Multigraded Regularity and Asymptotic Invariants

by

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To my family and my friends I met on both sides of the Atlantic

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER I

Introduction

This thesis consists of two mutually independent parts.

Chapters II and III constitute the first part, where our main concern is to study multigraded regularity on multiprojective spaces. The principal result is the bound we give on regularity for curves on biprojective spaces.

In the second part, Chapters IV and V, we study asymptotic invariants associated to linear series. This is joint work with C. Maclean and A. Küronya. In Chapter IV we focus on Okounkov bodies, which were first introduced by Lazarsfeld and Mustaţă in [34]. Our main concern is to ask what can be said about the set of convex bodies that appear as Okounkov bodies. We show first that the set of convex bodies appearing as the Okounkov bodies of a big Cartier divisor on a smooth projective variety with respect to an admissible flag is countable. We then give a complete characterization of the set of convex bodies which arise as Okounkov bodies of $\mathbb{R}$-divisors on smooth projective surfaces. We will show that such Okounkov bodies are always polygons, satisfying certain combinatorial criteria. In Chapter V, we study another asymptotic invariant, called the volume function associated to an irreducible projective variety or a multigraded linear series. In the classical case, we prove that the collection of functions which appear as the volume function of an irreducible projective variety is countable. By contrast, in the non-complete case we are able to show that there are uncountably many volume functions. More specifically, we prove that any continuous, log-concave and homogeneous function appears as the volume function of a multigraded linear series.

This thesis is adapted from the author's three papers [35], [28] and [29]. The last two preprints are joint work with Catriona Maclean and Alex Küronya.

In the remainder of this Chapter, we give a more detailed introduction to the contents of this thesis.

1.1 Regularity of smooth curves in biprojective spaces

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. Then Serre's Theorem ([24, Proposition III.5.3]) says that after twisting $\mathcal{F}$ by a sufficiently high multiple of the hyperplane line bundle, the higher cohomology groups of $\mathcal{F}$ vanishes. Mumford in [40] had the idea to introduce an invariant which gives a quantitative measure of how much one has to twist in order for the higher cohomology to vanish.

Specifically, recall that a coherent sheaf $\mathcal{F}$ is m-regular in the sense of Castelnuovo-Mumford if

$$H^i(\mathbb{P}^r, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(m-i)) \ = \ 0, \text{ for all } i > 0.$$

One then defines $\text{reg}(\mathcal{F}) \ = \ \min\{m \in \mathbb{Z} \mid \mathcal{F} \text{ is } m-\text{regular }\}$. Mumford showed that if $\mathcal{F}$ is m-regular, then $\mathcal{F}(m)$ is globally generated, and moreover it is also $(m+n)$-regular for all $n \in \mathbb{N}$. Thus this invariant bounds the cohomological complexity of the coherent sheaf. On the other hand, we shall see in §II.1 that it also bounds the algebraic complexity of the coherent sheaf, and for this reason regularity has been the focus of considerable recent activity, e.g. [11], [41], [18], [19], [17], [32], [13],[30], [31].

In §II.1 we give the definition and basic properties of Castelnuovo-Mumford regularity. We explain the complexity-theoretic meaning of this invariant and survey without proof several results giving bounds on regularity.

Inspired by the importance of Castelnuovo-Mumford regularity, both in algebraic geometry and commutative algebra, in recent years several authors extended the definition of regularity to the multigraded setting, e.g. [2], [36], [20], [51]. Notably, motivated by toric geometry, Maclagan and Smith in [36] introduced a multigraded variant of Castelnuovo-Mumford regularity.

Our main focus in Chapter II and III is to study multigraded regularity on multiprojective spaces. Suppose $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a multiprojective space, for some $n_1, \ldots, n_k \geq 1$ and $k \geq 1$, and $\mathcal{F}$ is a coherent sheaf on Y. If $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$ is an integral vector, then Maclagan and Smith say that $\mathcal{F}$ is $\mathbf{m}$-regular if

$$H^i(Y, \mathcal{F} \otimes \mathcal{O}_Y(\mathbf{m} - \mathbf{u})) \; = \; 0,$$

for all $i > 0$ and $\mathbf{u} \in \mathbb{N}^k$, such that $|\mathbf{u}| := u_1 + \cdots + u_k = i$. Inspired by Mumford's result, Hering, Schenck, and Smith prove in [23] its multigraded analogue, which in our setup says that if $\mathcal{F}$ is $\mathbf{m}$-regular, then $\mathcal{F}(\mathbf{m})$ is globally generated, and moreover it is also $(\mathbf{m} + \mathbf{u})$-regular for all $\mathbf{u} \in \mathbb{N}^k$. Thus, as in the classical case, the multigraded version of regularity of Maclagan and Smith gives a quantative measure of the cohomological complexity of a coherent sheaf. Also, if we define $\mathbf{reg}(\mathcal{F}) = \{\mathbf{m} \in \mathbb{Z}^k \mid \mathcal{F} \text{ is } \mathbf{m}-\text{regular}\}$, then the same result implies that this set is a union of positive cones

$$\mathbf{reg}(\mathcal{F}) \; = \bigcup_{\mathcal{F} \text{ is } \mathbf{m}-\text{regular}} (\mathbf{m} + \mathbb{N}^k)$$

By contrast to the classical case–where Mumford worked with a sheaf-theoretic geometric approach to regularity–the main work in the multigraded setting was done in the commutative algebra setting. In this context is easy to see that for finitely generated modules the regularity set is finitely generated (see Definition 2.24). In the geometric setting the question of when the regularity set is finitely generated is more complex. The answer to this question is the first result of this thesis. If we denote by

$$\pi_i : Y \to Y_i \ := \ \mathbb{P}^{n_1} \times \cdots \times \widehat{\mathbb{P}^{n_i}} \times \cdots \times \mathbb{P}^{n_k}$$

the projection that drops the i-th coordinate of Y, then inspired by a result of Hà, [20, Proposition 3.3.2], we prove in §II.4 the following theorem:

Theorem A. *Let $\mathcal{F}$ be a coherent sheaf on the multiprojective space Y. Then the following two conditions are equivalent*

(i) For each $i = 1, \dots, k$ there exists a point $x_i \in Y_i$ such that

$$\dim(\operatorname{supp}(\mathcal{F}|_{\pi_i^{-1}(x_i)})) > 0,$$

(ii) The set $\mathbf{reg}(\mathcal{F})$ is finitely generated [1] as a subset of $\mathbb{Z}^k$.

In the remainder of §II.2 and §II.3, we explain the algebraic approach to multigraded regularity given by Maclagan and Smith. In this setup we overview briefly the work of Hà, [20], which gives a relationship between algebraic and cohomological complexity. More specifically, Hà proves upper bounds on regularity in terms of degrees of syzygies. We end Chapter III by surveying some results about bounding regularity in terms of geometric data. We present first the work of Maclagan and Smith, which bounds regularity in terms of multigraded Hilbert polynomial. Also we show briefly how one can extend some results on bounding Castelnuovo-Mumford regularity to the multigraded case.

[1] We say that $\mathbf{reg}(\mathcal{F})$ is finitely generated as a subset of $\mathbb{Z}^k$ if there exists $\mathbf{m}_1, \dots, \mathbf{m}_l \in \mathbb{Z}^k$ such that $\mathbf{reg}(\mathcal{F}) = \bigcup_{i=1}^{i=l}(\mathbf{m}_i + \mathbb{N}^k)$.

With respect to bounding regularity in terms of geometric data, one question became the focus of considerable research. Inspired by the work of Castelnuovo [11], it is natural to seek upper bounds for the regularity of a smooth projective variety $X \subseteq \mathbb{P}^r$ in terms of its degree. For example, a celebrated result of Gruson, Lazarsfeld and Peskine [17] states that for an irreducible (possible singular) reduced nondegenerate curve $C \subseteq \mathbb{P}^r$ $(r \geq 3)$ of degree d, the ideal sheaf $\mathcal{I}_{C/\mathbb{P}^r}$ is $(d+2-r)$-regular.

In Chapter III, we give a multigraded counterpart to the result of [17]. More specifically, we prove the following theorem:

Theorem B. *Let $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ $(a, b \geq 2)$ be a smooth curve of bidegree (d_1, d_2) with nondegenerate birational projections. The the ideal sheaf $\mathcal{I}_{C|\mathbb{P}^a \times \mathbb{P}^b}$ is $(d_2 - b + 1, d_1 - a + 1)$-regular.*

As a corollary, Theorem B together with ([36, Theorem 1.4]) imply the inclusion:

$$\big((d_2 - b + 1, d_1 - a + 1) + \mathbb{N}^2\big) \subseteq \operatorname{reg}(\mathcal{I}_{C|\mathbb{P}^a \times \mathbb{P}^b})$$

By comparison to the methods used in [17], the proof of Theorem B, which will take most of Chapter III, uses generic projections and vector bundle techniques developed by Gruson, Peskine ([18], [19]), and Lazarsfeld ([32]) in the classical case. The main change is that instead of projecting to $\mathbb{P}^2$ we choose projections to $\mathbb{P}^1 \times \mathbb{P}^1$. The idea is to prove first that whenever $a \neq b$ or $r := a = b$ and the curve C is not included in the graph of an automorphism of $\mathbb{P}^r$, then there are plenty of projections to $\mathbb{P}^1 \times \mathbb{P}^1$, with the image of C having "nice" singularities. In §III.2 these projections will play an important role in establishing the bound in Theorem B. The remaining case, when the curve is included in the graph of an automorphism of $\mathbb{P}^r$, will be discussed in §III.3. There we will use the result of [17] to show that the same bound works. We end Chapter III with an example of a rational curve $C \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(3,3)$. This curve has the property that $(2,2) + \mathbb{N}^2 = \operatorname{reg}(C)$, showing in this case that the bounds we have in Theorem B are the best possible.

1.2 Okounkov bodies

In Chapter IV of this thesis we study an interesting construction of Lazarsfeld and Mustaţă in a recent paper [34], which was motivated by earlier works of Okounkov [44], [45].[2] The construction associates a convex body in $\mathbb{R}^n$, called the Okounkov body, to any big divisor D on a n-dimensional complex projective smooth variety X. Specifically, fix on X an admissible flag

$$Y_\bullet \ : \ X = Y_0 \ \supseteq \ Y_1 \ \supseteq \ Y_2 \ \supseteq \ \ldots \ \supseteq \ Y_{n-1} \ \supseteq \ Y_n = \{\mathrm{pt}\}$$

where Y_i is a smooth irreducible subvariety of codimension i in X. This flag determines a valuation $\nu : H^0(X, \mathcal{O}_X(mD)) \to \mathbb{Z}^n$ for all $m \in \mathbb{N}$, and the Okounkov body of D with respect to the flag $Y_\bullet$ is by definition the compact set

$$\Delta_{Y_\bullet}(X; D) \ := \ \text{closed convex hull}\Big(\bigcup_{m \geq 1} \frac{1}{m} \cdot \nu\big(H^0(X, \mathcal{O}_X(mD))\big) \Big) \subseteq \mathbb{R}^n \ .$$

(Sometimes it will be preferable to use the language of line bundles. If $\mathcal{L}$ is a line bundle on X, we write $\Delta_{Y_\bullet}(X; \mathcal{L}) \subseteq \mathbb{R}^n$ for the Okounkov body of a divisor D with $\mathcal{O}_X(D) = \mathcal{L}$.)

The Okounkov body encodes many asymptotic invariants of divisor D, and Lazarsfeld and Mustaţă link its properties to the geometry of D. For example, whenever D is big we have that

$$\operatorname{vol}_X(D) \ = \ n! \cdot \operatorname{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(X; D)) \tag{1.1}$$

[2] The same construction was studied independently by Kaveh and Khovanskii, [26]

where the right hand side is the Euclidean volume of $\Delta_{Y_\bullet}(X;D)$. This viewpoint renders transparent several basic properties about the volume of big divisors and will be used extensively in our study of the volume function in Chapter VI.

Our main concern here is to study the set of those convex bodies which appear as Okounkov bodies of line bundles on smooth varieties with respect to admissible flags. Our first theorem, proved in §IV.2, shows that this set is countable.

Theorem C. *Let $n \geq 1$ be a natural number. There then exists a countable set of bounded convex bodies $(\Delta_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n$ such that for any big divisor D on a n-dimensional complex smooth projective variety X and for any admissible flag $Y_\bullet$ on X, the Okounkov body $\Delta_{Y_\bullet}(X;D)$ is identified with Δ_i for some i.*

In [34] it was shown that for any smooth variety X equipped with an admissible flag $Y_\bullet$, the Okounkov bodies of big real classes on X with respect to $Y_\bullet$ fit together in a convex cone, called the global Okounkov cone. We prove Theorem C by analyzing the variation of global Okounkov cones in flat families.

The question then naturally arises whether this countable set of convex bodies can be characterized in small dimensions. We focus on the case of surfaces and try to give an affirmative answer in this setup. An explicit description of $\Delta(D)$ for any real divisor D on a smooth surface S with respect to a flag (C,x), where $C \subseteq S$ is a smooth curve and $x \in C$ a point, based on the Zariski decomposition, is given in [34, Theorem 6.4]. It was noted that it followed from this description that the Okounkov body was a possibly infinite polygon. In §IV.3, we give a complete characterization of Okounkov bodies on surfaces based on this work: these turn out to be finite polygons satisfying a few extra combinatorical conditions.

Theorem D. *The Okounkov body of an $\mathbb{R}$-divisor on a smooth surface with respect* *to some admissible flag is a finite polygon. Up to translation, a real polygon* $\Delta \subseteq \mathbb{R}^2_+$ *is the Okounkov body of an* $\mathbb{R}$*-divisor* D *on a smooth projective surface* S *with respect to a complete flag* (C, x) *if and only if*

$$\Delta = \{(t, y) \in \mathbb{R}^2 \mid \nu \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\}$$

for certain real numbers $\nu, \mu > 0$ *and certain piecewise linear functions* $\alpha, \beta : [\nu, \mu] \to \mathbb{R}_+$ *with rational slopes such that* β *is convex and* α *is increasing and concave.*

In the case when the divisor D is in fact a $\mathbb{Q}$-divisor, the proof of Theorem D implies that the break points of the functions α and β occur at rational points and that the number ν must be rational. As for the number μ it turns out that it might be irrational–we give an example of this–but it is at worst a quadratic irrational. As for the proof of Theorem D, which appears in §IV.3, it uses Zariski decomposition, as in [34, Theorem 6.4]; more precisely, Theorem D is proved via a detailed analysis of the variation of Zariski decomposition along a line segment. Conversely, we show that all convex bodies as in Theorem D are Okounkov bodies of divisors on smooth toric surfaces.

In contrast to the two dimensional case, in higher dimensions we cannot give a simple characterization of Okounkov bodies along the lines of Theorem D. In §IV.4 we give two examples of varieties equipped with flags, such that there exists line bundles with the property that the Okounkov body is non-polyhedral. Already in [34, Section 6.3] there appears an example of a non-polyhedral Okounkov body in higher dimensions. The novelty in both of our examples is that the varieties we deal with are Fano and Mori dream spaces respectively. Also, the variety in the first example is $\mathbb{P}^2 \times \mathbb{P}^2$, and we are able to give a complete description of a slice of the Okounkov body, which has a round shape. As for the second example, we find a

Mori dream space threefold with the property that for any ample divisor and any flag coming from a linear deformation of a certain given flag, the Okounkov body is round.

1.3 Volume functions

In the final part of this thesis, Chapter V, we focus on the volume function of projective varieties and multigraded linear series. If X is a complex projective smooth variety and D is a Cartier divisor on X, then the volume of D is defined to be

$$\mathrm{vol}_X(D) \;=\; \limsup_{k\to\infty} \frac{\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(kD))}{k^n/n!}.$$

By definition, D is big when $\mathrm{vol}_X(D) > 0$. The volume, and its restricted version, have played recently a crucial role in several important developments in higher dimensional geometry, e.g. [52], [21].

In the classical setting of ample divisors, the volume is simply the top self-intersection of D. Starting with the work of Fujita [14], Nakayama [43], and Tsuji [53], it became clear that the volume of a big divisor displays a surprising number of properties analoguous to those of ample ones. One can check [33, Section 2.2.C] that it depends only on the numerical class of D, that it is homogeneous of degree n, i.e. $\mathrm{vol}_X(pD) \;=\; p^n \cdot \mathrm{vol}_X(D)$ for any $p \in \mathbb{N}$, and that it satisfies a continuity property. These properties imply that one can extend uniquely the volume to a continuous function

$$\mathrm{vol}_X \;:\; N^1(X)_{\mathbb{R}} \;\longrightarrow\; \mathbb{R},$$

where we denote by $N^1(X)$ the Neron-Severi group of numerical equivalence classes of line bundles on X, and by $N^1(X)_{\mathbb{R}}$ the corresponding finite-dimensional real vector space. Besides continuity and homogeneity, another important feature of the volume function is log concavity, i.e. for any two classes $\xi, \xi' \in \mathrm{Big}(X)_{\mathbb{R}}$ we have

$$\mathrm{vol}_X(\xi + \xi')^{1/n} \;\geq\; \mathrm{vol}_X(\xi)^{1/n} \;+\; \mathrm{vol}_X(\xi')^{1/n}.$$

However, besides these properties, relatively little is known about the global behavior of the volume function, and understanding it more clearly remains a very important quest.[3]

Inspired by Theorem C on Okounkov bodies, the first question we tackle about the volume function is, how many functions like this can occur? In §V.2 we prove that there are only countably many of them.

Theorem E. *There exist countably many functions $f_j : \mathbb{R}^\rho \to \mathbb{R}$ with $j \in \mathbb{N}$, such that for any irreducible, projective and smooth variety X of dimension n and Picard number ρ there is an integral linear isomorphism*

$$\rho_X : \mathbb{R}^\rho \;\to\; N^1(X)_{\mathbb{R}}$$

with the property that $\mathrm{vol}_X \circ \rho_X = f_j$ *for some* $j \in \mathbb{N}$.

Also, in §V.2 we show that there are countably many closed convex cones $A_i \subseteq \mathbb{R}^\rho$ with $i \in \mathbb{N}$ such that $\rho_X^{-1}(\mathrm{Nef}(X)_{\mathbb{R}}) = A_i$ for some $i \in \mathbb{N}$. Thus, there exist countably many cones appearing as nef cones for all irreducible, projective and smooth projective varieties. The same statement can be easily deduced for the ample, big and pseudo-effective cones.

In (1.1) we have seen that the volume of a big Cartier divisor is proportional with the Euclidean volume of its Okounkov body. The result on the countability of global Okounkov cones implies the countability of the volume functions. As for the countability of the nef cones, the proof follows the same steps as the one given for

[3] In their interesting paper [10], Boucksom-Favre-Jonsson found a nice formula for the derivative of vol_X in any direction.

Theorem C, by analyzing the variation of volume function in families coming from multi-graded Hilbert schemes.

An interesting application of Theorem E concerns the set of volumes. If we define $\mathbb{V} \subseteq \mathbb{R}_+$ to be the set of all positive real numbers arising as the volume of a Cartier divisor on some complex projective smooth variety, then $\mathbb{V}$ has a structure of a countable multiplicative semigroup (see Corollary 5.15). Also we will see that all positive rational numbers are contained in $\mathbb{V}$, i.e. $\mathbb{Q}_+ \subseteq \mathbb{V}$. On the other hand in §V.2 we give an example of a four-fold whose volume function is given by a transcendental function. In particular one can find easily integral divisors on this four-fold whose volume is a transcendental number. Thus the set of volumes $\mathbb{V}$ contains transcendental numbers, deepening further the mystery surrounding the volume function in the classical case and in particular the structure of the set $\mathbb{V}$.

