-S)^2} \\
&= \left[\left(\frac{\bar{G}(x)}{g(x)}\right)'\Bigg|_{x=p_{2t}^B - A_t} - 1\right] \cdot \frac{dp_{2t}^B}{dS} \cdot G(p_{2t}^B - A_t) \\
&< 0
\end{aligned}$$

where the last inequality is by the facts that $G(\cdot)$ satisfies IFR, $\frac{dp_{2t}^B}{dS} > 0$, and $G(p_{2t}^B - A_t) = \frac{N_1+N_2-T}{N_1+N_2-S} > 0$.

Hence, $\pi_t^{AI,AD}(p_1, S, Q)$ is strictly concave in S.

Meanwhile, note that when $S = S_t^{=}$, $p_{2t}^B(S_t^{=}) = p_{2t}^U$ and by (B.12),

$$\begin{aligned}
\frac{d\pi_t^{AI,AD}(p_1,S,Q)}{dS}\Bigg|_{S=S_t^{=}} &= p_1 - c + (p_{2t}^U - c) \cdot G(p_{2t}^U - A_t) - (p_{2t}^U - c) \\
&\qquad \left(\text{recall that } p_{2t}^U \text{ satisfies } p_{2t}^U - c = \frac{\bar{G}(p_{2t}^U - A_t)}{g(p_{2t}^U - A_t)}\right) \\
&= p_1 - c - (p_{2t}^U - c)\bar{G}(p_{2t}^U - A_t)
\end{aligned}$$

Therefore, $\frac{d\pi_t^{AI,AD}(p_1,S,Q)}{dS}$ is continuous at $S = S_t^{=}$. This completes the proof. ∎

Corollary 4 *Suppose (p_1^*, S^*, Q^*) is an optimal solution to H-type's problem with $S^* > 0$ and $Q^* > 0$. Define $f(S)$ as a function of S:*

$$f(S) = \pi_L^{AI,AD}(p_1^*, S, Q^*) - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right), S \in [0, \min(T, N_1)]$$

$f(S)$ increases in $S \in [0, S^*]$.

Proof: By Lemma 19, S^* is a solution to $f(S) = 0$ on $[0, \min(T, N_1)]$ and $f(0) < 0$. In the meantime, by Lemma 20, $f(S)$ is concave. Therefore, there exist at most two solutions to $f(S) = 0$ on $(0, \min(T, N_1)]$. If S^* is the unique solution, then by concavity of $f(S)$ and $f(0) < 0$, $f(S)$ increases in $S \in [0, S^*]$. If there exists another solution S' on $(0, \min(T, N_1)]$, then $S^* < S'$. To see why, note that for any S satisfying $f(S) = 0$, $\pi_H^{AI,AD}(p_1^*, S, Q^*) = \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) - \pi_2^{*L}(S) + \pi_2^{*H}(S)$. Since (p_1^*, S^*, Q^*) is an optimal solution to H-type's problem, $\pi_H^{AI,AD}(p_1^*, S^*, Q^*) > \pi_H^{AI,AD}(p_1^*, S', Q^*)$. By Lemma 14, $S^* < S'$. Therefore, by concavity of $f(S)$, $f(S)$ increases in $S \in [0, S^*]$. ■

Lemma 21 *If (p_1^*, S^*, Q^*) is an optimal solution to H-type's problem where $S^* > 0$ and $Q^* > 0$, then $p_1^* = p_1^H(S^*)$.*

Proof: Prove by contradiction. Suppose $p_1^* < p_1^H(S^*)$. It suffices to find another feasible solution that strictly improves H type's total profit. If $p_1^* \geq p_1^H(0)$, by continuity of $p_1^H(S)$ in S, there exists a $S' \in [0, S^*)$ such that $p_1^* = p_1^H(S')$. Otherwise, let $S' = 0$. Define

$$
\begin{aligned}
g(S)|_{Q^*} &= \pi_L^{AI}\left(p_1^H(S), S, Q^*\right) - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\
&= (p_1^H(S) - c)S + \pi_2^{*L}(S) - Q^* - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right)
\end{aligned}
$$

Then by equation (B.8) and Corollary 4,

$$
\begin{aligned}
g(S^*)|_{Q^*} &= (p_1^H(S^*) - c)S^* + \pi_2^{*L}(S^*) - Q^* - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\
&> (p_1^* - c)S^* + \pi_2^{*L}(S^*) - Q^* - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\
&= f(S^*) = 0 \\
g(S')|_{Q^*} &= (p_1^H(S') - c)S' + \pi_2^{*L}(S') - Q^* - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\
&= (p_1^* - c)S' + \pi_2^{*L}(S') - Q^* - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\
&= f(S') \leq f(S^*) = 0
\end{aligned}
$$

Therefore, by continuity of $g(S)|_{Q^*}$, there exists a $S^{''} \in [S', S^*)$ such that $g(S^{''})|_{Q^*} = 0$. It is easy to check that $(p_1^H(S^{''}), S^{''}, Q^*)$ satisfies all the constraints: it clearly satisfies (3.11), (3.12), and (3.15). (3.14) holds since $0 \leq S' \leq S^{''} < S^* \leq \min(0, N_1)$, where the last inequality is from the feasibility of S^*. To see (3.13) also holds, note that by definition of $S^{''}$, condition (B.11), and Lemma 14,

$$
\begin{aligned}
&\pi_H^{AI,AD}(p_1^H(S^{''}), S^{''}, Q^*) - \pi_H^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\
&= \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) - \pi_2^{*L}(S^{''}) + \pi_2^{*H}(S^{''}) - \pi_H^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\
&= \pi_2^{*L}(S_L^{FI,OR}) - \pi_2^{*H}(S_L^{FI,OR}) - \pi_2^{*L}(S^{''}) + \pi_2^{*H}(S^{''}) > 0
\end{aligned}
$$

In the meantime,

$$
\pi_H^{AI}(p_1^H(S^{''}), S^{''}, Q^*) = \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) - \pi_2^{*L}(S^{''}) + \pi_2^{*H}(S^{''})
$$

By equation (B.9), Lemma 14 and the fact that $S^{''} < S^*$, $\pi_H^{AI}(p_1^H(S^{''}), S^{''}, Q^*) > \pi_H^{AI}(p_1^*, S^*, Q^*)$, which contradicts with the assumption that (p_1^*, S^*, Q^*) is optimal. ■

Proof of Theorem 14

It is clear that $Q_L = 0$. Suppose $Q_H = Q^* > 0$. It suffices to show that for $\epsilon > 0$ and $Q = Q^* - \epsilon > 0$, there exists a feasible solution strictly improving the objective function $\pi_H^{AI,AD}(p_1, S, Q)$. Similar to the function $g(S)|_Q^*$ in the proof of Lemma 21, define

$$g(S)|_{Q^*-\epsilon} = \pi_L^{AI,AD}\left(p_1^H(S), S, Q^* - \epsilon\right) - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right)$$

Clearly $g(0)|_{Q^*-\epsilon} < 0$ since otherwise $\pi_2^{*L}(0) > \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right)$, violating the assumption that $\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right)$ is L type's full-information strategy. In the meantime, by Lemma 21 and equation (B.8),

$$\begin{aligned} g(S^*)|_{Q^*-\epsilon} &= \pi_L^{AI,AD}\left(p_1^H(S^*), S^*, Q^* - \epsilon\right) - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\ &= \pi_L^{AI,AD}\left(p_1^*, S^*, Q^*\right) + \epsilon - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\ &= \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) + \epsilon - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) \\ &= \epsilon > 0 \end{aligned}$$

By continuity of $g(S)|_{Q^*-\epsilon}$ in S, there exists a $S' \in (0, S^*)$ such that $\pi_L^{AI,AD}\left(p_1^H(S'), S', Q^* - \epsilon\right) - \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) = 0$. It is easy to check that $\left(p_1^H(S'), S', Q^* - \epsilon\right)$ satisfies all constraints and

$$\pi_H^{AI,AD}\left(p_1^H(S'), S', Q^* - \epsilon\right) = \pi_L^{AI,AD}\left(p_{1L}^{FI,OR}, S_L^{FI,OR}, 0\right) - \pi_2^{*L}(S') + \pi_2^{*H}(S') \tag{B.13}$$

Comparing Equation (B.9) and (B.13), by Lemma 14 and the fact that $S' < S^*$, we have

$$\pi_H^{AI,AD}\left(p_1^H(S'), S', Q^* - \epsilon\right) > \pi_H^{AI,AD}(p_1^*, S^*, Q^*).$$

B.10 Proof of Theorem 15

We first prove that Lemma 13 still holds with costly quality.

Lemma 22 *The following results hold with costly quality:*

$$\bar{G}\left(p_{2H}^U - A_H\right) \geq \bar{G}\left(p_{2L}^U - A_L\right), \ (p_{2H}^U - c_H)\bar{G}\left(p_{2H}^U - A_H\right) > (p_{2L}^U - c_L)\bar{G}\left(p_{2L}^U - A_L\right)$$

Proof: We still define $\delta = \bar{G}\left(p_{2H}^U - A_H\right) - \bar{G}\left(p_{2L}^U - A_L\right)$, $\Delta = (p_{2H}^U - c)\bar{G}\left(p_{2H}^U - A_H\right) - (p_{2L}^U - c)\bar{G}\left(p_{2L}^U - A_L\right)$. We show $\delta \geq 0$ and $\Delta > 0$ in the following four cases:

- $c_L \leq \underline{c}_L$: In such a case, $c_H < c_L + A_H - A_L \leq \underline{c}_H$. By Equation (3.2), $p_{2L}^U = A_L + \alpha_L$ and $p_{2H}^U = A_L + \alpha_L$, which imply $\delta = 1 - 1 = 0$; $\quad \Delta = A_L + \alpha_L - c_H - (A_L + \alpha_L - c_L) = A_H - A_L - (c_H - c_L) > 0$.
- $\underline{c}_L < c_L < \bar{p}_L$ and $\underline{c}_H < c_H < \bar{p}_H$: By Equation (3.2), $p_{2L}^U \in (A_L + \alpha_L, A_L + \alpha_H)$, $p_{2H}^U \in (A_H + \alpha_L, A_H + \alpha_H)$ and

$$p_{2H}^U - A_H - \frac{\bar{G}(p_{2H}^U - A_H)}{g(p_{2H}^U - A_H)} = c_H - A_H < c_L - A_L = p_{2L}^U - A_L - \frac{\bar{G}(p_{2L}^U - A_L)}{g(p_{2L}^U - A_L)}$$

  By IFR property of $G(\cdot)$, $x - \frac{\bar{G}(x)}{g(x)}$ strictly increases in x. Therefore, $p_{2H}^U - A_H < p_{2L}^U - A_L$. By the fact that $g(\cdot) > 0$, $\bar{G}(\cdot)$ strictly decreases, which implies $\bar{G}(p_{2H}^U - A_H) > \bar{G}(p_{2L}^U - A_L)$ and $\delta > 0$. In the meantime, note that $\bar{G}(p_{2H}^U - A_H) > \bar{G}(p_{2L}^U - A_L) > 0$ and $p_{2H}^U - c_H \geq p_{2L}^U - c_L > 0$. Therefore,

$$(p_{2H}^U - c_H)\bar{G}(p_{2H}^U - A_H) > (p_{2H}^U - c_H)\bar{G}(p_{2L}^U - A_L) \geq (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L)$$

  That is, $\Delta > 0$.
- $\underline{c}_L < c_L < \bar{p}_L$ and $c_H \leq \underline{c}_H$: By Equation (3.2), $p_{2L}^U \in (A_L + \alpha_L, A_L + \alpha_H)$ and $p_{2H}^U = A_L + \alpha_L$. Therefore,

$$\begin{aligned}
&\delta = 1 - \bar{G}(p_{2L}^U - A_L) > 0;\\
&\Delta = A_L + \alpha_L - c_H - (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L)\\
&\quad = \underline{c}_H + \frac{1}{g(\alpha_L)} - c_H - (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L)\\
&\quad \geq \frac{1}{g(\alpha_L)} - (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L) \quad (\text{since } c_H \leq \underline{c}_H)\\
&\quad \geq \frac{\bar{G}(p_{2L}^U - A_L)}{g(p_{2L}^U - A_L)} - (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L) \left(\text{since } p_{2L}^U > A_L + \alpha_L \text{ and } \frac{\bar{G}(x)}{g(x)} \text{ decreases in } x\right)\\
&\quad = (p_{2L}^U - c_L) - (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L) \quad (\text{by Equation (3.2)})\\
&\quad = (p_{2L}^U - c_L)G(p_{2L}^U - A_L) > 0
\end{aligned}$$

- $c_L \geq \bar{p}_L$ and $c_H < \bar{p}_H$: By Equation (3.2), $p_{2L}^U = A_L + \alpha_H$ and $p_{2H}^U \in [A_L + \alpha_L, A_H + \alpha_H)$. Therefore, $\delta = \bar{G}(p_{2H}^U - A_H) - 0 > 0$ and $\Delta = (p_{2H}^U - c_H)\bar{G}(p_{2H}^U - A_H) - 0 > 0$. ∎

Proof of Theorem 15

We prove that (i) Lemma 7 and (ii) Lemma 8 still hold with costly quality. The proofs for Theorem 9 and 10 are almost exactly the same as the original proofs and hence are omitted.

(i) First note that for given A_t, both $p_2^t(S)$ and $p_1^t(S)$ increase in c. By this fact and Lemma 7,

$$p_2^H(S)|_{c_H} \geq p_2^H(S)|_{c_L} \geq p_2^L(S)|_{c_L}; \quad p_1^H(S)|_{c_H} \geq p_1^H(S)|_{c_L} > p_1^L(S)|_{c_L}.$$

(ii) To prove Lemma 8, it suffices to show that Lemma 14 still holds with costly quality. Similar to the original proof, define

$$
\begin{aligned}
W(S) &:= \pi_2^{*H}(S) - \pi_2^{*L}(S) \\
&= (p_2^H(S) - c_H)\min\left(T - S, (N_1 + N_2 - S)\bar{G}(p_2^H(S) - A_H)\right) \\
&\quad - (p_2^L(S) - c_L)\min\left(T - S, (N_1 + N_2 - S)\bar{G}(p_2^L(S) - A_L)\right)
\end{aligned}
$$

By Lemma 22, we show that $W(S)$ strictly decreases in S for each of the following three cases:

• $\frac{T-S}{N_1+N_2-S} \geq \bar{G}(p_{2H}^U - A_H)$

In such case, $p_2^H(S) = p_{2H}^U$, $p_2^L(S) = p_{2L}^U$, and

$$
\begin{aligned}
W(S) &= (N_1 + N_2 - S)\left(p_{2H}^U - c_H\right)\bar{G}\left(p_{2H}^U - A_H\right) - (N_1 + N_2 - S)\left(p_{2L}^U - c_L\right)\bar{G}(p_{2L}^U - A_L) \\
&= (N_1 + N_2)\left[(p_{2H}^U - c_H)\bar{G}(p_{2H}^U - A_H) - (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L)\right] \\
&\quad + \left[(p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L) - (p_{2H}^U - c_H)\bar{G}(p_{2H}^U - A_H)\right] S
\end{aligned}
$$

By Lemma 22, $W(S)$ strictly decreases in S.

• $\bar{G}(p_{2L}^U - A_L) < \frac{T-S}{N_1+N_2-S} \leq \bar{G}(p_{2H}^U - A_H)$

In such case, $p_2^H(S) = p_{2H}^B$, $p_2^L(S) = p_{2L}^U$, and

$$
\begin{aligned}
W(S) &= (p_{2H}^B - c_H)(T - S) - (N_1 + N_2 - S)(p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L) \\
&= (p_{2H}^B - c_H)(T - S) + (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L)S - (N_1 + N_2)(p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L)
\end{aligned}
$$

Similar to the original proof, it is straightforward to show that $W(S)$ is strictly concave in S by proving that the second-order derivative of $W(S)$ with respect to S is always negative. Meanwhile, at S's lower bound, i.e., when S satisfies $\frac{T-S}{N_1+N_2-S} = \bar{G}(p_{2H}^U - A_H)$, $p_{2H}^B = p_{2H}^U$ and

$$
\begin{aligned}
\frac{dW(S)}{dS} &= (p_{2H}^U - c_H) \cdot G(p_{2H}^U - A_H) - (p_{2H}^U - c_H) + (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L) \\
&\qquad \text{(recall that } p_{2H}^U \text{ satisfies } p_{2H}^U - c_H = \frac{\bar{G}(p_{2H}^U - A_H)}{g(p_{2H}^U - A_H)}) \\
&= -(p_{2H}^U - c_H)\bar{G}(p_{2H}^U - A_H) + (p_{2L}^U - c_L)\bar{G}(p_{2L}^U - A_L) \\
&< 0 \quad \text{(by Lemma 22)}
\end{aligned}
$$

By the strict concavity of $W(S)$, for S satisfying $\frac{T-S}{N_1+N_2-S} \leq \bar{G}(p_{2H}^U - A_H)$, $W(S)$ strictly decreases in S.

• $\frac{T-S}{N_1+N_2-S} < \bar{G}(p_{2L}^U - A_L)$

In such case, $p_2^H(S) = p_{2H}^B$, $p_2^L(S) = p_{2L}^B$, and

$$
W(S) = \left(p_{2H}^B(S) - c_H\right)(T - S) - \left(p_{2L}^B(S) - c_L\right)(T - S)
$$

Taking derivative w.r.t S,

$$
\begin{aligned}
\frac{dW(S)}{dS} &= \left[(p^B_{2L}(S) - c_L) - (p^B_{2H}(S) - c_H)\right] + (T - S)\left(\frac{dp^B_{2H}(S)}{dS} - \frac{dp^B_{2L}(S)}{dS}\right) \\
&= \left[(p^B_{2L}(S) - c_L) - (p^B_{2H}(S) - c_H)\right] \\
&\quad + \frac{N_1 + N_2 - T}{(N_1 + N_2 - S)} \cdot \left[\frac{\bar{G}(p^B_{2H}(S) - A_H)}{g(p^B_{2H}(S) - A_H)} - \frac{\bar{G}(p^B_{2L}(S) - A_L)}{g(p^B_{2L}(S) - A_L)}\right]
\end{aligned}
$$

By definition of $p^B_{2t}(S)$, $p^B_{2H}(S) - A_H = p^B_{2L}(S) - A_L = (\bar{G})^{-1}\left(\frac{T-S}{N_1+N_2-S}\right)$. Hence, $\frac{dW(S)}{dS} = p^B_{2L}(S) - p^B_{2H}(S) = A_L - c_L - (A_H - c_H) < 0$.

(b) First note that for given A_t, both $p^t_2(S)$ and $p^t_1(S)$ increase in c. By this fact and Lemma 13,

$$
p^H_2(S)|_{c_H} \geq p^H_2(S)|_{c_L} \geq p^L_2(S)|_{c_L}; \quad p^H_1(S)|_{c_H} \geq p^H_1(S)|_{c_L} > p^L_1(S)|_{c_L}.
$$

APPENDIX C

Proofs in Chapter 4

C.1 Proof of Theorem 16

Proof: The proof of the asymptotic result is similar to Cooper (2002) and extends the methodology by incorporating endogenous pricing decision and order cancellation.

In preparation, we define a sequence of deterministic problem:

$$\pi^D_{(k)} = \max_{\mathbf{p},\mathbf{o}} (\mathbf{p} - A^T\mathbf{c})^T\mathbf{o}, \quad \text{subject to:} \quad 0 \leq \mathbf{Ao} \leq k\mathbf{y}, 0 \leq \mathbf{o} \leq k\mathrm{E}[\mathbf{d}^{(1)}(\tau,\mathbf{p})], \mathbf{p} \geq A^T\mathbf{c} \qquad \text{(C.1)}$$

Let $\left(\mathbf{p}^D_{(k)}, \mathbf{o}^D_{(k)}\right)$ denote the optimal solution to the deterministic problem defined in equation (C.1). For the kth problem defined in section 4.3.3, consider a heuristic policy where the firm quote prices $\mathbf{p}^D_{(k)}$, accepts orders up to $\mathbf{o}^D_{(k)}$, and never cancels any accepted orders. Let Π^D_k denote the firm's expected profit from using this policy. Clearly, $\Pi^D_{(k)} = (\mathbf{p}^D_{(k)} - A^T\mathbf{c})^T\mathrm{E}[\min(\mathbf{o}^D_{(k)}, \mathbf{d}^{(k)}(\tau, \mathbf{p}^D_{(k)}))]$.

Next we follow three steps to prove the theorem: (i) $\lim_{k\to\infty} \Pi^D_{(k)}/\pi^D_{(k)} = 1$; (ii) for given k, $\Pi^D_{(k)} \leq \Pi^{ID,OO}_{(k)} \leq \Pi^*_{(k)} \leq \pi^D_{(k)}$; (iii) $\lim_{k\to\infty} \Pi^{ID,OO}_{(k)}/\pi^D_{(k)} = \lim_{k\to\infty} \Pi^*_{(k)}/\pi^D_{(k)} = \lim_{k\to\infty} \Pi^{ID,OO}_{(k)}/\Pi^*_{(k)} = 1$.

(i) First of all, it is straightforward to show that $\mathbf{p}^D_{(k)} = \mathbf{p}^D_{(1)}$ and $\mathbf{o}^D_{(k)} = k\mathbf{o}^D_{(1)}$. Hence,

$$\begin{aligned} \lim_{k\to\infty} \frac{\Pi^D_{(k)}}{\pi^D_{(k)}} &= \lim_{k\to\infty} \frac{(\mathbf{p}^D_{(1)} - A^T\mathbf{c})^T\mathrm{E}[\min(k\mathbf{o}^D_{(1)}, \mathbf{d}^{(k)}(\tau, \mathbf{p}^D_{(1)}))]}{k(\mathbf{p}^D_{(1)} - A^T\mathbf{c})^T\mathbf{o}^D_{(1)}} \\ &= \lim_{k\to\infty} \frac{(\mathbf{p}^D_{(1)} - A^T\mathbf{c})^T\mathrm{E}[\min(\mathbf{o}^D_{(1)}, k^{-1}\mathbf{d}^{(k)}(\tau, \mathbf{p}^D_{(1)}))]}{(\mathbf{p}^D_{(1)} - A^T\mathbf{c})^T\mathbf{o}^D_{(1)}} \end{aligned} \qquad \text{(C.2)}$$

By condition (4.8) and the fact that the function $f(x) = \min(\mathbf{o}^D_{(1)}, x)$ is continuous and bounded by $\mathrm{E}[\mathbf{d}^{(1)}(\tau, \mathbf{p}^D_{(1)})]$, we have

$$\lim_{k\to\infty} \mathrm{E}[\min(\mathbf{o}^D_{(1)}, k^{-1}\mathbf{d}^{(k)}(\tau, \mathbf{p}^D_{(1)}))] = \mathrm{E}[\min(\mathbf{o}^D_{(1)}, \mathbf{d}^{(1)}(\tau, \mathbf{p}^D_{(1)}))] = \mathbf{o}^D_{(1)}.$$

Applying this result to equation (C.2), we immediately have $\lim_{k\to\infty} \Pi^D_{(k)}/\pi^D_{(k)} = 1$.

(ii) The proof of $\Pi^D_{(k)} \leq \Pi^{ID,OO}_{(k)} \leq \Pi^*_{(k)}$ is trivial as the no-postponement heuristic policy with $\left(\mathbf{p}^D_{(k)}, \mathbf{o}^D_{(k)}\right)$ is feasible for both the model with partially-postponed fulfillment and that with dynamic fulfillment.

To show $\Pi^*_{(k)} \leq \pi^D_{(k)}$, let $(\mathbf{p}^*_{(k)}, \phi^*_{(k)})$ denote the optimal policy for the kth problem with dynamic order fulfillment. By equation (4.1), we have

$$\begin{aligned}
\Pi^*_{(k)} &= \mathrm{E}[(\mathbf{p}^*_{(k)} - A^T\mathbf{c})^T(N^{\phi^*_{(k)}}(\tau) - (1+\boldsymbol{\gamma})^T R^{\phi^*_{(k)}}(\tau))] \\
&\leq (\mathbf{p}^*_{(k)} - A^T\mathbf{c})^T \mathrm{E}[(N^{\phi^*_{(k)}}(\tau) - R^{\phi^*_{(k)}}(\tau))] \\
&\leq \pi^D_{(k)}
\end{aligned}$$

Here, the first inequality follows from the nonnegativity of $\boldsymbol{\gamma}$. To see why the second inequality holds, note that by the feasibility conditions, $0 \leq N^{\phi^*_{(k)}}(\tau) - R^{\phi^*_{(k)}}(\tau) \leq \mathbf{d}^{(k)}(\tau, \mathbf{p}^*_{(k)})$ and $A(N^{\phi^*_{(k)}}(\tau) - R^{\phi^*_{(k)}}(\tau)) \leq k\mathbf{y}$ almost surely. These facts, together with the assumption that demand always has finite mean, imply $0 \leq \mathrm{E}[N^{\phi^*_{(k)}}(\tau) - R^{\phi^*_{(k)}}(\tau)] \leq \mathrm{E}[\mathbf{d}^{(k)}(\tau, \mathbf{p}^*_{(k)})] = k\mathrm{E}[\mathbf{d}^{(1)}(\tau, \mathbf{p}^*_{(k)})]$ and $A\mathrm{E}[N^{\phi^*_{(k)}}(\tau) - R^{\phi^*_{(k)}}(\tau)] \leq k\mathbf{y}$. Hence, $(\mathbf{p}^*_{(k)}, \mathrm{E}[N^{\phi^*_{(k)}}(\tau) - R^{\phi^*_{(k)}}(\tau)])$ is feasible for the problem defined by equation (C.1) and the second inequality follows by definition of $\pi^D_{(k)}$.

Following the same logic, we can prove $\Pi^{ID,OO}_{(k)} \leq \pi^D_{(k)}$.

(iii) By part (ii), $\Pi^D_{(k)}/\pi^D_{(k)} \leq \Pi^{ID,OO}_{(k)}/\pi^D_{(k)} \leq 1$. By part (i), this immediately implies that $\Pi^{ID,OO}_{(k)}/\pi^D_{(k)}$ converges to one. Similarly, we can show the convergence for $\Pi^*_{(k)}/\pi^D_{(k)}$. These two convergence results together imply $\lim_{k\to\infty} \Pi^{ID,OO}_{(k)}/\Pi^*_{(k)} = 1$. ■

C.2 Proof of Lemma 10

Proof: Prove by contradiction. Suppose $Ao^*_1 + o^*_2 < y$. Let (z^*_1, z^*_2) denote the optimal demand rate under II,NO or II,OO strategy. Consider two cases: if $o^*_1 < z^*_1 + H_1$ (or $o^*_2 < z^*_2 + H_2$), then the seller can strictly improve his profit by slightly increasing o^*_1 (or o^*_2) and keeping everything else the same. This, however, contradicts with the optimality of $\mathbf{o}^*$; otherwise $o^*_1 \geq z^*_1 + H_1$ and $o^*_2 \geq z^*_2 + H_2$, which implies $\mathbf{z}^* = \mathbf{z}^U$. This further implies $y > Ao^*_1 + o^*_2 \geq A(z^U_1 + H_1) + z^U_2 + H_2 = \bar{y}$ and hence contradicts with the assumption $y < \bar{y}$. ■

C.3 Proof of Proposition 7

Proof: We first show that when $y < \bar{y}$, $\pi^{NO}(z^U_1, z^U_2, o_1, y - Ao_1)$ is concave in o_1 for $o_1 \in [0, y/A]$. Hence, to prove the existence of $\hat{y}_1$ and the structure of optimal policy, it suffices to show that $d\pi^{NO}(z^U_1, z^U_2, o_1, y - Ao_1)/do_1$ at $o_1 = z^U_1 + H_1$ is negative and at $o_1 = 0$ is nonnegative for all $0 < y < \bar{y}$, $d\pi^{NO}(z^U_1, z^U_2, o_1, y - Ao_1)/do_1$ at $o_1 = y/A$ is non-increasing in y, and $d\pi^{NO}(z^U_1, z^U_2, o_1, y - Ao_1)/do_1$ at $o_1 = y/A$ is nonnegative when $y = 0$ and negative when $y = \bar{y}$.

In preparation, notice that

$$\frac{d\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)}{do_1} = (p_1^U - AC)\bar{G}_1(o_1 - z_1^U) - A(p_2^U - C)\bar{G}_2(y - Ao_1 - z_2^U) \tag{C.3}$$

$$\frac{d^2\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)}{do_1^2} = -(p_1^U - AC)g_1(o_1 - z_1^U) - A^2(p_2^U - C)g_2(y - Ao_1 - z_2^U) \leq 0$$

The concavity of $\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)$ in o_1 is immediate.

Furthermore, by equation (C.3) and Assumption (R-5),

$$\begin{aligned}
\left.\frac{d\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)}{do_1}\right|_{o_1 = z_1^U + H} &= -A(p_2^U - C)\bar{G}_2(y - Az_1^U - AH_1 - z_2^U) \\
&< -A(p_2^U - C)\bar{G}_2(\bar{y} - Az_1^U - AH_1 - z_2^U) = 0, \\
\left.\frac{d\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)}{do_1}\right|_{o_1 = 0} &= (p_1^U - AC) - A(p_2^U - C)\bar{G}_2(y - z_2^U) \\
&\geq p_1^U - AC - A(p_2^U - C) \geq 0, \\
\left.\frac{d\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)}{do_1}\right|_{o_1 = y/A} &= (p_1^U - AC)\bar{G}_1(y/A - z_1^U) - A(p_2^U - C), \\
\frac{d\left\{\left.\frac{d\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)}{do_1}\right|_{o_1 = y/A}\right\}}{dy} &= -(p_1^U - AC)g_1(y/A - z_1^U)/A \leq 0, \\
\left.\frac{d\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)}{do_1}\right|_{o_1 = y/A, y = 0} &= (p_1^U - AC) - A(p_2^U - C) \geq 0 \\
\left.\frac{d\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)}{do_1}\right|_{o_1 = y/A, y = \bar{y}} &= (p_1^U - AC)\bar{G}_1(H_1 + (z_2^U + H_2)/A) - A(p_2^U - C) \\
&= -A(p_2^U - C) < 0.
\end{aligned}$$

Clearly, $\hat{y}_1$ satisfies $\left.\frac{d\pi^{NO}(z_1^U, z_2^U, o_1, y - Ao_1)}{do_1}\right|_{o_1 = y/A} = 0$ and $\hat{y}_1 = Az_1^U + A\bar{G}_1^{-1}\left(\frac{A(p_2^U - C)}{p_1^U - AC}\right)$. ■

C.4 Proof of Proposition 8

Proof: We prove the theorem in two steps: (i) For given o_1 (or o_2), find the optimal response function $o_2^*(o_1)$ (or $o_1^*(o_2)$); (ii) Depending on the values of p_1^U and p_2^U, find the intersection(s) of $o_2^*(o_1)$ and $o_1^*(o_2)$, i.e., the solution to $o_1 = o_1^*(o_2)$ and $o_2 = o_2^*(o_1)$. If the intersection(s) occur(s) along the line $Ao_1^* + o_2^*$, then it implies that overselling is not optimal and the optimal solution is the same with that in Proposition 7 for the no-overselling model. Otherwise, it is optimal to oversell and the optimal overselling quantities (o_1^*, o_2^*) are the coordinates of the intersection point(s) of $o_2^*(o_1)$ and $o_1^*(o_2)$.