With the emergence of Okounkov bodies, it became clear that in fact most of the properties of the vol_X are quite formal in nature, and can be extended to the (noncomplete) multigraded setting. Specifically, fix Cartier divisors $D_1, \ldots, D_p$ on X and write $\mathbf{m}D = m_1D_1 + \cdots + m_pD_p$ for $\mathbf{m} = (m_1, \ldots, m_p) \in \mathbb{N}^p$. A **multigraded linear series** $W_\bullet$ on X associated to $D_1, \ldots, D_p$ consists of subspaces

$$W_{\mathbf{m}} \ \subseteq \ H^0(X, \mathcal{O}_X(\mathbf{m}D))$$

such that $R(W_\bullet) = \oplus W_{\mathbf{m}}$ is a subalgebra of the section ring

$$R(X; D_1, \ldots, D_p) \ = \ \bigoplus_{\mathbf{m} \in \mathbb{N}^p} H^0(X, \mathcal{O}_X(\mathbf{m}D)).$$

The **support** of $W_\bullet$, $\mathrm{supp}(W_\bullet) \subseteq \mathbb{R}^p_+$, is the closed convex cone spanned by all indices $\mathbf{m} \in \mathbb{N}^p$ such that $W_{\mathbf{m}} \neq 0$. Now, given $\mathbf{m} \in \mathbb{N}^p$, set

$$\mathrm{vol}_{W_\bullet}(\mathbf{m}) \ := \ \limsup_{k \to \infty} \frac{\dim(W_{k \cdot \mathbf{m}})}{k^n/n!}.$$

Then as in the complete case, one can define the volume function of $W_\bullet$,

$$\mathrm{vol}_{W_\bullet} \;:\; \mathbb{N}^p \;\longrightarrow\; \mathbb{R}_+.$$

In [34] the authors associate a convex cone, called the Okounkov cone, to a multigraded linear series on a projective variety, generalizing the global construction mentioned above. They use convex geometry and semigroup theory to show that (under very mild hypothesis) the formal properties of the global volume function extend to the multigraded setting. Specifically, as in the global case, the function $\mathbf{m} \mapsto \mathrm{vol}_{W_\bullet}(\mathbf{m})$ extends uniquely to a continuous function

$$\mathrm{vol}_{W_\bullet} \;:\; \mathrm{int}(\mathrm{supp}(W_\bullet)) \;\longrightarrow\; \mathbb{R}_+$$

which is log-concave and homogeneous of degree $n = \dim(X)$ and extends continuously to all of $\mathrm{supp}(W_\bullet)$ (see Remark 5.29).

This definition is a natural extension of the volume function in the global case. If X is a smooth projective variety then the big cone, $\mathrm{Big}(X)_\mathbb{R}$, is pointed and vol_X vanishes outside of it. Choosing $D_1, \ldots, D_\rho$ integral divisors on X, whose classes in $N^1(X)_\mathbb{R}$ generate a cone containing $\mathrm{Big}(X)_\mathbb{R}$, then $\mathrm{vol}_X = \mathrm{vol}_{W_\bullet}$ on $\mathrm{Big}(X)_\mathbb{R}$, where $W_\bullet = \big(H^0(X, \mathcal{O}_X(\mathbf{m}D))\big)_{\mathbf{m}\in\mathbb{N}^\rho}$.

In this "in vitro" setting, we prove in a special case that any continuous, homogeneous and log-concave function in fact arises (up to scaling) as the volume function of a multigraded linear series:

Theorem F. *Let $K \subseteq \mathbb{R}^p_+$ be a closed convex cone with nonempty interior and suppose $f : K \to \mathbb{R}_+$ is a continuous function, which is non-zero, log-concave and homogeneous of degree p in the interior of K. Then there exists a smooth, projective variety Y of dimension p, a multigraded linear series $W_\bullet$ on Y and a positive constant k such that $\mathrm{vol}_{W_\bullet} \equiv k \cdot f$ on the interior of K. Moreover we have $\mathrm{supp}(W_\bullet) = K$.*

As a consequence of Theorem F, we observe that the volume function $\mathrm{vol}_{W_\bullet}$ of a multigraded linear series $W_\bullet$ can be wild. Alexandroff [1] showed that a function as in Theorem F is almost everywhere twice differentiable, but one can give examples of such functions which are nowhere three times differentiable (see Remark 5.27). This gives a positive answer to [34, Problem 7.2]. As for the proof of Theorem F, the linear series used are on $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (p times): first we prove that any pointed cone in $\mathbb{R}^p_+ \times \mathbb{R}^p_+$, modulo scaling, is the Okounkov cone of some multigraded linear series on $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. The volume of the corresponding multigraded linear series coincides with the Euclidean volume function of slices of its Okounkov cone. We finish by showing that any function as in Theorem F is the Euclidean volume function of a cone.

CHAPTER II

Multigraded Regularity

2.1 Castelnuovo-Mumford regularity.

In this section we review a few facts about Castelnuovo-Mumford regularity. General references for this are [40], [41], [33]. We will always work over the field of complex numbers. In the following $X \subseteq \mathbb{P}^r$ will be a projective subscheme of dimension n and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^r$.

In his efforts to simplify Grothendieck's work on the existence of Hilbert schemes, Mumford [40] realized the importance of vanishing theorems in order to establish the boundedness of the family of all subschemes of projective space with fixed Hilbert polynomial. He focused mainly on Serre vanishing, which states that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ there exists an integer m_0, depending on $\mathcal{F}$, such that

$$H^i(\mathbb{P}^r, \mathcal{F}(m)) \;=\; 0, \text{ for all } m \geq m_0,\; i \geq 1.$$

Mumford's idea was to give a quantitative measure of how much one has to twist so the higher cohomology vanishes. He proposed the following definition:

Definition 2.1 (Castelnuovo-Mumford regularity of a coherent sheaf)**.** Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbb{P}^r$, and let m be an integer. One says that $\mathcal{F}$ is *m-regular in the sense of Castelnuovo-Mumford* if

$$H^i(\mathbb{P}^r, \mathcal{F}(m-i)) \;=\; 0 \;\text{ for all } i > 0.$$

The *Castelnuovo-Mumford regularity* $\operatorname{reg}(\mathcal{F})$ of the coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ is the least integer m for which $\mathcal{F}$ is m-regular.

Remark 2.2. Let $S = \mathbb{C}[x_0, \ldots, x_r]$ be the coordinate ring of $\mathbb{P}^r$. Then one can associate to any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ a $\mathbb{Z}$-graded S module as follows

$$\Gamma_*(\mathcal{F}) \ = \ \bigoplus_{k\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{F}(k))$$

Now suppose that $H^0(\mathbb{P}^r, \mathcal{F}(k)) = 0$, for $k \ll 0$. Then Theorem 2.4 below implies that the graded S module is finitely generated. In this case it is not hard to see (see Lemma 2.32 in the multigraded case) that $\operatorname{reg}(\mathcal{F})$ exists and is finite.

Remark 2.3. Suppose $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^r$. If the dimension of $\operatorname{supp}(\mathcal{F})$ is at least one, then it is not hard to show that the number $\operatorname{reg}(\mathcal{F})$ exists and is finite.

While the formal definition may seem rather unintuitive, a result of Mumford gives a first indication that Castelnuovo-Mumford regularity measures the point at which cohomological complexities vanish.

Theorem 2.4 (Mumford's theorem)**.** *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$. If $\mathcal{F}$ is m-regular, then for $k \geq 0$:*

(i) $\mathcal{F}(m+k)$ is generated by its global sections.

(ii) The natural map

$$H^0(\mathbb{P}^r, \mathcal{F}(m)) \ \otimes \ H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \ \to \ H^0(\mathbb{P}^r, \mathcal{F}(m+k))$$

is surjective.

(iii) $\mathcal{F}$ is $(m+k)$-regular.

An important consequence of this theorem is that regularity governs the complexity of the algebraic objects associated to a coherent sheaf. As before let $S =$

$\mathbb{C}[x_0, \ldots, x_r]$ be the homogeneous coordinate ring of $\mathbb{P}^r$ and $\Gamma_*(\mathcal{F})$ be the corresponding $\mathbb{Z}$-graded S module of a coherent sheaf $\mathcal{F}$. For simplicity, assume the vanishing of the group of global sections $H^0(\mathbb{P}^r, \mathcal{F}(k))$ for all $k \ll 0$. In this setup, Theorem 2.4 implies that $\Gamma_*(\mathcal{F})$ is finitely generated $\mathbb{Z}$-graded S module. Thus, by general theory, $\Gamma_*(\mathcal{F})$ admits a minimal graded free resolution $E_\bullet$:

$$0 \rightarrow E_{n+1} \rightarrow E_n \rightarrow \ldots \rightarrow E_1 \rightarrow E_0 \rightarrow \Gamma_*(\mathcal{F}) \rightarrow 0.$$

where $E_p = \oplus_i S(-a_{p,i})$ is a free $\mathbb{Z}$-graded S-module. Here minimality means that the maps of $E_\bullet$ are given by matrices of homogeneous polynomials containing no non-zero constants as entries. The integers $a_{p,i}$-s specify the degrees of the generators of E_p and are uniquely determined by $\Gamma_*(\mathcal{F})$, hence by $\mathcal{F}$. Set

$$a_p = a_p(\mathcal{F}) = \max_i\{a_{p,i}\}$$

so that a_p is the largest degree of a generator of the p^{th} module of syzygies of $\Gamma_*(\mathcal{F})$.

With this in hand, the following result (see [33, Theorem 1.8.26]) shows the relationship between regularity and the syzygetic degrees of the sheaf $\mathcal{F}$:

Theorem 2.5. *Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbb{P}^r$. Then $\mathcal{F}$ is m-regular if and only if one of the following equivalent conditions is satisfied:*

(i) $\mathcal{F}$ is resolved by a long exact sequence

$$\ldots \rightarrow \oplus\mathcal{O}_{\mathbb{P}^r}(-m-2) \rightarrow \oplus\mathcal{O}_{\mathbb{P}^r}(-m-1) \rightarrow \oplus\mathcal{O}_{\mathbb{P}^r}(-m) \rightarrow \mathcal{F} \rightarrow 0$$

whose terms are direct sums of the indicated line bundles.

(ii) Each of the integers $a_p = a_p(\mathcal{F})$ satifies the inequality

$$a_p \leq p + m.$$

Remark 2.6. An easy application of Theorem 2.5 is that regularity has pleasant tensorial properties. If E_1 and E_2 are two vector bundles on $\mathbb{P}^r$, which are m and respectively n-regular, then $E_1 \otimes E_2$ is $(m+n)$-regular. This was used extensively in the problem of finding bounds of regularity of subvarieties of projective space.

Although Mumford's original interest was to apply regularity to the existence of Hilbert schemes, the fact that it governs the algebraic complexity of a coherent sheaf drew a considerable amount of work in the last thirty years, both in algebraic geometry and commutative algebra.

A particularly interesting case occurs when $\mathcal{F}$ is the ideal sheaf of a subvariety (or subscheme) of projective space:

Definition 2.7 (Regularity of a projective subvariety.)**.** We say that a subvariety (or a subscheme) $X \subseteq \mathbb{P}^r$ is m-regular if its ideal sheaf $\mathcal{I}_{X/\mathbb{P}^r}$ is. The *regularity* of X is the regularity $\operatorname{reg}(\mathcal{I}_{X/\mathbb{P}^r})$ of its ideal.

Remark 2.8. For any projective subscheme $X \subseteq \mathbb{P}^r$ denote by

$$I_X = \bigoplus_{k \in \mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{I}_{X/\mathbb{P}^r}(k))$$

the saturated homogeneous ideal of X. If X is m-regular, then Theorem 2.5 says that I_X is generated by forms of degree $\leq m$, and the p^{th} syzygies among these generators appear in degrees $\leq m+p$.

Remark 2.9. If $X \subseteq \mathbb{P}^r$ is a subscheme of dimension n, then (for $m > 0$) X is m-regular if and only if $H^i(\mathbb{P}^r, \mathcal{I}_{X/\mathbb{P}^r}(m-i)) = 0$ for $1 \leq i \leq n+1$.

As an example, if $C \subseteq \mathbb{P}^r$ is a smooth rational curve, embedded by a possibly incomplete linear series, then C is m-regular (for $m > 0$) if and only if hypersurfaces of degree $m-1$ cut out a complete linear series on C, i.e. if and only if the map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m-1)) \rightarrow H^0(C, \mathcal{O}_C(m-1))$$

is surjective. For example, if C is the image of the embedding

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3, [s,t] \mapsto [s^d, s^{d-1}t, st^{d-1}, t^d]$$

then C is $(d-1)$-regular but not $(d-2)$ regular.

The result of Theorem 2.5 tells us that regularity and the data given by a linear resolution of a coherent sheaf are interconnected. This emphasizes the importance in finding upper bounds to regularity of a projective scheme $X \subseteq \mathbb{P}^r$ in terms of geometric data. While the picture is not complete, the influential survey [6] of Bayer and Mumford reveals a fascinating difference between the case of smooth varieties and that of arbitrary schemes. On the one hand, for arbitrary schemes one can find examples that show that regularity grows exponentially as a function of input parameters. On the other hand, the regularity of smooth varieties is known or expected to grow linearly in terms of geometric invariants.

Gotzmann's bound. The earliest results bounded regularity in terms of Hilbert polynomials. Given a projective subscheme $X \subseteq \mathbb{P}^r$ write

$$P_X(k) = \chi(X, \mathcal{O}_X(k)).$$

Then it is known that $P_X \in \mathbb{Q}[t]$ is a polynomial with rational coefficients and is called the Hilbert polynomial of X. Gotzmann in [15] finds an optimal statement in this direction:

Theorem 2.10 (Gotzmann's regularity theorem)**.** *There exists an unique increasing sequence of positive integers a_i with $i = 1, \ldots, s$, such that one can write*

$$P_X(k) = \binom{k+a_1}{a_1} + \binom{k+a_2-1}{a_2} + \ldots + \binom{k+a_s-(s-1)}{a_s}$$

and then $\mathcal{I}_{X/\mathbb{P}^r}$ is s-regular.

Remark 2.11. Mumford [40] was actually the first to bound the regularity of X in terms of P_X. His interest was to show that all schemes of fixed Hilbert polynomial form a bounded family, i.e. are parametrized by finitely many irreducible varieties. Gotzmann instead used a different approach to prove Theorem 2.10.

Bounds from defining equations. Bayer remarks that in an actual computation the subcheme is described by explicit equations, hence the degrees of the generators of the ideal sheaf will be known. So, it is natural to bound regularity in terms of these degrees.

Definition 2.12. The *generating degree* $d(\mathcal{I})$ of an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^r}$ is the least integer d such that $\mathcal{I}(d)$ is globally generated.

For arbitrary subschemes of the projective space, Bayer and Mumford were first to find a bound on regularity in terms of the generating degree. In [6, Proposition 3.8], they give a very elementary proof of an essentially doubly exponential bound:

Theorem 2.13 (Bound for arbitrary ideals)**.** *Suppose $X \subseteq \mathbb{P}^r$ is an arbitrary projective subscheme. Then*

$$\operatorname{reg}(X) \;\leq\; (2d(\mathcal{I}_{X/\mathbb{P}^r}))^{r!}.$$

Remark 2.14. (1) Bayer and Mumford [6, Theorem 3.7] observed that work of Giusti and Galligo leads to the stronger bound $\operatorname{reg}(X) \leq (2d(\mathcal{I}_{X/\mathbb{P}^r}))^{2^{r-1}}$.
(2) Surprisingly, Bayer and Stillman [7] show that a construction of Mayr and Meyer give examples of projective subschemes whose regularity grows doubly exponential. The constraint is that these examples are combinatorial in nature, instead of geometric.

As we mentioned above, the picture for smooth subvarieties is expected to be completely different. Specifically, a result of Bertram, Ein, and Lazarsfeld [8] give a linear bound to regularity in terms of the generating degree.

Theorem 2.15 (Linear bounds for smooth ideals)**.** *Let $X \subseteq \mathbb{P}^r$ be an irreducible projective and smooth subvariety of dimension n and codimension $e = r - n$. If $d = d(\mathcal{I}_{X/\mathbb{P}^r})$ is the generating degree of the ideal sheaf of X, then*

$$H^i(\mathbb{P}^r, \mathcal{I}_{X/\mathbb{P}^r}(k)) = 0 \text{ for } i \geq 1 \text{ and } k \geq ed - r.$$

In particular, X is $(ed - e + 1)$-regular. Moreover, X fails to be $(ed - e)$-regular if and only if it is a transversal complete intersection of e hypersurfaces of degree d.

Castelnuovo-type of bounds. An important question is to find upper bounds of regularity of a projective subscheme $X \subseteq \mathbb{P}^r$ in terms of its degree. The appeal of this question rests on two points. First, it turned out to be a rather hard problem to tackle. Second, it is connected with a classical problem of Castelnuovo. In the early 1980-s several mathematicians proposed the following conjecture:

Conjecture 2.16 (Castelnuovo-type regularity conjecture)**.** *If $X \subseteq \mathbb{P}^r$ is a smooth non-degenerate subvariety of dimension n and degree d, then X is $(d + n + 1 - r)$-regular.*

Remark 2.17. If $i \geq 2$ and $k \geq -r$, then one has the isomorphism

$$H^i(\mathbb{P}^r, \mathcal{I}_{X/\mathbb{P}^r}(k)) \;=\; H^{i-1}(X, \mathcal{O}_X(k)),$$

In practice the vanishing of the group on the right can be handled relatively easily. Thus to give bounds for regularity one usually has to control the vanishing of the groups $H^1(\mathbb{P}^r, \mathcal{I}_{X/\mathbb{P}^r}(k))$. This groups measure the failure of hypersurfaces of degree k to cut out a complete linear series on X and the question of their vanishing leads to some classical results of Castelnuovo [11] for curves.

Let $C \subseteq \mathbb{P}^r$ be a smooth non-degenerate curve of degree d. Castelnuovo's approach to finding bounds, when hypersurfaces of degree k cut out a complete linear series on C, rests on the existence of general projections $p : \mathbb{P}^r \dashrightarrow \mathbb{P}^2$ away from C such that the image of C has "nice" singularities. This reduces the problem to one about the singularities of curves embedded in $\mathbb{P}^2$, and Castelnuovo finally is able to prove that hypersurfaces of degree $\geq d-2$ cut out a complete linear series on C. As for regularity bounds, one uses Castelnuovo's argument on bounding the genus of a space curve to handle the vanishing of the second cohomoly groups and deduces that C is $(d-1)$-regular.

In [17] Gruson, Lazarsfeld, and Peskine prove Conjecture 2.16 for any irreducible (possibly singular) reduced and non-degenerate curve. They also show that this bound is the best possible. Unlike Castelnuovo's arguments, their proof is essentially cohomological. They introduce a new interesting technique. The idea is that instead of taking a resolution of the ideal sheaf $\mathcal{I}_{C/\mathbb{P}^r}$, it is better to take a complex, which is exact outside of the curve C. Using a Beilinson-type construction, this complex helps them to express C as the locus where a matrix of linear forms drops rank. Taking the corresponding Eagon-Northcott complex, they arrive at complexes of simpler form of the ideal sheaf. Since the complex is exact away from a one-dimensional set they are able to read off the desired vanishings.

In the case of surfaces, Pinkham was first to attempt to generalize Castelunovo's arguments in order to find bounds on degrees which cut out complete linear series. In his work he takes $S \subseteq \mathbb{P}^5$ to be a smooth non-degenerate surface of degree d and using generic projections, reduces the problem to analyzing a finite birational map $X \to Y \subseteq \mathbb{P}^3$, where Y is possibly singular. Using this projection and Kodaira vanishing, Pinkham is able to find a curve $C \subseteq X$ such that for any $k \geq d-3$ the condition that hypersurfaces of degree k in $\mathbb{P}^5$ cut out a complete linear series on X is equivalent to the same condition imposed on C. Using the fact that the morphism $C \to D$, which is the restriction of the projection above to C, is generically a double cover, Pinkham is able to prove the following theorem:

Theorem 2.18. *([46]) Let $S \subseteq \mathbb{P}^5$ be a smooth non-degenerate surface of degree d. Then for any $k \geq d-2$, hypersurfaces of degree k cut out a complete linear series on S.*

Remark 2.19. Around the same time that Pinkham's work was published, Lazarsfeld [32] proved Conjecture 2.16 in the case of surfaces. As before, Lazarsfeld uses generic projections to $\mathbb{P}^3$, so the image of the surface has "nice" singularities. In comparison to Castelnuovo's approach, Lazarsfeld uses a vector bundle construction introduced for curves by Gruson and Peskine to obtain an optimal bound. This construction turns out to be very powerful. On the one hand, Kwak uses it to give linear bounds of regularity for smooth three-folds and four-folds in [30] and for smooth fivefolds and sixfolds in [31]. On the other hand, in Chapter 3 we will use this technique in the multigraded setting. This will help us to generalize the result of Gruson, Lazarsfeld and Peskine [17] in this setup.

2.2 Multigraded regularity

Inspired by the importance of classical Castelnuovo-Mumford regularity both in algebraic geometry and commutative algebra, in recent years the definition of regularity has been extended to the multigraded setting ([2], [36], [20], [25], [51]). In this section we will survey some of the definitions and results in this direction that have appeared in the literature.

In the previous section we have seen that in the classical case regularity can be defined either by vanishing of higher cohomology (Definition 2.1) or via bounds on the degrees of the syzygies in a minimal free resolution of the sheaf (Theorem 2.5). In the multigraded setting one does not expect an equivalence between syzygies and vanishing of higher cohomology. In view of this, the literature contains both approaches. On the one hand, Maclagan and Smith [36], inspired by Mumford's definition of regularity, used the cohomological approach to define *multigraded regularity*. On the other hand, Hà [20] and Sidman and Van Tuyl [50] define the *resolution regularity vector* by bounding the multigraded degrees of the syzygies in any free resolution of the sheaf.

Maclagan and Smith [36] work in the general toric setting and they use the vanishing of local cohomology to define their multigraded version of Castelnuovo-Mumford regularity. A related approach was followed by Hering, Schenck and Smith in [23] where they work sheaf-theoretically. We prefer to work mainly in this setup, and for simplicity we will focus on a product of projective spaces. Most of the features of the multigraded theory are visible here.

First let's introduce some notations, which will be used throughout the next three sections. Let $k \geq 1$ be a natural number and denote by

$$Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$$

the multi-projective space for some $n_1, \ldots, n_k \geq 1$. Notice that each line bundle on Y has the following form:

$$\mathcal{O}_Y(\mathbf{m}) \; := \; p_1^*(\mathcal{O}_{\mathbb{P}^{n_1}}(m_1)) \otimes \cdots \otimes p_k^*(\mathcal{O}_{\mathbb{P}^{n_k}}(m_k))$$

for some $\mathbf{m} := (m_1, \ldots, m_k) \in \mathbb{Z}^k$, where $p_i : Y \to \mathbb{P}^{n_i}$ is the projection on the i-th factor. In this setting the multigraded regularity can be defined as follows [23]:

Definition 2.20 (Geometric multigraded regularity)**.** Let $\mathcal{F}$ be a coherent sheaf on Y and $\mathbf{m} \in \mathbb{Z}^k$ an integral vector. One says that $\mathcal{F}$ is $\mathbf{m}$-*regular* if

$$H^i(Y, \mathcal{F} \otimes \mathcal{O}_Y(\mathbf{m} - \mathbf{u})) \ = 0,$$

for all $i > 0$ and all $\mathbf{u} \in \mathbb{N}^k$ with $|\mathbf{u}| := u_1 + \cdots + u_k = i$. Furthermore, we define the *multigraded regularity set* of $\mathcal{F}$ as follows

$$\mathbf{reg}(\mathcal{F}) = \{ \ \mathbf{m} \in \mathbb{Z}^k \mid \mathcal{F} \text{ is } \mathbf{m} - \text{regular } \}.$$

Remark 2.21*.* It is not hard to see, using the Künneth formula, that if $\mathcal{F} = \mathcal{O}_Y$ then it is $\mathbf{0}$-regular, where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}^k$.

Using the same techniques as in the classical case, Hering, Schenck, and Smith (see [23, Theorem 2.1]) prove a multigraded version of Mumford's Theorem, i.e. Theorem 2.4, indicating that this version of multigraded regularity still measures the point at which cohomological complexities vanish.

Theorem 2.22 (Multigraded version of Mumford's Theorem)**.** *If the coherent sheaf* $\mathcal{F}$ *is* $\mathbf{m}$*-regular, then for* $\mathbf{u} \in \mathbb{N}^k$*:*

(i) $\mathcal{F}(\mathbf{m} + \mathbf{u})$ *is globally generated.*

(ii) The map $H^0(Y, \mathcal{F}(\mathbf{m})) \otimes H^0(Y, \mathcal{O}_Y(\mathbf{u})) \to H^0(Y, \mathcal{F}(\mathbf{m} + \mathbf{u}))$ *is surjective.*

(iii) $\mathcal{F}$ *is* $(\mathbf{m} + \mathbf{u})$*-regular.*

Remark 2.23*.* Theorem 2.22 implies that the multigraded regularity set of $\mathcal{F}$ is a union of positive cones:

$$\mathbf{reg}(\mathcal{F}) \ = \bigcup_{\mathcal{F} \text{ is } \mathbf{m}-\text{regular}} (\mathbf{m} + \mathbb{N}^k).$$

An important distinction between Castelnuovo-Mumford regularity and its multigraded counterpart is that in the classical case is easy to describe those coherent sheaves whose Castelnuovo-Mumford regularity is bounded from below (see Remark 2.3). In the following we will see that in the multigraded setting the class of sheaves whose regularity set cannot be bounded from below is much larger and for this we will introduce the following definition:

Definition 2.24. Let $S \subseteq \mathbb{Z}^k$ be a subset such that there exists a set $J \subseteq \mathbb{Z}^k$ with the property that $S = \bigcup_{\mathbf{m} \in J}(\mathbf{m} + \mathbb{N}^k)$. We say that S is *finitely generated* as a subset of $\mathbb{Z}^k$ if there exists finitely many integral vectors $\mathbf{m}_1, \ldots, \mathbf{m}_l \in \mathbb{Z}^k$ such that $S = \bigcup_{i=1}^{i=l}(\mathbf{m}_i + \mathbb{N}^k)$.

Remark 2.25. The importance of this definition relies on the fact that if $\mathbf{reg}(\mathcal{F})$ is finitely generated as a subset of $\mathbb{Z}^k$, then the set $\mathbf{reg}(\mathcal{F})$ is given by finitely many data. At the same time Dickson Lemma says that a subset $S \subseteq \mathbb{Z}^k$, as in Definition 2.24, is finitely generated if and only if S is bounded from below, i.e. there exists a vector $\mathbf{m} \in \mathbb{Z}^k$ such that $S \subseteq (\mathbf{m} + \mathbb{N}^k)$.

Example 2.26 (Unbounded regularity set)**.** In the following example we construct a sheaf, which in the classical setting would have a bounded regularity, but in the multigraded setup the regularity set is unbounded.

Suppose $Y = \mathbb{P}^3 \times \mathbb{P}^3$ and let $C \subseteq Y$ be a smooth rational curve of bidegree $(3, 4)$. If $\mathcal{F} = \mathcal{O}_C$, then it is not hard to see that $\mathbf{reg}(\mathcal{F})$ is not finitely generared as a subset of $\mathbb{Z}^2$. Using the fact that the support of $\mathcal{F}$ is one dimensional, then to describe the multigraded regularity set $\mathbf{reg}(\mathcal{F})$ is enough to study the vanishings of the first cohomology groups. So it is not hard to see that the regularity set can be characterized as follows

$$\mathbf{reg}(\mathcal{O}_C) \;=\; \{\ (m_1, m_2) \in \mathbb{Z}^2 \mid 3m_1 + 4m_2 - 1 \geq 0\ \}.$$

and thus the set $\mathbf{reg}(\mathcal{O}_C)$ cannot be bounded from below.

Remark 2.27. Theorem A, the proof of which will be given in §2.4, describes all those sheaves, which have a finitely generated multigraded regularity set. The statement will be inspired by Remark 2.3.

Remark 2.28. In Example 2.44, we will see that even though the set $\mathbf{reg}(\mathcal{F})$ is bounded from below, usually this set as a subset of $\mathbb{Z}^k$ might not have a minimal element. This is the first main conceptual difference between Castelnuovo-Mumford regularity and its multigraded analogue.

Algebraic approach to regularity. We will now sketch the algebraic approach to the multigraded regularity given by Maclagan and Smith and we shall see the reason that it is more convenient to work in the algebraic setting in the general case of coherent sheaves. At the same time we will see the discrepancies with the classical case, where algebra and geometry interacted harmoniously.

The idea of Maclagan and Smith in [36] is to view Y as a simplicial toric variety. By general theory, for any toric variety Y one introduces the "homogeneous coordinate ring" S_Y of Y endowed with a semi-group grading. This construction is due to Cox [12] and as in the case of projective space there is a correspondence between coherent sheaves on Y and finitely generated graded S_Y modules. In our case the homogeneous coordinate ring of Y is the polynomial ring

$$S_Y \;=\; \mathbb{C}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$$

with the natural $\mathbb{Z}^k$-grading, where $\deg(x_{i,j}) = e_i$ is the i-th standard basis vector of $\mathbb{Z}^k$. Let $B_Y = \langle x_{1,0}, \ldots, x_{k,n_k} \rangle$ be the irrelevant ideal. With this in hand using local cohomology with respect to the irrelevant ideal B_Y, we give the algebraic analogue of multigraded regularity.

Definition 2.29 (Multigraded Castelnuovo-Mumford regularity)**.** For $\mathbf{m} \in \mathbb{Z}^k$, one says that a $\mathbb{Z}^k$-graded S_Y module M is $\mathbf{m}$-regular if the following conditions are satisfied:

(i) $H^i_{B_Y}(M)_{\mathbf{p}} = 0$ for all $i \geq 1$ and all $\mathbf{p} \in \bigcup(\mathbf{m} - \mathbf{u} + \mathbb{N}^k)$, where the union is taken over all integral vectors $\mathbf{u} \in \mathbb{N}^k$ with the property $|\mathbf{u}| = u_1 + \cdots + u_k = i - 1$;

(ii) $H^0_{B_Y}(M)_{\mathbf{u}} = 0$ for all $\mathbf{u} \in \bigcup_{j=1}^{j=k}(\mathbf{m} + e_i + \mathbb{N}^k)$, where as before e_i is the i-th standard basis vector of $\mathbb{Z}^k$.

Set $\mathbf{reg}(M) = \{\mathbf{m} \in \mathbb{Z}^k \mid M \text{ is } \mathbf{m}\text{-regular}\}$.

Remark 2.30. As in the geometric setting the regularity set is a union of positive cones

$$\mathbf{reg}(M) = \bigcup_{M \text{ is } \mathbf{m}\text{-regular}} (\mathbf{m} + \mathbb{N}^k)$$

On the other hand whenever M is a finitely generated S_Y module, we will prove in Lemma 2.32 that the set $\mathbf{reg}(M)$ is bounded from below, hence finitely generated as a subset of $\mathbb{Z}^k$. This is one of the main reasons why multigraded regularity have been studied in the algebraic setting.

Remark 2.31 (Local cohomology vs. sheaf cohomology). In order to explain the connection between the algebraic setting and the geometric one, choose $\mathcal{F}$ to be a coherent sheaf on Y. As in the projective space case, one can associate to $\mathcal{F}$ a $\mathbb{Z}^k$-graded S_Y module

$$\Gamma_*(\mathcal{F}) = \bigoplus_{\mathbf{m} \in \mathbb{Z}^k} H^0(Y, \mathcal{F}(\mathbf{m})).$$

For example, $\Gamma_*(\mathcal{O}_Y) = S_Y$ because $H^0(Y, \mathcal{O}_Y(\mathbf{m})) = (S_Y)_{\mathbf{m}}$ for any $\mathbf{m} \in \mathbb{Z}^k$. The scalar multiplicitation which makes $\Gamma_*(\mathcal{F})$ a $\mathbb{Z}^k$-graded S_Y module comes from the natural maps $H^0(Y, \mathcal{F}(\mathbf{m})) \otimes H^0(Y, \mathcal{O}_Y(\mathbf{u})) \to H^0(Y, \mathcal{F}(\mathbf{m}+\mathbf{u}))$. The correlation between algebra and geometry follows from the fact that the sheaf associated to $\Gamma_*(\mathcal{F})$ is isomorphic to $\mathcal{F}$ itself.

At the same time if M is a $\mathbb{Z}^k$-graded S_Y module whose associated sheaf is $\mathcal{F}$ then the multigraded Serre-Grothendieck correspondence gives the exact sequence

$$0 \to H^0_{B_Y}(M) \to M \to \bigoplus_{\mathbf{m}\in\mathbb{Z}^k} H^0(Y, \mathcal{F}(\mathbf{m})) \to H^1_{B_Y}(M) \to 0$$

and the isomorphisms

$$\bigoplus_{\mathbf{m}\in\mathbb{Z}^k} H^i(Y, \mathcal{F}(\mathbf{m})) \;\simeq\; H^{i+1}_{B_Y}(M)$$

for all $i \geq 1$. If $M = \Gamma_*(\mathcal{F})$, then by Theorem 2.22 one deduces that $\mathbf{reg}(\Gamma_*(\mathcal{F})) = \mathbf{reg}(\mathcal{F})$ for any coherent sheaf $\mathcal{F}$.

An interesting relation between algebra and geometry is the following lemma, which is a multigraded generalization of Remark 2.2.

Lemma 2.32. *In the same setting as before, let $\mathcal{F}$ be a coherent sheaf on the multi-projective space Y. If the associated module $\Gamma_*(\mathcal{F})$ is a finitely generated $\mathbb{Z}^k$-graded S_Y module then the set $\mathbf{reg}(\mathcal{F})$ is finitely generated as subset of $\mathbb{Z}^k$.*

Proof. Suppose that $\Gamma_*(\mathcal{F})$ is finitely generated $\mathbb{Z}^k$ graded S_Y module and let

$$\{f_1, \ldots, f_l\} \subseteq \Gamma_*(\mathcal{F})$$

be a set of generators and $\mathbf{m}_i = \deg(f_i)$, i.e. $f_i \in H^0(Y, \mathcal{F}(\mathbf{m}_i))$. Now choose $\mathbf{m} \in \mathbb{Z}^k$ such that for all i-s we have $\mathbf{m}_i \in \mathbf{m}+\mathbb{N}^k$. The idea is to show that $\mathbf{reg}(\mathcal{F}) \subseteq \mathbf{m}+\mathbb{N}^k$. As a result of finite generatedness we have that

$$\Gamma_*(\mathcal{F}) \;=\; \bigoplus_{\mathbf{u}\in\mathbf{m}+\mathbb{N}^k} H^0(Y, \mathcal{F}(\mathbf{u})).$$

This instead implies that $H^0(Y, \mathcal{F}(\mathbf{u})) = 0$ for all $\mathbf{u} \notin \mathbf{m} + \mathbb{N}^k$. So if $\mathbf{u} \in \text{reg}(\mathcal{F})$ such that $\mathbf{u} \notin \mathbf{m} + \mathbb{N}^k$, then by Theorem 2.22 the sheaf $\mathcal{F}(\mathbf{u})$ is globally generated and $H^0(Y, \mathcal{F}(\mathbf{u})) \neq 0$, thus contradicting our assumption. □

Example 2.33. The main difference between the algebraic setting and the geometric one is that the converse of Lemma 2.32 can fail. In fact, there are examples of coherent sheaves whose associated module is not finitely generated, but whose regularity set is finitely generated.

For example, let $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $X := \{x_0y_0z_0 - x_1y_1z_1 = 0\} \subseteq Y$. This is a smooth surface in the linear series defined by the line bundle $\mathcal{O}_Y(1,1,1)$. Take $\mathcal{F} = \mathcal{O}_X$ and we will use Theorem A to show first that the set $\mathbf{reg}(\mathcal{O}_C)$ is finitely generated.

Taking into account the statement of Theorem A, we only show conditon (i) for the first projection $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, as for the second and third the same ideas are valid. But for the first projection one notice easily that

$$\left(\pi_1^{-1}([1:0] \times [0:1])\right) \bigcap \left(\text{supp}(\mathcal{O}_C)\right) \ = \ \mathbb{P}^1 \times [1:0] \times [0:1].$$

Thus using Theorem A, we deduce that $\mathbf{reg}(\mathcal{O}_C)$ is finitely generated.

On the other hand, it is not hard to see that $\Gamma_*(\mathcal{O}_X)$ is not a finitely generated $\mathbb{Z}^3$-graded S_Y module. For this, consider the exact sequence

$$0 \to \mathcal{O}_Y(m-1, n-1, p-1) \to \mathcal{O}_Y(m,n,p) \to \mathcal{O}_X(m,n,p) \to 0.$$

If $m < 0$ then in cohomology one obtains the exact sequence

$$0 \to H^0(X, \mathcal{O}_X(m,n,p)) \to H^1(Y, \mathcal{O}_Y(m-1,n-1,p-1)) \to H^1(Y, \mathcal{O}_Y(m,n,p)).$$

Now whenever $m < 0$ and $n, p > 0$ one has

$$\dim_{\mathbb{C}}(H^1(Y, \mathcal{O}_Y(m-1,n-1,p-1))) \ = \ -mnp$$

and

$$\dim_{\mathbb{C}}(H^1(Y, \mathcal{O}_Y(m,n,p))) \;=\; (-m-1)(n+1)(p+1).$$

In particular, apply this to the 3-tuples $(m,n,p) = (-2^l, 2^l, 2^{2l})$, where $l \in \mathbb{N}$. The first group has dimension 2^{4l} and the second one $2^{4l} - 1$, implying that for 3-tuples like this one has

$$H^0(X, \mathcal{O}_X(m,n,p)) \neq 0.$$

So the $\mathbb{Z}^3$-graded S_Y module $\Gamma_*(\mathcal{O}_X)$ is not globally generated.

Multigraded regularity and syzygies. Both Theorem A and Example 2.33 give a good indication that in the geometric setting, the study of multigraded regularity in the most general case, that is, for any coherent sheaf on Y, has some drawbacks. The mere fact that there are coherent sheaves whose associated modules are not finitely generated, though the regularity set is fintely generated, is an important handicap, as one might expect, especially in the study of syzygies and their relation with regularity. From this viewpoint it is more natural to study regularity in the algebraic setting.

The starting point, in the algebraic setting, is to study the connections between multigraded regularity and syzygies is the multigraded version of Hilbert Syzygy Theorem [38, Proposition 8.18].

Proposition 2.34 (Multigraded Hilbert Syzygy Theorem)**.** *Let M be a finitely generated $\mathbb{Z}^k$-graded S_Y module, then M has a unique minimal $\mathbb{Z}^k$-graded finite free resolution $E_\bullet$*

$$0 \to E_s \to E_{s-1} \to \ldots \to E_1 \to E_0 \to M \to 0$$

where $E_i = \oplus_j S_Y(-\mathbf{b}^{i,j})$.

Remark 2.35. Minimality means that the maps of $E_\bullet$ are given by matrices of $\mathbb{Z}^k$-homogeneous polynomials containing no non-zero constant entries.

Inspired by the classical case, the idea of Há [20] and Sidman-Van Tuyl [50] is to define for each finitely generated S_Y module M, a vector bounding the degrees of each component in the minimal free resolution $E_\bullet$ of M.

Definition 2.36 (Resolution regularity vector)**.** Let M be a finitely generated $\mathbb{Z}^k$-graded S_Y module and $E_\bullet$ the minimal free resolution of M as in Proposition 2.34. Then for any $l = 1, \ldots, k$ define $b^i_l = \max_j\{b^{i,j}_l\}$, where each $b^{i,j}_l$ is the l-th component of the vector $\mathbf{b}^{i,j}$ coming from the resolution $E_\bullet$. If we denote by

$$\text{res-reg}_l(M) \;=\; \max_i\{b^i_l - i\},$$

then define the *resolution regularity vector* of M as follows

$$\textbf{res-reg}(M) \;=\; (\text{res-reg}_1(M), \ldots, \text{res-reg}_k(M)) \in \mathbb{Z}^k.$$

Remark 2.37. In the geometric setting, suppose $\mathcal{F}$ is a coherent sheaf on Y such that $\Gamma_*(\mathcal{F})$ is a finitely generated S_Y module. Let $E_\bullet$ be the minimal resolution of $\Gamma_*(\mathcal{F})$ as in Proposition 2.34. If we sheafify $E_\bullet$, then we obtain a free resolution of $\mathcal{F}$

$$0 \to \oplus_j \mathcal{O}_Y(-\mathbf{b}^{s,j}) \to \oplus_j \mathcal{O}_Y(-\mathbf{b}^{s-1,j}) \to \ldots \to \oplus_j \mathcal{O}_Y(-\mathbf{b}^{0,j}) \to \mathcal{F} \to 0$$

and define

$$\textbf{res-reg}(\mathcal{F}) \;=\; \textbf{res-reg}(\Gamma_*(\mathcal{F})).$$

The drawback is when $\Gamma_*(\mathcal{F})$ is not finitely generated, as it is hard to expect the resolution regularity vector of $\Gamma_*(\mathcal{F})$ to exist. Even when we take a finitely generated S_Y module M whose sheafification is $\mathcal{F}$ (which always exists), it still might happen that $\mathbf{reg}(M)$ and $\mathbf{reg}(\mathcal{F})$ do not coincide.