(i) By equation (4.11), for given $o_2 \in [0, \min(y, z_2^U + H_2)]$, the derivative of π^{OO} in o_1 for $o_1 \in$

$[0, \min(y/A, z_1^U + H_1)]$ is

$$\frac{\partial \pi^{OO}}{\partial o_1} = \begin{cases} (p_1^U - AC)\bar{G}_1(o_1 - z_1^U) & \text{if } o_1 \le (y - o_2)/A \\ \bar{G}_1(o_1 - z_1^U)[p_1^U - AC - A(p_2^U - C)(1+\gamma)\bar{G}_2(y - Ao_1 - z_2^U)] & \text{if } o_1 > (y - o_2)/A. \end{cases}$$

Consider two cases: if $p_1^U - AC \ge A(p_2^U - C)(1+\gamma)$, then π^{OO} is non-decreasing in o_1 for all o_1 and hence $o_1^* = \min(y/A, z_1^U + H_1)$; otherwise, π^{OO} is non-decreasing in o_1 for $o_1 \le (y - o_2)/A$ and then for $o_1 > (y - o_2)/A$, it is strictly quasi-concave. Further, by definition of $\hat{o}_1$, if $\hat{o}_1 \in ((y - o_2)/A, \min(y/A, z_1^U + H_1)]$, $\hat{o}_1$ satisfies $\frac{\partial \pi^{OO}}{\partial o_1} = 0$. This, together with the strict quasi-concavity, implies

$$\begin{aligned} o_1^*(o_2) &= \min\left(\max\left((y - o_2)/A, \hat{o}_1\right), \min\left(y/A, z_1^U + H_1\right)\right). \\ &= \min\left(\max\left((y - o_2)/A, \hat{o}_1\right), y/A, z_1^U + H_1\right) \\ &= \min\left(\min(\max\left((y - o_2)/A, \hat{o}_1\right), y/A), z_1^U + H_1\right) \end{aligned} \tag{C.4}$$

Note that by Assumption (R-5), $z_2^U \ge L_2$ and that by definition, $(\bar{G}_2)^{-1}(x) \ge -L_2$ for any $x \in [0,1]$. Hence, $\hat{o}_1 \le y/A$. Meanwhile, by the feasibility condition on o_2, $(y - o_2)/A \le \min(y/A, z_1^U + H_1)$. These two facts imply $\max((y - o_2)/A, \hat{o}_1) \le y/A$ and $(y - o_2)/A \le z_1^U + H_1$. Applying these results to equation (C.4), we have

$$\begin{aligned} o_1^*(o_2) &= \min\left(\max\left((y - o_2)/A, \hat{o}_1\right), z_1^U + H_1\right) \\ &= \max\left(\min((y - o_2)/A, z_1^U + H_1), \min(\hat{o}_1, z_1^U + H_1)\right) \\ &= \max\left((y - o_2)/A, \min(\hat{o}_1, z_1^U + H_1)\right) \end{aligned} \tag{C.5}$$

Symmetrically, by equation (4.11), for given $o_1 \in [0, \min(y/A, z_1^U + H_1)]$, the derivative of π^{OO} in o_2 for $o_2 \in [0, \min(y, z_2^U + H_2)]$ is

$$\frac{\partial \pi^{OO}}{\partial o_2} = \begin{cases} (p_2^U - C)\bar{G}_2(o_2 - z_2^U) & \text{if } o_2 \le y - Ao_1 \\ \bar{G}_2(o_2 - z_2^U)[p_2^U - C - (p_2^U - C)(1+\gamma)\bar{G}_1\left(\frac{y - o_2}{A} - z_1^U\right)] & \text{if } o_2 > y - Ao_1. \end{cases}$$

and the best response function is

$$o_2^*(o_1) = \begin{cases} \min(y, z_2^U + H_2) & \text{if } \gamma = 0 \\ \max\left(y - Ao_1, \min(\hat{o}_2, z_2^U + H_2)\right) & \text{if } \gamma > 0. \end{cases} \tag{C.6}$$

(ii) To find the intersections of $o_1^*(o_2)$ and $o_2^*(o_1)$, consider the following cases:

- $\gamma = 0$

  In such a case, $o_1^* = \min(y/A, z_1^U + H_1), o_2^* = \min(y, z_2^U + H_2)$. Since $0 < y < \bar{y}$, $Ao_1^* + o_2^* > y$.

- $0 < \gamma \leq \frac{p_1^U - Ap_2^U}{A(p_2^U - C)}$

  In such a case, $o_1^* = \min(y/A, z_1^U + H_1)$ and

$$\begin{aligned} o_2^* &= \max\left(y - Ao_1^*, \min(\hat{o}_2, z_2^U + H_2)\right) \\ &= \max\left(\max(0, y - A(z_1^U + H_1)), \min\left(\hat{o}_2, z_2^U + H_2\right)\right). \end{aligned}$$

  Hence, $Ao_1^* + o_2^* > y$ if and only if $\min\left(\hat{o}_2, z_2^U + H_2\right) > \max(0, y - A(z_1^U + H_1))$. Since $z_2^U + H_2 > \max(0, y - A(z_1^U + H_1))$ and $\hat{o}_2 > y - A(z_1^U + H_1)$, the condition translates to $\hat{o}_2 > 0$, i.e., $y > Az_1^U + A(\bar{G}_1)^{-1}\left(\frac{1}{1+\gamma}\right)$.

- $\gamma > \frac{p_1^U - Ap_2^U}{A(p_2^U - C)}$

  In such a case,

$$o_1^*(o_2) = \max\left((y - o_2)/A, \min(\hat{o}_1, z_1^U + H_1)\right), o_2^*(o_1) = \max\left(y - Ao_1, \min(\hat{o}_2, z_2^U + H_2)\right). \tag{C.7}$$

  Consider two sub-cases:

  - $A\hat{o}_1 + \hat{o}_2 \leq y$ $\left(\Leftrightarrow y \leq Az_1^U + z_2^U + A(\bar{G}_1)^{-1}\left(\frac{1}{1+\gamma}\right) + (\bar{G}_2)^{-1}\left(\frac{p_1^U - AC}{A(p_2^U - C)(1+\gamma)}\right)\right)$:

    In such a case, $\hat{o}_1 \leq (y - \hat{o}_2)/A = z_1^U + (\bar{G}_1)^{-1}\left(\frac{1}{1+\gamma}\right) \leq z_1^U + H_1$ and similarly, $\hat{o}_2 \leq z_2^U + H_2$. Therefore, equation (C.7) can be further simplified to

$$o_1^*(o_2) = \max\left((y - o_2)/A, \hat{o}_1\right), \quad o_2^*(o_1) = \max\left(y - Ao_1, \hat{o}_2\right).$$

    It is easy to check that $Ao_1^* + o_2^* = y$, where $o_1^* \in [\hat{o}_1, (y - \hat{o}_2)/A]$ and $o_2^* \in [\hat{o}_2, y - A\hat{o}_1]$.

  - $A\hat{o}_1 + \hat{o}_2 > y$:

    In such a case, it is easy to check that $A\min(\hat{o_1}, z_1^U + H_1) + \min(\hat{o}_2, z_2^U + H_2) > y$.[1] This implies that $o_1^* = \min(\hat{o}_1, z_1^U + H_1)$, $o_2^* = \min(\hat{o}_2, z_2^U + H_2)$, and $Ao_1^* + o_2^* > y$. ■

C.5 Proof of Lemma 11

Proof: Prove by contradiction, i.e., if any of the inequalities is violated, then we can construct another feasible solution which strictly increases the total profit and this contradicts with the optimality of (z_1^*, z_2^*, o_1^*).

- $-L_1 \leq o_1^* - z_1^*$

Suppose $-L_1 > o_1^* - z_1^*$, then with probability one, some demand of product 1 cannot be satisfied. Also note that since $q^* \geq 0$, $-L_1 > o_1^* - z_1^*$ implies $z_1^* > L_1$. The firm can strictly increases the

[1] Note that $A\hat{o_1} + z_2^U + H_2 > y$ and $A(z_1^U + H_1) + z_2^U + H_2 > y$, which imply $A\min(\hat{o_1}, z_1^U + H_1) + z_2^U + H_2 > y$. Similarly, we can show $A\min(\hat{o_1}, z_1^U + H_1) + \hat{o}_2 > y$. Therefore, $A\min(\hat{o_1}, z_1^U + H_1) + \min(\hat{o}_2, z_2^U + H_2) > y$.

total profit by slightly decreasing z_1, since by doing so, both products' prices are strictly increased, while both products' expected sales keep unchanged. Similarly, we can prove $-L_2 \leq y - Ao_1^* - z_2^*$.

• $o_1^* - z_1^* \leq H_1$

Suppose $o_1^* - z_1^* > H_1$, then with probability one, product 1's production schedule is higher than its demand. Consider two cases. If $y - Ao_1^* - z_2^* < H_2$, then the firm can strictly increases the total profit by slightly decreasing o_1, since by doing so, product 2's expected sales is strictly increased, while product 1's expected sales and both products' prices keep unchanged. If, however, $y - Ao_1^* - z_2^* \geq H_2$, this implies $\pi^{NO}(z^*, q^*) = (p_1(z_1^*, z_2^*) - AC)z_1^* + (p_2(z_1^*, z_2^*) - C)z_2^*$ and $\mathbf{z}^* = \mathbf{z}^U$. Therefore, $y > A(z_1^U + H_1) + (z_2^U + H_2)$. This, however, contradicts with the condition $y < \bar{y} = A(z_1^U + H_1) + (z_2^U + H_2)$, under which $\mathbf{o} \in \mathcal{EF}$ is optimal. The proof is then complete. Similarly, we can prove $y - Ao_1^* - z_2^* \leq H_2$.

Further, suppose $o_1^* - z_1^* = H_1$ and $y - Ao_1^* - z_2^* = H_2$. This clearly implies $\mathbf{z}^* = \mathbf{z}^U$ and $y = A(z_1^U + H_1) + (z_2^U + H_2)$, which contradicts with the condition $y < \bar{y}$. ■

C.6 Proof of Theorem 17

The proof uses eight lemmas: Lemma 23 establishes upper bounds for feasible z_1 and z_2; Lemma 24 through Lemma 27 focus on interior solutions, i.e., those satisfying first-order conditions for at least one variable. Lemma 24 proves some technical properties of interior solutions; Based on Lemma 24, Lemma 25, 26, and 27 show that the interior solutions satisfy super/sub- modularity, nonnegative-margin conditions, and second-order conditions, respectively. Lemma 28 and 29 are about boundary solutions, i.e., those on the boundary for at least one variable. Lemma 28 shows the first-order and second-order conditions for boundary solutions and Lemma 29 characterizes the boundary solutions when $o_1 = 0$ or $o_1 = y/A$. Lemma 30 presents a general technical result that under certain conditions, a function which is composed of two strictly quasi-concave functions is strictly quasi-concave itself. This result is important for the proof of strict quasi-concavity in Theorem 17 (i) and (ii). Lemma 31 is another technical lemma for the proof of Theorem 17 (ii).

The proof is organized as follows: we first claim and prove Lemma 23 through Lemma 30, and then prove Theorem 17 (i). Lemma 31 involves some notations defined in the proof of Theorem 17 (i) and hence is claimed and shown after it. Finally, Theorem 17 (ii) and (iii) are proved.

In preparation, define the first-order and second-order derivative functions of the profit function as following:

$$\Delta_1(z_1, z_2, o_1) = \frac{\partial \pi^{NO}(z_1, z_2, o_1)}{\partial z_1}$$

$$= -\frac{1}{b_1 b_2 - c_1 c_2}\{b_2 \mathrm{E}[\min(o_1, z_1 + \epsilon_1)] + c_2 \mathrm{E}[\min(y - Ao_1, z_2 + \epsilon_2)]\} + (p_1(\mathbf{z}) - AC)G_1(o_1 - z_1)$$

$$\Delta_2(z_1, z_2, o_1) = \frac{\partial \pi^{NO}(z_1, z_2, o_1)}{\partial z_2}$$

$$= -\frac{1}{b_1 b_2 - c_1 c_2}\{b_1 \mathrm{E}[\min(y - Ao_1, z_2 + \epsilon_2)] + c_1 \mathrm{E}[\min(o_1, z_1 + \epsilon_1)]\} + (p_2(\mathbf{z}) - C)G_2(y - Ao_1 - z_2)$$

$$\Delta_3(z_1, z_2, o_1) = \frac{\partial \pi^{NO}(z_1, z_2, o_1)}{\partial o_1} = (p_1(\mathbf{z}) - AC)\bar{G}_1(o_1 - z_1) - A(p_2(\mathbf{z}) - C)\bar{G}_2(y - Ao_1 - z_2)$$

$$\Delta_{11}(z_1, z_2, o_1) = \frac{\partial^2 \pi^{NO}(z_1, z_2, o_1)}{\partial z_1^2} = -\frac{2b_2}{b_1 b_2 - c_1 c_2}G_1(o_1 - z_1) - (p_1(\mathbf{z}) - AC)g_1(o_1 - z_1)$$

$$\Delta_{22}(z_1, z_2, o_1) = \frac{\partial^2 \pi^{NO}(z_1, z_2, o_1)}{\partial z_2^2} = -\frac{2b_1}{b_1 b_2 - c_1 c_2}G_2(y - Ao_1 - z_2) - (p_2(\mathbf{z}) - C)g_2(y - Ao_1 - z_2)$$

$$\Delta_{33}(z_1, z_2, o_1) = \frac{\partial^2 \pi^{NO}(z_1, z_2, o_1)}{\partial o_1^2} = -(p_1(\mathbf{z}) - AC)g_1(o_1 - z_1) - A^2(p_2(\mathbf{z}) - C)g_2(y - Ao_1 - z_2)$$

$$\Delta_{12}(z_1, z_2, o_1) = \frac{\partial^2 \pi^{NO}(z_1, z_2, o_1)}{\partial z_1 \partial z_2} = -\frac{c_1}{b_1 b_2 - c_1 c_2}G_1(o_1 - z_1) - \frac{c_2}{b_1 b_2 - c_1 c_2}G_2(y - Ao_1 - z_2)$$

$$\Delta_{13}(z_1, z_2, o_1) = \frac{\partial^2 \pi^{NO}(z_1, z_2, o_1)}{\partial z_1 \partial o_1}$$

$$= -\frac{b_2}{b_1 b_2 - c_1 c_2}\bar{G}_1(o_1 - z_1) + (p_1(\mathbf{z}) - AC)g_1(o_1 - z_1) + A\frac{c_2}{b_1 b_2 - c_1 c_2}\bar{G}_2(y - Ao_1 - z_2)$$

$$\Delta_{23}(z_1, z_2, o_1) = \frac{\partial^2 \pi^{NO}(z_1, z_2, o_1)}{\partial z_2 \partial o_1}$$

$$= A\frac{b_1}{b_1 b_2 - c_1 c_2}\bar{G}_2(y - Ao_1 - z_2) - A(p_2(\mathbf{z}) - C)g_2(y - Ao_1 - z_2) - \frac{c_1}{b_1 b_2 - c_1 c_2}\bar{G}_1(o_1 - z_1)$$

Lemma 23 *Any feasible policy satisfies* $z_1 \le \bar{z}_1 = a_1 - \frac{(b_1 b_2 - c_1 c_2)C - b_1(a_2 - L_2)}{c_2}$ *and* $z_2 \le \bar{z}_2 = a_2 - \frac{(b_1 b_2 - c_1 c_2)AC - b_2(a_1 - L_1)}{c_1}$. *Furthermore,* $\Delta_1(\bar{z}_1, z_2) < 0$ *and* $\Delta_2(\bar{z}_1, z_2) < 0$ *for all* $z_2 \ge L_2$*;* $\Delta_1(z_1, \bar{z}_2) < 0$ *and* $\Delta_2(z_1, \bar{z}_2) < 0$ *for all* $z_1 \ge L_1$.

Proof: It suffices to show that $p_1(\bar{z}_1, L_2) < AC$, $p_2(\bar{z}_1, L_2) = c$, $p_1(L_1, \bar{z}_2) = AC$, and $p_2(L_1, \bar{z}_2) < C$. Then by the fact that both $p_1(z_1, z_2)$ and $p_2(z_1, z_2)$ strictly decrease in both z_1 and z_2, $p_1(z_1, z_2) < AC$ and $p_2(z_1, z_2) < C$ for all z_1, z_2 satisfying $z_1 > \bar{z}_1$ and $z_2 \ge L_2$ or $z_1 \ge L_1$ and $z_2 > \bar{z}_2$. In the meantime, by the definitions of $\Delta_1(z_1, z_2, o_1)$ and $\Delta_2(z_1, z_2, o_1)$ and the fact that any feasible o_1 satisfies $0 \le o_1 \le y/A$, $\Delta_1(\bar{z}_1, z_2) < 0$ and $\Delta_2(\bar{z}_1, z_2) < 0$ for all $z_2 \ge L_2$, and $\Delta_1(z_1, \bar{z}_2) < 0$ and $\Delta_2(z_1, \bar{z}_2) < 0$ for all $z_1 \ge L_1$.

Following this logic, we note that

$$
\begin{aligned}
&p_1(\bar{z}_1, L_2) - AC \\
=&\frac{1}{b_1b_2 - c_1c_2}\left[b_2\left(a_1 - \left(a_1 - \frac{(b_1b_2 - c_1c_2)C - b_1(a_2 - L_2)}{c_2}\right)\right) + c_1(a_2 - L_2)\right] - AC \\
=&-\frac{a_2 - b_2C + c_2AC - L_2}{c_2} < 0, \quad \text{(since } z_2(AC, C) > L_2\text{)} \\
&p_2(\bar{z}_1, L_2) - C \\
=&\frac{1}{b_1b_2 - c_1c_2}\left[c_2\left(a_1 - \left(a_1 - \frac{(b_1b_2 - c_1c_2)C - b_1(a_2 - L_2)}{c_2}\right)\right) + b_1(a_2 - L_2)\right] - C = 0, \\
&p_1(L_1, \bar{z}_2) - AC \\
=&\frac{1}{b_1b_2 - c_1c_2}\left[b_2(a_1 - L_1) + c_1\left(a_2 - \left(a_2 - \frac{(b_1b_2 - c_1c_2)AC - b_2(a_1 - L_1)}{c_1}\right)\right)\right] - AC = 0, \\
&p_2(L_1, \bar{z}_2) - C \\
=&\frac{1}{b_1b_2 - c_1c_2}\left[c_2(a_1 - L_1) + b_1\left(a_2 - \left(a_2 - \frac{(b_1b_2 - c_1c_2)AC - b_2(a_1 - L_1)}{c_1}\right)\right)\right] - C \\
=&-\frac{a_1 - b_1AC + c_1C - L_1}{c_1} < 0 \quad \text{(since } z_1(AC, C) > L_1\text{)}.
\end{aligned}
$$

■

Lemma 24 *(Properties of Interior Solution)*

(i) Whenever $\Delta_1(z_1, z_2, o_1) = 0$ is satisfied at some $z_1 \geq L_1, z_2 \geq L_2$ and $o_1 > 0$,

$$(p_1(\boldsymbol{z}) - AC)g(o_1 - z_1) \geq \frac{b_2}{b_1b_2 - c_1c_2}\bar{G}_1(o_1 - z_1). \tag{C.8}$$

Moreover, the inequality in equation (C.8) is strict if the first-order condition is satisfied at some $o_1 \in (0, y/A)$.

(ii) Whenever $\Delta_2(z_1, z_2, o_1) = 0$ is satisfied at some $z_1 \geq L_1, z_2 \geq L_2$ and $o_1 < y/A$,

$$(p_2(\boldsymbol{z}) - C)g(y - Ao_1 - z_2) \geq \frac{b_1}{b_1b_2 - c_1c_2}\bar{G}_2(y - Ao_1 - z_2). \tag{C.9}$$

Moreover, the inequality in equation (C.9) is strict if the first-order condition is satisfied at some $o_1 \in (0, y/A)$.

Proof: (i) From the first-order condition $\Delta_1(z_1, z_2, o_1) = 0$, we have

$$
\begin{aligned}
0 =&\Delta_1(z_1, z_2, o_1) \\
=&-\frac{b_2}{b_1b_2 - c_1c_2}\mathrm{E}[\min(o_1, z_1 + \epsilon_1)] + (p_1(\mathbf{z}) - AC)G_1(o_1 - z_1) \\
&-\frac{c_2}{b_1b_2 - c_1c_2}\mathrm{E}[\min(y - Ao_1, z_2 + \epsilon_2)] \\
=&-\frac{b_2}{b_1b_2 - c_1c_2}\left[z_1 - L_1 + \int_{-L_1}^{o_1 - z_1}\bar{G}_1(x)dx\right] + (p_1(\mathbf{z}) - AC)G_1(o_1 - z_1) \\
&-\frac{c_2}{b_1b_2 - c_1c_2}\mathrm{E}[\min(y - Ao_1, z_2 + \epsilon_2)]
\end{aligned}
$$

Note that $z_1 - L_1 \geq 0$, $z_2 - L_2 \geq 0$, and $y - Ao_1 \geq 0$. Hence $\mathrm{E}[\min(y - Ao_1, z_2 + \epsilon_2)] \geq 0$ and

$$\int_{-L_1}^{o_1 - z_1} -\frac{b_2}{b_1 b_2 - c_1 c_2}\bar{G}_1(x) + (p_1(\mathbf{z}) - AC)g_1(x)dx \geq 0 \tag{C.10}$$

This is equivalent to

$$\int_{-L_1}^{o_1 - z_1} \bar{G}(x)\left(-\frac{b_2}{b_1 b_2 - c_1 c_2} + (p_1(\mathbf{z}) - AC)\frac{g_1(x)}{\bar{G}_1(x)}\right)dx \geq 0 \tag{C.11}$$

Note that the weak inequality in equation (C.11) holds as equality if and only $z_1 = L_1$ and $y - Ao_1 = 0$, which implies $o_1 > 0$ and $o_1 - z_1 > -L_1$. Therefore, $[-L_1, o_1 - z_1]$ is an interval with positive length. By equation (C.11) and $\bar{G}_1(x) \geq 0$, there exists at least one $\xi \in [-L_1, o_1 - z_1]$ such that $-\frac{b_2}{b_1 b_2 - c_1 c_2} + (p_1(\mathbf{z}) - AC)\frac{g_1(\xi)}{\bar{G}_1(\xi)} \geq 0$. In the meanwhile, since $G_1(\cdot)$ satisfies IFR property, $-\frac{b_2}{b_1 b_2 - c_1 c_2} + (p_1(\mathbf{z}) - AC)\frac{g_1(x)}{\bar{G}_1(x)}$ is nondecreasing in x. Therefore, $(p_1(\mathbf{z}) - AC)g_1(o_1 - z_1) \geq \frac{b_2}{b_1 b_2 - c_1 c_2}\bar{G}_1(o_1 - z_1)$.

Moreover, if $q < y/A$, $\mathrm{E}[\min(y - Ao_1, z_2 + \epsilon_2)] > 0$ and the inequality in equation (C.10) and (C.11) are strict. Therefore, there exists at least one $\xi \in [-L_1, o_1 - z_1]$ such that $-\frac{b_2}{b_1 b_2 - c_1 c_2} + (p_1(\mathbf{z}) - AC)\frac{g_1(\xi)}{\bar{G}_1(\xi)} > 0$. This fact, together with the IFR property, implies $(p_1(\mathbf{z}) - AC)g_1(o_1 - z_1) > \frac{b_2}{b_1 b_2 - c_1 c_2}\bar{G}_1(o_1 - z_1)$.

(ii) The proof is similar to the proof of (i). ■

Lemma 25 *(Super- and Sub- Modularity are Satisfied at Interior Solution)*

(i) Whenever $\Delta_1(z_1, z_2, o_1) \geq 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 > 0$, $\Delta_{13}(z_1, z_2, o_1) \geq 0$. *Furthermore, whenever* $\Delta_1(z_1, z_2, o_1) > 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 > 0$, *or* $\Delta_1(z_1, z_2, o_1) \geq 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $0 < o_1 < y/A$, $\Delta_{13}(z_1, z_2, o_1) > 0$.

(ii) Whenever $\Delta_2(z_1, z_2, o_1) \geq 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 < y/A$, $\Delta_{23}(z_1, z_2, o_1) \leq 0$. *Furthermore, whenever* $\Delta_2(z_1, z_2, o_1) > 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 < y/A$, *or* $\Delta_2(z_1, z_2, o_1) \geq 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $0 < o_1 < y/A$, $\Delta_{23}(z_1, z_2, o_1) < 0$.

Proof: (i) By $\Delta_1(z_1, z_2, o_1) \geq 0$, similarly to the proof of Lemma 24, we get $(p_1(\mathbf{z}) - AC)g_1(o_1 - z_1) \geq \frac{b_2}{b_1 b_2 - c_1 c_2}\bar{G}_1(o_1 - z_1)$. Hence,

$$\Delta_{13}(z_1, z_2, o_1) = -\frac{b_2}{b_1 b_2 - c_1 c_2}\bar{G}_1(o_1 - z_1) + (p_1(\mathbf{z}) - AC)g_1(o_1 - z_1) + A\frac{c_2}{b_1 b_2 - c_1 c_2}\bar{G}_2(y - Ao_1 - z_2) \geq 0$$

Furthermore, whenever $\Delta_1(z_1, z_2, o_1) > 0$ is satisfied at some $z_1 \geq L_1, z_2 \geq L_2$ and $o_1 > 0$, or $\Delta_1(z_1, z_2, o_1) \geq 0$ is satisfied at some $z_1 \geq L_1, z_2 \geq L_2$ and $0 < o_1 < y/A$, we have $(p_1(\mathbf{z}) - AC)g_1(o_1 - z_1) > \frac{b_2}{b_1 b_2 - c_1 c_2}\bar{G}_1(o_1 - z_1)$ and as a result, $\Delta_{13}(z_1, z_2, o_1) > 0$.

(ii) Similar to the proof of (i). ■

Lemma 26 *(Nonnegative-Margin Conditions are Satisfied at Interior Solution)*

(i) Whenever $\Delta_1(z_1, z_2, o_1) = 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 > 0$, $p_1(z_1, z_2) \geq AC$.

(ii) Whenever $\Delta_2(z_1, z_2, o_1) = 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 < y/A$, $p_2(z_1, z_2) \geq C$.

∎

Proof: Directly follow from equation (C.8) and (C.9) in Lemma 24.

Lemma 27 *(Second-order Conditions are Satisfied at Interior Solution)*

(i) Whenever $\Delta_1(z_1, z_2, o_1) = 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 > 0$, $\Delta_{11}(z_1, z_2, o_1) < 0$.

(ii) Whenever $\Delta_2(z_1, z_2, o_1) = 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 < y/A$, $\Delta_{22}(z_1, z_2, o_1) < 0$.

(iii) Whenever $\Delta_1(z_1, z_2, o_1) = \Delta_2(z_1, z_2, o_1) = 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 \in (0, y/A)$, $\Delta_{11}(z_1, z_2, o_1)\Delta_{22}(z_1, z_2, o_1) - (\Delta_{12}(z_1, z_2, o_1))^2 > 0$.

(iv) Whenever $\Delta_1(z_1, z_2, o_1) = \Delta_2(z_1, z_2, o_1) = \Delta_3(z_1, z_2, z_3) = 0$ *is satisfied at some* $z_1 \geq L_1, z_2 \geq L_2$ *and* $o_1 \in (0, y/A)$*, the Hessian matrix*

$$\mathcal{H} = \begin{pmatrix} \Delta_{11}(z_1, z_2, o_1) & \Delta_{12}(z_1, z_2, o_1) & \Delta_{13}(z_1, z_2, o_1) \\ \Delta_{12}(z_1, z_2, o_1) & \Delta_{22}(z_1, z_2, o_1) & \Delta_{23}(z_1, z_2, o_1) \\ \Delta_{13}(z_1, z_2, o_1) & \Delta_{23}(z_1, z_2, o_1) & \Delta_{33}(z_1, z_2, o_1) \end{pmatrix}$$

is negative definite.

Proof: (i) By Lemma 24, whenever $\Delta_1(z_1, z_2, o_1) = 0$ for some $z_1 \geq L_1$,

$$\begin{aligned} \Delta_{11}(z_1, z_2, o_1) &= -\frac{2b_2}{b_1 b_2 - c_1 c_2} G_1(o_1 - z_1) - (p_1(\mathbf{z}) - AC) g_1(o_1 - z_1) \\ &\leq -\frac{2b_2}{b_1 b_2 - c_1 c_2} G_1(o_1 - z_1) - \frac{b_2}{b_1 b_2 - c_1 c_2} \bar{G}_1(o_1 - z_1) \\ &= -\frac{b_2}{b_1 b_2 - c_1 c_2} G_1(o_1 - z_1) - \frac{b_2}{b_1 b_2 - c_1 c_2} < 0 \end{aligned}$$

(ii) The proof is similar to the proof of (i).