Unlike the $\mathbb{Z}$-graded case where different approaches agree to give a unique invariant, the relationship bewteen the multigraded regularity and the resolution regularity is not yet clear. Há [20, Theorem 4.2.1] investigates this relationship and proves the following theorem:

Theorem 2.38 (Syzygies and multigraded regularity)**.** *Let M be a finitely generated $\mathbb{Z}^k$-graded S_Y module and suppose that $\sigma = \text{proj-dim}(M)$. Then, one has the following inclusion*

$$\bigcup \left(\mathbf{res\text{-}reg}(M) + \sigma\mathbf{1} - \mathbf{m} + \mathbb{N}^k\right) \ \subseteq \ \mathbf{reg}(M)$$

where the union is taken over all $\mathbf{m} \in \mathbb{N}^k$ such that $|\mathbf{m}| = \sigma$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^k$.

Remark 2.39*.* In comparison to the classical case (Theorem 2.5) Há's Theorem gives an upper bound of the regularity set in terms of syzygies degrees. It would be interesting to know if one can give lower bounds.

Remark 2.40*.* It is not known if multigraded regularity has nice tensorial properties. In the case of $\mathbb{P}^1 \times \mathbb{P}^1$ we will prove in Chapter 4 that this is the case.

2.3 Multigraded regularity of a subvariety

As in the classical setting of Castelnuovo-Mumford regularity, the interesting case in the multigraded context is when $\mathcal{F}$ is the ideal sheaf of a subvariety (or a subscheme) of the multi-projective space.

Definition 2.41 (Multigraded regularity of a projective subvariety)**.** We say that a subvariety (or a subscheme) $X \subseteq Y$ is $\mathbf{m}$-regular if its ideal sheaf $\mathcal{I}_{X/Y}$ is. The *multigraded regularity set* of X is the multigraded regularity set $\mathbf{reg}(\mathcal{I}_{X/Y})$ of its ideal.

Remark 2.42. It is interesting to note that for a subscheme $X \subseteq Y$, there is no difference if one works either in the algebraic setting or the geometric one. This happens because the module $\Gamma_*(\mathcal{I}_{X/Y})$ is a submodule of S_Y, hence finitely generated. Thus the multigraded regularity in the algebraic setting coincides with the geometric one. Also by Lemma 2.32, this set is finitely generated.

Remark 2.43. In §3.4 we give an example of a rational curve $C \simeq \mathbb{P}^1 \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ such that the regularity set is described as follows

$$\mathbf{reg}(C) \;=\; (2,2) \;+\; \mathbb{N}^2.$$

This example will be important for us, as it shows that the bounds given in Theorem B are the best possible.

Example 2.44 (Non-minimality of multigraded regularity)**.** In comparison to the example above, the regularity set $\mathbf{reg}(X)$ is usually generated by more than one element. For example, take a rational curve $C \hookrightarrow Y = \mathbb{P}^3 \times \mathbb{P}^3$ of bidegree $(3,4)$, whose embedding is defined as follows:

$$[s:t] \in \mathbb{P}^1 \;\longrightarrow\; [s^3 : s^2t : st^2 : t^3] \times [s^4 : s^3t : st^3 : t^4] \in \mathbb{P}^3 \times \mathbb{P}^3.$$

Then it is not hard to check that the regularity set is described as follows

$$\mathbf{reg}(C) \;=\; \big((2,1) + \mathbb{N}^2\big) \cup \big((1,2) + \mathbb{N}^2\big).$$

This set is illustrated in Figure 2.1.

First, Theorem B implies that this curve is $(2,1)$-regular. In the following we will show that it is also $(1,2)$-regular. We claim that $H^1(\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3\times\mathbb{P}^3}(0,2)) = 0$, for which we use the following long exact sequence

$$H^0(Y, \mathcal{O}_Y(0,2)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)) \to H^1(Y, \mathcal{I}_{C/Y}(0,2)) \to H^1(Y, \mathcal{O}_Y(0,2)).$$

The way the embedding $C \subseteq Y$ was chosen, it is easy to notice that the first map is surjective. Using the Künneth formula and the vanishings of the higher cohomlogy of projective space, one deduces the vanishing of the the last group. Hence our group vanishes. The same ideas yield $H^1(\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3\times\mathbb{P}^3}(1,1)) = 0$. To prove that C is $(1,2)$-regular, we also need to establish that

$$H^2(Y, \mathcal{I}_{C/Y}(-1,2)) = H^2(Y, \mathcal{I}_{C/Y}(0,1)) = H^2(Y, \mathcal{I}_{C/Y}(1,-1)) = 0.$$

We show the vanishing of the first group as the rest follow from the same ideas. For this, our group is part of the following exact sequence

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(5)) \to H^2(Y, \mathcal{I}_{C/Y}(-1,2)) \to H^2(Y, \mathcal{O}_Y(-1,2)).$$

The vanishing of the first group follows from Serre duality, and the third group vanishes because of the Künneth formula.

Since C is of dimension one, then the vanishing of the rest of the H^i-s for $i \geq 3$ follows from the isomoprhisms

$$H^i(Y, \mathcal{O}_Y(m,n)) \;=\; H^i(Y, \mathcal{I}_{C/Y}(m,n))$$

and from the vanishings of the higher cohomology on $\mathbb{P}^3 \times \mathbb{P}^3$, which are obtained by using once more the Künneth formula and Serre duality. Thus the curve $C \subseteq Y$ is $(2,1)$ and $(1,2)$-regular.

On the other hand C is not $(1,1)$-regular. For this, it suffices to show that $H^1(Y, \mathcal{I}_{C/Y}(0,1))$ doesn't vanish. So, using the exact sequence

$$H^0(Y, \mathcal{O}_Y(0,1)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)) \to H^1(Y, \mathcal{I}_{C/Y}(0,1))$$

and the description of the embedding of $C \subseteq Y$, it is not hard to see that the first map is not surjective. Thus C is not $(1,1)$-regular. To show the description of the

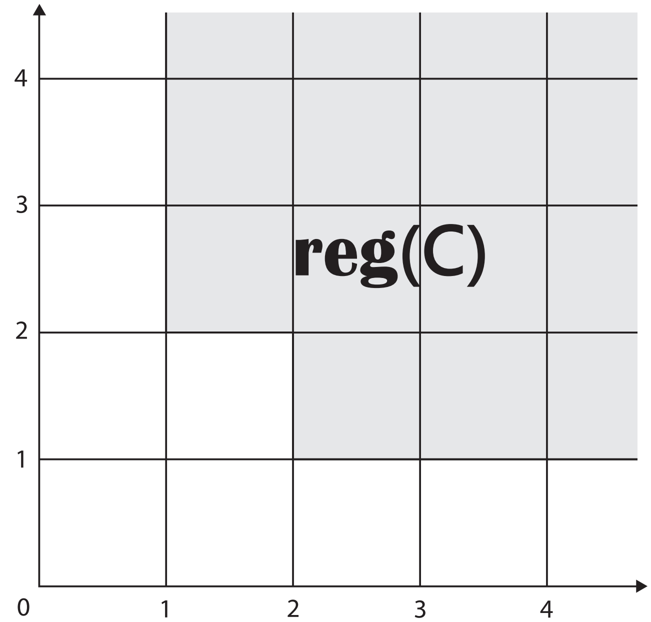

Figure 2.1: Regularity set of the curve C

regularity set $\mathbf{reg}(C)$ as above, it remains to prove that C fails to be $(0, m)$ and $(m, 0)$-regular for all $m \geq 2$. We prove that C is not $(0, m)$-regular, as one uses the same ideas to show that it is also not $(m, 0)$-regular. For this we use the following exact sequence:

$$H^0(Y, \mathcal{O}_Y(-1, m)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4m - 3)) \to H^1(Y, \mathcal{I}_{C/Y}(-1, m)).$$

Using the Künneth formula, we deduce that the first group vanishes. Now because $m \geq 2$ the second group does not vanish. Hence $H^1(Y, \mathcal{I}_{C/Y}(-1, m)) \neq 0$ and this tells us that C is not $(0, m)$-regular.

Gotzmann's bounds. As in the classical case it is interesting to ask for bounds on multigraded regularity. The earliest results bounded regularity in terms of the multigraded Hilbert polynomials. This was accomplished by Maclagan and Smith in [37].

In the following we will briefly describe the work of Maclagan and Smith. Let's start with the following definition:

Definition 2.45 (Multigraded Hilbert function)**.** Given a projective subscheme $X \subseteq Y$, one defines the *multigraded Hilbert function* as

$$P_X(\mathbf{m}) \;=\; \chi(X, \mathcal{O}_X(\mathbf{m})), \text{ for any } \mathbf{m} \in \mathbb{Z}^k.$$

Remark 2.46. As in the classical case, the multigraded Hilbert function P_X of X is a polynomial with rational coefficients and it takes the form

$$P_X(t_1, \ldots, t_k) \;=\; \sum_{\mathbf{m} \in A_{\mathbf{r}}} c_{\mathbf{m}} \cdot \prod_{i=1}^{i=k} \binom{t_i}{m_i},$$

for some $\mathbf{r} \in \mathbb{N}^k$, where $A_{\mathbf{r}} = \{\ \mathbf{m} \in \mathbb{N}^k \mid \mathbf{r} - \mathbf{m} \in \mathbb{N}^k\ \}$ and $c_{\mathbf{m}} \in \mathbb{Z}$. This follows inductively from the classical case, by taking hyperplane sections on components of Y.

As in their first paper, when they define multigraded regularity, the idea Maclagan and Smith use in [37] to bound multigraded regularity in terms of multigraded Hilbert polynomials is to think of the multiprojective space Y as a toric variety. So let

$$S_Y \;=\; \mathbb{C}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}] \;=\; \mathbb{C}[x_0, \ldots, x_N]$$

be the homogeneous ring of Y. If $\sigma \subseteq \{0, \ldots, N\}$ then it decomposes into a finite disjoint union

$$\sigma \;=\; \bigsqcup_{i=1}^{i=k} \sigma^i,$$

where σ^i consists of those $j \in \sigma$ such that the variable $x_j \in S$ corresponds to the the i-factor $\mathbb{P}^{n_i}$ of the multiprojective space Y. With this in hand one can associate to

σ a polynomial

$$P_\sigma(t_1, \ldots, t_k) \;=\; \prod_{i=1}^{i=k} \binom{t_i - |\sigma^i| + 1}{|\sigma^i| - 1}.$$

With this language in hand Maclagan and Smith introduce the following definition:

Definition 2.47 (Stanley filtrations)**.** Let $I \subseteq S_Y$ be a monomial ideal. A *Stanley filtration* is a set of pairs $\{(x^{\mathbf{u}_i}, \sigma_i) \mid 1 \leq i \leq m\ \}$, consisting of a monomial $x^{\mathbf{u}_i} \in S_Y$ and a subset $\sigma_i \subseteq \{0, \ldots, N\}$, such that the modules

$$M_i = S/\big(I + \langle x^{\mathbf{u}_{i+1}}, \ldots, x^{\mathbf{u}_m} \rangle\,\big)$$

form a filtration

$$\mathbb{C} = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_m = S_Y/I$$

with $M_i/M_{i+1} = \mathbb{C}[x_j \mid j \in \sigma_i]$. Moreover, if $X \subseteq Y$ is a projective subscheme, then we call a set $\{(x^{\mathbf{u}_i}, \sigma_i) \mid 1 \leq i \leq m\}$, a *Stanley filtration* of X, if it is a Stanley filtration for the initial ideal $\mathrm{in}(\Gamma_*(\mathcal{I}_{X/Y}))$, where we write $\mathrm{in}(I)$ for the initial ideal of $I \subseteq S_Y$ with respect to some monomial order.

Maclagan and Smith construct an algorithm for finding Stanley filtrations. Their first result is the following:

Proposition 2.48. *(i) For any multigraded polynomial $P(\mathbf{t})$, there are at most finitely many B_Y-saturated monomial ideals whose multigraded Hilbert polynomial is equal to $P(\mathbf{t})$.*

(ii) Let $X \subseteq Y$ be a projective subscheme and $\{(x^{\mathbf{u}_i}, \sigma_i) \mid 1 \leq i \leq m\}$ be a Stanley filtration for X. Then the multigraded Hilbert polynomial is equal to

$$P_X(\mathbf{t}) \;=\; \sum_{i=1}^{i=m} P_{\sigma_i}(\mathbf{t} - \deg(x^{\mathbf{m}})),$$

where each $P_{\sigma_i}(\mathbf{t})$ is defined as above.

To state the main result of Maclagan and Smith, we make the following definition:

Definition 2.49. If $X \subseteq Y$ is a projective subscheme, then the largest number of pairs in a Stanley filtration of X is called the *Gotzmann number* of $P_X(\mathbf{t})$.

Theorem 2.50 (Maclagan and Smith's theorem)**.** *Let $X \subseteq Y$ be a projective subscheme and suppose s is the Gotzmann's number of $P_X(\mathbf{t})$. Then X is $(s-1)\cdot\mathbf{1}$-regular, where $\mathbf{1} = (1,\ldots,1) \in \mathbb{Z}^k$.*

Remark 2.51. Using Theorem 2.50, Maclagan and Smith note that one can recover the original version of Gotzmann's Theorem in the classical case, with a slight improvement on the bound. For the proof of Theorem 2.50, they notice that the regularity of the inital ideal bounds the regularity of the ideal $\Gamma_*(\mathcal{I}_{X/Y})$. Using the algorithm they constructed to find Stanley filtrations of a monomial ideal, Maclagan and Smith are able to obtain the bound from Theorem 2.50 in the monomial setting. Remarkably, they obtain this bound using only the behaviour of multigraded regularity in short exact sequences.

Bounds from defining equations. As in the classical case, an interesting question is to find bounds on multigraded regularity in terms of the multidegrees of the generators of the ideal sheaf of a subscheme embedded in a multiprojective space. The main work in this direction was done only for zero dimensional subschemes, e.g. [50].

The question of giving upper bounds on regularity for arbitrary subschemes has not yet been tackled. As in the classical case (Theorem 2.13), it is presumed that for arbitrary subschemes one obtains exponential type bounds. The main issue when one wants to use the same ideas of Bayer and Mumford in order to exend Theorem 2.13 to the multigraded case is that on a multiprojective space Y, the line bundles

$\mathcal{O}_Y(e_i)$ are only base point free and not ample. Thus Serre vanishing does not apply in this case, as it might happen that there exists sheaves $\mathcal{F}$ on Y such that for $m \gg 0$ some higher cohomology group of $\mathcal{F}(m \cdot e_i)$ might not vanish.

In comparison to the case of arbitrary subschemes, the result of Bertram, Ein and Lazarsfeld [8] for smooth subvarieties can be easily generalized to the multigraded setting. As in the classical case, one obtain linear bounds.

Theorem 2.52 (Linear bounds for smooth ideals)**.** *Let $X \subseteq Y$ be a smooth subvariety of dimension n. Let $r = n_1 + \cdots + n_k$ be the dimension of Y and $e = r - n$ be the codimension of X in Y. If $\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k$, with each $d_i \geq 1$, is a vector which satisfies the property that the ideal sheaf $\mathcal{I}_{X/Y}(\mathbf{d})$ is globally generated, then X is $\mathbf{v} = (v_1, \ldots, v_k)$-regular, where each $v_i = ed_i - n_i + n$.*

Remark 2.53. The proof of the theorem uses the same ideas as that of Theorem 2.15 given in [33, Theorem 4.3.15] and we briefly explain it here.

Let $\mu : Y' \to Y$ be the blow-up of Y along X and E be the exceptional divisor of μ. For any $1 \leq i \leq n$ and $\mathbf{u} \in \mathbb{N}^k$ with $|\mathbf{u}| = i$ we will use the following isomorphism:

$$H^i(Y, \mathcal{I}_{X/Y}(\mathbf{v} - \mathbf{u})) \;=\; H^i(Y', \mu^*(\mathcal{O}_Y(\mathbf{v} - \mathbf{u})) \otimes \mathcal{O}_{Y'}(-E)).$$

The idea is to write $\mu^*(\mathcal{O}_Y(\mathbf{v} - \mathbf{u})) \otimes \mathcal{O}_{Y'}(-E) \;=\; \mathcal{O}_{Y'}(K_{Y'} + D)$, where D is a big and nef divisor on Y', so that one can apply Kawamata-Viehweg vanishing. In order to see that D is big and nef, one uses the fact that $\mu^*(\mathcal{O}_Y(\mathbf{d})) \otimes \mathcal{O}_{Y'}(-E)$ is base point free and $\mu^*(\mathcal{O}_Y(\mathbf{a}))$ is big and nef for any $\mathbf{a} = (a_1, \ldots a_k) \in \mathbb{N}^k$ with each $a_i \geq 1$.

As for the other vanishings, when $n + 1 \leq i \leq r$ one uses the isomorphism

$$H^i(Y, \mathcal{I}_{X/Y}(\mathbf{v} - \mathbf{u}) \;=\; H^i(Y, \mathcal{O}_Y(\mathbf{v} - \mathbf{u}))$$

Now by the Künneth formula and the fact that

$$ed_i - n_i + n \ \geq \ r - n - n_i + n \ = \ r - n_i,$$

the group on the right vanishes and the theorem follows.

Another important question is to find an upper bound on multigraded regularity of a projective subscheme $X \subseteq Y$ in terms of its multidegree. Inspired by the work of Gruson, Lazarsfeld and Peskine in the classical case, the first attempt was done by the author in [35], where he gives a linear bound in terms of its bidegree for smooth curves embedded in biprojective spaces. The main idea is to translate into the multigraded setting the techniques developed by Gruson and Peskine and effectively used by Lazarsfeld, on his work on bounding regularity for smooth surfaces [32]. This will be explained in full detail in Chapter 3.

2.4 Proof of Theorem A

In the final section of this chapter we give a proof of Theorem A, generalizing Remark 2.3 in the multigraded setting. We start with a general lemma, which will be helpful for our further discussion.

First let's fix some notation. Let $Z = \mathbb{P}^n \times X$, where X is a smooth projective variety, and $p_1 : Z \to \mathbb{P}^n$ and $p_2 : Z \to X$ are the projections to each factor. If we denote $\mathcal{O}_Z(1) = p_1^*(\mathcal{O}_{\mathbb{P}^n}(1))$ then the following lemma holds:

Lemma 2.54. *With the notations above, let $\mathcal{F}$ be a coherent sheaf on Z. Then the following two conditions are equivalent:*

(i) For any $x \in X$, the set $\operatorname{supp}(\mathcal{F}|_{p_2^{-1}(x)})$ is at most zero-dimensional.

(ii) For all $m \in \mathbb{Z}$ and any $i > 0$, $R^i p_{2*}(\mathcal{F}(m)) = 0$.

Proof. Suppose condition (i) holds. By the vanishing Theorem of Grothendieck (see [24, Theorem III.2.7]) for a coherent sheaf $\mathcal{M}$ on some projective scheme W such that $\mathcal{M}$ has a zero dimensional support, we have $H^i(W, \mathcal{M}) = 0$ for all $i > 0$. Now combining this with the Theorem on Formal Functions (see [24, Theorem III.11.1]), we deduce that for any point $x \in X$ we have

$$R^i p_{2*}\big(\mathcal{F}(m)\big)\widehat{\ }_x \;=\; 0, \text{ for any } i > 0 \text{ and all } m \in \mathbb{Z}.$$

Thus one has $R^i p_{2*}\big(\mathcal{F}(m)\big) = 0$ for any $i > 0$ and all $m \in \mathbb{Z}$.

Conversely, suppose condition (ii) is satisfied and that condition (i) is not. Let $V = \operatorname{supp}(\mathcal{F})$ and $x \in X$ be a point over which V has a maximal dimensional fiber. Then on $\mathbb{P}^n \times X$ we have the following short exact sequence of sheaves

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}|_{\mathbb{P}^n \times \{x\}} \to 0$$

where $\mathcal{K}$ is the kernel sheaf of the retriction map. Denote by a the dimension of the fiber of V over the point $x \in X$. Since $\operatorname{supp}(\mathcal{K}) \subseteq \operatorname{supp}(\mathcal{F})$, then for any $m \in \mathbb{Z}$ we have the following sequence on higher direct image sheaves:

$$\ldots \to R^a p_{2*}\big(\mathcal{K}(m)\big) \to R^a p_{2*}\big(\mathcal{F}(m)\big) \to R^a p_{2*}\big(\mathcal{F}(m)|_{\mathbb{P}^n \times \{x\}}\big) \to 0,$$

where the last map is surjective because the dimension of each fiber of $\operatorname{supp}(\mathcal{K})$ is at most a. Thus to contradict our assumption it is enough to show that there exists an integer $m \in \mathbb{Z}$ such that the sheaf $R^a p_{2*}\big(\mathcal{F}(m)|_{\mathbb{P}^n \times \{x\}}\big)$ does not vanish. As this sheaf is supported on the fiber $\mathbb{P}^n \times \{x\}$, we actually need to prove that for a sheaf $\mathcal{G}$ on $\mathbb{P}^n$, whose support is of dimension $a > 0$, there exists an integer $m \in \mathbb{Z}$ such that

$$H^a(\mathbb{P}^n, \mathcal{G}(m)) \;\neq\; 0.$$

Let W be the support of $\mathcal{G}$. If $W \subsetneqq \mathbb{P}^n$, then by taking a general projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^a$ we can find a finite branched covering $p : W \to \mathbb{P}^a$. Since this morphism is finite, we have that

$$H^i(\mathbb{P}^n, \mathcal{G}(m)) \;=\; H^i(\mathbb{P}^a, p_*(\mathcal{G})(m)).$$

for all $i \geq 0$ and all $m \in \mathbb{Z}$. Thus we can assume that $W = \mathbb{P}^a$.

Also we have a short exact sequence of sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{G} \to \mathcal{G}_2 \to 0$$

where $\mathcal{G}_1$ is the torsion sheaf of $\mathcal{G}$ and $\mathcal{G}_2$ is a torsion free sheaf. Since $\mathrm{supp}(\mathcal{G}_1) \subseteq \mathrm{supp}(\mathcal{G})$, then for any $m \in \mathbb{Z}$ the map

$$H^a(\mathbb{P}^a, \mathcal{G}(m)) \to H^a(\mathbb{P}^a, \mathcal{G}_2(m))$$

is surjective. Thus to prove the claim, we can also assume that $\mathcal{G}$ is a torsion free sheaf.

Now, let $\mathcal{G}^*$ be the dual of $\mathcal{G}$ and $r = \mathrm{rank}(\mathcal{G})$. Take $m_0 \gg 0$ such that the sheaf $\mathcal{G}^*(m_0)$ is globally generated. We can choose r global section of $\mathcal{G}^*(m_0)$ such that the map they define

$$\oplus_{i=1}^{i=r} \mathcal{O}_{\mathbb{P}^a} \to \mathcal{G}^*(m_0)$$

is an isomorphism at the generic point, i.e. the kernel and cokernel of this map have support of dimension at most $a-1$. As $\mathcal{G}$ is a torsion free sheaf, we have a natural inclusion $\mathcal{G} \subseteq \mathcal{G}^{**}$, which is an isomorphism on a nonempty open set. Thus, by dualizing the map above, we obtain a map

$$\mathcal{G} \to \oplus_{i=1}^{i=r} \mathcal{O}_{\mathbb{P}^a}(m_0)$$

which has the property that the kernel and the cokernel have support of dimension at most $a-1$. Thus for any $m \in \mathbb{Z}$ we obtain a surjective map

$$H^a(\mathbb{P}^a, \mathcal{G}(m)) \to \oplus_{i=1}^{i=r} H^a(\mathbb{P}^a, \mathcal{O}_{\mathbb{P}^a}(m+m_0))$$

If we take $m \ll 0$, then the group on the right does not vanish. Thus we prove our claim and this contradicts our assumption. $\square$

Another important tool we will need in order to prove Theorem A is the following application of the Leray spectral sequence:

Lemma 2.55. *Let $f : Z \to X$ be a morphism between two projective varieties. Suppose $\mathcal{F}$ is coherent sheaf on Z and we have that $H^j(X, R^k f_*(\mathcal{F})) = 0$ for all $j > 0$ and $k \geq 0$. Then we have the isomorphism*

$$H^i(Z, \mathcal{F}) = H^0(X, R^i f_*(\mathcal{F})) \text{ for all } i \geq 0.$$

Now, getting back to our setup, let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and let

$$\pi_i : Y \to Y_i \ := \ \mathbb{P}^{n_1} \times \cdots \times \widehat{\mathbb{P}^{n_i}} \times \cdots \times \mathbb{P}^{n_k}$$

the projection that drops the i-th coordinate of Y. With this in hand, by Remark 2.25, Theorem A is implied by the following proposition.

Proposition 2.56. *Let $\mathcal{F}$ be a coherent sheaf on the multiprojective space Y. Then the following two conditions are equivalent:*

(i) Suppose that for each $i = 1, \ldots, k$ there exists a point $x_i \in Y_i$ such that the support $\operatorname{supp}(\mathcal{F}|_{\pi_i^{-1}(x_i)})$ *is of dimension at least one.*

(ii) The set $\mathbf{reg}(\mathcal{F})$ is bounded from below as a subset of $\mathbb{Z}^k$.

Proof. Suppose that condition (i) takes place. In this case, we show that the existence of a closed point $x_1 \in Y_1$ such that the support $\mathbf{reg}(\mathcal{F}|_{\pi_1^{-1}(x_1)})$ is of dimension at least one, implies the existence of an integral vector $(m_1^0, \ldots, m_k^0) \in \mathbb{Z}^k$ such that for any $(m_2, \ldots, m_k) \in (m_2^0, \ldots, m_k^0) + \mathbb{N}^{k-1}$ there exists $i > 0$ such that

$$H^i(Y, \mathcal{F}(m_1^0, m_2, \ldots, m_k)) \neq 0.$$

It is easy to notice that by applying this reduction for each morhism π_i and Theorem 2.22, we deduce that the regularity set of $\mathcal{F}$ is bounded from below.

For this, first notice that by Lemma 2.55 we have $R^{i_0}\pi_{1*}(\mathcal{F}(m_1^0, 0, \ldots 0)) \neq 0$ for some $0 < i_0 \leq n_1$ and $m_1^0 \in \mathbb{Z}$. Now, there exists a vector $(m_2^0, \ldots, m_k^0) \in \mathbb{Z}^{k-1}$ such that for all $i \geq 0$ the sheaf $R^i\pi_{1*}(\mathcal{F}(m_1^0, 0, \ldots, 0))$ is $(m_2^0, \ldots, m_k^0)$-regular on Y_1. By Theorem 2.22, then for any $(m_2, \ldots, m_k) \in (m_2^0, \ldots, m_k^0) + \mathbb{N}^{k-1}$ we have

$$H^j(Y_1, R^i\pi_{1*}(\mathcal{F}(m_1^0, m_2, \ldots, m_k))) = 0, \text{ for all } i \geq 0 \text{ and } j > 0.$$

Also this implies that the sheaf $R^{i_0}\pi_{1*}(\mathcal{F}(m_1^0, m_2, \ldots m_k))$ is nonzero and globally generated. Now using Lemma 2.55 and the projection formula we obtain that

$$H^{i_0}(Y, \mathcal{F}(m_1^0, m_2, \ldots, m_k)) = H^0(Y_1, R^{i_0}\pi_{1*}(\mathcal{F}(m_1^0, m_2, \ldots, m_k))) \neq 0$$

for all $(m_2, \ldots, m_k) \in (m_2^0, \ldots, m_k^0) + \mathbb{N}^{k-1}$, and this proves the reduction.

Conversely, suppose that condition (ii) is satisfied and condition (i) is not. This says that for all $x \in Y_1$ the support $\text{supp}(\mathcal{F}|_{\pi_1^{-1}(x)})$ is at most zero dimensional. By Lemma 2.54, we have that

$$R^i\pi_{1*}(\mathcal{F}(m, 0, \ldots, 0)) = 0, \text{ for all } i > 0, m \in \mathbb{Z}.$$

By a degeneration of Leray spectral sequence and projection formula, then for all $(m_1, \ldots, m_k) \in \mathbb{Z}^k$ and $i > 0$ we have

$$H^i(Y, \mathcal{F}(m_1, \ldots, m_k)) = H^i(Y_1, \pi_{1*}(\mathcal{F}(m_1, 0, \ldots, 0)) \otimes \mathcal{O}_{Y_1}(m_2, \ldots, m_k)).$$

The idea is to prove that for any $m \in \mathbb{Z}$ there exists a vector $(m_2, \ldots, m_k) \in \mathbb{Z}^{k-1}$ such that $\mathcal{F}$ is $(m, m_2, \ldots, m_k)$-regular, which will contradict our assumption.

For this, denote by $s = n_2 + \cdots + n_k$ and choose $m \in \mathbb{Z}$. In this case, there exists a vector $(m_2, \ldots, m_k) \in \mathbb{Z}^{k-1}$ such that all the sheaves $\pi_{1*}(\mathcal{F}(m, 0, \ldots, 0))$, $\ldots$,$\pi_{1*}(\mathcal{F}(m-s, 0, \ldots, 0))$ are simultaneously $(m_2, \ldots, m_k)$-regular on Y_1. The idea is to show that $\mathcal{F}$ is $(m, m_2, \ldots, m_k)$-regular.

For this, take $i > 0$ and $\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{N}^k$ such that $|\mathbf{u}| = i$. Now, by Theorem 2.22 and the fact that $m - s \leq m - u_1 \leq m$, we deduce that the sheaf $\pi_{1*}(\mathcal{F}(m - u_1, 0 \ldots 0))$ is $(m_2 + u_1, m_3, \ldots, m_k)$-regular. Using the isomorphism

$$H^i(Y, \mathcal{F}(m - u_1, \ldots, m_k - u_k)) \;=\; H^i(Y_1, \pi_{1*}(\mathcal{F}(m - u_1, m_2 - u_2, \ldots, m_k - u_k)))$$

we deduce that the group on the right vanishes and this finishes the proof. □

CHAPTER III

Regularity of Smooth Curves in Biprojective Spaces

This chapter, which concludes the first part of this thesis, gives an upper bound on multigraded regularity for curves embedded in biprojective spaces. Here we give a proof to Theorem B.

The proof is adapted from [35] and we start with some notation and definitions. In this chapter $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ $(a, b \geq 2)$ will be a smooth irreducible curve. Denote by p_1 and p_2 the projections of $\mathbb{P}^a \times \mathbb{P}^b$ to each factor. We will only deal with curves $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ which have *nondegenerate birational projections*, meaning $p_1|_C$ and $p_2|_C$ are birational morphisms and have nondegenerate images, C_1 and C_2 respectively. Let $L_1 := \mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(1,0) \otimes \mathcal{O}_C$ and $L_2 := \mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(0,1) \otimes \mathcal{O}_C$, Then we say that C is of *bidegree* (d_1, d_2) if $d_1 = \deg_C(L_1)$ and $d_2 = \deg_C(L_2)$.

The main idea of the proof of Theorem B is the existence of good projections. In order to define them, let Λ_a and Λ_b be codimension two planes in $\mathbb{P}^a$ and $\mathbb{P}^b$ respectively. They define the projection maps: $\pi_a : \mathbb{P}^a \dashrightarrow \mathbb{P}^1$ and $\pi_b : \mathbb{P}^b \dashrightarrow \mathbb{P}^1$. If $\Lambda_a \cap C_1 = \emptyset$ and $\Lambda_b \cap C_2 = \emptyset$ then we have the diagram

$$\begin{array}{ccc} C & \subseteq & \mathbb{P}^a \setminus \{\Lambda_a\} \times \mathbb{P}^b \setminus \{\Lambda_b\} \\ \downarrow{\scriptstyle f_{a,b}} & & \downarrow{\scriptstyle \pi_a \times \pi_b} \\ \bar{C} & \subseteq & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

where $f_{a,b}$ is the restriction to C of $\pi_a \times \pi_b$. In this case we introduce the definition:

Definition 3.1. We will say that the pair (Λ_a, Λ_b) defines a *good projection* for C if $f_{a,b}$ is a birational morphism with fibers of length at most two, and the differential map of $f_{a,b}$ is injective for all $z \in C$.

As we said before, in order to prove that a smooth irreducible curve $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ of bidegree (d_1, d_2) with nondegenerate birational projections is $(d_2 - b + 1, d_1 - a + 1)$-regular, we first show the existence of good projections. This will be done in §3.1, where we show that whenever $a \neq b$, or $r := a = b$ and the curve C is not included in the graph of an automorphism of $\mathbb{P}^r$, there exist plenty of good projections to $\mathbb{P}^1 \times \mathbb{P}^1$. The idea will be based on Castelnuovo's approach, relying on the sort of monodromy argument developed by Harris [3, Chapter III]. Then in §3.2, whenever good projections exist, we are able to establish the bound in Theorem B, by using vector bundle techniques developed by Gruson, Peskine [18], [19] and Lazarsfeld [32]. In the case when one cannot construct good projections, we prove that the curve is included in the graph of an automorphism of $\mathbb{P}^r$. In this case we reduce the problem, in §3.3, to the classical setting of curves embedded in the projective space, and by making use of [17, Theorem 1.1], mentioned in §2.1, we finish the proof of Theorem B. We end this chapter, in §3.4, with an example of a rational curve $C \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(3, 3)$. This curve has the property that $\mathbf{reg}(C) = \big((2,2) + \mathbb{N}^2\big)$, showing that the bound in Theorem B is the best possible.

3.1 Existence of good projections

In this section, we prove the existence of good projections. Specifically our goal is to establish the following theorem:

Theorem 3.2 (Existence of good projections)**.** *Let $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ be a smooth curve with nondegenerate birational projections. Suppose that either $a \neq b$, or $r := a = b$*

and the curve C is not included in the graph of an automorphism of $\mathbb{P}^r$. *Then C has good projections to* $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 3.3. In our search for good projections we need to know that a general center $\Lambda_a \in \mathrm{Grass}(a-2, \mathbb{P}^a)$ is not contained in a hyperplane $H_a \subseteq \mathbb{P}^a$, such that the map p_2 projects at least two points of the set $(H_a \times \mathbb{P}^b) \cap C$ to the same one. To prove this, note that a hyperplane of this sort contains two points $(x_1, y_1), (x_2, y_2) \in (H_a \times \mathbb{P}^b) \cap C$ with $y_1 = y_2$ and $x_1 \neq x_2$. As a result the line $\overline{x_1 x_2}$ connecting both points is contained in the hyperplane H_a. On the other hand we assumed that $p_2|_C$ is birational to its image, so there exist only finitely many pairs of points on C having the same image under p_2. This implies that the family of these hyperplanes is of codimension at least two in $\mathbb{P}^{b*}$. As a consequence, the dimension of those $\Lambda_a \in \mathrm{Grass}(a-2, \mathbb{P}^a)$ contained in these hyperplanes is at most $a-2+a-1 = 2a-3$. But $\dim(\mathrm{Grass}(a-2, \mathbb{P}^a)) = 2a-2$ and the assertion follows immediately.

In order to prove Theorem 3.2, we introduce some notation which will simplify our exposition. If H_a and H_b are hyperplanes in $\mathbb{P}^a$ and $\mathbb{P}^b$ respectively, then denote by

$$(C.H_b) := p_1((\mathbb{P}^a \times H_b) \cap C) \subseteq \mathbb{P}^a,$$

$$(H_a.C) := p_2((H_a \times \mathbb{P}^b) \cap C) \subseteq \mathbb{P}^b$$

Also if $F_1, F_2 \subseteq \mathbb{P}^a$ then $\overline{F_1 F_2}$ will be the Zariski closure of the union of all lines connecting one point from F_1 and another one from F_2.

The proof of Theorem 3.2 uses the idea of "uniform position principle" developed by Harris, e.g. Chapter III of [3]. Specifically, we have the following lemma:

Lemma 3.4 (Uniform position principle). *If $\Lambda_a \subseteq \mathbb{P}^a$ is a general codimension two plane, then one of the following two situations must happen:*

(1_a) *For all hyperplanes H_a containing Λ_a, the set $(H_a.C)$ does not span $\mathbb{P}^b$.*

(2_a) *For a general hyperplane H_a containing Λ_a, any $b+1$ points of $(H_a.C)$ span $\mathbb{P}^b$.*

Proof. The curve C is the desingularization of C_1, so ([47, Theorem 1.1]) implies that the projection map $\pi_a|_C$ defined by a general codimension two plane $\Lambda_a \subseteq \mathbb{P}^a$ has the full symmetric group as its monodromy. If we set

$$U = \mathbb{P}^1 \setminus \{\text{Branch points of } \pi_a|_C\} \text{ and } V = (\pi_a|_C)^{-1}(U),$$

this says that $\forall y \in U$, every two points in the fiber $(\pi_a|_C)^{-1}(y)$ can be connected by a path in V lifted from a loop in U based at y. Now construct the following incidence correspondence:

$$I_a(b+1) \subseteq V \times .... \times V \times U$$

consisting of those tuples $(q_1, ..., q_{b+1}, y)$ where the points $q_1, ..., q_{b+1}$ are distinct and contained in the fiber $(\pi_a|_C)^{-1}(y)$. As the monodromy is the full symmetric group, $I_a(b+1)$ is connected. Now, the projection map $I_a(b+1) \to U$ is a covering map and U is irreducible, therefore $I_a(b+1)$ is an irreducible variety of dimension one. Otherwise, if it is reducible, then by connectivity of $I_a(b+1)$, two components have to intersect, forcing the existence of a point in U whose fiber contains fewer points than a general fiber of the map $I_a(b+1) \to U$.

Let's define the following closed subvariety:

$$\begin{aligned} J_a(b+1) &= \{(q_1, ..., q_{b+1}, y) \in I_a(b+1) : \\ &\qquad p_2(q_1), ..., p_2(q_{b+1}) \text{ don't span } \mathbb{P}^b\}. \end{aligned}$$

As $I_a(b+1)$ is irreducible, either $J_a(b+1) = I_a(b+1)$, or $\dim(J_a(b+1)) = 0$. Bearing in mind Remark 3.3 we have that the first case corresponds to (1_a), and the second one is equivalent with (2_a). □

Lemma 3.5. *Let $\Lambda_a \subseteq \mathbb{P}^a \setminus C_1$ be a general codimension two plane in the first factor. If condition (2_a) from Lemma 3.4 is satisfied, then for a general codimension two plane $\Lambda_b \subseteq \mathbb{P}^b$, the pair (Λ_a, Λ_b) defines a good projection for C.*

Proof. Since Λ_a is general, Remark 3.3 tells us that $\forall x \in \mathbb{P}^1$ both $(\overline{x\Lambda_a} \times \mathbb{P}^b) \cap C$ and $(\overline{x\Lambda_a}.C)$ have the same cardinality. With this in hand we can start the proof. First, notice that for a general choice of Λ_b, we need to show that the map $f_{a,b}$ has all the fibers of length at most two. The map $f_{a,b}$ has a fiber of length at least three only if there exist $x \in \mathbb{P}^1$ and a hyperplane H_b passing through Λ_b which contains at least three points from the set $(\overline{x\Lambda_a}.C)$. Condition (2_a) guarantees the existence of an open set $U \subseteq \mathbb{P}^1$, where $\forall x \in U$ any three points in $(\overline{x\Lambda_a}.C)$ span a plane in $\mathbb{P}^b$. Hence the family of those hyperplanes H_b which for some $x \in U$ contain at least three points from the set $(\overline{x\Lambda_a}.C)$ is of codimension two. Simultaneously Λ_b should not be included in a hyperplane which for some $x \in \mathbb{P}^1 \setminus U$ contains at least two points from $(\overline{x\Lambda_a}.C)$. Now, bearing in mind the ideas from Remark 3.3, the assertion follows immediately.

Next, for a general choice of Λ_b, we need to show that the map $f_{a,b}$ is birational to its image. The map is not birational if given a general hyperplane H_b containing Λ_b, there exists $x \in \mathbb{P}^1$ such that H_b contains at least two points from the set $(\overline{x\Lambda_a}.C)$. This forces Λ_b to intersect the line connecting these two points. The union of all lines connecting two points of $(\overline{x\Lambda_a}.C)$ when $x \in \mathbb{P}^1$ is of dimension two. Hence a general codimension two plane $\Lambda_b \subseteq \mathbb{P}^b$ intersects only finitely many lines, such that each one contains only two points of $(\overline{x\Lambda_a}.C)$ for some $x \in \mathbb{P}^1$. As Λ_b does not intersect C_2 inside $\mathbb{P}^b$ we deduce that a general hyperplane passing through Λ_b contains at most one point of $(\overline{x\Lambda_a}.C)$ for all $x \in \mathbb{P}^1$, and bearing in mind Remark 3.3 we obtain the birationality condition for a general Λ_b.