(iii) For the ease of notation, in the proof of (iii) and (iv), let $G_1 = G_1(o_1 - z_1)$, $G_2 = G_2(y - Ao_1 - z_2)$, $\bar{G}_1 = 1 - G_1$, $\bar{G}_2 = 1 - G_2$, $M_1 = (b_1 b_2 - c_1 c_2)(p_1(\mathbf{z}) - AC) g_1(o_1 - z_1)$, $M_2 = (b_1 b_2 - c_1 c_2)(p_2(\mathbf{z}) -$

$C)g_2(y - Ao_1 - z_2)$.

$$
\begin{aligned}
&\Delta_{11}(z_1, z_2, o_1)\Delta_{22}(z_1, z_2, o_1) - (\Delta_{12}(z_1, z_2, o_1))^2 \\
= &\left[-\frac{2b_2}{b_1b_2 - c_1c_2}G_1 - \frac{M_1}{b_1b_2 - c_1c_2}\right] \cdot \left[-\frac{2b_1}{b_1b_2 - c_1c_2}G_2 - \frac{M_2}{b_1b_2 - c_1c_2}\right] \\
&- \left[-\frac{c_1}{b_1b_2 - c_1c_2}G_1 - \frac{c_2}{b_1b_2 - c_1c_2}G_2\right]^2 \\
= &\frac{1}{(b_1b_2 - c_1c_2)^2}[4b_1b_2G_1G_2 + 2b_1M_1G_2 + 2b_2M_2G_1 + M_1M_2 - c_1^2(G_1)^2 - 2c_1c_2G_1G_2 - c_2^2(G_2)^2] \\
> &\frac{1}{(b_1b_2 - c_1c_2)^2}[4b_1b_2G_1G_2 + b_1b_2\bar{G}_1G_2 + b_1b_2\bar{G}_2G_1 + b_1M_1G_2 + b_2M_2G_1 + M_1M_2 - c_1^2(G_1)^2 \\
&\qquad - 2c_1c_2G_1G_2 - c_2^2(G_2)^2] \\
= &\frac{1}{(b_1b_2 - c_1c_2)^2}[2b_1b_2G_1G_2 + b_1b_2G_2 + b_1b_2G_1 + b_1M_1G_2 + b_2M_2G_1 + M_1M_2 - c_1^2(G_1)^2 \\
&\qquad - 2c_1c_2G_1G_2 - c_2^2(G_2)^2] \\
= &\frac{1}{(b_1b_2 - c_1c_2)^2}[2(b_1b_2 - c_1c_2)G_1G_2 + (b_1b_2 - c_2^2G_2)G_2 + (b_1b_2 - c_1^2G_1)G_1 + b_1M_1G_2 \\
&\qquad + b_2M_2G_1 + M_1M_2] \\
\geq &\ 0
\end{aligned}
$$

The first inequality is a direct application of Lemma 24. The second inequality follows from the facts that $\min(b_1, b_2) > \max(c_1, c_2)$, $G_1, G_2 \in [0, 1]$, and $M_1, M_2 > 0$.

(iv) To prove the negative definiteness, it suffices to show that the first and third order leading principal minors are negative and the second order leading principal minor is positive. That is, $\Delta_{11}(z_1, z_2, o_1) < 0$, $\Delta_{11}(z_1, z_2, o_1)\Delta_{22}(z_1, z_2, o_1) - (\Delta_{12}(z_1, z_2, o_1))^2 > 0$, and the determinant of the matrix $\mathcal{H}$ itself is negative. The first two are proved in (i) and (iii). In the following we prove that the determinant of the matrix $\mathcal{H}$ itself is negative.

We first explicitly write and simplify the determinant of the Hessian matrix.

$$
|\mathcal{H}| = (b_1b_2 - c_1c_2)^{-3} \begin{vmatrix} -2b_2G_1 - M_1 & -c_1G_1 - c_2G_2 & -b_2\bar{G}_1 + M_1 + Ac_2\bar{G}_2 \\ -c_1G_1 - c_2G_2 & -2b_1G_2 - M_2 & -c_1\bar{G}_1 + Ab_1\bar{G}_2 - AM_2 \\ -b_2\bar{G}_1 + M_1 + Ac_2\bar{G}_2 & -c_1\bar{G}_1 + Ab_1\bar{G}_2 - AM_2 & -M_1 - A^2M_2 \end{vmatrix}
$$

Adding the first row and (-A) multiplying the second row to the third row, and then performing the same operations to columns, we get

$$
\begin{aligned}
&(b_1b_2 - c_1c_2)^3|\mathcal{H}| \\
&= \begin{vmatrix} -2b_2G_1 - M_1 & -c_1G_1 - c_2G_2 & Ac_2 - b_2 + (Ac_1 - b_2)G_1 \\ -c_1G_1 - c_2G_2 & -2b_1G_2 - M_2 & Ab_1 - c_1 + (Ab_1 - c_2)G_2 \\ Ac_2 - b_2 + (Ac_1 - b_2)G_1 & Ab_1 - c_1 + (Ab_1 - c_2)G_2 & -2(A^2b_1 + b_2) + 2A(c_1 + c_2) \end{vmatrix} \\
&= (Ab_1 - c_1)^2(M_1 - M_1G_2) + (Ac_2 - b_2)^2(M_2 - M_2G_1) \\
&\quad + (Ab_1 - c_2)^2(M_1G_2^2 - M_1G_2) + (Ac_1 - b_2)^2(M_2G_1^2 - M_2G_1) \\
&\quad + (-4b_1b_2 + 2c_1c_2 + c_2^2 + c_1^2)M_1G_2 + A^2(-4b_1b_2 + 2c_1c_2 + c_1^2 + c_2^2)M_2G_1 \\
&\quad + 2A(Ab_1 - c_1)(b_1b_2 - c_1c_2)G_1 + 2(b_2 - Ac_2)(b_1b_2 - c_1c_2)G_2 + 2(b_1b_2 - c_1c_2)(-b_2 - A^2b_1)G_1G_2 \\
&\quad + (-2b_2 + 2Ac_2 + 2Ac_1 - 2A^2b_1)M_1M_2 + 2(b_2 - AC_1)(b_1b_2 - c_1c_2)(G_1^2G_2 - G_1G_2) \\
&\quad + 2A(Ab_1 - c_2)(b_1b_2 - c_1c_2)(G_1G_2^2 - G_1G_2) + 2Ac_1(b_1b_2 - c_1c_2)G_1^2 + 2Ac_2(b_1b_2 - c_1c_2)G_2^2 \\
&\leq (Ab_1 - c_1)^2M_1\bar{G}_2 + (Ac_2 - b_2)^2M_2\bar{G}_1 \\
&\quad + (-4b_1b_2 + 2c_1c_2 + c_2^2 + c_1^2)M_1G_2 + A^2(-4b_1b_2 + 2c_1c_2 + c_1^2 + c_2^2)M_2G_1 \\
&\quad + 2A(Ab_1 - c_1)(b_1b_2 - c_1c_2)G_1 + 2(b_2 - Ac_2)(b_1b_2 - c_1c_2)G_2 + 2(b_1b_2 - c_1c_2)(-b_2 - A^2b_1)G_1G_2 \\
&\quad + (-2b_2 + 2Ac_2 + 2Ac_1 - 2A^2b_1)M_1M_2 + 2Ac_1(b_1b_2 - c_1c_2)G_1^2 + 2Ac_2(b_1b_2 - c_1c_2)G_2^2 \\
&< (Ab_1 - c_1)^2M_1\bar{G}_2 + (Ac_2 - b_2)^2M_2\bar{G}_1 \\
&\quad + (-4b_1b_2 + 2c_1c_2 + c_2^2 + c_1^2)b_2\bar{G}_1G_2 + A^2(-4b_1b_2 + 2c_1c_2 + c_1^2 + c_2^2)b_1\bar{G}_2G_1 \\
&\quad + 2A(Ab_1 - c_1)(b_1b_2 - c_1c_2)G_1 + 2(b_2 - Ac_2)(b_1b_2 - c_1c_2)G_2 + 2(b_1b_2 - c_1c_2)(-b_2 - A^2b_1)G_1G_2 \\
&\quad + b_2(-b_2 + 2Ac_2 - A^2b_1)M_2\bar{G}_1 + b_1(-b_2 - A^2b_1 + 2Ac_1)M_1\bar{G}_2 \\
&\quad + 2Ac_1(b_1b_2 - c_1c_2)G_1^2 + 2Ac_2(b_1b_2 - c_1c_2)G_2^2 \\
&\leq (-2b_2)(b_1b_2 - c_1c_2)G_2 + (c_2^2 + c_1^2 - 2b_1b_2)b_2\bar{G}_1G_2 + (-2A^2b_1)(b_1b_2 - c_1c_2)G_1 \\
&\quad + A^2(c_1^2 + c_2^2 - 2b_1b_2)b_1\bar{G}_2G_1 + 2A(Ab_1 - c_1)(b_1b_2 - c_1c_2)G_1 + 2(b_2 - Ac_2)(b_1b_2 - c_1c_2)G_2 \\
&\quad + 2Ac_1(b_1b_2 - c_1c_2)G_1^2 + 2Ac_2(b_1b_2 - c_1c_2)G_2^2 \\
&= (c_2^2 + c_1^2 - 2b_1b_2)b_2\bar{G}_1G_2 + A^2(c_1^2 + c_2^2 - 2b_1b_2)b_1\bar{G}_2G_1 \\
&\quad + 2A(-c_1)(b_1b_2 - c_1c_2)G_1 - 2Ac_2(b_1b_2 - c_1c_2)G_2 + 2Ac_1(b_1b_2 - c_1c_2)G_1^2 + 2Ac_2(b_1b_2 - c_1c_2)G_2^2 \\
&\leq (c_2^2 + c_1^2 - 2b_1b_2)b_2\bar{G}_1G_2 + A^2(c_1^2 + c_2^2 - 2b_1b_2)b_1\bar{G}_2G_1 \leq 0
\end{aligned}
$$

The first and fourth inequalities above follow from $G_1^2 \leq G_1$ and $G_2^2 \leq G_2$. The second inequality is a direct application of Lemma 24. The third and last inequalities are implied by $\min(b_1, b_2) > \max(c_1, c_2)$. ■

Lemma 28 *(First-order and Second-order Conditions at Boundaries)*

(i) $\Delta_1(z_1, z_2, 0) < 0$ *for all* $z_1 \geq L_1, z_2 \geq L_2$*,* $\Delta_2(z_1, z_2, y/A) < 0$ *for all* $z_1 \geq L_1, z_2 \geq L_2$*.*

(ii) Given $z_1 = L_1$*, whenever* $\Delta_2(L_1, z_2, o_1) = \Delta_3(L_1, z_2, o_1) = 0$ *is satisfied at some* $z_2 \geq L_2$ *and* $o_1 < y/A$*,* $\Delta_{22}(L_1, z_2, o_1)\Delta_{33}(L_1, z_2, o_1) - (\Delta_{23}(L_1, z_2, o_1))^2 > 0$*.*

(iii) Given $z_2 = L_2$*, whenever* $\Delta_1(z_1, L_2, o_1) = \Delta_3(z_1, L_2, o_1) = 0$ *is satisfied at some* $z_1 \geq L_1$ *and* $o_1 > 0$*,* $\Delta_{11}(z_1, L_2, o_1)\Delta_{33}(z_1, L_2, o_1) - (\Delta_{13}(z_1, L_2, o_1))^2 > 0$*.*

(iv) Given $z_1 = L_1$ *and* $z_2 = L_2$*, whenever* $\Delta_3(L_1, L_2, o_1) = 0$ *is satisfied at some* $o_1 \in [0, y/A]$*,* $\Delta_{33}(L_1, L_2, o_1) < 0$*.*

Proof:

(i) For $z_1 \geq L_1$ and $z_2 \geq L_2$,

$$\begin{aligned}
\Delta_1(z_1, z_2, 0) = & -\frac{b_2}{b_1 b_2 - c_1 c_2}\mathrm{E}[\min(0, z_1 + \epsilon_1)] + (p_1(\mathbf{z}) - AC)G_1(-z_1) \\
& -\frac{c_2}{b_1 b_2 - c_1 c_2}\mathrm{E}[\min(y, z_2 + \epsilon_2)] \\
= & -\frac{c_2}{b_1 b_2 - c_1 c_2}\mathrm{E}[\min(y, z_2 + \epsilon_2)] < 0
\end{aligned}$$

Similarly, we can prove that $\Delta_2(z_1, z_2, y/A) < 0$ for all $z_1 \geq L_1, z_2 \geq L_2$.

(ii) For the ease of notation, still let $G_1 = G_1(o_1 - L_1)$, $G_2 = G_2(y - Ao_1 - z_2)$, $\bar{G}_1 = 1 - G_1$, $\bar{G}_2 = 1 - G_2$, $M_1 = (b_1 b_2 - c_1 c_2)(p_1(L_1, z_2) - AC)g_1(o_1 - L_1)$, $M_2 = (b_1 b_2 - c_1 c_2)(p_2(L_1, z_2) - C)g_2(y - Ao_1 - z_2)$. Then,

$$\begin{aligned}
& \Delta_{22}(L_1, z_2, o_1)\Delta_{33}(L_1, z_2, o_1) - (\Delta_{23}(L_1, z_2, o_1))^2 \\
= & (b_1 b_2 - c_1 c_2)^{-2}\left[(2b_1 G_2 + M_2)(M_1 + A^2 M_2) - (Ab_1\bar{G}_2 - AM_2 - c_1\bar{G}_1)^2\right] \\
= & (b_1 b_2 - c_1 c_2)^{-2}\left[(2b_1 G_2 + M_2)M_1 + 2A^2 b_1 M_2 + c_1\bar{G}_1(Ab_1\bar{G}_2 - AM_2 - c_1\bar{G}_1)\right. \\
& \left. -(Ab_1\bar{G}_2)^2 - AC_1\bar{G}_1(M_2 - b_1\bar{G}_2)\right] \\
= & (b_1 b_2 - c_1 c_2)^{-2}\left[(2b_1 G_2 + M_2)M_1 + A^2 b_1 M_2 + c_1\bar{G}_1(Ab_1\bar{G}_2 - AM_2 - c_1\bar{G}_1)\right. \\
& \left. +A^2 b_1^2\left(\bar{G}_2 - (\bar{G}_2)^2\right) + A(Ab_1 - c_1)(M_2 - b_1\bar{G}_2)\right] \\
= & (b_1 b_2 - c_1 c_2)^{-2}\left[(2b_1 G_2 + M_2)M_1 + (Ab_1 - c_1\bar{G}_1)AM_2 + c_1\bar{G}_1(Ab_1\bar{G}_2 - c_1\bar{G}_1)\right. \\
& \left. +A^2 b_1^2\bar{G}_2 G_2 + A(Ab_1 - c_1)(M_2 - b_1\bar{G}_2)\right]
\end{aligned} \tag{C.12}$$

Note that by Lemma 24, $M_2 \geq b_1\bar{G}_2 \geq 0$, and that by Lemma 26, $p_2(L_1, z_2) \geq C$. In the meanwhile, by $\Delta_3(L_1, z_2, o_1) = 0$, $(p_1(L_1, z_2) - AC)\bar{G}_1 = A(p_2(L_1, z_2) - C)\bar{G}_2$. Hence, $p_1(L_1, z_2) - AC \geq 0$ and $M_1 \geq 0$. Next, we consider two cases:

- $M_2 = b_1\bar{G}_2$

  By Lemma 24, this occurs only if $q = 0$. In such a case, $\bar{G}_1 = \bar{G}_1(-L_1) = 1$. Furthermore, by $\Delta_2(L_1, z_2, o_1) = 0$ and $q = 0$, $(p_2(L_1, z_2) - C)G_2 = b_1\text{E}[\min(y, z_2 + \epsilon_2)] > 0$. Hence, $p_2(L_1, z_2) > c$ and $G_2 > 0$. By $\frac{\partial \pi^{NO}(L_1,z_2,o_1)}{\partial o_1} = 0$ and $p_2(L_1, z_2) > c$,

$$
\begin{aligned}
Ab_1\bar{G}_2 - c_1\bar{G}_1 &= b_1\frac{(p_1(L_1, z_2) - AC)\bar{G}_1}{p_2(L_1, z_2) - C} - c_1\bar{G}_1 \\
&= \frac{\bar{G}_1}{p_2(L_1, z_2) - C}\left[b_1(p_1(L_1, z_2) - AC) - c_1(p_2(L_1, z_2) - C)\right] \\
&= \frac{\bar{G}_1}{p_2(L_1, z_2) - C}\left[b_1p_1(L_1, z_2) - c_1p_2(L_1, z_2) - Ab_1C + c_1c\right] \\
&= \frac{\bar{G}_1}{p_2(L_1, z_2) - C}\left[b_1\frac{b_2(a_1 - L_1) + c_1(a_2 - z_2)}{b_1b_2 - c_1c_2} - c_1\frac{c_2(a_1 - L_1) + b_1(a_2 - z_2)}{b_1b_2 - c_1c_2}\right. \\
&\qquad\qquad \left. - Ab_1C + c_1c\right] \\
&= \frac{\bar{G}_1}{p_2(L_1, z_2) - C}\left[a_1 - L_1 - Ab_1C + c_1c\right] \\
&> 0 \quad \text{(by } z_1(AC, C) > L_1\text{)}.
\end{aligned}
$$

Therefore, $Ab_1\bar{G}_2 > c_1\bar{G}_1 = c_1 > 0$, which implies $\bar{G}_2 > 0$. In the meantime, recall that $Ab_1 > c_1 \geq C_1\bar{G}_1$ and $G_1, G_2, \bar{G}_1, \bar{G}_2 \in [0, 1]$. Hence, all the terms inside the brackets in equation (C.12) are nonnegative and in particular, $A^2b_1^2\bar{G}_2G_2$ is positive. Therefore, $\Delta_{22}(L_1, z_2, o_1)\Delta_{33}(L_1, z_2, o_1) - (\Delta_{23}(L_1, z_2, o_1))^2 > 0$.

- $M_2 > b_1\bar{G}_2$

  In such a case, $M_2 > 0$ and it implies $p_2(L_1, z_2) - C > 0$. Similarly to the first case, we can show $Ab_1\bar{G}_2 - c_1\bar{G}_1 > 0$. In the meantime, recall that $Ab_1 > c_1 \geq C_1\bar{G}_1$ and $G_1, G_2, \bar{G}_1, \bar{G}_2 \in [0, 1]$. Hence, all the terms inside the brackets in equation (C.12) are nonnegative and in particular, both $(Ab_1 - c_1\bar{G}_1)AM_2$ and $A(Ab_1 - c_1)(M_2 - b_1\bar{G}_2)$ are positive. Therefore, $\Delta_{22}(L_1, z_2, o_1)\Delta_{33}(L_1, z_2, o_1) - (\Delta_{23}(L_1, z_2, o_1))^2 > 0$.

(iii) Similar to the proof of (ii).

(iv) Recall

$$
\begin{aligned}
\Delta_3(L_1, L_2, o_1) &= (p_1(L_1, L_2) - AC)\bar{G}_1(o_1 - L_1) - A(p_2(L_1, L_2) - C)\bar{G}_2(y - Ao_1 - L_2) \\
\Delta_{33}(L_1, L_2, o_1) &= -(p_1(L_1, L_2) - AC)g_1(o_1 - L_1) - A^2(p_2(L_1, L_2) - C)g_2(y - Ao_1 - L_2)
\end{aligned}
$$

To show that whenever $\Delta_3(L_1, L_2, o_1) = 0$, $\Delta_{33}(L_1, L_2, o_1) < 0$, first note that since $g_1(o_1 - L_1) \geq 0$ and $g_2(y - Ao_1 - L_2) \geq 0$, $\Delta_{33}(L_1, L_2, o_1) \leq 0$ for all feasible o_1. Hence, it suffices to show that $\Delta_{33}(L_1, L_2, o_1)$ and $\Delta_3(L_1, L_2, o_1)$ cannot both be zero.

For $\Delta_{33}(L_1, L_2, o_1)$, since $p_1(L_1, L_2) > AC$ and $p_2(L_1, L_2) > c$, $\Delta_{33}(L_1, L_2, o_1) = 0$ if and only if $g_1(o_1 - L_1) = g_2(y - Ao_1 - L_2) = 0$. Recall that for $j = 1, 2$, $g_j(x) > 0$ for $x \in (-L_j, H_j)$ and that from the feasibility constraints following Lemma 11, $-L_1 \leq o_1 - L_1 \leq H_1$, $-L_2 \leq y - Ao_1 - L_2 \leq H_2$, and at least one of $o_1 - L_1 \leq H_1$ and $y - Ao_1 - L_2 \leq H_2$ is strict. Therefore, $\Delta_{33}(L_1, L_2, o_1) = 0$ only if one of the three scenarios occurs: first, $g_1(H_1) = g_2(-L_2) = 0$ and $o_1 - L_1 = H_1$ and $y - Ao_1 - L_2 = -L_2$; second, $g_1(-L_1) = g_2(-L_2) = 0$ and $o_1 - L_1 = -L_1$ and $y - Ao_1 - L_2 = -L_2$, or third, $g_1(-L_1) = g_2(H_2) = 0$ and $o_1 - L_1 = -L_1$ and $y - Ao_1 - L_2 = H_2$. However, in the first and third scenarios, $\Delta_3(L_1, L_2, o_1)$ is nonzero, and the second scenario contradicts with the condition $y > 0$. Therefore, for $y \in (0, \bar{y})$, $\Delta_{33}(L_1, L_2, o_1)$ and $\Delta_3(L_1, L_2, o_1)$ cannot both be zero and the proof is complete. ∎

Lemma 29 *(Boundary Solution)*

When $o_1 = 0$, there exists a unique pair of (z_1, z_2), denoted by $(z_1^*(0), z_2^*(0))$, which maximizes the profit function $\pi^{NO}(z_1, z_2, 0)$ subject to $z_1 \geq L_1, z_2 \geq L_2, p_1(z_1, z_2) \geq AC, p_2(z_1, z_2) \geq C$. Specifically, $z_1^*(0) = L_1$ and

$$z_2^*(0) = \begin{cases} L_2 & \text{if } \Delta_2(L_1, L_2, 0) < 0 \\ \text{the unique solution to } \Delta_2(L_1, z_2, 0) = 0 & \text{otherwise.} \end{cases}$$

When $o_1 = y/A$, there exists a unique pair of (z_1, z_2), denoted by $(z_1^*(y/A), z_2^*(y/A))$, which maximizes the profit function $\pi^{NO}(z_1, z_2, y/A)$ subject to $z_1 \geq L_1, z_2 \geq L_2, p_1(z_1, z_2) \geq AC, p_2(z_1, z_2) \geq C$. Specifically, $z_2^*(y/A) = L_2$ and

$$z_1^*(y/A) = \begin{cases} L_1 & \text{if } \Delta_1(L_1, L_2, y/A) < 0 \\ \text{the unique solution to } \Delta_1(z_1, L_2, y/A) = 0 & \text{otherwise.} \end{cases}$$

Proof: When $q = 0$, $z_1^*(0) = L_1$ by Lemma 28 (i). Furthermore, by Lemma 27, $\pi^{NO}(L_1, z_2, 0)$ is strictly quasi-concave in z_2 for $z_2 \geq L_2$. Hence, there exists a unique $z_2^*(0)$ maximizing $\pi^{NO}(L_1, z_2, 0)$ subject to $z_2 \geq L_2$. Note that for $z_2 > \bar{z}_2$, $p_2(L_1, z_2) < C$ and $\Delta_2(L_1, z_2, 0) < 0$. Therefore, if $\Delta_2(L_1, L_2, 0) < 0$, $z_2^*(0) = L_2$; otherwise, $z_2^*(0) \in [L_2, \bar{z}_2)$ and is the unique solution to the first order condition. In either case, it is easy to check that $(z_1^*(0), z_2^*(0))$ satisfies $p_1(z_1, z_2) \geq AC$ and $p_2(z_1, z_2) \geq C$: if $\Delta_2(L_1, L_2, 0) < 0$, $z_1^*(0) = L_1, z_2^*(0) = L_2$, and by assumption, $p_1(L_1, L_2) \geq AC$ and $p_2(L_1, L_2) \geq C$; otherwise, by Lemma 26, we have $p_2(L_1, z_2^*(0)) \geq C$. Furthermore,

$$\begin{aligned}
&b_1(p_1(L_1, z_2^*(0)) - AC) - c_1(p_2(L_1, z_2^*(0)) - C) \\
&= b_1 p_1(L_1, z_2^*(0)) - c_1 p_2(L_1, z_2^*(0)) - Ab_1 C + c_1 c \\
&= b_1 \frac{b_2(a_1 - L_1) + c_1(a_2 - z_2^*(0))}{b_1 b_2 - c_1 c_2} - c_1 \frac{c_2(a_1 - L_1) + b_1(a_2 - z_2^*(0))}{b_1 b_2 - c_1 c_2} - Ab_1 C + c_1 C \\
&= a_1 - L_1 - Ab_1 C + c_1 C \\
&> 0 \quad (\text{by } z_1(AC, C) > L_1).
\end{aligned}$$

Therefore, $p_1(L_1, z_2^*(0)) - AC > \frac{c_1}{b_1}(p_2(L_1, z_2^*(0)) - C) \geq 0$. This completes the proof for the case $q = 0$. The proof for the case $o_1 = y/A$ is similar and omitted for conciseness. ∎

Lemma 30 *Let $f(x)$ be a continuous real-valued function defined on $[a, b]$, where $f(x) = f_1(x)$ for $x \in [a, c]$ and $f(x) = f_2(x)$ for $x \in [c, b]$. Then $f(x)$ is continuously differentiable and strictly quasi-concave on $[a, b]$ if the following conditions are satisfied:*

(i) $f_1(x)$ and $f_2(x)$ are continuously differentiable and strictly quasi-concave on $[a, c]$ and $[c, b]$, respectively. Let $f_1'(x)$ and $f_2'(x)$ be the derivative of $f_1(x)$ and $f_2(x)$, respectively,

(ii) $f_1'(c) = f_2'(c)$,

(iii) If $f_1'(c) = 0$, the left-hand derivative of $f_1'(x)$ at $x = c$ and the right-hand derivative of $f_2'(x)$ at $x = c$ both exist and are both negative.

Proof: To show that $f(x)$ is continuously differentiable on $[a, b]$, note that by (i), it suffices to show that $f(x)$ is differentiable at $x = c$ and the derivative of $f(x)$ is continuous at $x = c$. By (i), $f_1'(x)$ and $f_2'(x)$ are continuous. Hence, by (ii),

$$\lim_{\delta \to 0^+} \frac{f(c+\delta) - f(c)}{\delta} = \lim_{\delta \to 0^+} f_1'(c+\delta) = f_1'(c) = f_2'(c) = \lim_{\delta \to 0^-} f_2'(c+\delta) = \lim_{\delta \to 0^-} \frac{f(c+\delta) - f(c)}{\delta}$$

This implies that $f(x)$ is differentiable at $x = c$ and the derivative of $f(x)$ is continuous at $x = c$.

To show the strict quasi-concavity of $f(x)$ on $[a, b]$, by its continuous differentiability on $[a, b]$ and (i), it suffices to show that if $f'(c) = 0$, $f(x)$ strictly increases in x at a left-hand neighborhood of c and strictly decreases in x at a right-hand neighborhood of c. By (iii), let

$$A = \lim_{x \to c^-} [f_1'(x) - f_1'(c)]/(x - c), \quad B = \lim_{x \to c^+} [f_2'(x) - f_2'(c)]/(x - c)$$

By (iii), $A < 0$ and $B < 0$. By definition of limit, for $\epsilon = -A/2 > 0$, there exists a $\delta > 0$, such that for all $0 < c - x < \delta$, $|[f_1'(x) - f_1'(c)]/(x - c) - A| \leq \epsilon$. That is, for all $x \in (c - \delta, c)$, $f_1'(x) \geq (A - \epsilon)(x - c) + f_1'(c) = A(x - c)/2 > 0$. This implies that $f(x)$ strictly increases in x at $(c - \delta, c)$. Following the same idea, we can prove the existence of a right-hand neighborhood of c in which $f_2'(x) < 0$. ∎

Proof of Theorem 17 (i)

Proof: The case when $o_1 = 0$ or $o_1 = y/A$ has been proved in Lemma 29. Hence, we focus on the case when $0 < o_1 < y/A$. The proof follows three steps: (i-1) For given $o_1 \in (0, y/A)$ and $z_1 \in [L_1, \bar{z}_1]$, there exists a unique z_2, denoted by $z_2^*(z_1, o_1)$, maximizing $\pi^{NO}(z_1, z_2, o_1)$ subject to $z_2 \geq L_2$; (i-2) For given $o_1 \in (0, y/A)$, $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ is continuously differentiable and strictly quasi-concave in z_1 for $z_1 \in [L_1, \bar{z}_1]$; (i-3) For given $o_1 \in (0, y/A)$, there exists a unique z_1, denoted by $z_1^*(o_1)$, maximizing $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ subject to $z_1 \geq L_1, z_2^*(z_1, o_1) \geq L_2, p_1(z_1, z_2^*(z_1, o_1)) \geq$

$AC, p_2(z_1, z_2^*(z_1, o_1)) \geq C$. By (i-1) through (i-3), $(z_1^*(o_1), z_2^*(o_1))$ uniquely exists and $z_2^*(o_1) = z_2^*(z_1^*(o_1), o_1)$.

(i-1) By Lemma 27 (ii), $\pi^{NO}(z_1, z_2, o_1)$ is strictly quasi-concave in z_2 for $z_2 \geq L_2$. Therefore, there exists a unique z_2 maximizing $\pi^{NO}(z_1, z_2, o_1)$ subject to $z_2 \geq L_2$. Specifically, if $\Delta_2(z_1, L_2, o_1) < 0$, $z_2^*(z_1, o_1) = L_2$; otherwise, by $\Delta_2(z_1, L_2, o_1) \geq 0$ and $\Delta_2(z_1, \bar{z}_2, o_1) < 0$, $z_2^*(z_1, o_1)$ lies in $[L_2, \bar{z}_2)$ and is the unique solution to $\Delta_2(z_1, z_2, o_1) = 0$. In such a case, denote $z_2^*(z_1, o_1)$ by $z_2^I(z_1, o_1)$, where the superscript I represents interior solution.

(i-2) By (i-1), we have

$$\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1) = \begin{cases} \pi^{NO}(z_1, L_2, o_1) & \text{if } \Delta_2(z_1, L_2, o_1) < 0 \\ \pi^{NO}(z_1, z_2^I(z_1, o_1), o_1) & \text{otherwise.} \end{cases} \tag{C.13}$$

It is easy to derive

$$\frac{\partial \pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)}{\partial z_1} = \begin{cases} \Delta_1(z_1, L_2, o_1) & \text{if } \Delta_2(z_1, L_2, o_1) < 0 \\ \Delta_1(z_1, z_2^I(z_1, o_1), o_1) & \text{if } \Delta_2(z_1, L_2, o_1) > 0. \end{cases}$$

First note that $\pi^{NO}(z_1, z_2, o_1)$, $\Delta_2(z_1, z_2, o_1)$, and $\Delta_1(z_1, z_2, o_1)$ are all continuous in (z_1, z_2), which implies that $z_2^I(z_1, o_1)$ is continuous in z_1 and both $\pi^{NO}(z_1, L_2, o_1)$ and $\pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)$ are continuously differentiable in z_1.