Lastly we need the differential map of $f_{a,b}$ to be injective for all $z \in C$. This means that there should not exist a hyperplane H_b passing through Λ_b and a point $x \in \mathbb{P}^1$ such that $\overline{x\Lambda_a} \times H_b$ contains the tangent direction at some point on C. The fact that Λ_a is general implies that for only finitely many $x \in \mathbb{P}^1$ we have that $\overline{x\Lambda_a} \times \mathbb{P}^b$ is tangent to C at some point. Therefore a general Λ_b does not intersect the projection to $\mathbb{P}^b$ of the tangent direction at these points, so the differential map of $f_{a,b}$ is injective for all $z \in C$. □

Lemma 3.4 and Lemma 3.5 imply that one might fail to obtain good projections, only in the case wherein for a general codimension two plane $\Lambda_a \subseteq \mathbb{P}^a$, condition (1_a) in Lemma 3.4 is satisfied. This means that for a general hyperplane $H_a \subseteq \mathbb{P}^a$, the set $(H_a.C)$ does not span $\mathbb{P}^b$, and as this condition is closed, it is true for all hyperplanes. Similarly, the same is true when we deal with codimension two planes in $\mathbb{P}^b$. Thus we deduce that the only case when there fail to exist good projections is when the curve C satisfies the following property:

(*) For all hyperplanes $H_a \subseteq \mathbb{P}^a$ and $H_b \subseteq \mathbb{P}^b$, the finite sets $(H_a.C)$ and $(C.H_b)$ fail to span $\mathbb{P}^b$ and $\mathbb{P}^a$ respectively.

The last fact needed to prove Theorem 3.2 is the following lemma:

Lemma 3.6. *Let $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ be a curve which satisfies $(*)$. Then for a general hyperplane $H_a \subseteq \mathbb{P}^a$ we have:*

(i) The set $H_a \cap C_1$ spans the hyperplane H_a.

(ii) The set of points $(H_a.C)$ spans a unique hyperplane H_b in $\mathbb{P}^b$.

(iii) $(H_a \times \mathbb{P}^b) \cap C = (\mathbb{P}^a \times H_b) \cap C$.

Proof. First, the uniform position principle [3] states that if $C_1 \subseteq \mathbb{P}^a$ is an irreducible, nondegenerate curve, then for a general hyperplane $H_a \subseteq \mathbb{P}^a$, the set $H_a \cap C_1$ spans H_a. Secondly, we want to show that $(*)$ forces these hyperplanes to satisfy (ii) in the lemma. Choose a hyperplane H_a which satisfies (i) and suppose that the set $(H_a.C)$ generates a plane $\Pi_b \subseteq \mathbb{P}^b$ of codimension at least two. Hence for all hyperplanes H_b passing through Π_b we have $(H_a.C) \subseteq H_b$ and therefore $H_a \cap C_1 \subseteq (C.H_b)$. Now $(*)$ says that the set $(C.H_b)$ lies in a hyperplane. Together with (i) we get that $(C.H_b) \subseteq H_a$, for all hyperplanes H_b containing Π_b. As the union of all hyperplanes containing Π_b covers $\mathbb{P}^b$ we have that $C \subseteq H_a \times \mathbb{P}^b$, which is a contradiction. Lastly, for (iii) notice that because $(H_a.C)$ spans the hyperplane $H_b \subseteq \mathbb{P}^b$, we have the inclusion

$$(H_a \times \mathbb{P}^b) \cap C \ \subseteq (\mathbb{P}^a \times H_b) \cap C$$

This tells us that $H_a \cap C_1 \subseteq (C.H_b)$ inside $\mathbb{P}^a$ and by $(*)$ combined with (i) we deduce that the set $(C.H_b)$ spans H_a and obtain the reverse inclusion,

$$(H_a \times \mathbb{P}^b) \cap C \supseteq (\mathbb{P}^a \times H_b) \cap C,$$

completing the proof. □

Proof of Theorem 3.2. The paragraph following Lemma 3.5 says that we might fail to obtain good projections only if the curve $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ satisfies property $(*)$. Now Lemma 3.6 and $(*)$ implies that there exist open sets of hyperplanes $H_a \subseteq \mathbb{P}^a$ and $H_b \subseteq \mathbb{P}^b$ such that condition (iii) in Lemma 3.6 is satisfied. Now choose H_a and H_b where $H_a \times \mathbb{P}^b$ and $\mathbb{P}^a \times H_b$ intersect the curve C transversally at every point. We deduce, by condition (iii) in Lemma 3.6, that the line bundles L_1 and L_2 are isomorphic: denote this common bundle by L. Now, let $W_i \subseteq H^0(C, L)$ be the linear subseries defining the restriction map $p_i|_C$ for each $i = 1, 2$. Since condition (iii)

from Lemma 3.6 is an open condition, there exist an open set of sections in W_1 and an open set of sections in W_2 that correspond to each other. This forces $W_1 = W_2$ inside $H^0(C, L)$, therefore $a = b$ and the curve C is included in the graph of an automorphism, so Theorem 3.2 is proved. □

Remark 3.7. Choose a general codimension two plane Λ_a and suppose it satisfies condition (2_a) from Lemma 3.4. As this condition is open, then the same property is satisfied by other general codimension two planes in the first factor in $\mathbb{P}^a$. Therefore for a curve $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ as in Theorem 3.2 we have plenty of pairs (Λ_a, Λ_b) which define a good projection for C.

3.2 Regularity bounds in the general case

In this section our goal is to prove that the bound on the multigraded regularity given in Theorem B holds for all curves $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ which satisfy the conditions in Theorem 3.2.

Theorem 3.8 (Regularity bound). *Let $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ be a smooth curve of bidegree (d_1, d_2) as in Theorem 3.2. Then the ideal sheaf $\mathcal{I}_{C/\mathbb{P}^a \times \mathbb{P}^b}$ is $(d_2 - b + 1, d_1 - a + 1)$-regular.*

The key idea for the proof of Theorem 3.8 is the following result, which allows us to connect the regularity of the ideal sheaf of C with the regularity of a certain vector bundle on $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 3.9. *If the pair (Λ_a, Λ_b) defines a good projection for C then on $\mathbb{P}^1 \times \mathbb{P}^1$ we have the following short exact sequence:*

$$0 \to \mathcal{E} \to V_a \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, 0) \oplus V_b \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \xrightarrow{\epsilon} (f_{a,b})_*(\mathcal{O}_C) \to 0$$

where $\mathcal{E}$ is a vector bundle of rank $a+b-1$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and V_a, V_b are vector spaces of dimension $a-1$ and $b-1$ respectively.

Proof. Blow-up $\mathbb{P}^a$ along Λ_a and $\mathbb{P}^b$ along Λ_b to get the diagram:

$$\begin{array}{ccc} & C \subseteq Y := Bl_{\Lambda_a}(\mathbb{P}^a) \times Bl_{\Lambda_b}(\mathbb{P}^b) & \\ \swarrow^{\mu_a \times \mu_b} & & \searrow^{p_a \times p_b} \\ C \subseteq \mathbb{P}^a \times \mathbb{P}^b & & \bar{C} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

The morphism $p_a \times p_b$ will resolve the projection map $\pi_a \times \pi_b$, whose restriction to C is a good projection for the curve. Now set

$$A_1 := (\mu_a \times \mu_b)^*(\mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(1,0)) \text{ and } A_2 := (\mu_a \times \mu_b)^*(\mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(0,1)).$$

As $\Lambda_a \cap C_1 = \emptyset$ and $\Lambda_b \cap C_2 = \emptyset$, we have that $\mathcal{I}_{C/Y} = (\mu_a \times \mu_b)^*(\mathcal{I}_{C/\mathbb{P}^a \times \mathbb{P}^b})$. Using the diagram and the notations we made, for each $i = 1, 2$ we get an exact sequence on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$0 \to (p_a \times p_b)_*(\mathcal{I}_{C/Y}(A_i)) \to (p_a \times p_b)_*(\mathcal{O}_Y(A_i)) \xrightarrow{\epsilon_i} (f_{a,b})_*(L_i)$$

The idea is to describe the points $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$, where the stalk of either ϵ_1 or ϵ_2 is surjective. For ϵ_1, by Nakayama's lemma, it suffices to show that the map:

$$\begin{array}{ccc} (p_a \times p_b)_*(A_1) \otimes \mathbb{C}(x,y) & \xrightarrow{\epsilon_1 \otimes \mathbb{C}(x,y)} & (f_{a,b})_*(L_1) \otimes \mathbb{C}(x,y) \\ \downarrow \simeq & & \downarrow \simeq \\ H^0(\overline{\Lambda_a x} \times \overline{\Lambda_b y}, \mathcal{O}_{\overline{\Lambda_a x} \times \overline{\Lambda_b y}}(1,0)) & \longrightarrow & H^0(\mathcal{O}_{C \cap (\overline{\Lambda_a x} \times \overline{\Lambda_b y})}(1,0)) \end{array}$$

is surjective. Equivalently, it is enough to study the surjectiveness of the bottom horizontal map. We know that the pair (Λ_a, Λ_b) defines a good projection, hence the intersection $C \cap \overline{\Lambda_a x} \times \overline{\Lambda_b y}$ consists of at most two points. If it is a point then clearly the bottom horizontal map is surjective. If it consists of (x^1, y^1) and (x^2, y^2) then the bottom horizontal map is surjective whenever $x^1 \neq x^2$. Symmetrically we have an analogous picture for ϵ_2 and we deduce that for all $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ the stalk of at least one of the maps ϵ_1 or ϵ_2 is surjective.

Note that $Bl_{\Lambda_a}(\mathbb{P}^a) = \mathbb{P}(V_a \otimes \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and $Bl_{\Lambda_b}(\mathbb{P}^b) = \mathbb{P}(V_b \otimes \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, where V_a and V_b are vector spaces of dimension $a-1$ and $b-1$ respectively. This allows one to have the isomorphisms:

$$(p_a \times p_b)_*(A_1) \simeq V_a \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0),$$

$$(p_a \times p_b)_*(A_2) \simeq V_b \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1).$$

The ideas above tells us that if we tensor ϵ_1 by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,0)$, ϵ_2 by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,-1)$, and sum them together then we get the following surjective map:

$$V_a \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus V_b \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,-1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \xrightarrow{\epsilon_0} (f_{a,b})_*(\mathcal{O}_C).$$

Notice that the second and the fourth component of the domain of ϵ_0 have the same image, so we actually get the following short exact sequence:

$$0 \to \mathcal{E} \to V_a \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,0) \oplus V_b \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,-1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \xrightarrow{\epsilon} (f_{a,b})_*(\mathcal{O}_C) \to 0$$

with $\mathcal{E} = \mathrm{Ker}(\epsilon)$. Now provided that $(f_{a,b})_*(\mathcal{O}_C)$ is Cohen-Macaulay sheaf with support of codimension 1 we have that $\mathcal{E}$ is a vector bundle of rank $a+b-1$ and this ends the proof. □

Now the idea is to find bounds on the multigraded regularity of the vector bundle $\mathcal{E}$, but before that we have the following proposition:

Proposition 3.10. *Under the assumptions of Theorem 3.2, we can make a choice of a good projection so that the dual vector bundle $\mathcal{E}^*$ is $(-1,-1)$-regular.*

Proof. Serre duality and the short exact sequence from Proposition 3.9 imply the vanishings:

$$H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^*(-3,-1)) = H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^*(-1,-3)) = 0.$$

As for the group $H^2(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E}^*(-2,-2))$, notice that by Serre duality it is isomorphic to $H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E})$. Now the idea is to use the following exact sequence in cohomolgy:

$$H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E}) \to H^0(V_a\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0)\oplus V_b\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1)\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}) \to H^0(C, \mathcal{O}_C).$$

As C is irreducible, the latter map is an isomorphism between two one dimensional spaces. This imply that $H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E}) = 0$. Also notice that the same argument shows that $H^1(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E}) = 0$, which we will need later.

It remains to show that both $H^1(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E}^*(-1,-2))$ and $H^1(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E}^*(-2,-1))$ vanish. We will prove that the first group is zero, as the second vanishing follows from the same ideas.

First, by Serre duality the group in question is isomorphic to $H^1(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E}(-1,0))$. Now, using the vanishings of $H^1(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E})$ we proved above and the exact sequence

$$0 \longrightarrow \mathcal{E}(-1,0) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_{\{x\}\times\mathbb{P}^1} \longrightarrow 0$$

we obtain the isomomorphism

$$H^0(\{x\}\times\mathbb{P}^1, \mathcal{E}|_{\{x\}\times\mathbb{P}^1}) = H^1(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{E}(-1,0)),$$

for all $x \in \mathbb{P}^1$. The morphism $p_1|_C$ has a nondegenerate image, so the multiplication map by x

$$(f_{a,b})_*(\mathcal{O}_C)(-1,0) \longrightarrow (f_{a,b})_*(\mathcal{O}_C)$$

is injective. Thus the Snake lemma implies that $\forall x \in \mathbb{P}^1$ we have the exact sequence:

$$0 \to \mathcal{E}|_{\{x\}\times\mathbb{P}^1} \to \big(V_a\oplus\mathbb{C}\big)\otimes\mathcal{O}_{\{x\}\times\mathbb{P}^1}\oplus V_b\otimes\mathcal{O}_{\{x\}\times\mathbb{P}^1}(-1) \to (f_{a,b})_*(\mathcal{O}_C)|_{\{x\}\times\mathbb{P}^1} \to 0.$$

To finish the proof it remains to show that there exists a point $x \in \mathbb{P}^1$ such that the map

$$V_a\otimes H^0(\mathcal{O}_{\{x\}\times\mathbb{P}^1})\oplus H^0(\mathcal{O}_{\{x\}\times\mathbb{P}^1}) \stackrel{l_1\oplus l_2}{\longrightarrow} H^0((f_{a,b})_*(\mathcal{O}_C)|_{\{x\}\times\mathbb{P}^1})$$

is injective. For this purpose, suppose the projection map is given by the formula:

$$(\pi_a \times \pi_b)\big([x_0 : \ldots : x_a] \times [y_0 : \ldots : y_b]\big) = [x_0 : x_1] \times [y_0 : y_1].$$

Remark 3.7 implies that we can choose $x \in \mathbb{P}^1$ which satisfies

$$\big(\overline{\Lambda_a x} \times \mathbb{P}^b \cap C\big) = \{P_1, \ldots, P_{d_1}\} \text{ with } P_i = (x^i, y^i) \text{ and } x^i \neq x^j \text{ for } i \neq j.$$

Finally assume that $P_1 \in \{x_2 = \ldots = x_a = 0\} \times \mathbb{P}^b$. In this case we have

$$(f_{a,b})_*(\mathcal{O}_C)|_{\{x\}\times\mathbb{P}^1} = \oplus_{i=1}^{d_1} \mathbb{C}_{P_i}$$

and we can write $l_2(1) = (1, \ldots, 1)$ and $l_1(e_i) = (x_i|_{P_1}, \ldots, x_i|_{P_{d_1}})$ for some basis $\{e_2, \ldots, e_a\}$ of V_a. As $x_i|_{P_1} = 0$, for all $i = 2, \ldots, a$, it is enough to prove that l_1 is injective. Suppose the opposite. Then there exists $(u_2, \ldots, u_a) \in \mathbb{C}^{a-1}$ such that

$$(\sum_{i=2}^{a} u_i x_i|_{P_1}, \ldots, \sum_{i=2}^{a} u_i x_i|_{P_{d_1}}) = (0, \ldots, 0).$$

This means that the set $\{x^1, \ldots, x^{d_1}\} \subseteq \{\sum_{i=2}^{a} u_i x_i = 0\} \cap \overline{x\Lambda_a}$, therefore the points $x^1, \ldots, x^{d_1}$ span a plane Π_a of codimension at least two. Choose $(x, y) \in C$ with $x \notin \Pi_a$ and a hyperplane H_a containing $\overline{\Lambda_a x}$. Now the intersection $H_a \times \mathbb{P}^b \cap C$ consists of at least $d_1 + 1$ points. This is a contradiction as we assumed that $p_1|_C$ has a nondegenerate image. Therefore l_1 is injective and the vector bundle $\mathcal{E}^*$ is $(-1, -1)$-regular. □

The last ingredient necessary for Theorem 3.8 is the following lemma, which will later allow us to connect the multigraded regularity of $\mathcal{E}$ to the one of $\mathcal{E}^*$:

Lemma 3.11 (Tensorial property of regularity). *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two locally free sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. If $\mathcal{G}_1$ is (p, q)-regular and $\mathcal{G}_2$ is (m, n)-regular then $\mathcal{G}_1 \otimes \mathcal{G}_2$ is $(p + m, q + n)$-regular. In particular as we work over complex numbers for all $k \in \mathbb{N}$ we have that $\bigwedge^k(\mathcal{G}_1)$ is (kp, kq)-regular.*

Proof. It is enough to consider the case when $\mathcal{G}_1$ and $\mathcal{G}_2$ are $(0,0)$-regular. We will only prove the vanishing of $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{G}_1 \otimes \mathcal{G}_2(-1,0))$, as the other ones follow from the same ideas. By Theorem 2.22, we obtain that both $\mathcal{G}_1$ and $\mathcal{G}_2$ are globally generated. One therefore has the two short exact sequences,

$$0 \longrightarrow \mathcal{M}_i \longrightarrow \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{G}_i \longrightarrow 0,$$

where the $\mathcal{M}_i$ are locally free sheaves for $i = 1, 2$. Now tensor the second sequence with $\mathcal{G}_1(-1,0)$ and get the exact sequence in cohomology:

$$\oplus H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{G}_1(-1,0)) \to H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{G}_1 \otimes \mathcal{G}_2(-1,0)) \to H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{G}_1 \otimes \mathcal{M}_2(-1,0))$$

As $\mathcal{G}_1$ is $(0,0)$-regular, the left group vanishes and is enough to prove that the right one also does. For this, tensor the first short exact sequence with $\mathcal{M}_2(-1,0)$ and in cohomology we obtain

$$\oplus H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{M}_2(-1,0)) \to H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{M}_2 \otimes \mathcal{G}_1(-1,0)) \to 0.$$

This implies that it remains to show the vanishing of $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{M}_2(-1,0))$. Going back to the second short exact sequence, tensor it with $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,0)$ and get the exact sequence:

$$H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{G}_2(-1,0)) \to H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{M}_2(-1,0)) \to \oplus H^2(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,0)).$$

As $\mathcal{G}_2$ is $(0,0)$-regular this implies our vanishing, and the proof is done. □

Proof of Theorem 3.8. Theorem 3.2 and Remark 3.7 imply the existence of plenty of good projections. Proposition 3.9 allows us to construct a vector bundle $\mathcal{E}$, whose dual $\mathcal{E}^*$ is $(-1,-1)$-regular by Proposition 3.10. Using the isomorphism

$$\mathcal{E} \simeq \bigwedge^{a+b-2} (\mathcal{E}^*) \otimes \det(\mathcal{E})$$

with Lemma 3.11 and the exact sequence from Proposition 3.9 we obtain that $\mathcal{E}$ is $(d_2 - b + 1, d_1 - a + 1)$-regular. It remains to show that this implies that the ideal sheaf $\mathcal{I}_{C/\mathbb{P}^a \times \mathbb{P}^b}$ is also $(d_2 - b + 1, d_1 - a + 1)$-regular.