We also show that for given o_1, both $\pi^{NO}(z_1, L_2, o_1)$ and $\pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)$ are strictly quasi-concave in z_1. For $\pi^{NO}(z_1, L_2, o_1)$, it is strictly quasi-concave in z_1 for $z_1 \geq L_1$ by Lemma 27 (i). For $\pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)$, it is easy to see that

$$\begin{aligned}
&\frac{\partial \pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)}{\partial z_1} = \Delta_1(z_1, z_2^I(z_1, o_1), o_1) \\
&\frac{\partial^2 \pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)}{\partial z_1^2} \\
=&\Delta_{11}(z_1, z_2^I(z_1, o_1), o_1) + \Delta_{12}(z_1, z_2^I(z_1, o_1), o_1)\frac{\partial z_2^I(z_1, o_1)}{\partial z_1} \\
=&\left\{ \frac{1}{\Delta_{22}(z_1, z_2, o_1)} \left[\Delta_{11}(z_1, z_2, o_1)\Delta_{22}(z_1, z_2, o_1) - (\Delta_{12}(z_1, z_2, o_1))^2 \right] \right\} \Big|_{z_2 = z_2^I(z_1, o_1)}
\end{aligned}$$

Whenever $\Delta_1(z_1, z_2^I(z_1, o_1), o_1) = 0$ for some $z_1 \geq L_1$ and $z_2^I(z_1, o_1) \geq L_2$, it implies that z_1 and $z_2^I(z_1, o_1)$ satisfy $\Delta_1(z_1, z_2, o_1) = \Delta_2(z_1, z_2, o_1) = 0$. By Lemma 27 (iii), $\frac{\partial \Delta_1(z_1, z_2^I(z_1, o_1), o_1)}{\partial z_1} < 0$. Therefore, $\pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)$ is strictly quasi-concave in z_1 for $z_1 \geq L_1$ and $z_2^I(z_1, o_1) \geq L_2$.

Now, to show the continuous differentiability and strict quasi-concavity of $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ in z_1, first note that the proof is trivial if for all $z_1 \in [L_1, \bar{z}_1]$, $\Delta_2(z_1, L_2, o_1)$ is always positive or is always negative. Now assume that there exists a $z_1 \in [L_1, \bar{z}_1]$, denoted by $\hat{z}_1$, such that $\Delta_2(\hat{z}_1, z_2, o_1) = 0$. To show the continuous differentiability and strict quasi-concavity of $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ for all z_1, by Lemma 30, it suffices to show that first, $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ is continuous at $\hat{z}_1$; second, $\frac{\partial \pi^{NO}(z_1,L_2,o_1)}{\partial z_1} = \frac{\partial \pi^{NO}(z_1,z_2^I(z_1,o_1),o_1)}{\partial z_1}$ at $z_1 = \hat{z}_1$, third, if $\frac{\partial \pi^{NO}(z_1,z_2^*(z_1,o_1),o_1)}{\partial z_1} = 0$ at $z_1 = \hat{z_1}$, $\lim_{z_1 \to \hat{z}_1^-} \frac{\partial^2 \pi^{NO}(z_1,L_2,o_1)}{\partial z_1^2} < 0$ and $\lim_{z_1 \to \hat{z}_1^+} \frac{\partial^2 \pi^{NO}(z_1,z_2^I(z_1,o_1),o_1)}{\partial z_1^2} < 0$.

The continuity of $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ in z_1 at $\hat{z}_1$ is clear since by definition of $\hat{z}_1$, $z_2^I(\hat{z}_1, o_1) = L_2$. In the meanwhile, by envelop theorem,

$$\left.\frac{\partial \pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)}{\partial z_1}\right|_{z_1=\hat{z}_1} = \Delta_1(\hat{z}_1, z_2^I(\hat{z}_1, o_1), o_1) = \left.\frac{\partial \pi^{NO}(z_1, L_2, o_1)}{\partial z_1}\right|_{z_1=\hat{z}_1}.$$

To prove the third point, note that $z_2^I(z_1, o_1)$ is continuous in z_1 and all of the second-order and cross derivative of $\pi^{NO}(z_1, z_2, o_1)$ are continuous in (z_1, z_2). Hence,

$$\lim_{z_1 \to \hat{z}_1^-} \frac{\partial^2 \pi^{NO}(z_1, L_2, o_1)}{\partial z_1^2} = \Delta_{11}(\hat{z}_1, L, o_1)$$

$$\lim_{z_1 \to \hat{z}_1^+} \frac{\partial^2 \pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)}{\partial z_1^2} = \left.\frac{\partial^2 \pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)}{\partial z_1^2}\right|_{z_1=\hat{z}_1}$$

By Lemma 27(i) and (iii), if $\frac{\partial \pi^{NO}(z_1,z_2^*(z_1,o_1),o_1)}{\partial o_1} = 0$ at $z_1 = \hat{z_1}$, $\lim_{z_1 \to \hat{z}_1^-} \frac{\partial^2 \pi^{NO}(z_1,L_2,o_1)}{\partial z_1^2} < 0$ and $\lim_{z_1 \to \hat{z}_1^+} \frac{\partial^2 \pi^{NO}(z_1,z_2^I(z_1,o_1),o_1)}{\partial z_1^2} < 0$. This completes the proof of (i-2).

(i-3) By (i-2), there exists a unique $z_1^*(o_1)$ maximizing $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ subject to $z_1 \geq L_1$. Specifically, if $\left.\frac{\partial \pi^{NO}(z_1,z_2^*(z_1,o_1),o_1)}{\partial z_1}\right|_{z_1=L_1} < 0$, $z_1^*(o_1) = L_1$; otherwise, note that by (i-2),

$$\frac{\partial \pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)}{\partial z_1} = \Delta_1(z_1, z_2^*(z_1, o_1), o_1)$$

Hence, $\left.\frac{\partial \pi^{NO}(z_1,z_2^*(z_1,o_1),o_1)}{\partial z_1}\right|_{z_1=\bar{z}_1} = \Delta_1(\bar{z}_1, z_2^*(\bar{z}_1, o_1), o_1) < 0$ by Lemma 23. Therefore, in such a case, $z_1^*(o_1)$ lies in $[L, \bar{z}_1)$ and is the unique solution to $\frac{\partial \pi^{NO}(z_1,z_2^*(z_1,o_1),o_1)}{\partial z_1} = 0$. In such a case, denote $z_1^*(o_1)$ by $z_1^I(o_1)$.

Next, we show that $z_1^*(o_1)$ satisfies all the constraints. By the definitions of $z_1^*(o_1)$ and $z_2^*(z_1, o_1)$, it clearly satisfies $z_1^*(o_1) \geq L_1$ and $z_2^*(z_1^*(o_1), o_1) \geq L_2$. Also notice that in optimum, only one of the following four scenarios can occur, and we show that $z_1^*(o_1)$ satisfies $p_1(z_1^*(o_1), z_2^*(z_1^*(o_1), o_1)) \geq AC$ and $p_2(z_1^*(o_1), z_2^*(z_1^*(o_1), o_1)) \geq C$ in each of the four scenarios:

- $z_1^*(o_1) = L_1$, $z_2^*(z_1^*(o_1), o_1) = L_2$

  By assumption, $p_1(L_1, L_2) \geq AC$ and $p_2(L_1, L_2) \geq C$.

- $z_1^*(o_1) = L_1$, $z_2^*(z_1^*(o_1), o_1) = z_2^I(L_1, o_1)$

  In such a case, $\Delta_2(L_1, z_2^I(L_1, o_1), o_1) = 0$. By Lemma 26 (ii), $p_2(L_1, z_2^I(L_1, o_1)) \geq C$. Furthermore,

$$\begin{aligned}
& b_1(p_1(L_1, z_2^I(L_1, o_1)) - AC) - c_1(p_2(L_1, z_2^I(L_1, o_1)) - C) \\
&= b_1 p_1(L_1, z_2^I(L_1, o_1)) - c_1 p_2(L_1, z_2^I(L_1, o_1)) - Ab_1 C + c_1 C \\
&= b_1 \frac{b_2(a_1 - L_1) + c_1(a_2 - z_2^I(L_1, o_1))}{b_1 b_2 - c_1 c_2} - c_1 \frac{c_2(a_1 - L_1) + b_1(a_2 - z_2^I(L_1, o_1))}{b_1 b_2 - c_1 c_2} - Ab_1 C + c_1 C \\
&= a_1 - L_1 - Ab_1 C + c_1 C \\
&> 0 \quad (\text{by } z_1(AC, C) > L_1).
\end{aligned}$$

Therefore, $p_1(L_1, z_2^I(L_1, o_1)) - AC > \frac{c_1}{b_1}(p_2(L_1, z_2^I(L_1, o_1)) - C) \geq 0$.

- $z_1^*(o_1) = z_1^I(o_1)$, $z_2^*(z_1^*(o_1), o_1) = L_2$

  In such a case, $\Delta_1(z_1^I(o_1), L_2, o_1) = 0$. By Lemma 26 (i), $p_1(z_1^I(o_1), L_2) \geq AC$. Furthermore,

$$\begin{aligned}
& b_2(p_2(z_1^I(o_1), L_2) - C) - c_2(p_1(z_1^I(o_1), L_2) - AC) \\
&= b_2 p_2(z_1^I(o_1), L_2) - c_2 p_1(z_1^I(o_1), L_2) - b_2 C + Ac_2 C \\
&= b_2 \frac{c_2(a_1 - z_1^I(o_1)) + b_1(a_2 - L_2)}{b_1 b_2 - c_1 c_2} - c_2 \frac{b_2(a_1 - z_1^I(o_1)) + c_1(a_2 - L_2)}{b_1 b_2 - c_1 c_2} - b_2 C + Ac_2 C \\
&= a_2 - L_2 - b_2 C + Ac_2 C \\
&> 0 \quad (\text{by } z_2(AC, C) > L).
\end{aligned}$$

  Therefore, $p_2(z_1^I(o_1), L_2) - C > \frac{c_2}{b_2}(p_1(z_1^I(o_1), L_2) - AC) \geq 0$.

- $z_1^*(o_1) = z_1^I(o_1)$, $z_2^*(z_1^*(o_1), o_1) = z_2^I(z_1^I(o_1), o_1)$

  In such a case, $\Delta_1(z_1^I(o_1), z_2^I(z_1^I(o_1), o_1), o_1) = \Delta_2(z_1^I(o_1), z_2^I(z_1^I(o_1), o_1), o_1) = 0$. By Lemma 26, $p_1(z_1^I(o_1), z_2^I(z_1^I(o_1), o_1)) \geq AC$ and $p_2(z_1^I(o_1), z_2^I(z_1^I(o_1), o_1)) \geq C$.

This completes the proof of (i-3) and (i). ■

Lemma 31 *(Technical Lemma for Theorem 17 (ii))*

(i) If $\Delta_2(L_1, L_2, 0) < 0$*,* $\Delta_2(L_1, L_2, o_1) < 0$ *for all* $o_1 \in [0, y/A]$*. Otherwise, there exists a critical number* $\bar{o}_1 \in [0, y/A)$ *such that* $\Delta_2(L_1, L_2, o_1) \geq 0$ *if* $o_1 \in [0, \bar{o}_1)$*,* $\Delta_2(L_1, L_2, \bar{o}_1) = 0$*, and* $\Delta_2(L_1, L_2, o_1) < 0$ *if* $o_1 \in (\bar{o}_1, y/A]$*.*

Symmetrically, if $\Delta_1(L_1, L_2, y/A) < 0$*,* $\Delta_1(L_1, L_2, o_1) < 0$ *for all* $o_1 \in [0, y/A]$*. Otherwise, there exists a critical number* $\underline{o}_1 \in [0, y/A]$ *such that* $\Delta_1(L_1, L_2, o_1) < 0$ *if* $o_1 \in [0, \underline{o}_1)$*,* $\Delta_1(L_1, L_2, \underline{o}_1) = 0$*, and* $\Delta_1(L_1, L_2, o_1) \geq 0$ *if* $o_1 \in [\underline{o}_1, y/A]$*.*

(ii) - (iv) below assume $\Delta_2(L_1, L_2, 0) \geq 0$*.*

(ii) If $\Delta_1(L_1, z_2^I(L_1, o_1), o_1) > 0$ *for some* $o_1^0 \in [0, \bar{o}_1]$*, then* $\Delta_1(L_1, z_2^I(L_1, o_1), o_1) > 0$ *for all* $o_1 \in [o_1^0, \bar{o}_1]$*.*

(iii) For $o_1 \in [0, \bar{o}_1]$*, there exists a function* $\hat{z}_1(o_1) \in [L_1, \bar{z}_1)$ *such that* $\Delta_2(z_1, L_2, o_1) \geq 0$ *for* $z_1 \in [L_1, \hat{z}_1(o_1))$*,* $\Delta_2(\hat{z}_1(o_1), L_2, o_1) = 0$*, and* $\Delta_2(z_1, L_2, o_1) < 0$ *for* $z_1 \in (\hat{z}_1(o_1), \bar{z}_1]$*. Furthermore,* $\hat{z}_1(o_1)$ *is continuous and non-increasing in* o_1 *for* $o_1 \in [0, \bar{o}_1]$ *and* $\hat{z}_1(\bar{o}_1) = L_1$*.*

(iv) If $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1) \leq 0$ *for some* $o_1^0 \in [0, \bar{o}_1]$*, then* $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1) \leq 0$ *for all* $o_1 \in [0, o_1^0]$*.*

(v) If $z_1^*(o_1) = L_1$ *and* $z_2^*(o_1) = L_2$ *for* $o_1 \in [\alpha, \beta] \subseteq [0, y/A]$*, then* $\pi^{NO}(L_1, L_2, o_1)$ *is continuously differentiable and strictly quasi-concave in* o_1 *for* $o_1 \in [\alpha, \beta]$*.*

(vi) If $z_1^*(o_1) = L_1$ *for* $o_1 \in [\alpha, \beta] \subseteq [0, y/A]$*, then* $\pi^{NO}(L_1, z_2^*(o_1), o_1)$ *is continuously differentiable and strictly quasi-concave in* o_1 *for* $o_1 \in [\alpha, \beta]$*.*

(vii) If $z_2^*(o_1) = L_2$ *for* $o_1 \in [\alpha, \beta] \subseteq [0, y/A]$*, then* $\pi^{NO}(z_1^*(o_1), L_2, o_1)$ *is continuously differentiable and strictly quasi-concave in* o_1 *for* $o_1 \in [\alpha, \beta]$*.*

(viii) If $z_1^*(o_1) = z_1^I(o_1)$ *and* $z_2^*(o_1) = z_2^I(z_1^I(o_1), o_1)$ *for* $o_1 \in [\alpha, \beta] \subseteq [0, y/A]$*, then* $\pi^{NO}(z_1^I(o_1), z_2^I(z_1^I(o_1), o_1), o_1)$ *is continuously differentiable and strictly quasi-concave in* o_1 *for* $o_1 \in [\alpha, \beta]$*.*

Proof: (i) To prove the results related to $\Delta_2(L_1, L_2, o_1)$, noting that by Lemma 28 (i), $\Delta_2(L_1, L_2, y/A) < 0$, it suffices to show that $\Delta_2(L_1, L_2, o_1)$ is quasi-convex in o_1, which is to show that if the derivative of $\Delta_2(L_1, L_2, o_1)$ with respect to o_1 is nonnegative for some o_1^0, then it is nonnegative for all $o_1 \geq o_1^0$. Recall the derivative of $\Delta_2(L_1, L_2, o_1)$ with respect to o_1:

$$\begin{aligned}
&\frac{1}{do_1} d\left[\Delta_2(L_1, L_2, o_1)\right] \\
&= A\frac{b_1}{b_1b_2 - c_1c_2}\bar{G}_2(y - Ao_1 - L_2) - A(p_2(L_1, L_2) - C)g_2(y - Ao_1 - L_2) - \frac{c_1}{b_1b_2 - c_1c_2}\bar{G}_1(o_1 - L_1) \\
&= \bar{G}_2(y - Ao_1 - L_2)\left[A\frac{b_1}{b_1b_2 - c_1c_2} - A(p_2(L_1, L_2) - C)\frac{g_2(y - Ao_1 - L_2)}{\bar{G}_2(y - Ao_1 - L_2)}\right. \\
&\qquad\qquad \left. - \frac{c_1}{b_1b_2 - c_1c_2}\frac{\bar{G}_1(o_1 - L_1)}{\bar{G}_2(y - Ao_1 - L_2)}\right]
\end{aligned}$$

Since $G(x)$ satisfies IFR property, $\frac{g_2(y - Ao_1 - L_2)}{\bar{G}_2(y - Ao_1 - L_2)}$ is non-increasing in o_1. In the meantime, since $\bar{G}(x)$ is non-increasing in x, $\frac{\bar{G}_1(o_1 - L_1)}{\bar{G}_2(y - Ao_1 - L_2)}$ is non-increasing in o_1. Therefore, $\frac{1}{do_1} d\left[\Delta_2(L_1, L_2, o_1)\right]$ is the product of a nonnegative term and a term non-decreasing in o_1. Therefore, as o_1 increases, it can only cross zero from below and once it crosses zero, it stays nonnegative. In other words, as o_1 increases, $\Delta_2(L_1, L_2, o_1)$ is first non-increasing and then non-decreasing in o_1.

By symmetry, the results related to $\Delta_1(L_1, L_2, y/A)$ can be proved following the same logic.

(ii) We follow three steps to prove the result: (ii-1) show that for given $z_2 \geq L_2$, if $\Delta_1(L_1, z_2, o_1) > 0$ for some $o_1' \in [0, y/A]$, then $\Delta_1(L_1, z_2, o_1) > 0$ for all $o_1 \in [o_1', y/A]$; (ii-2) prove that $z_2^I(L_1, o_1)$ is greater than or equal to L and is non-increasing in o_1 for $o_1 \in [0, \bar{o}_1)$. (ii-3) By (ii-1), (ii-2), and the facts $\Delta_1(L_1, z_2^I(L_1, o_1^0), o_1^0) > 0$ and that $\Delta_1(z_1, z_2, o_1)$ is non-increasing in z_2 for $z_2 \geq L_2$, for all $o_1 \in [o_1^0, \bar{o}_1)$,

$$\Delta_1(L_1, z_2^I(L_1, o_1), o_1) \geq \Delta_1(L_1, z_2 = z_2^I(L_1, o_1^0), o_1) > 0.$$

(ii-1) By Lemma 25 (i), both $\Delta_1(L_1, z_2, o_1)$ and $\Delta_{13}(L_1, z_2, o_1)$ are positive for all $q \in [o_1', y/A]$.

(ii-2) First note that by definition of $\bar{o}_1$, for all $o_1 \in [0, \bar{o}_1)$, $\Delta_2(L_1, L_2, o_1) \geq 0$ and hence by part (i-1) in the proof of Theorem 17 , $z_2^*(L_1, o_1) = z_2^I(L_1, o_1) \geq L_2$. The monotonicity of $z_2^I(L_1, o_1)$ in o_1 is a direct corollary of Lemma 25 (ii).

(iii) The existence of $\hat{z}_1(o_1)$ is directly from Lemma 23, the definition of $\bar{o}_1$, and the fact that for given z_2, $\Delta_2(z_1, z_2, o_1)$ is non-increasing in z_1. The continuity of $\hat{z}_1(o_1)$ in o_1 is implied by the continuity of $\Delta_2(z_1, z_2, o_1)$ in (z_1, o_1).

Next, we prove the monotonicity of $\hat{z}_1(o_1)$ in o_1 by contradiction. Suppose $0 \leq q < q' < \bar{o}_1$ and $\hat{z}_1(q) < \hat{z}_1(q')$. Then for given $z_1 \in (\hat{z}_1(q), \hat{z}_1(q'))$, $\Delta_2(z_1, L_2, q) < 0$ and $\Delta_2(z_1, L_2, q') \geq 0$. Since $\Delta_2(z_1, L_2, o_1)$ is continuous and differentiable in o_1, there must exist some $q'' \in (q, q'] \subset (0, y/A)$ such that $\Delta_2(z_1, L_2, o_1'') = 0$ and $\Delta_{23}(z_1, L_2, q'') \geq 0$. However, this contradicts with Lemma 25 (ii).

Furthermore, when $o_1 = \bar{o}_1$, by definitions of $\bar{o}_1$ and $\hat{z}_1(o_1)$, $\hat{z}_1(\bar{o}_1) = L_1$. This completes the proof for (iii).

(iv) Prove by contradiction. Suppose $\Delta_1(\hat{z}_1(o_1'), z_2^I(\hat{z}_1(o_1'), o_1'), o_1') > 0$ for some $o_1' \in [0, o_1^0]$. By Lemma 25 (i), both $\Delta_1(\hat{z}_1(o_1'), z_2^I(\hat{z}_1(o_1'), o_1'), o_1) > 0$ and $\Delta_{13}(\hat{z}_1(o_1'), z_2 = z_2^I(\hat{z}_1(o_1'), o_1'), o_1) > 0$ for all $o_1 \in [o_1', o_1^0]$. Similarly to (ii-2), we can show that $z_2^I(\hat{z}_1(o_1'), o_1)$ is greater than or equal to L and is non-increasing in o_1 for $o_1 \in [o_1', o_1^0]$. Therefore, for all $o_1 \in [o_1', o_1^0]$,

$$\Delta_1(\hat{z}_1(o_1'), z_2^I(\hat{z}_1(o_1'), o_1), o_1) \geq \Delta_1(\hat{z}_1(o_1'), z_2^I(\hat{z}_1(o_1'), o_1'), o_1) > 0 \tag{C.14}$$

Furthermore, by part (i-2) in the proof of Theorem 17, $\pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)$ is strictly quasi-concave in z_1. Therefore, by equation (C.14), for all $z_1 \leq \hat{z}_1(o_1')$, $\Delta_1(z_1, z_2^I(z_1, o_1), o_1) > 0$. Now, by (iii), for all $o_1 \in [o_1', o_1^0]$, $\hat{z}_1(o_1) \leq \hat{z}_1(o_1')$. Therefore, for all $o_1 \in [o_1', o_1^0]$,

$$\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1) > 0$$

This is a direct contradiction to $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1) \leq 0$ at $o_1 = o_1^0$. The proof of (iv) is thus complete.

(v) Clearly $\pi^{NO}(L_1, L_2, o_1)$ is continuously differentiable. By Lemma 28 (vi), $\pi^{NO}(L_1, L_2, o_1)$ is strictly quasi-concave.

(vi) By Equation (C.13),

$$\pi^{NO}(L_1, z_2^*(o_1), o_1) = \begin{cases} \pi^{NO}(L_1, L_2, o_1) & \text{if } \Delta_2(L_1, L_2, o_1) < 0 \\ \pi^{NO}(L_1, z_2^I(L_1, o_1), o_1) & \text{otherwise.} \end{cases}$$

By (i), consider the following cases:

- $\alpha \geq \bar{o}_1$

  In such a case, $\pi^{NO}(L_1, z_2^*(o_1), o_1) = \pi^{NO}(L_1, L_2, o_1)$ for all $o_1 \in [\alpha, \beta]$. By (v), $\pi^{NO}(L_1, z_2^*(o_1), o_1)$ is continuously differentiable and strictly quasi-concave.

- $\beta \leq \bar{o}_1$

  In such a case, $\pi^{NO}(L_1, z_2^*(o_1), o_1) = \pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)$ for all $o_1 \in [\alpha, \beta]$. Since both $\pi^{NO}(L_1, z_2, o_1)$ and $\Delta_2(z_1, z_2, o_1)$ are continuous in (z_2, o_1), both $z_2^I(L_1, o_1)$ and $\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)$ are continuous in o_1. In the meantime,

  $$\frac{d\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1} = \Delta_3(L_1, z_2^I(L_1, o_1), o_1)$$

  Since $\Delta_3(L_1, z_2, o_1)$ is continuous in (z_2, o_1) and $z_2^I(L_1, o_1)$ is continuous in o_1, $\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)$ is continuously differentiable in o_1.

  To show $\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)$ is strictly quasi-concave in o_1, it suffices to show that when $\frac{d\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1} = 0$, $\frac{d^2\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1^2} < 0$. Note that

  $$\begin{aligned} &\frac{d^2\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1^2} \\ &= \Delta_{33}(L_1, z_2^I(L_1, o_1), o_1) + \Delta_{23}(L_1, z_2^I(L_1, o_1), o_1) \cdot \frac{dz_2^I(L, o_1)}{do_1} \\ &= \left\{ \frac{1}{\Delta_{22}(L_1, z_2, o_1)} \left[\Delta_{33}(L_1, z_2, o_1)\Delta_{22}(L_1, z_2, o_1) - (\Delta_{23}(L_1, z_2, o_1))^2 \right] \right\} \Bigg|_{z_2 = z_2^I(L_1, o_1)} \end{aligned}$$

  When $\frac{d\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1} = 0$, o_1 and $z_2^I(L_1, o_1)$ satisfy $\Delta_2(L_1, z_2, o_1) = \Delta_3(L_1, z_2, o_1) = 0$. By Lemma 28(i), such o_1 is less than y/A. Therefore, by Lemma 27 (ii) and Lemma 28 (ii), $\frac{d^2\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1^2} < 0$.

- $\alpha < \bar{o}_1 < \beta$

  In such a case, $\bar{o}_1 \in (0, y/A)$ and

  $$\pi^{NO}(L_1, z_2^*(o_1), o_1) = \begin{cases} \pi^{NO}(L_1, L_2, o_1) & \text{if } \bar{o}_1 < o_1 \leq \beta \\ \pi^{NO}(L_1, z_2^I(L_1, o_1), o_1) & \text{if } \alpha \leq o_1 \leq \bar{o}_1. \end{cases} \tag{C.15}$$

  To show that $\pi^{NO}(L_1, z_2^*(o_1), o_1)$ is continuously differentiable and strictly quasi-concave for $o_1 \in [\alpha, \beta]$, note that by Lemma 30 and the two cases proved earlier, it suffices to show that first, $\pi^{NO}(L_1, z_2^*(o_1), o_1)$ is continuous at $o_1 = \bar{o}_1$; second, $\frac{d\pi^{NO}(L_1, L_2, o_1)}{do_1} = \frac{d\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1}$ at

$o_1 = \bar{o}_1$; and third, when $\frac{d\pi^{NO}(L_1, z_2^*(L_1,o_1),o_1)}{do_1} = 0$ at $o_1 = \bar{o}_1$, $\lim_{o_1 \to \bar{o}_1^-} \frac{d^2\pi^{NO}(L_1, z_2^*(L_1,o_1),o_1)}{do_1^2} < 0$ and $\lim_{o_1 \to \bar{o}_1^+} \frac{d^2\pi^{NO}(L_1, z_2^*(L_1,o_1),o_1)}{do_1^2} < 0$.

First note that by definition of $\bar{o}_1$, $z_2^*(\bar{o}_1) = L$ and hence $\pi^{NO}(L_1, z_2^*(o_1), o_1)$ is continuous at $o_1 = \bar{o}_1$. In the meanwhile,

$$\left.\frac{d\pi^{NO}(L_1, L_2, o_1)}{do_1}\right|_{o_1=\bar{o}_1} = \Delta_3(L_1, L_2, \bar{o}_1) = \left.\frac{d\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1}\right|_{o_1=\bar{o}_1}$$

To show the third point, by equation (C.15) and the fact that $z_2^I(L_1, o_1)$ is continuous in o_1 and all the second-order and cross derivatives of $\pi^{NO}(L_1, z_2, o_1)$ are continuous in (z_2, o_1),

$$\begin{aligned}
&\lim_{o_1 \to \bar{o}_1^-} \frac{d^2\pi^{NO}(L_1, z_2^*(L_1, o_1), o_1)}{do_1^2} = \lim_{o_1 \to \bar{o}_1^-} \frac{d^2\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1^2} \\
&= \lim_{o_1 \to \bar{o}_1^-} \left.\left\{ \frac{1}{\Delta_{22}(L_1, z_2, o_1)} \left[\Delta_{33}(L_1, z_2, o_1)\Delta_{22}(L_1, z_2, o_1) - (\Delta_{23}(L_1, z_2, o_1))^2 \right] \right\}\right|_{z_2 = z_2^I(L_1, o_1)} \\
&= \left.\left\{ \frac{1}{\Delta_{22}(L_1, z_2, o_1)} \left[\Delta_{33}(L_1, z_2, o_1)\Delta_{22}(L_1, z_2, o_1) - (\Delta_{23}(L_1, z_2, o_1))^2 \right] \right\}\right|_{z_2 = z_2^I(L_1, \bar{o}_1), o_1 = \bar{o}_1}
\end{aligned}$$

When $\frac{d\pi^{NO}(L_1, z_2^*(L_1,o_1),o_1)}{do_1} = 0$ at $o_1 = \bar{o}_1$, $\bar{o}_1$ and $z_2^I(L_1, \bar{o}_1)$ satisfy $\Delta_2(L_1, z_2, o_1) = \Delta_3(L_1, z_2, o_1) = 0$. Also recall that $\bar{o}_1 \in (0, y/A)$. By Lemma 27 (ii) and Lemma 28 (ii),

$$\lim_{o_1 \to \bar{o}_1^-} \frac{d^2\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1^2} < 0.$$

In the meanwhile,

$$\lim_{o_1 \to \bar{o}_1^+} \frac{d^2\pi^{NO}(L_1, z_2^*(L_1, o_1), o_1)}{do_1^2} = \lim_{o_1 \to \bar{o}_1^+} \frac{d^2\pi^{NO}(L_1, L_2, o_1)}{do_1^2} = \Delta_{33}(L_1, L_2, \bar{o}_1)$$

When $\frac{d\pi^{NO}(L_1, z_2^*(L_1,o_1),o_1)}{do_1} = 0$ at $o_1 = \bar{o}_1$, $\bar{o}_1$ satisfies $\frac{d\pi^{NO}(L_1,L_2,o_1)}{do_1} = 0$. By (v), $\Delta_{33}(L_1, L_2, \bar{o}_1) < 0$. This completes the proof of the case $\alpha < \bar{o}_1 < \beta$, as well as the proof of (vi).

(vii) By the second half of (i), (vii) can be proved following the same logic as the proof of (vi).

(viii) In such a case, $z_1^*(o_1)$ and $z_2^*(o_1)$ satisfy $\Delta_1(z_1, z_2, o_1) = \Delta_2(z_1, z_2, o_1) = 0$. Since both $\Delta_1(z_1, z_2, o_1)$ and $\Delta_2(z_1, z_2, o_1)$ are continuous in (z_1, z_2, o_1), $z_1^*(q)$ and $z_2^*(q)$ are both continuous in o_1. In the meantime,

$$\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} = \Delta_3(z_1^*(o_1), z_2^*(o_1), o_1)$$

Since $\Delta_3(z_1, z_2, o_1)$ is continuous in (z_1, z_2, o_1), $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuously differentiable.

To show the strict quasi-concavity of $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$, it suffices to show that when $\frac{d\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1} = 0$, $\frac{d^2\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1^2} < 0$. Note that

$$\begin{aligned}\frac{d^2\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1^2} &= \Delta_{33}(z_1^*(o_1), z_2^*(o_1), o_1) + \Delta_{13}(z_1^*(o_1), z_2^*(o_1), o_1) \cdot \frac{dz_1^*(o_1)}{do_1} \\ &\quad + \Delta_{23}(z_1^*(o_1), z_2^*(o_1), o_1) \cdot \frac{dz_2^*(o_1)}{do_1} \end{aligned} \tag{C.16}$$

Recall that $z_1^*(o_1)$ and $z_2^*(o_1)$ satisfy $\Delta_1(z_1^*(o_1), z_2^*(o_1), o_1) = \Delta_2(z_1^*(o_1), z_2^*(o_1), o_1) = 0$. By the rule of implicit differentiation,

$$\begin{cases} \frac{dz_1^*(o_1)}{do_1}\Delta_{11} + \Delta_{13} + \Delta_{12}\frac{dz_2^*(o_1)}{do_1} = 0 \\ \frac{dz_2^*(o_1)}{do_1}\Delta_{22} + \Delta_{23} + \Delta_{12}\frac{dz_1^*(o_1)}{do_1} = 0 \end{cases}$$

Solving this system of equations, we get

$$\left(\frac{dz_1^*(o_1)}{do_1}, \frac{dz_2^*(o_1)}{do_1}\right) = \left(\frac{\Delta_{23}\Delta_{12} - \Delta_{13}\Delta_{22}}{\Delta_{11}\Delta_{22} - (\Delta_{12})^2}, \quad \frac{\Delta_{13}\Delta_{12} - \Delta_{23}\Delta_{11}}{\Delta_{11}\Delta_{22} - (\Delta_{12})^2}\right)\Bigg|_{z_1=z_1^*(o_1), z_2=z_2^*(o_1)}.$$

Apply this into equation (C.16), we have

$$\frac{d^2\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1^2} = \left[\frac{|\mathcal{H}|}{\Delta_{11}\Delta_{22} - (\Delta_{12})^2}\right]\Bigg|_{z_1=z_1^*(o_1), z_2=z_2^*(o_1)} \tag{C.17}$$

where $\mathcal{H}$ is defined in Lemma 27.

When $\frac{d\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1} = 0$ for some $o_1 \in [\alpha, \beta]$, $z_1^*(o_1)$, $z_2^*(o_1)$, and o_1 satisfy $\Delta_1(z_1, z_2, o_1) = \Delta_2(z_1, z_2, o_1) = \Delta_3(z_1, z_2, o_1) = 0$. By Lemma 28 (i), such o_1 cannot equal to 0 or y/A. Therefore, by Lemma 27 (iii) and (iv), $\frac{d^2\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1^2} < 0$. ■

Proof of Theorem 17 (ii) and (iii)

Proof: To prove (ii), we first specify the expression of $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ in four cases and then prove that in each case, $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuously differentiable and strictly quasi-concave.

To this end, first from (i), for given $o_1 \in [0, y/A]$, we summarize $(z_1^*(o_1), z_2^*(o_1))$ as following:

- If $\Delta_2(L_1, L_2, o_1) < 0$

  In such a case, $\Delta_2(z_1, z_2, o_1) < 0$ for all $z_1 \geq L_1, z_2 \geq L_2$. Hence, $z_2^*(z_1, o_1) = L_2$ for all $z_1 \in [L_1, \bar{z}_1]$. Therefore, if $\Delta_1(L_1, L_2, o_1) < 0$, $z_1^*(o_1) = L_1$; otherwise, $z_1^*(o_1) = z_1^I(o_1)$.

- If $\Delta_2(L_1, L_2, o_1) \geq 0$

  In such a case, it is implied that $\Delta_2(L_1, L_2, 0) \geq 0$ by Lemma 25 (i). By Lemma 31 (i) and (iii), $0 \leq o_1 \leq \bar{o}_1$ and there exists a function $\hat{z}_1(o_1) \in [L_1, \bar{z}_1]$ such that $\Delta_2(z_1, L_2, o_1) \geq 0$ for $z_1 \in [L_1, \hat{z}_1(o_1)]$ and $\Delta_2(z_1, L_2, o_1) < 0$ for $z_1 \in (\hat{z}_1(o_1), \bar{z}_1]$. Clearly, if $z_1 \in [L_1, \hat{z}_1(o_1)]$, $z_2^*(z_1, o_1) = z_2^I(z_1, o_1)$, and if $z_1 \in (\hat{z}_1(o_1), \bar{z}_1]$, $z_2^*(z_1, o_1) = L_2$.

– if $\Delta_1(L_1, z_2^I(L_1, o_1), o_1) \leq 0$

In such a case, $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ strictly decreases in z_1 for $z_1 > L_1$. Hence, $z_1^*(o_1) = L_1$. Furthermore, since $\Delta_2(L_1, L_2, o_1) \geq 0$, $z_2^*(o_1) = z_2^I(L_1, o_1)$.

– if $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1) > 0$

In such a case, $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ strictly increases in z_1 for $z_1 \leq \hat{z}_1(o_1)$. Hence, $z_2^*(o_1) = L_2$; if $\Delta_1(L_1, L_2, o_1) < 0$, $z_1^*(o_1) = L_1$, otherwise, $z_1^*(o_1) = z_1^I(o_1)$.

– if $\Delta_1(L_1, z_2^I(L_1, o_1), o_1) > 0$ and $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1) \leq 0$

In such a case, $\pi^{NO}(z_1, z_2^*(z_1, o_1), o_1)$ has a positive derivative in z_1 at $z_1 = L_1$ and strictly decreases in z_1 for $z_1 > \hat{z}_1(o_1)$. Hence, $z_1^*(o_1) = z_1^I(o_1)$ and $z_2^*(o_1) = z_2^I(z_1^I(o_1), o_1)$.

According to the $(z_1^*(o_1), z_2^*(o_1))$ summarized above and Lemma 31, we specify the expression of $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ for the following four cases:

(ii-1) $\Delta_2(L_1, L_2, 0) < 0$

For all $o_1 \in [0, y/A]$, $z_2^*(o_1) = L_2$ and $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1) = \pi^{NO}(z_1^*(o_1), L_2, o_1)$. By Lemma 31 (vii), $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuously differentiable and strictly quasi-concave in o_1 for $o_1 \in [0, y/A]$.

(ii-2) $\Delta_2(L_1, L_2, 0) \geq 0$ and $\Delta_1(L_1, z_2^I(L_1, \bar{o}_1), \bar{o}_1) \leq 0$

By Lemma 31 (ii), $\Delta_1(L_1, z_2^I(L_1, o_1), o_1) \leq 0$ for all $o_1 \in [0, \bar{o}_1]$. Then we have

$$\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1) = \begin{cases} \pi^{NO}(L_1, z_2^*(o_1), o_1) & \text{if } o_1 \in [0, \bar{o}_1] \\ \pi^{NO}(z_1^*(o_1), L_2, o_1) & \text{if } o_1 \in (\bar{o}_1, y/A] \end{cases}$$

By Lemma 30 and Lemma 31 (vi) and (vii), to show $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuously differentiable and strictly quasi-concave in o_1 on $[0, y/A]$, it suffices to show that $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuous at $\bar{o}_1$, $\left.\frac{d\pi^{NO}(L_1, z_2^*(o_1), o_1)}{do_1}\right|_{o_1=\bar{o}_1} = \left.\frac{d\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1}\right|_{o_1=\bar{o}_1}$, and if $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} = 0$ at $o_1 = \bar{o}_1$, the left-hand and right-hand derivatives of $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1}$ at $o_1 = \bar{o}_1$ are both negative.

By definition of $\bar{o}_1$ and $\Delta_1(L_1, z_2^I(L_1, \bar{o}_1), \bar{o}_1) \leq 0$, $\lim_{o_1 \to \bar{o}_1^+} z_1^*(o_1) = L_1$ and $\lim_{o_1 \to \bar{o}_1^-} z_2^*(o_1) = L_2$. This implies that $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuous at $\bar{o}_1$. To show $\left.\frac{d\pi^{NO}(L_1, z_2^*(o_1), o_1)}{do_1}\right|_{o_1=\bar{o}_1} = \left.\frac{d\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1}\right|_{o_1=\bar{o}_1}$, first note that by the definition of $\bar{o}_1$, $z_2^*(o_1) = z_2^I(L_1, o_1)$ for $o_1 \in [0, \bar{o}_1]$. Hence, $\left.\frac{d\pi^{NO}(L_1, z_2^*(o_1), o_1)}{do_1}\right|_{o_1=\bar{o}_1} = \Delta_3(L_1, z_2^*(\bar{o}_1), \bar{o}_1) = \Delta_3(L_1, L_2, \bar{o}_1)$. To find $\left.\frac{d\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1}\right|_{o_1=\bar{o}_1}$, note that $\Delta_1(L_1, L_2, \bar{o_1}) = \Delta_1(L_1, z_2^I(L_1, \bar{o}_1), \bar{o}_1) \leq 0$. Consider two cases: $\Delta_1(L_1, L_2, \bar{o_1}) = 0$ and $\Delta_1(L_1, L_2, \bar{o_1}) < 0$. If $\Delta_1(L_1, L_2, \bar{o_1}) = 0$, by Lemma 28 (i) and Lemma 25 (i), $\Delta_1(L_1, L_2, o_1) \geq$

0 and $z_1^*(o_1) = z_1^I(o_1)$ for all $o_1 \geq \bar{o}_1$. Hence,

$$\left.\frac{d\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1}\right|_{o_1=\bar{o}_1} = \left.\frac{d\pi^{NO}(z_1^I(o_1), L_2, o_1)}{do_1}\right|_{o_1=\bar{o}_1} = \Delta_3(z_1^I(\bar{o}_1), L, \bar{o}_1) = \Delta_3(L_1, L_2, \bar{o}_1).$$

Otherwise, by $\Delta_1(L_1, L_2, \bar{o_1}) < 0$ and the continuity of $\Delta_1(L_1, L_2, o_1)$ in o_1, $\Delta_1(L_1, L_2, o_1) < 0$ for o_1 in a right-hand neighborhood of $\bar{o}_1$. Hence,

$$\left.\frac{d\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1}\right|_{o_1=\bar{o}_1} = \left.\frac{d\pi^{NO}(L_1, L_2, o_1)}{do_1}\right|_{o_1=\bar{o}_1} = \Delta_3(L_1, L_2, \bar{o}_1).$$

That is, in both cases, $\left.\frac{d\pi^{NO}(L_1, z_2^*(o_1), o_1)}{do_1}\right|_{o_1=\bar{o}_1} = \left.\frac{d\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1}\right|_{o_1=\bar{o}_1}$.

To show that if $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} = 0$ at $o_1 = \bar{o}_1$, the left-hand and right-hand derivatives of $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1}$ at $o_1 = \bar{o}_1$ are both negative, it suffices to show that in such a case, $\lim_{o_1 \to \bar{o}_1^-} \frac{d^2\pi^{NO}(L_1, z_2^*(o_1), o_1)}{do_1^2} < 0$ and $\lim_{o_1 \to \bar{o}_1^+} \frac{d^2\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1^2} < 0$. By definition of $\bar{o}_1$ and the discussion above,

$$\begin{aligned}
&\lim_{o_1 \to \bar{o}_1^-} \frac{d^2\pi^{NO}(L_1, z_2^*(o_1), o_1)}{do_1^2} = \left.\frac{d^2\pi^{NO}(L_1, z_2^I(L_1, o_1), o_1)}{do_1^2}\right|_{o_1=\bar{o}_1} \\
&= \Delta_{33}(L_1, z_2^I(L_1, o_1), o_1) + \Delta_{23}(L_1, z_2^I(L_1, o_1), o_1) \cdot \frac{dz_2^I(L_1, o_1)}{do_1} \\
&= \left.\left\{ \frac{1}{\Delta_{22}(L_1, z_2, o_1)} \left[\Delta_{33}(L_1, z_2, o_1)\Delta_{22}(L_1, z_2, o_1) - (\Delta_{23}(L_1, z_2, o_1))^2 \right] \right\}\right|_{z_2 = z_2^I(L_1, o_1)}
\end{aligned}$$

If $\Delta_1(L_1, L_2, \bar{o_1}) = 0$, by Lemma 28(i), it implies $\bar{o}_1 > 0$ and

$$\begin{aligned}
&\lim_{o_1 \to \bar{o}_1^+} \frac{d^2\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1^2} = \left.\frac{d^2\pi^{NO}(z_1^I(o_1), L_2, o_1)}{do_1^2}\right|_{o_1=\bar{o}_1} \\
&= \Delta_{33}(z_1^I(o_1), L_2, o_1) + \Delta_{13}(z_1^I(o_1), L_2, o_1) \cdot \frac{dz_1^I(o_1)}{do_1} \\
&= \left.\left\{ \frac{1}{\Delta_{11}(z_1, L_2, o_1)} \left[\Delta_{33}(z_1, L_2, o_1)\Delta_{11}(z_1, L_2, o_1) - (\Delta_{13}(z_1, L_2, o_1))^2 \right] \right\}\right|_{z_1 = z_1^I(o_1)}
\end{aligned}$$

If $\Delta_1(L_1, L_2, \bar{o_1}) < 0$

$$\lim_{o_1 \to \bar{o}_1^+} \frac{d^2\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1^2} = \left.\frac{d^2\pi^{NO}(L_1, L_2, o_1)}{do_1^2}\right|_{o_1=\bar{o}_1} = \Delta_{33}(L_1, L_2, \bar{o}_1)$$

By Lemma 28 (ii) through (iv), if $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} = 0$ at $o_1 = \bar{o}_1$, $\lim_{o_1 \to \bar{o}_1^-} \frac{d^2\pi^{NO}(L_1, z_2^*(o_1), o_1)}{do_1^2} < 0$ and $\lim_{o_1 \to \bar{o}_1^+} \frac{d^2\pi^{NO}(z_1^*(o_1), L_2, o_1)}{do_1^2} < 0$. This completes the proof of (ii-2).

(ii-3) $\Delta_2(L_1, L_2, 0) \geq 0$, $\Delta_1(L_1, z_2^I(L_1, \bar{o}_1), \bar{o}_1) > 0$, and $\Delta_1(\hat{z}_1(\bar{o}_1), z_2^I(\hat{z}_1(\bar{o}_1), \bar{o}_1), \bar{o}_1) \leq 0$.

By Lemma 28(i), $\Delta_1(L_1, z_2^I(L_1, 0), 0) < 0$. Since $\Delta_1(L_1, z_2^I(L_1, \bar{o}_1), \bar{o}_1) > 0$, from the continuity of $\Delta_1(L_1, z_2^I(L_1, o_1), o_1)$ in o_1[2] and Lemma 31(ii), there exists a $\hat{o}_1 \in (0, \bar{o}_1)$ such that

[2]Note that the continuity of $\Delta_1(L_1, z_2^I(L_1, o_1), o_1)$ is implied by the continuity of $z_2^I(L_1, o_1)$ in o_1 and $\Delta_1(L_1, z_2, o_1)$ in (z_2, o_1).

$\Delta_1(L_1, z_2^I(L_1, o_1), o_1) \leq 0$ if $o_1 \in [0, \hat{o}_1)$, $\Delta_1(L_1, z_2^I(L_1, \hat{o}_1), \hat{o}_1) = 0$, and $\Delta_1(L_1, z_2^I(L_1, o_1), o_1) > 0$ if $o_1 \in (\hat{o}_1, \bar{o}_1]$. Furthermore, since $\Delta_1(\hat{z}_1(\bar{o}_1), z_2^I(\hat{z}_1(\bar{o}_1), \bar{o}_1), \bar{o}_1) \leq 0$, by Lemma 31 (iv), $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1) \leq 0$ for all $o_1 \in (\hat{o}_1, \bar{o}_1]$.

Therefore,

$$\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1) = \begin{cases} \pi^{NO}(L_1, z_2^*(o_1), o_1) & \text{if } o_1 \in [0, \hat{o}_1] \\ \pi^{NO}(z_1^I(o_1), z_2^I(z_1^I(o_1), o_1), o_1) & \text{if } o_1 \in (\hat{o}_1, \bar{o}_1] \\ \pi^{NO}(z_1^*(o_1), L_2, o_1) & \text{if } o_1 \in (\bar{o}_1, y/A]. \end{cases}$$

By Lemma 30 and Lemma 31 (vi) through (viii), to show $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuously differentiable and strictly quasi-concave in o_1 on $[0, y/A]$, it suffices to show that both $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ and its derivative in o_1 are continuous at both $\hat{o}_1$ and $\bar{o}_1$, and that if $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} = 0$ at $o_1 = \hat{o}_1$ (or at $o_1 = \bar{o}_1$), the left-hand and right-hand derivatives of $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1}$ at $o_1 = \hat{o}_1$ (or at $o_1 = \bar{o}_1$) are both negative. The proof of these results follow the same idea as the proof in (ii-2) and here we only show the continuity at both $\hat{o}_1$ and $\bar{o}_1$:

By definition of $\hat{o}_1$ and the facts that $\hat{o}_1 \leq \bar{o}_1$ and both $z_1^I(o_1)$ and $z_2^I(z_1^I(o_1), o_1)$ are continuous in o_1,[3] $\lim_{o_1 \to \hat{o}_1^+} z_1^I(o_1) = z_1^I(\hat{o}_1) = L_1$, $\lim_{o_1 \to \hat{o}_1^+} z_2^I(z_1^I(o_1), z_2^I(z_1^I(o_1), o_1), o_1) = z_2^I(L_1, \hat{o}_1)$, and $\lim_{o_1 \to \hat{o}_1^-} z_2^*(o_1) = z_2^I(L_1, \hat{o}_1)$. By the continuity of $\pi^{NO}(z_1, z_2, o_1)$ in (z_1, z_2, o_1), $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuous at $\hat{o}_1$.

By definition of $\bar{o}_1$ and $\Delta_1(L_1, z_2^I(L_1, \bar{o}_1), \bar{o}_1) = \Delta_1(L_1, L_2, \bar{o}_1) > 0$, $\lim_{o_1 \to \bar{o}_1^+} z_1^*(o_1) = z_1^I(\bar{o}_1)$, $\lim_{o_1 \to \bar{o}_1^-} z_1^I(o_1) = z_1^I(\bar{o}_1)$, and $\lim_{o_1 \to \bar{o}_1^-} z_2^I(z_1^I(o_1), o_1) = z_2^I(z_1^I(\bar{o}_1), \bar{o}_1) = z_2^I(L_1, \bar{o}_1) = L$. By the continuity of $\pi^{NO}(z_1, z_2, o_1)$ in (z_1, z_2, o_1), $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuous at $\bar{o}_1$.

(ii-4) $\Delta_2(L_1, L_2, 0) \geq 0$, $\Delta_1(L_1, z_2^I(L_1, \bar{o}_1), \bar{o}_1) > 0$, and $\Delta_1(\hat{z}_1(\bar{o}_1), z_2^I(\hat{z}_1(\bar{o}_1), \bar{o}_1), \bar{o}_1) > 0$.

Similarly to (ii-3), we can show the existence of $\hat{o}_1 \in (0, \bar{o}_1)$. In the meantime, by the definition of $\hat{o}_1$, (i-2), and Lemma 31 (iii), $\Delta_1(\hat{z}_1(\hat{o}_1), z_2^I(\hat{z}_1(\hat{o}_1), \hat{o}_1), \hat{o}_1) < 0$.

Since $\Delta_1(\hat{z}_1(\bar{o}_1), z_2^I(\hat{z}_1(\bar{o}_1), \bar{o}_1), \bar{o}_1) > 0$, from the continuity of $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1)$ in o_1[4] and Lemma 31 (iv), there exists a $\check{o}_1 \in (\hat{o}_1, \bar{o}_1)$ such that $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1) \leq 0$ if $o_1 \in (\hat{o}_1, \check{o}_1)$, $\Delta_1(\hat{z}_1(\check{o}_1), z_2^I(\hat{z}_1(\check{o}_1), \check{o}_1), \check{o}_1) = 0$, and $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1) > 0$ if $o_1 \in (\check{o}_1, \bar{o}_1]$.

By definitions of $\bar{o}_1$, $\hat{o}_1$, and $\check{o}_1$, we can summarize $(z_1^*(o_1), z_2^*(o_1)$ as following: if $o_1 \in [0, \hat{o}_1]$, $z_1^*(o_1) = L_1$; if $o_1 \in (\hat{o}_1, \check{o}_1]$, $z_1^*(o_1) = z_1^I(o_1)$ and $z_2^*(o_1) = z_2^I(z_1^I(o_1), o_1)$; if $o_1 \in (\check{o}_1, y/A]$,

[3]Note that $z_1^I(o_1)$ is the unique solution to $\Delta_1(z_1, z_2^I(z_1, o_1), o_1) = 0$. From the continuity of $z_2^I(z_1, o_1)$ in (z_1, o_1) and $\Delta_1(z_1, z_2, o_1)$ in (z_1, z_2, o_1), both $z_1^I(o_1)$ and $z_2^I(z_1^I(o_1), o_1)$ are continuous in o_1.

[4]Note that the continuity of $\Delta_1(\hat{z}_1(o_1), z_2^I(\hat{z}_1(o_1), o_1), o_1)$ in o_1 is implied by the continuity of $\hat{z}_1(o_1)$ in o_1 (Lemma 31 (iii)) and $z_2^I(z_1, o_1)$ in (z_1, o_1) and $\Delta_1(z_1, z_2, o_1)$ in (z_1, z_2, o_1).

$z_2^*(o_1) = L_2$. Therefore,

$$\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1) = \begin{cases} \pi^{NO}(L_1, z_2^*(o_1), o_1) & \text{if } o_1 \in [0, \hat{o}_1] \\ \pi^{NO}(z_1^I(o_1), z_2^I(z_1^I(o_1), o_1), o_1) & \text{if } o_1 \in (\hat{o}_1, \check{o}_1] \\ \pi^{NO}(z_1^*(o_1), L_2, o_1) & \text{if } o_1 \in (\check{o}_1, y/A]. \end{cases}$$

By Lemma 30 and Lemma 31 (vi) through (viii), to show $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuously differentiable and strictly quasi-concave in o_1 on $[0, y/A]$, it suffices to show that both $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ and its derivative in o_1 are continuous at both $\hat{o}_1$ and $\check{o}_1$, and that if $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} = 0$ at $o_1 = \hat{o}_1$ (or at $o_1 = \check{o}_1$), the left-hand and right-hand derivatives of $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1}$ at $o_1 = \check{o}_1$ (or at $o_1 = \check{o}_1$) are both negative. The proof of these results follow the same idea as the proof in (ii-2) and here we only show the continuity at $\check{o}_1$:

By definition of $\check{o}_1$ and the facts that $\pi^{NO}(z_1, z_2^I(z_1, o_1), o_1)$ is strictly quasi-concave in o_1, $\hat{z_1}(o_1)$ is non-increasing in o_1, and $\hat{z_1}(\bar{o}_1) = L_1$,

$$\Delta_1(L_1, z_2^I(L_1, \check{o}_1), \check{o}_1) \geq \Delta_1(\hat{z}_1(\check{o}_1), z_2^I(\hat{z}_1(\check{o}_1), \check{o}_1), \check{o}_1) = 0.$$

This fact, together with Lemma 25 (i), implies that for all $o_1 \geq \hat{o}_1$, $\Delta_1(L_1, z_2^I(L_1, o_1), o_1) \geq 0$. Hence, $\lim_{o_1 \to \check{o}_1^+} z_1^*(o_1) = \lim_{o_1 \to \check{o}_1^+} z_1^I(o_1) = z_1^I(\check{o}_1)$. On the other hand, $\lim_{o_1 \to \check{o}_1^-} z_1^I(o_1) = z_1^I(\check{o}_1)$, which implies $\lim_{o_1 \to \check{o}_1^-} z_1^*(o_1) = \lim_{o_1 \to \check{o}_1^-} z_1^I(o_1)$. In the meantime, by the definition of $\check{o}_1$, $z_1^I(\check{o}_1) = \hat{z}_1(\check{o}_1)$ and hence, $\lim_{o_1 \to \check{o}_1^-} z_2^I(z_1^I(o_1), o_1) = z_2^I(z_1^I(\check{o}_1), \check{o}_1) = z_2^I(\hat{z}_1(\check{o}_1), \check{o}_1) = L_2$. By the continuity of $\pi^{NO}(z_1, z_2, o_1)$ in (z_1, z_2, o_1), $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuous at $\check{o}_1$.

(iii) By (ii), there exists a unique $o_1^* \in [0, y/A]$ maximizing $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$, where by (i), $z_1^*(o_1), z_2^*(o_1)$ maximizes $\pi^{NO}(z_1, z_2, o_1)$ subject to $z_1 \geq L_1, z_2 \geq L_2, p_1(z_1, z_2) \geq AC, p_2(z_1, z_2) \geq C$. Therefore, (z_1^*, z_2^*, o_1^*) uniquely exists and equals to $(z_1^*(o_1^*), z_2^*(o_1^*), o_1^*)$. ■

C.7 Proof of Theorem 18

Proof: (i) By Theorem 17 (ii), $o_1^* = y/A$ if and only if $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} \geq 0$ at $o_1 = y/A$. Also note that when $y \geq \bar{y}$, $o_1^* = z_1^U + H_1 \in (0, y/A)$. It then suffices to show that $\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1}\Big|_{o_1 = y/A}$ is non-increasing in y and has a positive limit as y approaches 0 from above. By Lemma 29 and the continuity of $\Delta_1(z_1, z_2, o_1)$ and $\Delta_2(z_1, z_2, o_1)$ in (z_1, z_2, o_1),

$$\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1}\Big|_{o_1 = y/A} = \begin{cases} \Delta_3(L_1, L_2, y/A) & \text{if } \Delta_1(L_1, L_2, y/A) < 0 \\ \Delta_3(z_1^I(y/A), L, y/A) & \text{if } \Delta_1(L_1, L_2, y/A) \geq 0 \end{cases};$$

where $z_1^I(y/A)$ is the unique solution to $\Delta_1(z_1, L_2, y/A) = 0$.

To show that $\frac{d\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1}\Big|_{o_1=y/A}$ is non-increasing in y, noting that $\Delta_1(L_1, L_2, y/A)$ is continuous in y, it then suffices to show that both $\Delta_3(L_1, L_2, y/A)$ and $\Delta_3(z_1^I(y/A), L_2, y/A)$ are non-increasing in y. To prove the latter result, we examine the derivative of each function with respect to y:

$$\begin{aligned}
&\frac{d\Delta_3(L_1, L_2, y/A)}{dy}\\
=&\frac{d\left[(p_1(L_1, L_2) - AC)\bar{G}_1(y/A - L) - A(p_2(L_1, L_2) - C)\right]}{dy}\\
=&-(p_1(L_1, L_2) - AC)g_1(y/A - L)/A \leq 0;\\
&\frac{d\Delta_3(z_1^I(y/A), L_2, y/A)}{dy}\\
=&\frac{\partial\Delta_3(z_1, L_2, y/A)}{\partial y}\Big|_{z_1=z_1^I(y/A)} + \frac{\partial\Delta_3(z_1, L_2, y/A)}{\partial z_1}\Big|_{z_1=z_1^I(y/A)} \cdot \frac{dz_1^I(y/A)}{dy}\\
=&\left[\frac{\partial\Delta_3(z_1, L_2, y/A)}{\partial y} + \Delta_{13}(z_1, L_2, y/A) \cdot \left(-\frac{d(\Delta_1(z_1, L_2, y/A))/dy}{\Delta_{11}(z_1, L_2, y/A)}\right)\right]\Big|_{z_1=z_1^I(y/A)}
\end{aligned}$$

For ease of notation, let $M_1 = (b_1b_2 - c_1c_2)(p_1(z_1^I(y/A), L_2) - AC)g(y/A - z_1^I(y/A))$, $G_1 = G(y/A - z_1^I(y/A))$, and $\bar{G}_1 = 1 - G_1$. Then we have

$$\begin{aligned}
&\frac{d\Delta_3(z_1^I(y/A), L, y/A)}{dy}\\
=&\frac{1}{A(b_1b_2 - c_1c_2)}\left\{-M_1 + \left[M_1 + Ac_2 - b_2\bar{G}_1\right] \cdot \frac{M_1 - b_2\bar{G}_1}{2b_2G_1 + M_1}\right\}\\
=&\frac{1}{A(b_1b_2 - c_1c_2)(2b_2G_1 + M_1)}\left\{-(b_2 - Ac_2)M_1 - Ac_2b_2\bar{G}_1 - b_2(M_1 - b_2(\bar{G}_1)^2)\right\}
\end{aligned}$$

Since $y/A > 0$, $\bar{G}_1 \in [0, 1]$, and $z_1^I(y/A)$ satisfies $\Delta_1(z_1, L_2, y/A) = 0$, by Lemma 24 (i), $M_1 \geq b_2\bar{G}_1 \geq b_2(\bar{G}_1)^2$. In the meantime, $G_1, \bar{G}_1, M_1 \geq 0$ and $b_2 \geq Ac_2$. Therefore, $\frac{d\Delta_3(z_1^I(y/A),L,y/A)}{dy} \leq 0$.