Writing $l := d_2 - b$ and $k := d_1 - b + 1$, then the regularity of $\mathcal{E}$ we proved above, implies that the map

$$H^0(V_1 \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(l-1,k) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(l,k-1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(l,k)) \xrightarrow{g} H^0(\mathcal{O}_C(l,k))$$

is surjective. Assume again that the projection map is given by the formula

$$(\pi_{\Lambda_a} \times \pi_{\Lambda_b})\big([x_0 : \ldots : x_a] \times [y_0 : \ldots : y_b]\big) = [x_0 : x_1] \times [y_0 : y_1].$$

Thus the fact that g is surjective says that $H^0(C, O_C(l,k))$ is generated by the restriction to C of polynomials of the type $x_i F$, $y_j G$ and H, for $i = 2, \ldots, a$ and $j = 2, \ldots, b$, where $F \in H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(l-1,k))$, $G \in H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(l,k-1))$ and $H \in H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(l,k))$. Hence the map

$$H^0(\mathbb{P}^a \times \mathbb{P}^b, \mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(d_2 - b, d_1 - a + 1)) \to H^0(C, \mathcal{O}_C(d_2 - b, d_1 - a + 1))$$

is surjective and $H^1(\mathbb{P}^a \times \mathbb{P}^b, \mathcal{I}_{C/\mathbb{P}^a \times \mathbb{P}^b}(d_2 - b, d_1 - a + 1))$ vanishes. Likewise it can be shown that

$$H^1(\mathbb{P}^a \times \mathbb{P}^b, \mathcal{I}_{C/\mathbb{P}^a \times \mathbb{P}^b}(d_2 - b + 1, d_1 - a)) = 0.$$

Now let's use the sequence

$$0 \longrightarrow \mathcal{I}_{C/\mathbb{P}^a \times \mathbb{P}^b} \longrightarrow \mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

together with the exact sequence from Proposition 3.9 to get that

$$H^2(\mathbb{P}^a \times \mathbb{P}^b, \mathcal{I}_{C/\mathbb{P}^a \times \mathbb{P}^b}(m,n)) = H^1(C, \mathcal{O}_C(m,n)) = H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}(m,n))$$

for all $m, n \geq -1$. This implies the vanishings of the second cohomology groups we need, because $d_2 - b + 1 \geq 1$ and $d_1 - a + 1 \geq 1$.

Using the fact that C is of dimension one, then the other higher cohomology groups of the ideal sheaf also vanish. Thus we conclude that $\mathcal{I}_{C/\mathbb{P}^a\times\mathbb{P}^b}$ is (d_2-b+1, d_1-a+1)-regular. □

3.3 Regularity bounds in a special case

In order to finish the proof of Theorem B, it remains to study the case when $r := a = b$ and C is included in the graph of an automorphism of $\mathbb{P}^r$. This is discussed in the following proposition:

Proposition 3.12. *Let $C \subseteq \mathbb{P}^r \times \mathbb{P}^r$ be a smooth curve of bidegree (d, d). If C is contained in the diagonal $\Delta_{\mathbb{P}^r}$ and is nondegenerate, then $\mathcal{I}_{C/\mathbb{P}^r\times\mathbb{P}^r}$ is $(d-r+1, d-r+1)$-regular.*

Proof. Denote by $J := \mathcal{I}_{C/\mathbb{P}^r\times\mathbb{P}^r}$ and $k := d-r+1$. As C is included in the diagonal we have $O_{\mathbb{P}^r\times\mathbb{P}^r}(1,0)|_C \simeq \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^r}(0,1)|_C$ and denote this line bundle of degree d by L. To prove that $H^2(\mathbb{P}^r\times\mathbb{P}^r, J(k-2,k)) = 0$ use the following short exact sequence:

$$0 \longrightarrow J(k-2,k) \longrightarrow \mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^r}(k-2,k) \longrightarrow L^{\otimes 2k-2} \longrightarrow 0.$$

As $C \subseteq \mathbb{P}^r$ is nondegenerate then $k \geq 1$ and we have the vanishings

$$H^1(\mathbb{P}^r\times\mathbb{P}^r, \mathcal{O}(k-2,k)) \;=\; H^2(\mathbb{P}^r\times\mathbb{P}^r, \mathcal{O}(k-2,k)) \;=\; 0.$$

This vanishings and the long exact sequence in cohomology of the sequence above, tells us that we have the isomorphism

$$H^1(C, L^{\otimes 2k-2}) \simeq H^2(\mathbb{P}^r\times\mathbb{P}^r, J(k-2,k)).$$

Now [17, Theorem 1.1] says that for a nondegenerate curve $C \subseteq \mathbb{P}^r$ of degree d we have that $L^{\otimes n}$ is non-special for all $n \geq d-r$, where $L := \mathcal{O}_{\mathbb{P}^r}(1)|_C$. In our case

$2k-2 = 2d-2r \geq d-r$ and the vanishing follows. Using exactly the same ideas it is easy to show

$$H^2(\mathbb{P}^r \times \mathbb{P}^r, J(k-1,k-1)) = H^2(\mathbb{P}^r \times \mathbb{P}^r, J(k,k-2)) = 0$$

It remains to show $H^1(\mathbb{P}^r \times \mathbb{P}^r, J(k-1,k)) = 0$. This vanishing is equivalent to the surjectiveness of the map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(k-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(C, L^{\otimes 2k-1}).$$

Now this map can be factored as follows:

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}^r}(k-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(k)) & \longrightarrow & H^0(C, L^{\otimes 2k-1}) \\ \downarrow l & \nearrow u & \\ H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2k-1)) & & \end{array}$$

As l is surjective we need only to prove that u is surjective. For this we use a result from [17], which states that for a nondegenerate curve $C \subseteq \mathbb{P}^r$ of degree d we have

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to H^0(C, L^{\otimes n}) \text{ is surjective } \forall n \geq d-r+1.$$

In our case $n = 2k-1 = 2d-2r+1 \geq d-r+1$ and this finishes the proof. □

3.4 An example

We end this chapter with an example, showing that the bound in Theorem B is the best possible. The curve, from the following example, has the property that

$$\mathbf{reg}(C) \;=\; (d_2 - b + 1, d_1 - a + 1) + \mathbb{N}^2.$$

Thus by Theorem B, $(d_2 - b + 1, d_1 - a + 1) + \mathbb{N}^2$ is the maximal set contained in $\mathbf{reg}(C)$ for all curves $C \subseteq \mathbb{P}^a \times \mathbb{P}^b$ of bidegree (d_1, d_2) with nondegenerate birational projections.

The idea is to find examples of curves with high order "secant lines". In our case we will consider the "secant lines" of the type

$$l \times [y_0 : ... : y_b] \subseteq \mathbb{P}^a \times \mathbb{P}^b,$$

where $l \subseteq \mathbb{P}^a$ is a line. Suppose that $l \times [y_0 : ... : y_b] \cap C$ consists of k points. If $s \in H^0(\mathbb{P}^a \times \mathbb{P}^b, O_{\mathbb{P}^a \times \mathbb{P}^b}(k-1, n))$ is a hypersurface, vanishing along the curve C, then the index of intersection with the "secant line" $l \times [y_0 : ... : y_b]$ is $k-1$, forcing s to vanish along $l \times [y_0 : ... : y_b]$. It follows that for any $n \in \mathbb{N}$, the ideal sheaf $\mathcal{I}_{C/\mathbb{P}^a \times \mathbb{P}^b}(k-1, n)$ is not globally generated, and therefore by the multigraded version of Mumford's Theorem 2.22, the ideal sheaf is not $(k-1, n)$-regular.

Example 3.13. Let the morphism $\psi : C = \mathbb{P}^1 \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ be given by the formula

$$\psi([s:t]) = [t^2 s - 4s^3 : t^3 - 4s^2 t : t^2 s - 3s^3] \times [s^2 t - t^3 : s^3 - st^2 : t^3].$$

First notice that ψ defines an embedding, such that the curve C is of bidegree $(3,3)$ with nondegenerate birational projections. At the same time we have the following:

$$C \cap \big(\{x_2 = 0\} \times [0:0:1]\big) \;=\; \{[1:1], [1:-1]\}$$

$$C \cap \big([0:0:1] \times \{4y_0 + 3y_2 = 0\}\big) \;=\; \{[1:2], [1:-2]\}.$$

Bearing in mind the ideas above we obtain that the ideal sheaf $\mathcal{I}_{C/\mathbb{P}^2 \times \mathbb{P}^2}$ is not $(1, s)$ and $(t, 1)$-regular for all $s, t \in \mathbb{N}$. Theorem B states that this ideal is $(2,2)$-regular and we conclude that $\text{reg}(C) = \big(\big((2,2) + \mathbb{N}^2\big)$.

It is easy to notice that we can generalize this example, i.e. find a rational curve $C \subseteq \mathbb{P}^r \times \mathbb{P}^r$ of bidegree $(r+1, r+1)$ and nondegenerate birational projections such that $\text{reg}(C) = \big((2,2) + \mathbb{N}^2\big)$.

CHAPTER IV

Okounkov Bodies

4.1 Introduction

We now begin the second part of this thesis, where our main focus will be *Okounkov bodies*. These are convex bounded bodies associated to Cartier divisors and were introduced by Lazarsfeld and Mustaţă [34], based on the work of Okounkov [44], [45] (see also [26] for an independent development). In this chapter we consider the following question: what can be said about the set of convex bodies that appear as Okounkov bodies? In §4.1 we give a brief introduction to the construction of Okounkov bodies and present the basic properties they satisfy, with a special emphasis on the connection between them and the volume function, which will be used fruitfully in the next chapter. In §4.2 we give a proof of Theorem C and show that the set of all convex bodies appearing as the Okounkov bodies of big Cartier divisors on projective varieties with respect to any admissible flag is countable. Then, in §4.3, we give a full description of the Okounkov body of a $\mathbb{R}$-divisor on some smooth projective surface. More precisely, in Theorem D we show that they are polygons, closely given by rational data. In sharp contrast, in §4.4 we give a technique that enables us to find examples of Okounkov bodies in higher dimensions whose geometry becomes more complicated. This chapter is based on [29], which is a joint work with Alex Küronya and Catriona Maclean.

Before introducing the construction of Okounkov bodies we fix some notation that we will use throughout the next two chapters. If X is an irreducible projective variety of dimension n, then $N^1(X)$ will be the *Neron-Severi group* of numerical equivalences classes of divisors on X. We will denote by $N^1(X)_{\mathbb{R}}$ the corresponding finite dimensional vector space and call it the Neron-Severi space. We will say that X has *Picard number* ρ if $\dim_{\mathbb{R}}(N^1(X)_{\mathbb{R}}) = \rho$.

Inside the Neron-Severi space $N^1(X)_{\mathbb{R}}$ we have four cones.

The first one is called the *ample cone* of X:

$$\mathrm{Amp}(X)_{\mathbb{R}} \;\subseteq\; N^1(X)_{\mathbb{R}}.$$

It is the convex cone of all ample $\mathbb{R}$-divisor classes on X. Equivalently, one could define $\mathrm{Amp}(X)_{\mathbb{R}}$ to be the convex cone in $N^1(X)_{\mathbb{R}}$ spanned by the classes of all ample integral (or rational) divisors. The ample cone is open and its closure is the *nef cone* $\mathrm{Nef}(X)_{\mathbb{R}}$. It is a basic Theorem of Kleiman (see [27]) that

$$\mathrm{Nef}(X)_{\mathbb{R}} \;=\; \{D \in N^1(X)_{\mathbb{R}} \mid (D.C) \geq 0 \text{ for all effective curves } C \text{ on } X\}.$$

We define the *pseudo-effective cone* of X

$$\overline{\mathrm{Eff}}(X)_{\mathbb{R}} \;\subseteq\; N^1(X)_{\mathbb{R}}$$

to be the closed convex cone generated by those integral classes which can be represented by an integral divisor $D \in N^1(X)$ which satisfies the property that there exists an integer $m \in \mathbb{N}$ such that $H^0(X, \mathcal{O}_X(mD)) \neq 0$. The interior of the pseudo-effective cone $\overline{\mathrm{Eff}}(X)_{\mathbb{R}}$ is an open cone denoted by $\mathrm{Big}(X)_{\mathbb{R}}$ and called the *big cone* of X. This cone is the convex cone generated by the classes of big integral divisors, where an integral divisor $D \in N^1(X)$ is called *big* if it satisfies the property

$$\dim_{\mathbb{C}}(H^0(X, \mathcal{O}_X(mD))) \;\geq\; C \cdot m^n$$

for sufficiently large $m \in \mathbb{N}$ and some positive constant $C > 0$. Besides the previous natural inclusions between these cones, we have two others:

$$\overline{\mathrm{Eff}}(X)_{\mathbb{R}} \supseteq \mathrm{Nef}(X)_{\mathbb{R}},$$

$$\mathrm{Big}(X)_{\mathbb{R}} \supseteq \mathrm{Amp}(X)_{\mathbb{R}}.$$

4.2 Okounkov bodies

In this section we recall the construction of Okounkov bodies and some relevant facts about them. This material is adapted from [34].

Let X be a smooth projective variety of dimension n. The construction of Okounkov bodies starts with the choice of an *admissible flag*:

Definition 4.1 (Admissible flag)**.** An *admissible flag* on X is a flag $Y_\bullet$ of irreducible subvarieties

$$Y_\bullet \ : \ Y_0 = X \ \supseteq \ Y_1 \ \supseteq \ Y_2 \ \supseteq \ \ldots \ \supseteq \ Y_{n-1} \ \supseteq \ Y_n = \{\mathrm{pt}\}$$

such that each Y_i is smooth and $\mathrm{codim}_X(Y_i) = i$.

Remark 4.2. In [34], Lazarsfeld and Mustaţă define Okounkov bodies in a more general setup. The only difference from our setup is that they ask each element Y_i of the flag to be smooth only at the point Y_n. For simplicity, we will consider the case when each Y_i is globally smooth.

The purpose of this admissible flag is that it determines a map

$$\nu_{Y_\bullet} \ : \ H^0(X, \mathcal{O}_X(D)) - \{0\} \to \mathbb{Z}^n, \ s \to (\nu_1(s), \ldots, \nu_n(s))$$

where D is any Cartier divisor on X. The map is defined recursively. For this let $s \in H^0(X, \mathcal{O}_X(D))$ be a non-zero section and define

$$\nu_1(s) \ = \ \mathrm{ord}_{Y_1}(s).$$

Now s determines a nonzero section

$$\tilde{s}_1 \in H^0(X, \mathcal{O}_X(D - \nu_1(s)Y_1))$$

that does not vanish identically along Y_1. So after restriction we get a nonzero section

$$0 \neq s_1 \in H^0(Y_1, \mathcal{O}_X(D - \nu_1(s)Y_1)|_{Y_1})$$

and we set

$$\nu_2(s) = \mathrm{ord}_{Y_2}(s_1).$$

Now continue in this manner and define all the $\nu_i(s)$.

Definition 4.3 (Valuation attached to a flag)**.** We call the map $\nu_{Y_\bullet}$ defined above *the valuation map* attached to the admissible flag $Y_\bullet$. We set $\nu_{Y_\bullet}(s) = \infty$ if $s = 0$.

Remark 4.4. Strictly speaking $\nu_{Y_\bullet}$ is not a valuation, as it is defined on the spaces of global sections of different line bundles, but it satisfies valuation-like properties. On the one hand, from the construction, it follows that for any two sections $s, t \in H^0(X, \mathcal{O}_X(D))$ we have

$$\nu_{Y_\bullet}(s + t) \geq \min_{\text{lex order}}\{\nu_{Y_\bullet}(s), \nu_{Y_\bullet}(t)\}.$$

On the other hand, if $s_1 \in H^0(X, \mathcal{O}_X(D_1))$ and $s_2 \in H^0(X, \mathcal{O}_X(D_2))$ then

$$\nu_{Y_\bullet}(s_1 \otimes s_2) = \nu_{Y_\bullet}(s_1) + \nu_{Y_\bullet}(s_2).$$

Example 4.5. Let $X = \mathbb{P}^n$ be the projective space of dimension n and choose $Y_\bullet$ to be the admissible flag of linear spaces defined in homogeneous coordinates $x_0, \ldots, x_n$ by $Y_i = \{x_1 = \ldots = x_i = 0\}$. Let $|D|$ be the linear system of hypersurfaces of degree m. Then $\nu_{Y_\bullet}$ is the lexicographic valuation determined on monomials of degree m by

$$\nu_{Y_\bullet}(x_0^{a_0} \cdot x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}) = (a_1, a_2, \ldots, a_n).$$

In other words, $\nu_{Y_\bullet}(\sum c_{\mathbf{a}} x^{\mathbf{a}}) = \min_{\text{lex-order}}\{\mathbf{a} \mid c_{\mathbf{a}} \neq 0\}$, where $\mathbf{a} = (a_0, \ldots, a_n)$.

The valuation $\nu_{Y_\bullet}$ attached to the admissible flag $Y_\bullet$ satisfies the following basic property (see [34, Lemma 1.4]):

Lemma 4.6. *Let $W \subseteq H^0(X, \mathcal{O}_X(D))$ be a subspace. Then the number of valuation vectors arising from sections in W is equal to the dimension of W:*

$$\#\Big(\mathrm{Im}\big((W - \{0\}) \xrightarrow{\nu_{Y_\bullet}} \mathbb{Z}^n\big)\Big) \;=\; \dim(W)$$

The idea of Lazarsfeld and Mustaţă is that one can associate to a Cartier divisor a semigroup, using the valuation map discussed before.

Definition 4.7 (Graded semigroup of a Cartier divisor)**.** The *graded semigroup* of D is the sub-semigroup

$$\Gamma_{Y_\bullet}(X; D) \;=\; \{(\nu_{Y_\bullet}(s), m) \mid 0 \neq s \in H^0(X, \mathcal{O}_X(mD)), m \geq 0\}$$

of $\mathbb{N}^n \times \mathbb{N} = \mathbb{N}^{n+1}$.

Writing $\Gamma = \Gamma_{Y_\bullet}(X; D)$, denote by $\Sigma(\Gamma) \;\subseteq\; \mathbb{R}^{n+1}$ the closed convex cone spanned by Γ. The Okounkov body of D is then the base of this cone:

Definition 4.8 (Okounkov body)**.** The *Okounkov body* of D with respect to the flag $Y_\bullet$ is the convex set

$$\Delta_{Y_\bullet}(X; D) \;=\; \Sigma(\Gamma) \cap (\mathbb{R}^n \times \{1\})$$

and we view $\Delta_{Y_\bullet}(X; D)$ in the natural way as a closed convex subset of $\mathbb{R}^n$.

Example 4.9. Let $X = \mathbb{P}^n$ and take $Y_\bullet$ to be the linear flag appearing in Example 4.5. Also take D to be the divisor defined by a hyperplane in $\mathbb{P}^n$. Then it follows immediately from that example that the Okounkov body of D with respect to the flag $Y_\bullet$ is the simplex

$$\Delta_{Y_\bullet}(\mathbb{P}^n; D) \;=\; \{(z_1, \ldots, z_n) \in \mathbb{R}^n \mid z_1 \geq 0, \ldots, z_n \geq 0, \sum z_i \leq 1\}.$$

In §4.4 we give examples of Okounkov bodies with a more complicated description.

In the following proposition we state the basic properties of Okounkov bodies (see [34, Lemma 1.11] and [34, Proposition 4.1]).

Proposition 4.10 (Variational properties)**.** *Let D be a big divisor on X and $Y_\bullet$ an admissible flag.*

(i) The Okounkov body $\Delta_{Y_\bullet}(X;D)$ is a compact convex set of $\mathbb{R}^n$ which depends only on the numerical equivalence class of D.

(ii) For any integer $p > 0$, one has

$$\Delta_{Y_\bullet}(X, pD) \;=\; p \cdot \Delta_{Y_\bullet}(X;D)$$

where the expression on the right denotes the homothetic image of $\Delta_{Y_\bullet}(X;D)$ under scaling by the factor p.

Remark 4.11 (Rational classes). Take a big rational class $\xi \in \mathrm{Big}(X)_{\mathbb{Q}}$ and an integer $p \gg 0$ such that $p \cdot \xi$ is an integral class of $N^1(X)$. Suppose D is a Cartier divisor representing this integral class. Then we can define the Okounkov body of the class ξ to be

$$\Delta_{Y_\bullet}(X;\xi) \;=\; \frac{1}{p} \cdot \Delta_{Y_\bullet}(X;D) \;\subseteq \mathbb{R}^n.$$

The proposition above implies that this definition is independent of the choice of D and p.

An important consequence of Lemma 4.6 is the connection between the Okounkov body of a Cartier divisor and its volume, [34, Theorem A].

Theorem 4.12 (Okounkov body and the volume of a divisor)**.** *If D is a big Cartier divisor on X, then*

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(X;D)) \;=\; \frac{1}{n!} \cdot \mathrm{vol}_X(D),$$

where the left side is the standard Euclidean volume in $\mathbb{R}^n$*.*

Remark 4.13. The quantity on the right is the *volume* of D, which we will study in Chapter 5. It is important to note that this theorem tells us that by varying only the flag one might obtain different Okounkov bodies for a fixed integral divisor, but their volume remains constant.