To show that $\frac{d\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1}\Big|_{o_1=y/A}$ has a nonnegative limit as y approaches 0 from above, first note that it is easy to show that $\Delta_1(L_1, L_2, y/A)$ is quasi-convex in y and equals to zero when y is zero. If $\frac{d\Delta_1(L_1,L_2,y/A)}{dy}$ is negative at $y = 0$, $\Delta_1(L_1, L_2, y/A)$ is negative at a right neighborhood of $y = 0$. Hence,

$$\begin{aligned}
&\lim_{y\to 0^+}\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1}\Big|_{o_1=y/A} = \lim_{y\to 0^+}\Delta_3(L_1, L_2, y/A)\\
=&\lim_{y\to 0^+}(p_1(L_1, L_2) - AC)\bar{G}_1(y/A - L) - A(p_2(L_1, L_2) - C)\\
=&(p_1(L_1, L_2) - AC) - A(p_2(L_1, L_2) - C) = p_1(L_1, L_2) - Ap_2(L_1, L_2) > 0
\end{aligned}$$

If, however, $\frac{d\Delta_1(L_1,L_2,y/A)}{dy}$ is nonnegative at $y = 0$, $\Delta_1(L_1, L_2, y/A)$ is nonnegative for all $y > 0$.

Also note that by $\Delta_1(L_1, L_2, y/A) = 0$ at $y = 0$, we know $\lim_{y \to 0^+} z_1^I(y/A) = L$. Hence,

$$\lim_{y \to 0^+} \left. \frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} \right|_{o_1 = y/A} = \lim_{y \to 0^+} \Delta_3(z_1^I(y/A), L, y/A)$$

$$= \Delta_3(L_1, L_2, y/A)\big|_{y=0} = p_1(L_1, L_2) - Ap_2(L_1, L_2) > 0$$

This completes the proof of the existence of $\acute{y}$.

If $p_1(L_1, L_2) < Ap_2(L_1, L_2)$, the existence of $\grave{y}$ can be proved following the same logic: we can show that $\left. \frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} \right|_{o_1 = 0}$ is non-decreasing in y and has a negative limit as y approaches zero from above.

If $p_1(L_1, L_2) = Ap_2(L_1, L_2)$, we see that the derivatives of $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ in o_1 at $o_1 = 0$ and $o_1 = y/A$ both approach to zero as y approaches zero from above. By the monotonicity of both in y, $\left. \frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} \right|_{o_1 = 0} > 0$ and $\left. \frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} \right|_{o_1 = y/A} < 0$ for all $y > 0$. Therefore, $0 < o_1^* < y/A$ for all $y > 0$.

(ii) It suffices to show that $\left. \frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} \right|_{o_1 = 0}$ is non-increasing in A. By Lemma 29,

$$\left. \frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} \right|_{o_1 = 0} = \begin{cases} \Delta_3(L_1, L_2, 0) & \text{if } \Delta_2(L_1, L_2, 0) < 0 \\ \Delta_3(L_1, z_2^I(0), 0) & \text{if } \Delta_2(L_1, L_2, 0) \geq 0 \end{cases};$$

where $z_2^I(0)$ is the unique solution to $\Delta_2(L_1, z_2, 0) = 0$. Notice that fixing everything else, $\Delta_2(L_1, z_2, 0)$ is independent of A, which implies that $z_2^I(0)$ is also independent of A. To show the monotonicity of $\left. \frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} \right|_{o_1 = 0}$ in A, since it is continuous in A, it thus suffices to show that for given z_2, both $\Delta_3(L_1, L_2, 0)$ and $\Delta_3(L_1, z_2^I(0), 0)$ are non-increasing in A, which is straightforward to prove by noting that $\frac{d\Delta_3(L_1, L_2, 0)}{dA} = -C - (p_2(L_1, L_2) - C)\bar{G}_2(y - L) < 0$ and $\frac{d\Delta_3(L_1, z_2^I(0), 0)}{dA} = -C - (p_2(L_1, z_2^I(0)) - C)\bar{G}_2(y - z_2^I(0)) < 0$. ■

C.8 Proof of Theorem 19

Proof: (i) To prove that o_1^* is non-decreasing in y, by Theorem 17 (ii) and its proof, it suffices to show that (i-1) $\Delta_3(L_1, L_2, o_1)$, (i-2) $\Delta_3(z_1^I(L_2, o_1), L_2, o_1)$, (i-3) $\Delta_3(L_1, z_2^I(L_1, o_1), o_1)$, and (i-4) $\Delta_3(z_1^I(o_1), z_2^I(o_1), o_1)$ are all non-decreasing in y, where $z_1^I(L_2, o_1)$, $z_2^I(L_1, o_1)$, and $(z_1^I(o_1), z_2^I(o_1))$ satisfy $\Delta_1(z_1, L_2, o_1) = 0$, $\Delta_2(L_1, z_2, o_1) = 0$, and $\Delta_1(z_1, z_2, o_1) = \Delta_2(z_1, z_2, o_1) = 0$, respectively. We prove these results by examining the derivatives of those functions with respect to y, for given o_1. In preparation, let $\Delta_{1y}(z_1, z_2, o_1) = \frac{\partial \Delta_1(z_1, z_2, o_1)}{\partial y}$, $\Delta_{2y}(z_1, z_2, o_1) = \frac{\partial \Delta_2(z_1, z_2, o_1)}{\partial y}$, and $\Delta_{3y}(z_1, z_2, o_1) = \frac{\partial \Delta_3(z_1, z_2, o_1)}{\partial y}$. For ease of notation, also define the following functions: $M_1(z_1, z_2) = (b_1 b_2 - c_1 c_2)(p_1(z_1, z_2) - AC)g_1(o_1 - z_1)$, $M_2(z_1, z_2) = (b_1 b_2 - c_1 c_2)(p_2(z_1, z_2) - C)g_2(y - Ao_1 - z_2)$,

$G_1(z_1) = G_1(o_1 - z_1)$, $\bar{G}_1(z_1) = 1 - G_1(z_1)$, $G_2(z_2) = G_2(y - Ao_1 - z_2)$, and $\bar{G}_2(z_2) = 1 - G(z_2)$. We suppress the variables in the functions whenever no confusion is caused.

(i-1)

$$\frac{d\Delta_3(L_1, L_2, o_1)}{dy} = \Delta_{3y}(L_1, L_2, o_1) = A(p_2(L_1, L_2) - C)g_2(y - Ao_1 - L_2) \geq 0,$$

(i-2) By definition of $z_1^I(L_2, o_1)$ and Lemma 28(i), $o_1 > 0$. Furthermore,

$$\begin{aligned}
&\frac{d\Delta_3(z_1^I(L_2, o_1), L_2, o_1)}{dy} \\
=&\Delta_{3y}(z_1^I(L_2, o_1), L_2, o_1) + \Delta_{13}(z_1^I(L_2, o_1), L_2, o_1) \cdot \frac{dz_1^I(L_2, o_1)}{dy} \\
=&\left[\Delta_{3y}(z_1, z_2, o_1) - \Delta_{13}(z_1, z_2, o_1) \cdot \frac{\Delta_{1y}(z_1, z_2, o_1)}{\Delta_{11}(z_1, z_2, o_1)}\right]\Bigg|_{z_1 = z_1^I(L_2, o_1), z_2 = L} \\
=&\left[\frac{\Delta_{3y}(z_1, z_2, o_1)\Delta_{11}(z_1, z_2, o_1) - \Delta_{13}(z_1, z_2, o_1) \cdot \Delta_{1y}(z_1, z_2, o_1)}{\Delta_{11}(z_1, z_2, o_1)}\right]\Bigg|_{z_1 = z_1^I(L_2, o_1), z_2 = L} \\
=&\left[\frac{AM_2(2b_2G_1 + M_1) - c_2\bar{G}_2(-b_2\bar{G}_1 + M_1 + Ac_2\bar{G}_2)}{2b_2G_1 + M_1}\right]\Bigg|_{z_1 = z_1^I(L_2, o_1), z_2 = L} \\
=&\left[\frac{2Ab_2M_2G_1 + AM_1M_2 + c_2b_2\bar{G}_2\bar{G}_1 - M_1c_2\bar{G}_2 - Ac_2^2(\bar{G}_2)^2}{2b_2G_1 + M_1}\right]\Bigg|_{z_1 = z_1^I(L_2, o_1), z_2 = L} \\
\geq&\left[\frac{2Ab_2b_1G_1\bar{G}_2 + AM_1b_1\bar{G}_2 + c_2b_2\bar{G}_2\bar{G}_1 - M_1c_2\bar{G}_2 - Ac_2^2(\bar{G}_2)^2}{2b_2G_1 + M_1}\right]\Bigg|_{z_1 = z_1^I(L_2, o_1), z_2 = L} \\
=&\left[\frac{(2Ab_2b_1 - Ac_2^2)G_1\bar{G}_2 + (Ab_1 - c_2)M_1\bar{G}_2 + c_2(b_2 - Ac_2)\bar{G}_2\bar{G}_1 + Ac_2^2[\bar{G}_2 - (\bar{G}_2)^2]}{2b_2G_1 + M_1}\right] \\
\geq&0
\end{aligned}$$

where the first inequality is by definition of $z_1^I(L_2, o_1)$ and Lemma 24, and the second inequality follows from Assumption (ID-2) and the fact $\bar{G}_1, \bar{G}_2 \in [0, 1]$.

(i-3) By definition of $z_2^I(L_1, o_1)$ and Lemma 28(i), $o_1 < y/A$. Furthermore,

$$\begin{aligned}
&\frac{d\Delta_3(L_1, z_2^I(L_1,o_1), o_1)}{dy} \\
=&\Delta_{3y}(L_1, z_2^I(L_1,o_1), o_1) + \Delta_{23}(L_1, z_2^I(L_1,o_1), o_1) \cdot \frac{dz_2^I(L_1,o_1)}{dy} \\
=&\left[\Delta_{3y}(z_1,z_2,o_1) - \Delta_{23}(z_1,z_2,o_1) \cdot \frac{\Delta_{2y}(z_1,z_2,o_1)}{\Delta_{22}(z_1,z_2,o_1)}\right]\Bigg|_{z_1=L_1, z_2=z_2^I(L_1,o_1)} \\
=&\left[\frac{\Delta_{3y}(z_1,z_2,o_1)\Delta_{22}(z_1,z_2,o_1) - \Delta_{23}(z_1,z_2,o_1) \cdot \Delta_{2y}(z_1,z_2,o_1)}{\Delta_{22}(z_1,z_2,o_1)}\right]\Bigg|_{z_1=L_1, z_2=z_2^I(L_1,o_1)} \\
=&\left[\frac{AM_2(2b_1G_2+M_2) + (Ab_1\bar{G}_2 - AM_2 - c_1\bar{G}_1)(-b_1\bar{G}_2 + M_2)}{2b_1G_2+M_2}\right]\Bigg|_{z_1=L_1, z_2=z_2^I(L_1,o_1)} \\
=&\left[\frac{2Ab_1M_2G_2 + AM_2^2 + (-Ab_1^2(\bar{G}_2)^2 + b_1c_1\bar{G}_1\bar{G}_2 + 2Ab_1M_2\bar{G}_2 - AM_2^2 - c_1M_2\bar{G}_1)}{2b_1G_2+M_2}\right] \\
=&\left[\frac{2Ab_1M_2 - Ab_1^2(\bar{G}_2)^2 + b_1c_1\bar{G}_1\bar{G}_2 - c_1M_2\bar{G}_1}{2b_1G_2+M_2}\right]\Bigg|_{z_1=L_1, z_2=z_2^I(L_1,o_1)} \\
=&\left[\frac{(Ab_1 - c_1\bar{G}_1)M_2 + Ab_1M_2 - Ab_1^2(\bar{G}_2)^2 + b_1c_1\bar{G}_1\bar{G}_2}{2b_1G_2+M_2}\right]\Bigg|_{z_1=L_1, z_2=z_2^I(L_1,o_1)} \\
\geq&\left[\frac{(Ab_1 - c_1\bar{G}_1)M_2 + Ab_1^2[\bar{G}_2 - (\bar{G}_2)^2] + b_1c_1\bar{G}_1\bar{G}_2}{2b_1G_2+M_2}\right]\Bigg|_{z_1=L_1, z_2=z_2^I(L_1,o_1)} \\
\geq& 0
\end{aligned}$$

where the first inequality is by definition of $z_2^I(L_1, o_1)$ and Lemma 24, and the second inequality follows from Assumption (ID-2) and the fact $\bar{G}_1, \bar{G}_2 \in [0,1]$.

(i-4) By definition of $(z_1^I(o_1), z_2^I(o_1))$ and Lemma 28(i), $0 < o_1 < y/A$. Furthermore,

$$\begin{aligned}
\frac{d\Delta_3(z_1^I(o_1), z_2^I(o_1), o_1)}{dy} &= \Delta_{3y}(z_1^I(o_1), z_2^I(o_1), o_1) + \Delta_{13}(z_1^I(o_1), z_2^I(o_1), o_1) \cdot \frac{dz_1^I(o_1)}{dy} \\
&\quad + \Delta_{23}(z_1^I(o_1), z_2^I(o_1), o_1) \cdot \frac{dz_2^I(o_1)}{dy}
\end{aligned}$$

By definition of $(z_1^I(o_1), z_2^I(o_1))$ and the chain rule,

$$\begin{cases} \frac{dz_1^I(o_1)}{dy}\Delta_{11} + \Delta_{1y} + \Delta_{12}\frac{dz_2^I(o_1)}{dy} = 0 \\ \frac{dz_2^I(o_1)}{dy}\Delta_{22} + \Delta_{2y} + \Delta_{12}\frac{dz_1^I(o_1)}{dy} = 0 \end{cases}$$

Solving this system of equations, we get

$$\left(\frac{dz_1^I(o_1)}{dy}, \frac{dz_2^I(o_1)}{dy}\right) = \left(\frac{\Delta_{2y}\Delta_{12} - \Delta_{1y}\Delta_{22}}{\Delta_{11}\Delta_{22} - (\Delta_{12})^2}, \quad \frac{\Delta_{1y}\Delta_{12} - \Delta_{2y}\Delta_{11}}{\Delta_{11}\Delta_{22} - (\Delta_{12})^2}\right)\Bigg|_{z_1=z_1^I(o_1), z_2=z_2^I(o_1)}.$$

Applying this into the expression of $\frac{d\Delta_3(z_1^I(o_1),z_2^I(o_1),o_1)}{dy}$ and simplifying, we have

$$
\begin{aligned}
&\frac{d\Delta_3(z_1^I(o_1),z_2^I(o_1),o_1)}{dy}\\
=&\frac{(b_1b_2-c_1c_2)^{-2}}{\Delta_{11}\Delta_{22}-(\Delta_{12})^2}\big\{(2Ab_1-c_1-c_2)M_1M_2+(b_1c_1-b_1c_2)M_1\bar{G}_2\\
&+(b_1c_2-Ab_1^2)M_1(\bar{G}_2)^2+(4Ab_1b_2-AC_1^2-Ac_2^2-2Ac_1c_2)M_2\\
&+(2Ac_1^2+b_2c_2-b_2c_1-4Ab_1b_2+2Ac_1c_2)M_2\bar{G}_1+(b_2c_1-AC_1^2)M_2(\bar{G}_1)^2\\
&+(c_1+c_2)(b_1b_2-c_1c_2)\bar{G}_1\bar{G}_2+(2Ab_1-c_2)(b_1b_2-c_1c_2)\bar{G}_1(\bar{G}_2)^2\\
&-c_1(b_1b_2-c_1c_2)(\bar{G}_1)^2\bar{G}_2-2Ab_1(b_1b_2-c_1c_2)(\bar{G}_2)^2\big\}\big|_{z_1=z_1^I(o_1),z_2=z_2^I(o_1)}\\
\geq&\frac{(b_1b_2-c_1c_2)^{-2}}{\Delta_{11}\Delta_{22}-(\Delta_{12})^2}\big\{(Ab_1-c_1)b_1M_1\bar{G}_2+(b_1c_1-b_1c_2)M_1\bar{G}_2+(b_1c_2-Ab_1^2)M_1(\bar{G}_2)^2\\
&+(4Ab_1b_2-AC_1^2-Ac_2^2-2Ac_1c_2)M_2+(Ab_1-c_2)b_2M_2\bar{G}_1\\
&+(2Ac_1^2+b_2c_2-b_2c_1-4Ab_1b_2+2Ac_1c_2)M_2\bar{G}_1\\
&+(b_2c_1-AC_1^2)M_2(\bar{G}_1)^2+(c_1+c_2)(b_1b_2-c_1c_2)\bar{G}_1\bar{G}_2+(2Ab_1-c_2)(b_1b_2-c_1c_2)\bar{G}_1(\bar{G}_2)^2\\
&-c_1(b_1b_2-c_1c_2)(\bar{G}_1)^2\bar{G}_2-2Ab_1(b_1b_2-c_1c_2)(\bar{G}_2)^2\big\}\big|_{z_1=z_1^I(o_1),z_2=z_2^I(o_1)}\\
=&\frac{(b_1b_2-c_1c_2)^{-2}}{\Delta_{11}\Delta_{22}-(\Delta_{12})^2}\big\{(Ab_1^2-b_1c_2)M_1[\bar{G}_2-(\bar{G}_2)^2]\\
&+(Ab_1b_2+Ac_1^2-Ac_2^2-b_2c_1)M_2+(-2Ac_1^2+b_2c_1+3Ab_1b_2-2Ac_1c_2)(M_2-M_2\bar{G}_1)\\
&+(b_2c_1-AC_1^2)M_2(\bar{G}_1)^2+c_2(b_1b_2-c_1c_2)[\bar{G}_1\bar{G}_2-\bar{G}_1(\bar{G}_2)^2]\\
&+\ 2Ab_1(b_1b_2-c_1c_2)[\bar{G}_1(\bar{G}_2)^2-(\bar{G}_2)^2]+c_1(b_1b_2-c_1c_2)[\bar{G}_1\bar{G}_2-(\bar{G}_1)^2\bar{G}_2]\big\}\big|_{z_1=z_1^I(o_1),z_2=z_2^I(o_1)}\\
\geq&\frac{(b_1b_2-c_1c_2)^{-2}}{\Delta_{11}\Delta_{22}-(\Delta_{12})^2}\{(-2Ac_1^2+b_2c_1+3Ab_1b_2-2Ac_1c_2)M_2G_1\\
&\qquad-2Ab_1(b_1b_2-c_1c_2)G_1(\bar{G}_2)^2\}\big|_{z_1=z_1^I(o_1),z_2=z_2^I(o_1)}\\
=&\frac{(b_1b_2-c_1c_2)^{-2}}{\Delta_{11}\Delta_{22}-(\Delta_{12})^2}\{(-2Ac_1^2+b_2c_1+Ab_1b_2)M_2G_1\\
&\qquad+2A(b_1b_2-c_1c_2)G_1[M_2-b_1(\bar{G}_2)^2]\}\big|_{z_1=z_1^I(o_1),z_2=z_2^I(o_1)}\\
\geq&0
\end{aligned}
$$

where the first inequality is by definition of $(z_1^I(o_1), z_2^I(o_1))$ and Lemma 24, and both the second and the last inequalities follow from Assumption (ID-2) and the facts that $M_1, M_2 \geq 0$ and $\bar{G}_1, \bar{G}_2 \in [0,1]$.

(ii) By the symmetry of the problem, (ii) can be proved following the same logic as (i). ■

C.9 Proof of Proposition 9

In preparation, note that when $c_1 = c_2 = 0$, there is no price substitution and demand for a product is only dependent on its own price. Let $\pi_i(z,o) = (p_i(z) - A_i c)\mathrm{E}[\min(o, z+\epsilon_i)]$ and clearly, $\pi^{NO}(z_1, z_2, o_1) = \pi_1(z_1, o_1) + \pi_2(z_2, y - Ao_1)$. Define $\Delta_1\pi_1(z_1,o_1) = \frac{\partial \pi_1(z_1,o_1)}{\partial z_1}$, $\Delta_2\pi_1(z_1,o_1) = \frac{\partial \pi_1(z_1,o_1)}{\partial o_1}$, $\Delta_{11}\pi_1(z_1,o_1) = \frac{\partial^2 \pi_1(z_1,o_1)}{\partial z_1^2}$, $\Delta_{12}\pi_1(z_1,o_1) = \frac{\partial^2 \pi_1(z_1,o_1)}{\partial z_1 o_1}$, $\Delta_{22}\pi_1(z_1,o_1) = \frac{\partial^2 \pi_1(z_1,o_1)}{\partial o_1^2}$, and similar notations for the derivatives of π_2.

The proof uses the following two lemmas.

Lemma 32 *(i) For given $o_1 > 0$, there exists a unique z_1, denoted by $z_1^*(o_1)$, which maximizes the profit function $\pi_1^{PP}(z_1, o_1)$ subject to $z_1 \geq L_1, p_1(z_1) \geq AC$. Specifically,*

$$z_1^*(o_1) = \begin{cases} L_1 & \text{if } \Delta_1\pi_1(L_1, o_1) < 0 \\ z_1^I(o_1) & \text{otherwise} \end{cases},$$

where $z_1^I(o_1)$ is the unique solution to $\Delta_1\pi_1(z_1, o_1) = 0$. For $o_1 = 0$, $\pi_1^{PP}(z_1, 0) = 0$ for all $z_1 \geq L_1$. In such a case, we define $z_1^*(0) = L_1$.

Symmetrically, for given $o_1 < y/A$, there exists a unique z_2, denoted by $z_2^*(o_1)$, which maximizes the profit function $\pi_2^{PP}(z_2, y - Ao_1)$ subject to $z_2 \geq L_2, p_2(z_2) \geq C$. Specifically,

$$z_2^*(o_1) = \begin{cases} L_2 & \text{if } \Delta_1\pi_2(L_2, y - Ao_1) < 0 \\ z_2^I(o_1) & \text{otherwise} \end{cases},$$

where $z_1^I(o_1)$ is the unique solution to $\Delta_1\pi_2(z_2, y - Ao_1) = 0$. For $o_1 = y/A$, $\pi_2^{PP}(z_2, 0) = 0$ for all $z_2 \geq L_2$. In such a case, we define $z_2^*(0) = L_2$.

Furthermore, $z_1^*(o_1)$ is non-decreasing in o_1 and $z_2^*(o_1)$ is non-increasing in o_1.

(ii) $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuously differentiable and strictly quasi-concave in o_1 for $o_1 \in [0, y/A]$.

(iii) There exists a unique solution o_1^* maximizing the profit function $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ subject to $0 \leq o_1 \leq y/A$.

Proof: (i) By the symmetric nature of the problem, it suffices to show the results on z_1 and then the results on z_2 can be proved following the same logic. To see the existence of unique z_1^* for $o_1 > 0$, notice that similar to Theorem 17 (i), we can show that $\pi_1(z_1, o_1)$ is strictly quasi-concave in z_1 for given $o_1 > 0$, which implies the uniqueness and the expression of optimal z_1.

Next, to see the monotonicity of $z_1^*(o_1)$ in o_1, first notice that similarly to Lemma 31(i), we can show that there exists a $\underline{o}_1 \in [0, y/A]$ such that $\Delta_1\pi_1(L_1, 0) < 0$ for $0 < o_1 \leq \underline{o}_1$ and $\Delta_1\pi_1(L_1, 0) \geq 0$ for $\underline{o}_1 < o_1 \leq y/A$. Hence, $z_1^*(o_1) = L_1$ for $0 \leq o_1 \leq \underline{o}_1$ and $z_1^*(o_1) = z_1^I(o_1)$ for $\underline{o}_1 < o_1 \leq y/A$. On the other hand, similarly to Lemma 25 (i), we can show that if $\Delta_1\pi_1(z_1, o_1) \geq 0$ for some $z_1 \geq L_1$

and $o_1 > 0$, then $\Delta_{12}\pi_1(z_1, o_1) \geq 0$. This implies that $z_1^I(o_1)$ is non-decreasing in o_1, which in turn implies that $z_1^*(o_1)$ is non-decreasing in o_1.

(ii) Since $\Delta_1\pi_1(z_1, o_1)$ and $\Delta_2\pi_2(z_2, y - Ao_1)$ are continuous in (z_1, o_1) and (z_2, o_1), respectively, $z_1^*(o_1)$ and $z_2^*(o_1)$ are both continuous in o_1. In the meanwhile, it is clear that

$$\frac{d\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1} = \Delta_2\pi_1(z_1^*(o_1), o_1) - A\Delta_2\pi_2(z_2^*(o_1), y - Ao_1) \tag{C.18}$$

Therefore, $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ is continuously differentiable in o_1. Furthermore, by the expression and monotonicity of $z_1^*(o_1)$ and $z_2^*(o_1)$ in o_1, $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$ can only take one of the following two forms:

$$\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1) = \begin{cases} \pi^{NO}(L_1, z_2^I(o_1), o_1) & \text{if } o_1 \in [0, o_1'] \\ \pi^{NO}(z_1^I(o_1), z_2^I(o_1), o_1) & \text{if } o_1 \in (o_1', o_1''] \\ \pi^{NO}(z_1^I(o_1), L_2, o_1) & \text{if } o_1 \in (o_1'', y/A] \end{cases},$$

or

$$\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1) = \begin{cases} \pi^{NO}(L_1, z_2^I(o_1), o_1) & \text{if } o_1 \in [0, o_1'] \\ \pi^{NO}(L_1, L_2, o_1) & \text{if } o_1 \in (o_1', o_1''] \\ \pi^{NO}(z_1^I(o_1), L_2, o_1) & \text{if } o_1 \in (o_1'', y/A] \end{cases}.$$

By the continuously differentiability, to prove the strict quasi-concavity of $\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)$, it suffices to show that $\pi^{NO}(L_1, L_2, o_1)$, $\pi^{NO}(L_1, z_2^I(o_1), o_1)$, $\pi^{NO}(z_1^I(o_1), L_2, o_1)$, and $\pi^{NO}(z_1^I(o_1), z_2^I(o_1), o_1)$ are all strictly quasi-concave in o_1, and the left-hand and right-hand second-order derivatives at each boundary point are negative if the first-order derivative is zero at the point. The proof follows the same logic as the proof of Lemma 31 (v)-(viii) and Theorem 17 (ii), and thus is omitted for conciseness.

(iii) Directly implied by (ii). ■

Lemma 33 *Let* $o_2 = y - Ao_1$. $\pi_1(z_1^*(o_1), o_1)$ *and* $\pi_2(z_2^*(o_2), o_2)$ *are non-decreasing and concave in* o_1 *and* o_2*, respectively.*

Proof: Notice that $z_1^*(o_1)$ is non-decreasing in o_1 and $\pi_1(z_1, o_1)$ is non-decreasing in both z_1 and o_1. Hence, $\pi_1(z_1^*(o_1), o_1)$ is non-decreasing in o_1. To prove the concavity of $\pi_1(z_1^*(o_1), o_1)$ in o_1, note that it is clearly continuously differentiable in o_1 and by the existence of $\underline{o}_1$ in the proof of Lemma 32(i), we only need to show that both $\pi_1(L_1, o_1)$ and $\pi_1(z_1^I(o_1), o_1)$ are concave in o_1. The concavity of both functions is clear by examining the second-order derivative of each function in o_1:

$$\frac{d^2\pi_1(L_1,o_1)}{do_1^2}$$
$$=-\left(p_1(L_1)-AC\right)g_1(o_1-L_1)\le 0$$
$$\frac{d^2\pi_1(z_1^I(o_1),o_1)}{do_1^2}$$
$$=\Delta_{22}\pi_1(z_1^I(o_1),o_1)-\Delta_{12}\pi_1(z_1^I(o_1),o_1)\cdot\frac{\Delta_{12}\pi_1(z_1^I(o_1),o_1)}{\Delta_{11}\pi_1(z_1^I(o_1),o_1)}$$
$$=\left[\frac{-(p_1(z_1)-AC)g(o_1-z_1)/b_1-[(p_1(z_1)-AC)g(o_1-z_1)/b_1-(\bar{G}(o_1-z_1))^2/b_1^2]}{2G(o_1-z_1)/b_1+(p_1(z_1)-AC)g(o_1-z_1)}\right]\Bigg|_{z_1=z_1^I(o_1)}\le 0$$

where the second inequality follows from the fact shown in the proof of Lemma 32 (i): as $z_1^I(o_1)$ satisfies $\Delta_1\pi_1(z_1,o_1)=0$ for some $o_1>0$, it also satisfies $\Delta_{12}\pi_1(z_1,o_1)=(p_1(z_1)-AC)g(o_1-z_1)-\bar{G}(o_1-z_1)/b_1\ge 0$.

The monotonicity and concavity of $\pi_2(z_2^*(o_2),o_2)$ in o_2 follow from the symmetry of the problem. ■

Proof of Proposition 9

Proof: (i) Let $o_2=y-Ao_1$. First note that, by the proof of Lemma 32, z_1^* depends on y solely through o_1^* and z_2^* depends on y solely through $y-Ao_1^*$. That is, for given o_1 and o_2, $z_1^*(o_1)$ and $z_2^*(o_2)$ are independent of y. Similarly, for given o_2, $z_2^*(o_2)$ is independent of A. On the other hand, $z_1^*(o_1)$ indeed depends on A since $\Delta_1\pi_1(z_1,o_1)$ is a function of A. By equation (C.18), $\frac{d\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1}$ is continuous in both y and A. Furthermore,

$$\frac{d\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1}$$
$$=\Delta_2\pi_1(z_1^*(o_1),o_1)-A\Delta_2\pi_2(z_2^*(o_1),y-Ao_1)$$
$$=(p_1(z_1^*(o_1))-AC)\bar{G}(o_1-z_1^*(o_1))-A\left[(p_2(z_2^*(o_2))-C)\bar{G}(o_2-z_2^*(o_2))\right]\big|_{o_2=y-Ao_1}$$
$$\frac{d^2\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1dy}$$
$$=-A\left.\frac{d^2\pi_2(z_2^*(o_2),o_2)}{do_2^2}\right|_{o_2=y-Ao_1}\ge 0\quad\text{(by Lemma 33)}$$

If $\Delta\pi_1(L_1,o_1)\ge 0$ at some A_0 and its neighborhood,

$$\left.\frac{d^2\pi^{NO}(z_1^*(o_1),z_2^*(o_1),o_1)}{do_1dA}\right|_{A=A_0}$$
$$=-C\bar{G}_1(o_1-z_1^I(o_1))-\Delta_{12}\pi_1(z_1^I(o_1),o_1)\frac{-cG_1(o_1-z_1^I(o_1))}{\Delta_{11}\pi_1(z_1^I(o_1),o_1)}$$
$$+\left[-\frac{d\pi_2(z_2^*(o_2),o_2)}{do_2}+A^2\frac{d^2\pi_2(z_2^*(o_2),o_2)}{do_2^2}\right]\Bigg|_{o_2=y-Ao_1,A=A_0}\le 0\quad\text{(by Lemma 33)}$$

otherwise,

$$\frac{d^2\pi^{NO}(z_1^*(o_1), z_2^*(o_1), o_1)}{do_1 dA}$$
$$= -C\bar{G}_1(o_1 - L_1) + \left[-\frac{d\pi_2(z_2^*(o_2), o_2)}{do_2} + A^2 \frac{d^2\pi_2(z_2^*(o_2), o_2)}{do_2^2} \right]\Bigg|_{o_2 = y - Ao_1} \leq 0$$

Therefore, o_1^* is non-decreasing in y and non-increasing in A. The monotonicity of o_2^* in y is implied by the symmetry of the problem.

(ii) The monotonicity of o_1^* and o_2^* in y implies the monotonicity of z_1^* and z_2^* in y. To see the monotonicity of z_1^* in A, notice that for given o_1, $\Delta_1\pi_1(z_1, o_1)$ decreases in A, implying that $z_1^*(o_1)$ decreases in A. This fact, together with (i) and Lemma 32 (i), implies $\frac{dz_1^*}{dA} = \frac{dz_1^*(o_1^*)}{dA} = \frac{\partial z_1^*(o_1)}{\partial o_1}\Big|_{o_1 = o_1^*} \cdot \frac{do_1^*}{dA} + \frac{\partial z_1^*(o_1)}{\partial A}\Big|_{o_1 = o_1^*} \leq 0$.

(iii) Since p_1^* depends on y and A solely through z_1^*, the monotonicity of p_1^* in y and A follows from the monotonicity of z_1^* in y and A. The same logic applies to p_2^*. ■

C.10 Bimodal Profit Function for Full-Overselling Model

When there is no penalty for cancelling orders, obviously the firm does not ration orders, i.e., $o_1^* = \min(y/A, z_1 + H_1)$ and $o_2^* = \min(y, z_2 + H_2)$. It is easy to check that o_1^* and o_2^* satisfy the conditions in Lemma 12. In such a case, the decision problem becomes

Ex-post:

$$\pi(z_1, z_2, \epsilon_1, \epsilon_2) = \max_{q_1, q_2} (p_1(z_1, z_2) - AC)q_1 + (p_2(z_1, z_2) - C)q_2$$

subject to: $\quad 0 \leq q_1 \leq \min(z_1 + \epsilon_1, y/A, z_1 + H_1), 0 \leq q_2 \leq \min(z_2 + \epsilon_2, y, z_2 + H_2), 0 \leq Ao_1 + q_2 \leq y$

Note that for any nonnegative q_1 and q_2 satisfying $0 \leq Ao_1 + q_2 \leq y$, the conditions $q_1 \leq y/A$ and $q_2 \leq y$ are automatically met. Furthermore, by definition, $\epsilon_1 \leq H_1$ and $\epsilon_2 \leq H_2$. Therefore, the post-demand problem is equivalent to

$$\pi(z_1, z_2, \epsilon_1, \epsilon_2) = \max_{q_1, q_2} (p_1(z_1, z_2) - AC)q_1 + (p_2(z_1, z_2) - C)q_2$$

$$\text{subject to:} \quad 0 \leq q_1 \leq z_1 + \epsilon_1, 0 \leq q_2 \leq z_2 + \epsilon_2, 0 \leq Aq_1 + q_2 \leq y$$

Ex-ante:

$$\max_{z_1, z_2} \pi^{FO}(z_1, z_2) = \mathrm{E}_{\epsilon_1, \epsilon_2} \pi(z_1, z_2, \epsilon_1, \epsilon_2) \quad \text{subject to: } z_1 \geq L_1, z_2 \geq L_2, p_1(z_1, z_2) \geq AC, p_2(z_1, z_2) \geq C$$

It is straightforward to derive the expression of $\pi^{FO}(z_1, z_2)$ for the following two cases:

- If $p_1(z_1, z_2) \geq Ap_2(z_1, z_2)$

$$
\begin{aligned}
\pi^{FO}(z_1, z_2) &= (p_1(z_1, z_2) - AC)\mathrm{E}_{\epsilon_1}[\min(z_1 + \epsilon_1, y/A)] \\
&\quad + (p_2(z_1, z_2) - C)\mathrm{E}_{\epsilon_1,\epsilon_2}[\min(z_2 + \epsilon_2, (y - A(z_1 + \epsilon_1))^+)] \\
&= (p_1(z_1, z_2) - AC)\mathrm{E}_{\epsilon_1}[\min(z_1 + \epsilon_1, y/A)] + (p_2(z_1, z_2) - C)\mathrm{E}_{\epsilon_2}[\min(z_2 + \epsilon_2, y)] \\
&\quad - (p_2(z_1, z_2) - C)\mathrm{E}_{\epsilon_1,\epsilon_2}[(\min(A(z_1 + \epsilon_1), y) + \min(z_2 + \epsilon_2, y) - y)^+] \qquad \text{(C.19)}
\end{aligned}
$$

- If $p_1(z_1, z_2) < Ap_2(z_1, z_2)$

$$
\begin{aligned}
\pi^{FO}(z_1, z_2) &= (p_1(z_1, z_2) - AC)\mathrm{E}_{\epsilon_1,\epsilon_2}[\min(z_1 + \epsilon_1, (y - (z_2 + \epsilon_2))^+/A)] \\
&\quad + (p_2(z_1, z_2) - C)\mathrm{E}_{\epsilon_2}[\min(z_2 + \epsilon_2, y)] \\
&= (p_1(z_1, z_2) - AC)\mathrm{E}_{\epsilon_1}[\min(z_1 + \epsilon_1, y/A)] + (p_2(z_1, z_2) - C)\mathrm{E}_{\epsilon_2}[\min(z_2 + \epsilon_2, y)] \\
&\quad - (p_1(z_1, z_2)/A - C)\mathrm{E}_{\epsilon_1,\epsilon_2}[(\min(A(z_1 + \epsilon_1), y) + \min(z_2 + \epsilon_2, y) - y)^+] \\
&\qquad \text{(C.20)}
\end{aligned}
$$

For this problem, we can further refine the feasible region, as in Lemma 34.

Lemma 34 *(Refining Feasible Region for (ID,FO) Strategy) The optimal z_1^* and z_2^* satisfy $-(AL_1 + L_2) \leq y - Az_1^* - z_2^* \leq AH_1 + H_2$, $y - Az_1^* \geq -AL_1$, and $y - z_2^* \geq -L_2$.*

Without loss of optimality, we impose the conditions in Lemma 34 on any feasible policies. Within this refined feasible region, we show that the profit function $\pi^{FO}(z_1, z_2)$ is bimodal in the two demand rates, i.e., it can have at most two local maxima. A numerical example of a bimodal profit function is illustrated in Figure C.1.

Theorem 20 *When both $g_1(x)$ and $g_2(x)$ are non-increasing in x, $\pi^{FO}(z_1, z_2)$ is bimodal in the refined feasible region $\{(z_1, z_2) : z_1 \geq L_1, z_2 \geq L_2, p_1(z_1, z_2) \geq AC, p_2(z_1, z_2) \geq C, -(AL_1 + L_2) \leq y - Az_1 - z_2 \leq AH_1 + H_2, y - Az_1 \geq -AL_1, y - z_2 \geq -L_2\}$. Specifically, there can exist one local optimum in the region $\{(z_1, z_2) : p_1(z_1, z_2) \geq Ap_2(z_1, z_2)\}$ and another one in the region $\{(z_1, z_2) : p_1(z_1, z_2) < Ap_2(z_1, z_2)\}$.*

Theorem 20 and the example in Figure C.1 help build the intuition about multiple local optima. When the firm's pricing decision is endogenous, so is the prioritization of the two products at the order-fulfillment stage. For a given product ordering (e.g., in the region $\{(z_1, z_2) : p_1(z_1, z_2) \geq Ap_2(z_1, z_2)\}$), the profit function can still be well-behaved and have at most one local optimum (ref. Theorem 20). Nevertheless, along the switching line at which the two products have equal priority ($p_1(z_1, z_2) = Ap_2(z_1, z_2)$), the profit function is not differentiable and its partial derivative may jump up from negative to positive, which results in a bimodal function. To see why, note that by equation (C.19) and (C.20), under the two different orderings, the profit functions differ only in the

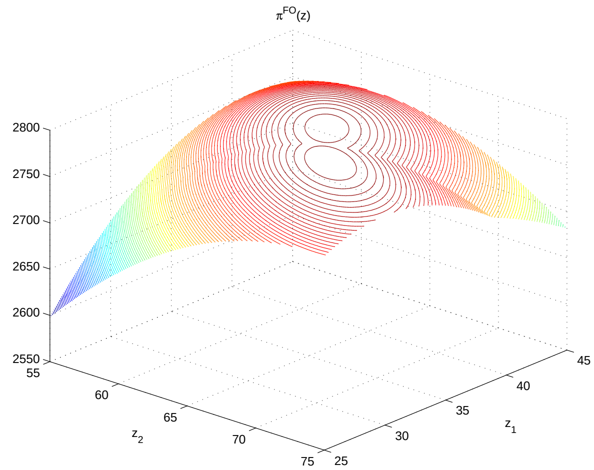


Figure C.1. An example where $\pi^{FO}(\mathbf{z})$ has two local maxima: $a_1 = 100$, $a_2 = 200$, $b_1 = 2.7$, $b_2 = 5$, $c_1 = c_2 = 0.5$, $C = 0$, $A = 1$, $y = 100$, $\gamma = 0$, $\epsilon_1, \epsilon_2 \sim$ Uniform$[-20, 20]$.

relative margin that the firm loses by cancelling each lower-priority order. Further, when a product's demand rate, say z_1, increases, product 1's relative margin decreases faster than product 2 does[5] and thus product 1's priority also decreases. Therefore, for given z_2, as z_1 crosses the switching line from below, product 1 becomes the lower-priority product and the rate at which the profit function changes in z_1 jumps up.

In short, compared to the (ID,NO) model, both (ID,FO) and (ID,OO) models may have multiple optimal solutions due to the non-differentiability of the profit function, which arises from ex-post optimization of the order-fulfillment decisions.

The proofs of Lemma 34 and Theorem 20 are provided in the following subsections.

C.10.1 Proof of Lemma 34

Proof: Let $(q_1^*(z_1, z_2, \epsilon_1, \epsilon_2), q_2^*(z_1, z_2, \epsilon_1, \epsilon_2))$ denote the optimal post-demand production strategy as a function of z_1, z_2, ϵ_1, and ϵ_2.

We prove the first property by contradiction and then show that the first property implies the second and third properties. Suppose $y - Az_1^* - z_2^* < -(AL_1 + L_2)$. Consider two cases: $p_1(z_1^*, z_2^*) - AC \geq p_2(z_1^*, z_2^*) - C$ and $p_1(z_1^*, z_2^*) - AC < p_2(z_1^*, z_2^*) - C$. In the first case, product 1 has higher priority than product 2. Therefore, for any ϵ_1 and ϵ_2, if demand for product 2 is fully satisfied, demand for product 1 must also be fully satisfied. However, demand for both products cannot be all satisfied since $A(z_1^* + \epsilon_1) + z_2^* + \epsilon_2 \geq Az_1^* + z_2^* - (AL_1 + L_2) > y$. Consequently, demand for product 2 can only be partially satisfied, i.e., $q_2^*(z_1^*, z_2^*, \epsilon_1, \epsilon_2) < z_2^* + \epsilon_2$. In such a case, fixing

[5] By equation (4.10) and Assumption (ID-2), $\left|\frac{\partial[p_1(z_1,z_2)/A]}{dz_1}\right| = \frac{b_2}{A(b_1 b_2 - c_1 c_2)} > \left|\frac{\partial p_2(z_1,z_2)}{dz_1}\right| = \frac{c_2}{(b_1 b_2 - c_1 c_2)}$.

$\mathbf{q}^*(\mathbf{z}^*, \boldsymbol{\epsilon})$ and z_1^*, the seller can slightly decrease z_2^* and strictly improve his total expected profit. This contradicts with the optimality of z_1^*, z_2^*. Similarly, such a contradiction can be reached for the case when $p_1(z_1^*, z_2^*) - AC < p_2(z_1^*, z_2^*) - C$. The proof of $y - Az_1^* - z_2^* \geq -(AL_1 + L_2)$ is thus complete.

Now, suppose $y - Az_1^* - z_2^* > AH_1 + H_2$. It is easy to see that in such a case, $q_1^*(z_1^*, z_2^*, \epsilon_1, \epsilon_2) = z_1^* + \epsilon_1$ and $q_2^*(z_1^*, z_2^*, \epsilon_1, \epsilon_2) = z_2^* + \epsilon_2$. Hence, $\pi^{FO}(z_1^*, z_2^*) = (p_1(z_1^*, z_2^*) - AC)z_1^* + (p_2(z_1^*, z_2^*) - C)z_2^*$, which implies $z_1^* = z_1^U$ and $z_2^* = z_2^U$. By the hypothetical assumption, $y > \bar{y}$. However, this contradicts with the condition $y < \bar{y}$.

Moreover, by the first property and the feasibility constraints $z_1 \geq L_1, z_2 \geq L_2$, we have $y - Az_1^* \geq -AL_1 + (z_2^* - L_2) \geq -AL_1$ and $y - z_2^* \geq A(z_1^* - L_1) - L_2 \geq -L_2$. ■

C.10.2 Proof of Theorem 20

Proof: Recall the expression of $\pi^{FO}(z_1, z_2)$:

(i) If $p_1(z_1, z_2) - AC \geq A(p_2(z_1, z_2) - C)$

$$\begin{aligned}&\pi^{FO}(z_1, z_2)\\=&(p_1(z_1, z_2) - AC)\mathrm{E}_{\epsilon_1}[\min(z_1 + \epsilon_1, y/A)] + (p_2(z_1, z_2) - C)\mathrm{E}_{\epsilon_1,\epsilon_2}[\min(z_2 + \epsilon_2, (y - A(z_1 + \epsilon_1))^+)]\end{aligned}$$

(ii) If $p_1(z_1, z_2) - AC < A(p_2(z_1, z_2) - C)$

$$\begin{aligned}&\pi^{FO}(z_1, z_2)\\=&(p_1(z_1, z_2) - AC)\mathrm{E}_{\epsilon_1,\epsilon_2}[\min(z_1 + \epsilon_1, (y - (z_2 + \epsilon_2))^+/A)] + (p_2(z_1, z_2) - C)\mathrm{E}_{\epsilon_2}[\min(z_2 + \epsilon_2, y)]\end{aligned}$$

To show that $\pi^{FO}(z_1, z_2)$ is bimodal, it suffices to show that it is unimodal in each of the two cases (i) and (ii) above. Furthermore, by the same logic as that used in the proof of Theorem 17(i), it suffices to show that in either case (i) or (ii), the followings are true:

(a) Whenever $\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_1} = 0$ for some feasible z_1 and z_2, $\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1^2} < 0$;

(b) Whenever $\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_2} = 0$ for some feasible z_1 and z_2, $\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2} < 0$;

(c) Whenever $\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_1} = \frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_2} = 0$ for some feasible z_1 and z_2, $\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1^2}\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2} - \left(\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 > 0$.

Case (i): $p_1(\mathbf{z}) - AC \geq A(p_2(\mathbf{z}) - C)$

Perform a change of variables: let $y - Az_1 = M$ and $y - Az_1 - z_2 = N$. Then, $z_1 = \frac{y-M}{A}$, $z_2 = M - N$, and

$$\begin{pmatrix} p_1(M,N) \\ p_2(M,N) \end{pmatrix} = \frac{1}{b_1 b_2 - c_1 c_2}\begin{bmatrix} b_2(a_1 - y/A) + c_1 a_2 + (b_2/A - c_1)M + c_1 N \\ c_2(a_1 - y/A) + b_1 a_2 + (c_2/A - b_1)M + b_1 N \end{bmatrix}$$

and

$$\pi^{FO}(M,N) =(p_1(M,N)-AC)\mathrm{E}_{\epsilon_1}[\min(z_1(M)+\epsilon_1, z_1(M)+M/A)] + (p_2(M,N)-C)\left[\int_{-L_1}^{M/A} z_2(M,N) - \mathrm{E}_{\epsilon_2}[\max(Ax-N+\epsilon_2,0)]\, dG_1(x)\right]$$

By the feasibility conditions of z_1 and z_2, it is easy to derive the feasibility conditions of M and N:

$$-AL_1 \le M \le y - AL_1, L_2 \le M-N \le y+L_2, -(AL_1+L_2) \le N \le AH_1+H_2$$

$$p_1(M,N) \ge AC, p_2(M,N) \ge C, p_1(M,N)-AC \ge A(p_2(M,N)-C).$$

Let $\Delta_M(M,N)$, $\Delta_N(M,N)$, $\Delta_{MM}(M,N)$, $\Delta_{MN}(M,N)$, and $\Delta_{NN}(M,N)$ denote the first-order and second-order derivative functions of $\pi^{FO}(M,N)$, respectively. In the following, we first prove a technical lemma, Lemma 35, which is used to prove some properties of the interior solutions (i.e., those satisfying at least one of the first-order conditions) of $\pi^{FO}(M,N)$ in Lemma 36. These properties further imply (a) through (c), as shown in Proposition 12.

Lemma 35 *(Technical Lemma for Lemma 36) Whenever $\Delta_N(M,N)=0$ for some feasible M and N,*

$$\int_{-L_1}^{M/A}\left[(p_2(M,N)-C)g_2(N-Ax) - \frac{b_1}{b_1b_2-c_1c_2}\bar{G}_2(N-Ax)\right]\, dG_1(x) \ge 0$$

Proof: It is easy to derive

$$\begin{aligned}&\Delta_N(M,N)\\ =&\frac{c_1}{b_1b_2-c_1c_2}\mathrm{E}_{\epsilon_1}[\min(y/A, z_1(M)+\epsilon_1)] - (p_2(M,N)-C)\int_{-L_1}^{M/A} G_2(N-Ax)\, dG_1(x)\\ &+\frac{b_1}{b_1b_2-c_1c_2}\int_{-L_1}^{M/A}\mathrm{E}_{\epsilon_2}[\min(M-Ax, M-N+\epsilon_2)]\, dG_1(x)\end{aligned}$$

For $c_1>0$ and $z_1(M)\ge L$, when $\Delta_N(M,N)=0$,

$$\begin{aligned}&(p_2(M,N)-C)\int_{-L_1}^{M/A} G_2(N-Ax)\, dG_1(x)\\ >&\frac{b_1}{b_1b_2-c_1c_2}\int_{-L_1}^{M/A}\mathrm{E}_{\epsilon_2}[\min(M-Ax, M-N+\epsilon_2)]\, dG_1(x) \ge 0\end{aligned}$$

This implies

$$p_2(M,N)-C > \frac{b_1}{b_1b_2-c_1c_2}\frac{\int_{-L_1}^{M/A}\mathrm{E}_{\epsilon_2}[\min(M-Ax, M-N+\epsilon_2)]\, dG_1(x)}{\int_{-L_1}^{M/A} G_2(N-Ax)\, dG_1(x)}$$

Hence,

$$\begin{aligned}&\int_{-L_1}^{M/A}\left[(p_2(M,N)-C)g_2(N-Ax) - \frac{b_1}{b_1b_2-c_1c_2}\bar{G}_2(N-Ax)\right]\, dG_1(x)\\ \ge&\frac{b_1}{b_1b_2-c_1c_2}\int_{-L_1}^{M/A}\left[g_2(N-Ay)\frac{\int_{-L_1}^{M/A}\mathrm{E}_{\epsilon_2}[\min(M-Ax, M-N+\epsilon_2)]\, dG_1(x)}{\int_{-L_1}^{M/A} G_2(N-Ax)\, dG_1(x)} - \bar{G}_2(N-Ay)\right]\, dG_1(y)\end{aligned}$$

Define a function of N:

$$
\begin{aligned}
&f(N)\\
:=&\int_{-L_1}^{M/A}\int_{-L_1}^{M/A}\left[g_2(N-Ay)\mathrm{E}_{\epsilon_2}[\min(M-Ax,M-N+\epsilon_2)]-\bar{G}_2(N-Ay)G_2(N-Ax)\right]\,dG_1(x)dG_1(y)
\end{aligned}
$$

Since $\int_{-L_1}^{M/A}G_2(N-Ax)\,dG_1(x)>0$, to show

$$
\int_{-L_1}^{M/A}\left[(p_2(M,N)-C)g_2(N-Ax)-\frac{b_1}{b_1b_2-c_1c_2}\bar{G}_2(N-Ax)\right]\,dG_1(x)\geq 0,
$$

it suffices to show that $f(N)\geq 0$ for any N which is feasible and satisfies $\Delta_N(M,N)=0$. Notice that

$$
\begin{aligned}
&f'(N)\\
=&\int_{-L_1}^{M/A}\int_{-L_1}^{M/A}\left[g_2'(N-Ay)\mathrm{E}_{\epsilon_2}[\min(M-Ax,M-N+\epsilon_2)]-\bar{G}_2(N-Ay)G_2(N-Ax)\right]dG_1(x)dG_1(y)\\
\leq&0
\end{aligned}
$$

Furthermore, by Lemma 34 (ii), $N\leq AH_1+H_2$. When $N=AH_1+H_2$, for any $x,y\leq H_1$, $N-Ax,N-Ay\geq H_2$. Hence, $f(AH_1+H_2)=0$. This, together with $f'(N)\leq 0$, implies $f(N)\geq 0$ for all $N\leq AH_1+H_2$. ■

Lemma 36 *In case (i),*

(a) whenever $\Delta_M(M,N)+\Delta_N(M,N)=0$ *for some feasible* M *and* N, $\Delta_{MM}(M,N)+2\Delta_{MN}(M,N)+\Delta_{NN}(M,N)<0$.

(b) whenever $\Delta_N(M,N)=0$ *for some feasible* M *and* N, $\Delta_{NN}(M,N)<0$.

(c) whenever $\Delta_M(M,N)=0$ *and* $\Delta_N(M,N)=0$ *for some feasible* M *and* N,
$\Delta_{MM}(M,N)\Delta_{NN}(M,N)-(\Delta_{MN}(M,N))^2>0$.

Proof: We derive the proof into two cases: (i-1) $M>AH_1$, and (i-2) $M\leq AH_1$.

Case (i-1): $p_1(M,N)-AC\geq A(p_2(M,N)-C)$ and $M>AH_1$

In such a case,

$$\pi^{FO}(M,N) = (p_1(M,N) - AC)z_1(M)$$
$$+ (p_2(M,N) - C)\int_{-L_1}^{H_1} z_2(M,N) - \mathrm{E}_{\epsilon_2}[\max(\epsilon_2 + Ax - N, 0)]\, dG_1(x)$$
$$\Delta_M(M,N) = \frac{b_2/A - c_1}{b_1b_2 - c_1c_2} z_1(M) - (p_1(M,N) - AC)/A + (p_2(M,N) - C)$$
$$+ \frac{c_2/A - b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{H_1} \mathrm{E}_{\epsilon_2}[\min(M - Ax, M - N + \epsilon_2)]\, dG_1(x)$$
$$\Delta_N(M,N) = \frac{c_1}{b_1b_2 - c_1c_2} z_1(M) - (p_2(M,N) - C)\int_{-L_1}^{H_1} G_2(N - Ax)\, dG_1(x)$$
$$+ \frac{b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{H_1} \mathrm{E}_{\epsilon_2}[\min(M - Ax, M - N + \epsilon_2)]\, dG_1(x)$$
$$\Delta_{MM}(M,N) = -\frac{2(b_2 - AC_1 - Ac_2 + A^2b_1)}{A^2(b_1b_2 - c_1c_2)}$$
$$\Delta_{MN}(M,N) = \frac{b_1 - c_1/A}{b_1b_2 - c_1c_2} - \frac{c_2/A - b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{H_1} G_2(N - Ax)\, dG_1(x)$$
$$\Delta_{NN}(M,N) = -\frac{2b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{H_1} G_2(N - Ax)\, dG_1(x) - (p_2(M,N) - C)\int_{-L_1}^{H_1} G_2(N - Ax)\, dG_1(x)$$

(i-1-a) It is easy to derive

$$\Delta_{MM}(M,N) + 2\Delta_{MN}(M,N) + \Delta_{NN}(M,N)$$
$$= -\frac{2(b_2 - Ac_2)}{A^2(b_1b_2 - c_1c_2)} - \frac{2c_2/A}{b_1b_2 - c_1c_2}\int_{-L_1}^{H_1} G_2(N - Ax)\, dG_1(x)$$
$$- (p_2(M,N) - C)\int_{-L_1}^{H_1} G_2(N - Ax)\, dG_1(x) < 0$$

(i-1-b) By Lemma 35,

$$\Delta_{NN}(M,N) \le -\frac{b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{H_1} G_2(N - Ax)\, dG_1(x) - \frac{b_1}{b_1b_2 - c_1c_2} < 0 \tag{C.21}$$

(i-1-c) By equation (C.21),

$$\Delta_{MM}(M,N)\Delta_{NN}(M,N)$$
$$\ge \frac{2(b_2 - AC_1 - Ac_2 + A^2b_1)}{A^2(b_1b_2 - c_1c_2)}\left[\frac{b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{H_1} G_2(N - Ax)\, dG_1(x) + \frac{b_1}{b_1b_2 - c_1c_2}\right]$$

In the meantime, by the facts $0 \le G_2(N - Ax) \le 1$, $Ab_1 > \max(c_1, c_2)$, and $b_2 > A\max(c_1, c_2)$,

$$0 \le \Delta_{MN}(M,N) \le \frac{2b_1 - c_1/A - c_2/A}{b_1b_2 - c_1c_2} < \frac{2(b_2 - AC_1 - Ac_2 + A^2b_1)}{A^2(b_1b_2 - c_1c_2)}$$

and

$$0 \le \Delta_{MN}(M,N) < \frac{b_1}{b_1b_2 - c_1c_2} + \frac{b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{H_1} G_2(N - Ax)\, dG_1(x).$$

Therefore, $\Delta_{MM}(M,N)\Delta_{NN}(M,N) - (\Delta_{MN}(M,N))^2 > 0$.

Case (i-2): $p_1(M,N) - AC \geq A(p_2(M,N) - C)$ and $M \leq AH_1$

In such a case,

$$\begin{aligned}
\pi^{FO}(M,N) = &(p_1(M,N) - AC)z_1(M) - (p_1(M,N) - AC)\mathrm{E}_{\epsilon_1}[\max(\epsilon_1 - M/A, 0)] \\
&+ (p_2(M,N) - C)\int_{-L_1}^{M/A} z_2(M,N) - \mathrm{E}_{\epsilon_2}[\max(\epsilon_2 + Ax - N, 0)]\, dG_1(x) \\
\Delta_M(M,N) = &\frac{b_2/A - c_1}{b_1b_2 - c_1c_2}\mathrm{E}_{\epsilon_1}[\min(y/A, z_1(M) + \epsilon_1)] - (p_1(M,N) - AC)G_1(M/A)/A \\
&+ \frac{c_2/A - b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{M/A} \mathrm{E}_{\epsilon_2}[\min(M - Ax, M - N + \epsilon_2)]\, dG_1(x) \\
&+ (p_2(M,N) - C)G_1(M/A) \\
\Delta_N(M,N) = &\frac{c_1}{b_1b_2 - c_1c_2}\mathrm{E}_{\epsilon_1}[\min(y/A, z_1(M) + \epsilon_1)] - (p_2(M,N) - C)\int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x) \\
&+ \frac{b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{M/A} \mathrm{E}_{\epsilon_2}[\min(M - Ax, M - N + \epsilon_2)]\, dG_1(x) \\
\Delta_{MM}(M,N) = &-\frac{2(b_2/A - c_1 - c_2 + Ab_1)}{A(b_1b_2 - c_1c_2)}G_1(M/A) \\
&- [(p_1(M,N) - AC) - A(p_2(M,N) - C)]G_1(M/A)/A^2 \\
\Delta_{MN}(M,N) = &\frac{b_1 - c_1/A}{b_1b_2 - c_1c_2}G_1(M/A) + \frac{b_1 - c_2/A}{b_1b_2 - c_1c_2}\int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x) \\
\Delta_{NN}(M,N) = &-\frac{2b_1}{b_1b_2 - c_1c_2}\int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x) \\
&- (p_2(M,N) - C)\int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x)
\end{aligned}$$

(i-2-a) It is easy to derive

$$\begin{aligned}
\Delta_M(M,N) + \Delta_N(M,N) = &\frac{b_2/A}{b_1b_2 - c_1c_2}\mathrm{E}_{\epsilon_1}[\min(y/A, z_1(M) + \epsilon_1)] - (p_1(M,N) - AC)G_1(M/A)/A \\
&+ \frac{c_2/A}{b_1b_2 - c_1c_2}\int_{-L_1}^{M/A} \mathrm{E}_{\epsilon_2}[\min(M - Ax, M - N + \epsilon_2)]\, dG_1(x) \\
&+ (p_2(M,N) - C)\int_{-L_1}^{M/A} \bar{G}_2(N - Ax)\, dG_1(x)
\end{aligned}$$

Notice that when $M = -AL$, $\Delta_M(M,N) + \Delta_N(M,N) = \frac{b_2}{A(b_1b_2 - c_1c_2)}y/A > 0$. This implies that when $\Delta_M(M,N) + \Delta_N(M,N) = 0$, $M > -AL$. This fact further implies

$$\begin{aligned}
&\Delta_{MM}(M,N) + 2\Delta_{MN}(M,N) + \Delta_{NN}(M,N) \\
= & -\frac{2(b_2/A - c_1 - c_2 + Ab_1)}{A(b_1b_2 - c_1c_2)} G_1(M/A) - [(p_1(M,N) - AC) - A(p_2(M,N) - C)]G_1(M/A)/A^2 \\
& -\frac{2c_2/A}{b_1b_2 - c_1c_2} \int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x) - (p_2(M,N) - C) \int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x) \\
< & \; 0
\end{aligned}$$

(i-2-b) Notice that if $M = -AL$, $\Delta_N(M,N) = \frac{c_1}{b_1b_2 - c_1c_2} y/A > 0$. Hence, when $\Delta_N(M,N) = 0$, $M > -AL$, implying $G_1(M/A) > 0$. This fact, together with Lemma 35, implies

$$\begin{aligned}
\Delta_{NN}(M,N) &\leq -\frac{b_1}{b_1b_2 - c_1c_2} \int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x) - \frac{b_1}{b_1b_2 - c_1c_2} G_1(M/A) \\
&\leq -\frac{b_1}{b_1b_2 - c_1c_2} \int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x) - \frac{b_1}{b_1b_2 - c_1c_2} G_1(M/A) \qquad \text{(C.22)} \\
&< 0
\end{aligned}$$

(i-2-c) By equation (C.22) and the assumption $p_1(M,N) - AC \geq A(p_2(M,N) - C)$,

$$\Delta_{MM}(M,N) \leq -\frac{2(b_2/A - c_1 - c_2 + Ab_1)}{A(b_1b_2 - c_1c_2)} G_1(M/A)$$

and

$$\begin{aligned}
&\Delta_{MM}(M,N)\Delta_{NN}(M,N) \\
\geq & \frac{2(b_2/A - c_1 - c_2 + Ab_1)}{A(b_1b_2 - c_1c_2)} G_1(M/A) \cdot \frac{b_1}{b_1b_2 - c_1c_2} \left[\int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x) + G_1(M/A) \right]
\end{aligned}$$

In the meantime, it is proved in (i-2-b) that when $\Delta_N(M,N) = 0$, $G_1(M/A) > 0$. This fact, together with Assumption (A-4), implies

$$0 \leq \Delta_{MN}(M,N) \leq \frac{2b_1 - c_1/A - c_2/A}{b_1b_2 - c_1c_2} G_1(M/A) < \frac{2(b_2/A - c_1 - c_2 + Ab_1)}{A(b_1b_2 - c_1c_2)} G_1(M/A),$$

and

$$0 \leq \Delta_{MN}(M,N) < \frac{b_1}{b_1b_2 - c_1c_2} \left[\int_{-L_1}^{M/A} G_2(N - Ax)\, dG_1(x) + G_1(M/A) \right]$$

Therefore, $\Delta_{MM}(M,N)\Delta_{NN}(M,N) - (\Delta_{MN}(M,N))^2 > 0$. ■

Proposition 12 *In case (i),*

(a) Whenever $\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_1} = 0$ *for some feasible* z_1 *and* z_2, $\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1^2} < 0$;

(b) Whenever $\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_2} = 0$ *for some feasible* z_1 *and* z_2, $\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2} < 0$;

(c) Whenever $\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_1} = \frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_2} = 0$ *for some feasible* z_1 *and* z_2, $\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1^2} \frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2} - \left(\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1 \partial z_2} \right)^2 > 0$.

Proof: By $z_1 = \frac{y-M}{A}$ and $z_2 = M - N$,

$$\begin{aligned}
\Delta_M(M,N) &= -\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_1} \cdot \frac{1}{A} + \frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_2} \\
\Delta_N(M,N) &= -\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_2} \\
\Delta_{MM}(M,N) &= \frac{1}{A^2}\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1^2} - \frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2 \partial z_1} \cdot \frac{2}{A} + \frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2} \\
\Delta_{MN}(M,N) &= \frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2 \partial z_1} \cdot \frac{1}{A} - \frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2} \\
\Delta_{NN}(M,N) &= \frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2}
\end{aligned}$$

These imply

$$\begin{aligned}
\Delta_M(M,N) + \Delta_N(M,N) &= -\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_1} \cdot \frac{1}{A} \\
\Delta_{MM}(M,N) + 2\Delta_{MN}(M,N) + \Delta_{NN}(M,N) &= \frac{1}{A^2}\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1^2} \\
\Delta_{MM}(M,N)\Delta_{NN}(M,N) - (\Delta_{MN}(M,N))^2 &= \frac{1}{A^2}\left[\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1^2}\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2} - \left(\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2 \partial z_1}\right)^2\right]
\end{aligned}$$

(a) Whenever $\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_1} = 0$, $\Delta_M(M,N) + \Delta_N(M,N) = 0$. By Lemma 36 (a), $\Delta_{MM}(M,N) + 2\Delta_{MN}(M,N) + \Delta_{NN}(M,N) < 0$, which implies $\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1^2} < 0$.

(b) Whenever $\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_2} = 0$, $\Delta_N(M,N) = 0$. By Lemma 36 (b), $\Delta_{NN}(M,N) < 0$, which implies $\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2} < 0$.

(c) Whenever $\frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_1} = \frac{\partial \pi^{FO}(z_1,z_2)}{\partial z_2} = 0$, $\Delta_M(M,N) = \Delta_N(M,N) = 0$. By Lemma 36 (c), $\Delta_{MM}(M,N)\Delta_{NN}(M,N) - (\Delta_{MN}(M,N))^2 > 0$, which implies

$$\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_1^2}\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2^2} - \left(\frac{\partial^2 \pi^{FO}(z_1,z_2)}{\partial z_2 \partial z_1}\right)^2 > 0.$$

■

Case (ii): $p_1(\mathbf{z}) - AC < A(p_2(\mathbf{z}) - C)$

By the symmetry of the problem,[6] following exactly the same logic as in case (i), we can show that Proposition 12 also holds for case (ii). This thus proves the bimodality of $\pi^{FO}(z_1, z_2)$. ■

C.11 Proof of Lemma 12

Proof: We prove $A(z_1^* + H_1) + z_2^* + H_2 > y$ by contradiction. Suppose $A(z_1^* + H_1) + z_2^* + H_2 \leq y$. Similar to (i), it can be shown that $\mathbf{o}^* = \mathbf{z}^* + \mathbf{H}$ and $\mathbf{o}^*(\mathbf{z}^*, \mathbf{z}^* + \mathbf{H}, \boldsymbol{\epsilon}) = \mathbf{z}^* + \boldsymbol{\epsilon}$ for any realization

[6] We can switch the labels and unit component consumptions of the two products such that case (ii) is a modified version of case (i): product 1 uses one unit of component per product, product 2 uses A unit of component per product, and product 1's margin per component is higher than product 2's.

of $\boldsymbol{\epsilon}$. These, however, imply $\mathbf{z}^* = \mathbf{z}^U$, which, together with the hypothetical assumption, indicates $A(z_1^U + H_1) + z_2^U + H_2 = \bar{y} \le y$, obviously violating the condition $y < \bar{y}$.

To show $o_1^* \ge (y - (z_2^* + H_2))/A$, suppose $o_1^* < (y - (z_2^* + H_2))/A$. This implies that for any realization of ϵ_1 and ϵ_2, $A\min(o_1^*, z_1^* + \epsilon_1) + \min(o_2^*, z_2^* + \epsilon_2) \le Ao_1^* + z_2^* + H_2 < y$. This implies that $q_1^*(z_1^*, z_2^*, o_1^*, o_2^*, \epsilon_1, \epsilon_2) = \min(o_1^*, z_1^* + \epsilon_1)$ and $q_2^*(z_1^*, z_2^*, o_1^*, o_2^*, \epsilon_1, \epsilon_2) = \min(o_2^*, z_2^* + \epsilon_2)$. Hence, $\pi^{OO}(z_1^*, z_2^*, o_1^*, o_2^*) = (p_1(z_1^*, z_2^*) - AC)\mathrm{E}[\min(o_1^*, z_1^* + \epsilon_1)] + p_2((z_1^*, z_2^*) - C)\mathrm{E}[\min(o_2^*, z_2^* + \epsilon_2)]$. Clearly, in such a case the seller can strictly increase his total expected profit by slightly increasing o_1^* and keeping everything else the same. This, however, contradicts with the optimality of $(z_1^*, z_2^*, o_1^*, o_2^*)$ and proves $o_1^* \ge (y - (z_2^* + H_2))/A$. By the symmetry of the problem, $o_2^* \ge y - A(z_1^* + H_1)$ can be proved in a similar way. ■

C.12 Proof of Proposition 11

Proof: (i) Let $\hat{o}(z) = y - z - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$. By Proposition 10 and the symmetry of the problem, the optimal symmetric overselling strategy is:

if $\gamma = 0$, $o^*(z) = \min(y, z + H)$; if $\gamma > 0$, $o^*(z) = z + H$ if $y \ge \bar{y}$ and if $y < \bar{y}$,

$$o^*(z) = \begin{cases} y/2 & \text{if } z \ge y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right) \\ \hat{o}(z) & \text{if } z \in \left[y/2 - H/2 - \frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right), y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right) \\ z + H & \text{if } z \in \left(y/2 - H, y/2 - H/2 - \frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right). \end{cases} \tag{C.23}$$

Clearly $o^*(z)$ is continuous in z. If $\gamma = 0$, since $2(z + H) > y$ for any feasible z and $y \in (0, \bar{y})$, $o^*(z) > y/2$ for all $y \in (0, \bar{y})$. If $\gamma > 0$, $o^*(z) > y/2$ for $z \in \left(y/2 - H, y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)$.

(ii) For $\gamma = 0$,

$$\begin{aligned} \pi^{OO}(z, o^*(z)) &= (p(z) - C)\left\{2\mathrm{E}[\min(y, z + \epsilon_1)] - \mathrm{E}[\max(\min(y, z + \epsilon_1) + \min(y, z + \epsilon_2) - y, 0)]\right\} \\ &= (p(z) - C)\mathrm{E}[\min(\min(y, z + \epsilon_1) + \min(y, z + \epsilon_2), y)] \end{aligned}$$

To show that $\pi^{OO}(z, o^*(z))$ is strict quasi-concave, consider two cases: If $z \ge y + L$, $z + \epsilon_1 \ge y$ for any $\epsilon_1 \ge -L$. Hence, $\pi^{OO}(z, o^*(z)) = (p(z) - C)y$ and strictly decreases in z. Otherwise, it is easy to derive the first-order and second-order derivatives of $\pi^{OO}(z, o^*(z))$:

$$\begin{aligned} \frac{d\pi^{OO}(z, o^*(z))}{dz} &= -\frac{1}{b - c}\mathrm{E}[\min(\min(y, z + \epsilon_1) + \min(y, z + \epsilon_2), y)] \\ &\quad + (p(z) - C)\int_{-L}^{y-z} G(y - 2z - x)dG(x) \\ \frac{d^2\pi^{OO}(z, o^*(z))}{dz^2} &= -\frac{2}{b - c}\int_{-L}^{y-z} G(y - 2z - x)dG(x) - 2(p(z) - C)\int_{-L}^{y-z} g(y - 2z - x)dG(x) \end{aligned}$$

When $\frac{d\pi^{OO}(z,o^*(z))}{dz} = 0$, $\int_{-L}^{y-z} G(y-2z-x)dG(x) > 0$, which implies $\frac{d^2\pi^{OO}(z,o^*(z))}{dz^2} < 0$ and proves the strict quasi-concavity of $\pi^{OO}(z, o^*(z))$.

For $\gamma > 0$, by equation (C.23), to show the strict quasi-concavity of $\pi^{OO}(z, o^*(z))$, it suffices to show that $\pi^{OO}(z, o^*(z))$ is continuously differentiable and that $\pi^{OO}(z, y/2)$, $\pi^{OO}(z, \hat{o}(z))$, and $\pi^{OO}(z, z+H)$ are all strict quasi-concave in z. To this end, first note the first-order and second-order derivatives of these functions in z as follows:

$$\begin{aligned}
\frac{d\pi^{OO}(z,y/2)}{dz} &= -\frac{2}{b-c}\mathrm{E}[\min(y/2, z+\epsilon_1)] + 2(p(z)-C)G(y/2-z) \\
\frac{d^2\pi^{OO}(z,y/2)}{dz^2} &= -\frac{4}{b-c}G(y/2-z) - 2(p(z)-C)g(y/2-z) \\
\frac{d\pi^{OO}(z,\hat{o}(z))}{dz} &= -\frac{1}{b-c}\{2\mathrm{E}[\min(\hat{o}(z), z+\epsilon_1)] - (1+\gamma)\mathrm{E}[\max(\min(\hat{o}(z), z+\epsilon_1) \\
&\qquad + \min(\hat{o}(z), z+\epsilon_2) - y, 0)]\} \\
&\quad + 2(p(z)-C)\left\{G(\hat{o}(z)-z) - (1+\gamma)\int_{y-\hat{o}(z)-z}^{\hat{o}(z)-z} \bar{G}(y-2z-x)dG(x)\right\} \\
\frac{d^2\pi^{OO}(z,\hat{o}(z))}{dz^2} &= -\frac{4}{b-c}\left\{G(\hat{o}(z)-z) - (1+\gamma)\int_{y-\hat{o}(z)-z}^{\hat{o}(z)-z} \bar{G}(y-2z-x)dG(x)\right\} \\
&\quad - 4(p(z)-C)(1+\gamma)\int_{y-\hat{o}(z)-z}^{\hat{o}(z)-z} g(y-2z-x)\, dG(x)
\end{aligned}$$

For $z > y/2 - H$,

$$\begin{aligned}
\frac{d\pi^{OO}(z,z+H)}{dz} &= -\frac{1}{b-c}\{2z - (1+\gamma)\mathrm{E}[\max(\epsilon_1+\epsilon_2+2z-y, 0)]\} \\
&\quad + 2(p(z)-C)\left\{1 - (1+\gamma)\int_{y-2z-H}^{H} \bar{G}(y-2z-x)dG(x)\right\} \\
\frac{d^2\pi^{OO}(z,z+H)}{dz^2} &= -\frac{4}{b-c}\left\{1 - (1+\gamma)\int_{y-2z-H}^{H} \bar{G}(y-2z-x)dG(x)\right\} \\
&\quad - 4(p(z)-C)(1+\gamma)\int_{y-2z-H}^{H} g(y-2z-x)\, dG(x)
\end{aligned}$$

For $z \leq y/2 - H$,

$$\begin{aligned}
\frac{d\pi^{OO}(z,z+H)}{dz} &= -\frac{2z}{b-c} + 2(p(z)-C) \\
\frac{d^2\pi^{OO}(z,z+H)}{dz^2} &= -\frac{4}{b-c} < 0
\end{aligned}$$

It is straightforward to show that at $z = y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$, $\frac{d\pi^{OO}(z,y/2)}{dz} = \frac{d\pi^{OO}(z,\hat{o}(z))}{dz}$, and that at $z = y/2 - H/2 - \frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$, $\frac{d\pi^{OO}(z,\hat{o}(z))}{dz} = \frac{d\pi^{OO}(z,z+H)}{dz}$, and that $\frac{d\pi^{OO}(z,z+H)}{dz}$ is continuous at $z = y/2 - H$. Hence, $\pi^{OO}(z, o^*(z))$ is continuously differentiable.

To show the strict quasi-concavity of all three functions, it suffices to show that for each function, whenever the first-order derivative is zero, the second-order derivative is negative:

For $\pi^{OO}(z, y/2)$, whenever $d\pi^{OO}(z, y/2)/dz = 0$ for some $z \geq L$, $p(z) - C > 0$ and $G(y/2 - z) > 0$, which imply $d^2\pi^{OO}(z, y/2)/dz^2 < -2/(b-c) < 0$.

For $\pi^{OO}(z, \hat{o}(z))$, first note that since $\hat{o}(z) > y/2$ for $z \in \left[y/2 - H/2 - \frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right), y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)$, $\hat{o}(z) - z > y - \hat{o}(z) - z = (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$, which implies

$$G(\hat{o}(z) - z) > (1+\gamma)\int_{y-\hat{o}(z)-z}^{\hat{o}(z)-z} \bar{G}(y - 2z - x)dG(x) \text{ and } \int_{y-\hat{o}(z)-z}^{\hat{o}(z)-z} g(y - 2z - x)\, dG(x) > 0$$

Meanwhile, by feasibility condition, $p(z) - C \geq 0$. These facts jointly imply that whenever $d\pi^{OO}(z, \hat{o}(z))/dz = 0$, $d^2\pi^{OO}(z, \hat{o}(z))/dz^2 < 0$.

For $\pi^{OO}(z, z + H)$ with $z \in \left(y/2 - H, y/2 - H/2 - \frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right]$, the proof is similar to that for $\pi^{OO}(z, \hat{o}(z))$ once we note that for z in such a range, $\bar{G}(y - 2z - H) \leq \frac{1}{1+\gamma}$.

For $\pi^{OO}(z, z + H)$ with $z \leq y/2 - H$, the function has a negative second-order derivative and hence is strictly concave, which is automatically strictly quasi-concave. This completes the proof of (ii).

(iii) By (ii), $z^{ID,OO}$ is the unique maximizer of $\pi^{OO}(z, o^*(z))$ subject to $z \geq L$ and $p(z) \geq C$, and $o^{ID,OO} = o^*\left(z^{ID,OO}\right)$.

(iv) If $\gamma = 0$, by (i), clearly $\tilde{y}_1$ exists and equals to zero. If $\gamma > 0$, the case $y \geq \bar{y}$ is trivial: obviously $z^{ID,OO} = z^U$ and $o^{ID,OO} = z^U + H \leq y/2$. For $y < \bar{y}$, note that by (i) and (iii), $o^{ID,OO} > y/2$ if and only if $z^{ID,OO} \in \left(y/2 - H, y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)$. Furthermore, by (ii), $z^{ID,OO} \in \left(y/2 - H, y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)$ if and only if $L < y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ and $\frac{d\pi^{OO}(z,o^*(z))}{dz}$ is negative at $z = y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$.[7] To show the existence of $\tilde{y}_1$, it then suffices to show that $\frac{d\pi^{OO}(z,o^*(z))}{dz}$ at $z = y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ strictly decreases in y and is negative when $y = \bar{y}$.

At $z = y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$,

$$\begin{aligned} &\frac{d\pi^{OO}(z, o^*(z))}{dz} \\ =& -\frac{2}{b-c}\mathrm{E}[\min(y/2, z + \epsilon_1)] + 2(p(z) - C)G(y/2 - z) \\ =& -\tfrac{2}{b-c}\left\{y/2 + \mathrm{E}\left[\min\left(0, -(\bar{G})^{-1}\left(\tfrac{1}{1+\gamma}\right) + \epsilon_1\right)\right]\right\} + \tfrac{2\gamma}{1+\gamma}\left(p\left(y/2 - (\bar{G})^{-1}\left(\tfrac{1}{1+\gamma}\right)\right) - C\right) \end{aligned} \quad \text{(C.24)}$$

Since $p(z)$ strictly decreases in z, clearly $\frac{d\pi^{OO}(z,o^*(z))}{dz}$ at $z = y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ strictly decreases

[7] Note that the condition $p(z) \geq C$ is implied. To see why, consider two cases: if $\frac{d\pi^{OO}(z,o^*(z))}{dz}$ is nonnegative at $z = L$, then $z^{ID,OO}$ is the solution to $\frac{d\pi^{OO}(z,o^*(z))}{dz} = 0$ between L and $y/2 - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ and it is easy to check that the solution to first-order condition always satisfies $p(z) \geq C$; if, however, $\frac{d\pi^{OO}(z,o^*(z))}{dz}$ is negative at $z = L$, then clearly $z^{ID,OO} = L$ and by assumption, $p(L) \geq C$.

in y. Further, when $y=\bar{y}$, $y/2-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right) > z^U = z^{ID,OO}$. This fact and (ii) jointly implies that $\frac{d\pi^{OO}(z,o^*(z))}{dz} < 0$ at $z = y/2-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ when $y=\bar{y}$. This completes the proof of (iv).

(v) If $\gamma = 0$, by (i), clearly $\tilde{y}_2$ exists and equals to zero.

If $\gamma > 0$, the case $y \geq \bar{y}$ is trivial: obviously $o^{ID,OO} = z^U + H = \min(y, z^U+H)$. For $y < \bar{y}$, note that by (i), $o^{ID,OO}$ either equals to $y/2$ or $\min(\hat{o}, z^{ID,OO}+H)$, which is always less than y since $\hat{o} < y$ for $\gamma > 0$ and $z^{ID,OO} \geq L$. Furthermore, by the feasibility condition of $z^{ID,OO}$, $z^{ID,OO}+H > y/2$. Therefore, $o^{ID,OO} = \min(y, z^{ID,OO}+H)$ if and only if $o^{ID,OO} = z^{ID,OO}+H$. Further, by (i), $o^{ID,OO} = z^{ID,OO}+H$ if and only if $z^{ID,OO} \leq y/2 - H/2 - \frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$, which by (ii), occurs if and only if $L \leq y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ and $\frac{d\pi^{OO}(z,z+H)}{dz}$ at $z = y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ is non-positive. To show the existence of $\tilde{y}_2$, it then suffices to show that $\frac{d\pi^{OO}(z,z+H)}{dz}$ at $z = y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ is non-increasing in y and negative when $y = \bar{y}$. Also note that it is easy to see that $o^{ID,OO} = \min(y, z^{ID,OO}+H)$ only if $o^{ID,OO} > y/2$, which further implies that if $\tilde{y}_2$ exists, $\tilde{y}_2 \geq \tilde{y}_1$.

By (ii), when $z = y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right) > y/2-H$,

$$\begin{aligned}\frac{d\pi^{OO}(z,z+H)}{dz} = & -\frac{1}{b-c}\Bigg\{y-H-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\\ & -(1+\gamma)\mathrm{E}\left[\max\left(\epsilon_1+\epsilon_2-H-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right),0\right)\right]\Bigg\}\\ & +2\left(p\left(y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)-C\right)\cdot\\ & \cdot\left\{1-(1+\gamma)\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H}\bar{G}\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)dG(x)\right\}\end{aligned}$$

Since $p(z)$ strictly decreases in z, clearly $\frac{d\pi^{OO}(z,z+H)}{dz}$ at $z = y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ strictly decreases in y. Further, when $y=\bar{y}$, $y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right) > z^U = z^{ID,OO}$. This fact and (ii) jointly implies that $\frac{d\pi^{OO}(z,z+H)}{dz} < 0$ at $z = y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ when $y=\bar{y}$. This completes the proof of (v).

(vi) The case when $\gamma = 0$ has been proved in the proof of (iv) and (v).

To show the monotonicity of $\tilde{y}_1$ in γ, recall that in (iv), $\tilde{y}_1$ equals to either $2L+2(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ or the infimum of the set of all y in $[0,\bar{y})$ satisfying $\frac{d\pi^{OO}(z,o^*(z))}{dz} < 0$ at $z = y/2-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$, whichever is greater. To show the monotonicity of $\tilde{y}_1$ in γ, since $2L+2(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ strictly increases in γ, it then suffices to show that for given $y < \bar{y}$, $\frac{d\pi^{OO}(z,o^*(z))}{dz}$ at $z = y/2-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ is non-decreasing in γ. Since $y/2-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ is non-increasing in γ, by equation (C.24), $\frac{d\pi^{OO}(z,o^*(z))}{dz}$ at $z = y/2-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ is clearly non-decreasing in γ.

Similarly, to show the monotonicity of $\tilde{y}_1$ in γ, following similar logic, it suffices to show that for given $y < \bar{y}$, $\frac{d\pi^{OO}(z,z+H)}{dz}$ at $z = y/2 - H/2 - \frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ is non-decreasing in γ. By (ii),

$$\frac{d\left\{\left.\frac{d\pi^{OO}(z,z+H)}{dz}\right|_{z=y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}\right\}}{d\gamma}$$
$$= \left.\frac{d^2\pi^{OO}(z,z+H)}{dz^2}\right|_{z=y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)} \cdot \frac{d\left\{y/2 - H/2 - \frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right\}}{d\gamma}$$
$$+ \left.\frac{d^2\pi^{OO}(z,z+H)}{dzd\gamma}\right|_{z=y/2-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}$$

where

$$\left.\frac{d^2\pi^{OO}(z,z+H)}{dz^2}\right|_{z=y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}$$
$$= -\frac{4}{b-c}\left\{1 - (1+\gamma)\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H} \bar{G}\left(H + (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right) - x\right) dG(x)\right\}$$
$$- 4(p(z) - C)(1+\gamma)\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H} g\left(H + (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right) - x\right)\, dG(x),$$

$$\frac{d\left\{y/2 - H/2 - \frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right\}}{d\gamma} = -\frac{1}{2(1+\gamma)^2 g\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)},$$

and

$$\left.\frac{d^2\pi^{OO}(z,z+H)}{dzd\gamma}\right|_{z=y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}$$
$$= \frac{1}{b-c}\mathrm{E}\left[\max\left(\epsilon_1 + \epsilon_2 - H - (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right), 0\right)\right]$$
$$- 2(p(z) - C)\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H} \bar{G}\left(H + (\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right) - x\right) dG(x).$$

Hence,

$$\frac{d\left\{\left.\frac{d\pi^{OO}(z,z+H)}{dz}\right|_{z=y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}\right\}}{d\gamma}$$

$$= \frac{2}{(b-c)(1+\gamma)^2 g\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)}\left\{1-(1+\gamma)\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H}\bar{G}\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)dG(x)\right\}$$

$$+2\frac{p(z)-C}{(1+\gamma)g\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)}\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H} g\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)\,dG(x)$$

$$+\frac{1}{b-c}\mathrm{E}\left[\max\left(\epsilon_1+\epsilon_2-H-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right),0\right)\right]$$

$$-2(p(z)-C)\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H}\bar{G}\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)dG(x)$$

$$= \frac{2}{(b-c)(1+\gamma)^2 g\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)}\left\{1-(1+\gamma)\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H}\bar{G}\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)dG(x)\right\}$$

$$+\frac{1}{b-c}\mathrm{E}\left[\max\left(\epsilon_1+\epsilon_2-H-(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right),0\right)\right]$$

$$+2(p(z)-C)\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H}\frac{\bar{G}\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)g\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)}{g\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)}$$

$$-\bar{G}\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)\,dG(x)$$

Since $G(x)$ has the IFR property, $\frac{g\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)}{\bar{G}\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)}$ is non-increasing in x. Hence for $x \le H$,

$$\frac{g\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)}{\bar{G}\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)} \ge \frac{g\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)}{\bar{G}\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)}$$

Therefore,

$$\int_{(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}^{H}\frac{\bar{G}\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)g\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)}{g\left((\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)\right)}-\bar{G}\left(H+(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)-x\right)\,dG(x)\ge 0.$$

and hence all the terms in the expression of $d\left\{\left.\frac{d\pi^{OO}(z,z+H)}{dz}\right|_{z=y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)}\right\}\Big/d\gamma$ are non-negative, implying that $\frac{d\pi^{OO}(z,z+H)}{dz}$ at $z=y/2-H/2-\frac{1}{2}(\bar{G})^{-1}\left(\frac{1}{1+\gamma}\right)$ is non-decreasing in γ. This completes the proof of (vi). ∎

C.13 Calculation of Consumer Surplus and Social Surplus

First we briefly review the calculation of consumer surplus in Economics. As illustrated in Figure C.2, when the supply q is sufficient, consumer surplus is the area under the maximum price $p(d)$ (calculated from aggregated demand curve $d(p)$) and above the actual price p. That is, $CS = (p(0)-$

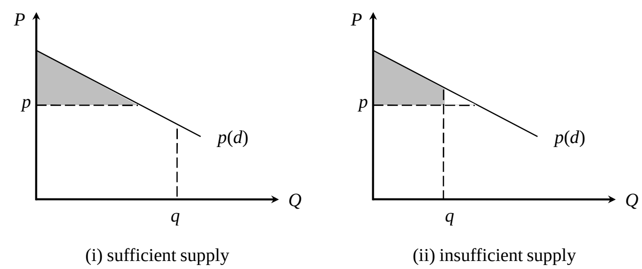


Figure C.2. Consumer surplus (the shaded areas)

$p)d(p)/2$. If the supply is limited, i.e., the supply is lower than the quantity that customers are willing to buy at current price ($p(q) > p$), consumer surplus is the area under the maximum price, above the actual price, and to the left of the quantity actually supplied. That is, $CS = (p(0) + p(q) - 2p)q/2$. Combining these two cases, consumer surplus is

$$CS = (p(0) - p + (p(q) - p)^+) \min(d(p), q)/2. \tag{C.25}$$

When overselling occurs, consumer surplus is composed of two parts: for customers who obtain a product, their aggregate consumer surplus can be calculated using equation (C.25) with q being the actual number of orders fulfilled; for customers whose orders are initially accepted but later cancelled, their aggregate consumer surplus is the cancellation penalty received per order multiplied by the total number of cancelled orders.

We now calculate the consumer surplus for given prices p_1, p_2 and overselling quantities o_1, o_2, and realized demand shocks ϵ_1, ϵ_2. From the demand function $d_i(p_i) = z_i + \epsilon_i = a_i - b_i p_i + c_i p_j + \epsilon_i$, we can derive the maximum price for each product: $p_i(d_i) = (a_i + c_i p_j + \epsilon_i - d_i)/b_i$, $i = 1, 2$, $j = 3 - i$. For each product, the number of orders actually fulfilled depends on the priority of the product. We thus consider the following two cases:

- $p_1 - AC + s_1 \geq A(p_2 - C + s_2)$

  In such a case, product 1 has higher priority and thus for product 1, the number of orders fulfilled equals to the number of orders accepted, i.e., $q_1 = \min(o_1, z_1 + \epsilon_1)$. For product 2, some of the orders may be cancelled: the number of orders fulfilled is $q_2 = \min(\min(o_2, z_2 + \epsilon_2), y - A\min(o_1, z_1 + \epsilon_1))$ and the number of orders cancelled is $\min(o_2, z_2 + \epsilon_2) - q_2 = \max(A\min(o_1, z_1 + \epsilon_1) + \min(o_2, z_2 + \epsilon_2) - y, 0)$. Hence, total consumer surplus is

$$\begin{aligned} CS(\epsilon_1, \epsilon_2) = {} & (p_1(0) - p_1 + (p_1(q_1) - p_1)^+) \min(d_1(p_1), q_1)/2 \\ & + (p_2(0) - p_2 + (p_2(q_2) - p_2)^+) \min(d_2(p_2), q_2)/2 \\ & + s_2(\min(o_2, z_2 + \epsilon_2) - q_2) \end{aligned}$$

- $p_1 - AC + s_1 < A(p_2 - C + s_2)$

  Similarly to the first case, we can derive that $q_1 = \min(\min(o_1, z_1+\epsilon_1), (y-\min(o_2, z_2+\epsilon_2))/A)$, $q_2 = \min(o_2, z_2 + \epsilon_2)$, and

$$\begin{aligned} CS(\epsilon_1, \epsilon_2) = & (p_1(0) - p_1 + (p_1(q_1) - p_1)^+) \min(d_1(p_1), q_1)/2 \\ & + (p_2(0) - p_2 + (p_2(q_2) - p_2)^+) \min(d_2(p_2), q_2)/2 \\ & + s_1(\min(o_1, z_1 + \epsilon_1) - q_1) \end{aligned}$$

Before the realization of demand shocks, the expected consumer surplus is $CS = \mathrm{E}_{\epsilon_1,\epsilon_2}[CS(\epsilon_1, \epsilon_2)]$. The social surplus, by definition, is the sum of expected consumer surplus and firm's expected profit.

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