The construction of Okounkov bodies can be generalized easily to the multigraded setting and even more to the non-complete case. For that fix divisors $D_1, \ldots, D_p$ on X and write $\mathbf{m}D = m_1D_1 + \ldots + m_pD_p$ for $\mathbf{m} = (m_1, \ldots, m_p) \in \mathbb{N}^p$. Then we have the following definition:

Definition 4.14 (Multigraded linear series)**.** A *multigraded linear series* $W_\bullet$ on X associated to $D_1, \ldots, D_p$ consists of subspaces

$$W_{\mathbf{m}} \ \subseteq \ H^0(X, \mathcal{O}_X(\mathbf{m}D))$$

such that $R(W_\bullet) = \oplus W_{\mathbf{m}}$ is a subalgebra of the section ring $R(D_1, \ldots, D_p) = \oplus H^0(X, \mathcal{O}_X(\mathbf{m}D))$. The *support* of $W_\bullet$, $\mathrm{supp}(W_\bullet) \subseteq \mathbb{R}^p_+$, is the closed convex cone spanned by all indices $\mathbf{m} \in \mathbb{N}^p$ such that $W_{\mathbf{m}} \neq 0$.

As in the complete case, one associates a semigroup to $W_\bullet$ with respect to an admissible flag $Y_\bullet$.

Definition 4.15 (Multigraded semigroup of linear series)**.** Let $W_\bullet$ be a multigraded linear series on X and $Y_\bullet$ be an admissible flag. We define the *multigraded semigroup* of $W_\bullet$ with respect to the flag $Y_\bullet$ to be

$$\Gamma_{Y_\bullet}(W_\bullet) \ = \ \{(\nu_{Y_\bullet}(s), \mathbf{m}) \mid 0 \neq s \in W_{\mathbf{m}}, \mathbf{m} \in \mathbb{N}^p\}$$

Now we can associate to $W_\bullet$ a cone:

Definition 4.16 (Okounkov cone of a multigraded linear series)**.** Let $W_\bullet$ be a multigraded linear series on X and $Y_\bullet$ an admissible flag. Then set

$$\Delta_{Y_\bullet}(X; W_\bullet) \subseteq \mathbb{R}^n \times \mathbb{R}^p_+$$

to be the closed convex cone generated by $\Gamma_{Y_\bullet}(W_\bullet)$ and call it the *Okounkov cone* of $W_\bullet$ with respect to $Y_\bullet$. If $\pi_2 : \mathbb{R}^n \times \mathbb{R}^p_+ \to \mathbb{R}^p_+$ is the second projection then notice that $\pi_2(\Delta_{Y_\bullet}(X; W_\bullet)) = \mathrm{supp}(W_\bullet)$.

Example 4.17. Let $Y = \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ (p-times). Now take on Y the following admissible flag

$$(4.1)\quad Y_\bullet : Y_0 = Y \supseteq Y_1 = [0:1] \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \supseteq \ldots \supseteq Y_p = [0:1] \times \ldots \times [0:1]$$

and denote by $V_\bullet$ the complete multigraded linear series, whose graded pieces are

$$V_{\mathbf{m}} = H^0(Y, \mathcal{O}_Y(\mathbf{m})) \text{ for all } \mathbf{m} \in \mathbb{N}^p.$$

Then the multigraded semigroup of $V_\bullet$ has the following description:

$$\Gamma(V_\bullet) = \{(n_1, \ldots, n_p, m_1, \ldots, m_p) \in \mathbb{N}^p \times \mathbb{N}^p \mid 0 \le n_i \le m_i \text{ for all } i = 1 \ldots p\}.$$

Given this, the Okounkov cone of $V_\bullet$ with respect to $Y_\bullet$ is

$$\Delta(V_\bullet) = \{(z_1, \ldots, z_{2p}) \mid 0 \le z_i \le z_{p+i} \text{ for all } i = 1, \ldots, p\} \subseteq \mathbb{R}^p_+ \times \mathbb{R}^p_+.$$

We will use this example extensively in in Chapter 5.

An important consequence of the construction of the Okounkov cone for multigraded linear series is that all the Okounkov bodies of rational big classes $\xi \in \mathrm{Big}(X)_\mathbb{Q}$ fit together in a closed convex cone.

This happens because the pseudo-effective cone $\overline{\mathrm{Eff}}(X)_\mathbb{R}$ is a pointed cone inside the vector space $N^1(X)_\mathbb{R}$, ([34, Lemma 4.6]). So one can choose integral divisors

$D_1, \ldots, D_\rho$ on X whose classes form a $\mathbb{Z}$-basis for $N^1(X)$ and every effective divisor is numerically equivalent to a $\mathbb{N}$-linear combination of the divisors D_i. Now construct the association

$$\mathbb{N}^\rho \xrightarrow{\simeq} \mathbb{N}D_1 \oplus \ldots \oplus \mathbb{N}D_\rho,\ \mathbf{m} \mapsto \mathbf{m}D$$

where $\mathbf{m}D = m_1 D_1 + \ldots + m_\rho D_\rho$. If we take $V_\bullet$ to be the complete multigraded linear series, whose graded pieces are $V_{\mathbf{m}} = H^0(Y, \mathcal{O}_Y(\mathbf{m}D))$, then taking into account the association above we denote the Okounkov cone of $V_\bullet$ by

$$\Delta_{Y_\bullet}(X) \ \subseteq \ \mathbb{R}^n \times N^1(X)_{\mathbb{R}}.$$

The following theorem [34, Theorem B] connects this cone with the Okounkov bodies of rational classes on X.

Theorem 4.18 (Global Okounkov cone)**.** *There exists a closed convex cone, called the global Okounkov cone*

$$\Delta_{Y_\bullet}(X) \ \subseteq \ \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$$

satisfying the property that for any big class $\xi \in N^1(X)_{\mathbb{Q}}$, the fiber

$$\Delta_{Y_\bullet}(X) \cap \big(\mathbb{R}^n \times \{\xi\}\big) = \Delta_{Y_\bullet}(X; \xi).$$

4.3 Countability of Okounkov bodies

In this section we prove Theorem C, showing the countability of Okounkov bodies. Theorem 4.18 points out that it is enough to prove the countability of global Okounkov cones and this can be stated as follows:

Theorem 4.19 (Countability of global Okounkov cones)**.** *There exist countably many closed convex cones $(\Delta_i)_{i \in \mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}^\rho$ such that for any complex smooth* *and projective variety* X *of dimension* n *with Picard number* ρ *equipped with an admissible flag* $Y_\bullet$*, there is an integral linear isomorphism*

$$\rho_X : \mathbb{R}^\rho \;\rightarrow\; N^1(X)_{\mathbb{R}}$$

with the property that $(id_{\mathbb{R}^n} \times \rho_X^{-1})(\Delta_{Y_\bullet}(X))$ *is equal to* Δ_i *for some* $i \in \mathbb{N}$.

Remark 4.20. (1) We say that ρ_X is integral if $\rho_X(\mathbb{Z}^\rho) \subseteq N^1(X)$.
(2) In [34], the Okounkov bodies were defined in more general setup. The Y_i were not assumed to be smooth, but merely irreducible and smooth at the point Y_n. The statement of Theorem C can easily be generalised to flags of this form. Suppose that Theorem C holds under the additional hypothesis that each component of the flag is smooth.

Consider now a smooth variety with a flag $Y_\bullet$ of irreducible subvarieties of X, which are smooth at the point Y_n. Choose a proper birational map $\mu : X' \to X$, isomorphic in some neighborhood of Y_n, such that the proper transform Y_i' of each Y_i is smooth and irreducible. The flag $Y_\bullet'$ is then admissible in our sense, and hence for any divisor D on X there is an $i \in \mathbb{N}$ such that $\Delta_{Y_\bullet'}(X'; \mu^*(D)) = \Delta_i$. The fact that X is smooth, then by Zariski's Main Theorem, $\mu_*(\mathcal{O}_{X'}) = \mathcal{O}_X$, and hence

$$H^0(X, \mathcal{O}_X(mD)) \;=\; H^0(X', \mu^*(\mathcal{O}_X(mD))),$$

for any $m \in \mathbb{N}$. Since μ is an isomorphism in a neighborhood of Y_n it follows that the convex body $\Delta_{Y_\bullet}(X; D)$ is the same as $\Delta_{Y_\bullet'}(X'; \mu^*(D))$, and hence it is one of the convex bodies Δ_i from Theorem C.

We start by collecting some definitions and technical prerequisites we need for the proof of Theorem 4.19. The idea is to use the multigraded Hilbert Flag functor, parametrizing flags contained in a multiprojective space. First, denote

$$Y \;=\; \underbrace{\mathbb{P}^{2n+1} \times \ldots \times \mathbb{P}^{2n+1}}_{\rho \text{ times}} \,.$$

Next, let $p_i : Y \to \mathbb{P}^{2n+1}$ be the projection map to the i-th factor. Each Cartier divisor can be described as

$$\mathbf{m}D \;=\; m_1 D_1 + \ldots + m_\rho D_\rho,$$

for some $\mathbf{m} := (m_1, \ldots, m_\rho) \in \mathbb{Z}^\rho$, where each D_i is the pullback by the projection map p_i of a hyperplane section on the i-factor of Y.

For any closed smooth variety $X \subseteq Y$ we denote by ρ_X the map $\rho_X : \mathbb{Z}^\rho \to N^1(X)$ given by $\rho_X(\mathbf{m}) = (\mathbf{m}D)|_X$. We also denote by ρ_X the induced map $\rho_X : \mathbb{R}^\rho \to N^1(X)_{\mathbb{R}}$.

Now, for any closed subscheme $Z \subseteq Y$ we can define its multigraded Hilbert function as follows

$$P_Z(\mathbf{m}) \;=\; \chi(Z, (\mathcal{O}_Y(\mathbf{m}D))|_Z), \text{ for all } \mathbf{m} \in \mathbb{N}^\rho.$$

With this in hand, suppose we are given an $(n+1)$-tuple of numerical functions

$$\mathbf{P}(\mathbf{m}) \;=\; (P_0(\mathbf{m}), \ldots, P_n(\mathbf{m})).$$

Then the multigraded Hilbert Flag functor $\mathcal{H}_{\mathbf{P}}$ assigns to any scheme T the set of all flat families of closed flags

$$Y \times T \;\supseteq\; \mathcal{X}_0 \;\supseteq\; \mathcal{X}_1 \;\supseteq \ldots \supseteq\; \mathcal{X}_{n-1} \;\supseteq\; \mathcal{X}_n$$

where for each i the projection map $\phi_i : \mathcal{X}_i \to T$ is flat and surjective. Also, any closed point $t \in T$ defines a flag

$$Y \times \{t\} \;\supseteq\; X_{0,t} \;\supseteq\; X_{1,t} \;\supseteq\; \ldots \;\supseteq\; X_{n-1,t} \;\supseteq\; X_{n,t}$$

such that the multigraded Hilbert function $P_{X_{i,t}}(\mathbf{m}) = P_i(\mathbf{m})$. The following proposition says that this functor is represented by a projective scheme.

Proposition 4.21. *Suppose we are given an n-tuple of numerical functions $\mathbf{P} = (P_0, \dots, P_n)$, where $P_i : \mathbb{Z}^\rho \to \mathbb{Z}$ for all i. There then exists a projective scheme $H_\mathbf{P}$, a closed subscheme $\mathcal{X}_\mathbf{P} \subset W \times H_\mathbf{P}$ and a flag of closed subschemes $\mathcal{Y}_{\bullet,\mathbf{P}} : \mathcal{X}_\mathbf{P} = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \dots \supset \mathcal{Y}_n$ such that*

1. *the induced projection map $\phi_i : \mathcal{Y}_i \to H_\mathbf{P}$ is flat and surjective for all i,*
2. *for all i and all $t \in H_\mathbf{P}$ we have that $P_{\mathcal{Y}_{i,t}} = P_i$,*
3. *for any projective subvariety $X \subseteq W$ and a flag of subvarieties $X = Y_0 \supset Y_1 \supset Y_2 \supset \dots \supset Y_n$ such that $P_{Y_i} = P_i$ there is a $t \in H_\mathbf{P}$ and an isomorphism $\beta : \mathcal{X}_t \to X$ such that $\beta(\mathcal{Y}_{i,t}) = Y_i$ for all i.*

Proof(Sketch): By [22, Corollary 1.2], there exists for each i a multigraded Hilbert scheme H_{P_i}, equipped with a flat surjective family $\mathcal{Y}'_i \subset W \times H_{P_i}$ such that for any $Y'_i \subset W$ such that $P_{Y'_i} = P_i$, there is a t such that $\mathcal{Y}'_{i,t} = Y'_i$.

We consider H_{P_i} and $\mathcal{Y}'_i$ with their reduced scheme structure. We now define

$$H_\mathbf{P} \subset H_{P_0} \times H_{P_1} \times \dots \times H_{P_n}$$

to be given by the incidence relation: $t = (h_0, \dots, h_n) \in H_\mathbf{P}$ if and only if $\mathcal{Y}'_{i,h_i} \subset \mathcal{Y}'_{i-1,h_{i-1}}$ for all i. Each element $\mathcal{Y}_i$ of the flag $\mathcal{Y}_{\bullet,\mathbf{P}}$ is defined to be $\mathcal{Y}_i = \pi_i^*(\mathcal{Y}'_i)$, where $\pi_i : H_\mathbf{P} \to H_{P_i}$ is the projection onto the factor H_{P_i}. By definition, $\mathcal{Y}_i \subset \mathcal{Y}_{i-1}$ for all i and $\mathcal{Y}_i \to H_\mathbf{P}$ is surjective and flat because $\mathcal{Y}'_i$ is. Condition (1) therefore holds. Condition (2) is immediate. By the universal property of multigraded Hilbert schemes H_{P_i}, Condition (3) is also satisfied. □

Remark 4.22. It is not hard to check the universality of the multigraded Flag Hilbert scheme. For Theorem 4.19 we use only the fact that all smooth varieties and all admissible flags on them with a fixed $(n+1)$-tuple of multigraded Hilbert functions can fit together in a projective scheme. The construction we give in the proof of Proposition 4.21 does satisfy this property, because of the universality of the multigraded Hilbert scheme proved in [22, Corollary 1.2].

The following proposition will play an important role in the proof of Theorem 4.19.

Proposition 4.23. *Let $\phi : \mathcal{X} \to T$ be a smooth projective and surjective morphism with reduced and irreducible fibers of relative dimension n between two quasi-projective varieties. Let $D_1, \ldots, D_\rho$ be Cartier divisors on $\mathcal{X}$. If there is a closed point $t_0 \in T$ such that the restrictions $D_1|_{X_{t_0}}, \ldots, D_\rho|_{X_{t_0}}$ form a basis for the Neron-Severi space $N^1(X_{t_0})_{\mathbb{R}}$, then for all $t \in T$ the divisors $D_1|_{X_t}, \ldots, D_\rho|_{X_t}$ are linear independent in $N^1(X_t)_{\mathbb{R}}$.*

Proof. First notice that $D_1|_{X_t}, \ldots, D_\rho|_{X_t}$ are Cartier divisor on X_t. If one wants to show that they are linearly dependent in $N^1(X_t)_{\mathbb{R}}$, then it is enough to prove that they are linear dependent over $\mathbb{Z}$. Therefore we only need to show that given a Cartier divisor D on $\mathcal{X}$ such that $D|_{X_{t_0}} \not\equiv_{\text{num}} 0$, then $D|_{X_t} \not\equiv_{\text{num}} 0$ for any $t \in T$.

The idea is to use induction on $n = \dim(\mathcal{X}) - \dim(T)$. Assume first that $n = 1$. In this case X_{t_0} is a smooth irreducible curve and the condition $D|_{X_{t_0}} \not\equiv_{\text{num}} 0$ is equivalent with $(D.X_{t_0}) \neq 0$. As ϕ is flat and T irreducible then for any $t_1 \in T \setminus \{t_0\}$, we have

$$(D.X_{t_1}) \ = \ (D.X_{t_0}) \ \neq \ 0,$$

implying that $D|_{X_{t_1}} \neq 0$.

In the general case, when $n \geq 2$, let $t_1 \in T \setminus \{t_0\}$. Also choose a Cartier divisor A on $\mathcal{X}$ such that its corresponding line bundle $\mathcal{O}_{\mathcal{X}}(A)$ is very ample relative to the map ϕ. The Theorems of Bertini and generic smoothness imply that for a general section W of A, the fiber $W_t = W \cap X_t$ is smooth and irreducible for all t in some open neighborhood of t_0. The same statement holds for t_1, and using the fact that T is irreducible, one can choose a general section W and an open neighborhood $U \subseteq T$, containing both t_0 and t_1, such that W_t is smooth and irreducible for all $t \in U$. Now, as W is general, the map

$$\phi_W^U \;=\; \phi|_{W\cap\phi^{-1}(U)} \;:\; W \cap \phi^{-1}(U) \longrightarrow U$$

is flat and of relative dimension $n-1$, and because each fiber of ϕ_W^U is smooth, we can assume without loss of generality that the map is also smooth. With this in hand, suppose first that $D|_{W_{t_0}} \not\equiv_{\text{num}} 0$. Then by applying induction on the family ϕ_W^U we have $D|_{W_{t_1}} \not\equiv_{\text{num}} 0$. This in turn implies that $D|_{X_{t_1}} \not\equiv_{\text{num}} 0$. Now, whenever $D|_{W_{t_0}} \equiv_{\text{num}} 0$, we have two cases. In the first one, when $n = 2$, use the fact that W_{t_0} is an ample section of X_{t_0}, and by the Hodge Index Theorem on surfaces one deduces that $(D|_{X_{t_0}})^2 < 0$. Hence by flatness, one finds that $(D|_{X_{t_1}})^2 < 0$ and therefore $D|_{X_{t_1}} \not\equiv_{\text{num}} 0$. For the second case, $n \geq 3$, one can use a higher dimensional version of the Hodge Index Theorem (Corollary I.4.2 [27]) and deduce that the condition $D|_{W_{t_0}} \equiv_{\text{num}} 0$ implies that $D|_{X_{t_0}} \equiv_{\text{num}} 0$. This contradicts our assumptions and ends the proof. □

In the proof of Theorem 4.19, T will be a reduced and irreducible quasiprojective variety and $\mathcal{X} \subseteq Y \times T$ will be a closed subscheme such that the induced projection $\phi : \mathcal{X} \to T$ will be a flat, projective and surjective morphism. We assume we are given a flag of closed subschemes of $\mathcal{X}$

$$\mathcal{Y}_\bullet \; : \; \mathcal{Y}_0 \; = \; \mathcal{X} \; \supseteq \; \mathcal{Y}_1 \; \supseteq \; \ldots \; \supseteq \; \mathcal{Y}_{n-1} \; \supseteq \; \mathcal{Y}_n$$

such that the restriction maps $\phi_i = \phi|_{\mathcal{Y}_i} : \mathcal{Y}_i \to T$ are flat, projective and surjective. We say that $t \in T$ admits an *admissible fiber* if the fiber X_t is smooth and irreducible and the flag $Y_{t,\bullet}$ is admissible. We call the morphism $\phi : \mathcal{X} \to T$ a *family of admissible flags* if each $t \in T$ admits an admissible fiber.

Proof of Theorem 4.19. Let X be a smooth and irreducible variety of dimension n and Picard number ρ, equipped with an admissible flag $Y_\bullet$. We start by showing that X can be embedded in Y in such a way that the induced map of real vector spaces ρ_X is a linear integral isomorphism.

For this, choose ρ very ample Cartier divisors $D_{1,X}, \ldots, D_{\rho,X}$ on X forming a $\mathbb{Q}$-basis of the Néron-Severi space $N^1(X)_{\mathbb{Q}}$. As X is smooth, [48, Theorem 5.4.9] says that for each i there exists an embedding $X \subseteq \mathbb{P}^{2n+1}$ such that $D_{i,X}$ is the pullback of a hyperplane section on $\mathbb{P}^{2n+1}$. We can embed X in Y in the following way:

$$X \subseteq \underbrace{X \times \ldots \times X}_{\rho \text{ times}} \subseteq Y, \tag{4.2}$$

where the first embedding is given by the diagonal. Notice that the restriction map $\rho_X : \mathbb{R}^\rho \to N^1(X)_{\mathbb{R}}$ is an integral linear isomorphism. Also each Y_i is embedded in Y the same way as X and the restrictions $D_{j,X}|_{Y_i}$ remain very ample Cartier divisors. Thus each multigraded Hilbert function P_{Y_i} given by the embedding of Y_i in Y is a polynomial with rational coefficients and of degree $\dim(Y_i)$.

This tells us that for all smooth varieties of dimension n and Picard number ρ, and for all admissible flags on them, we have countably many $(n+1)$-tuples of multigraded Hilbert functions. By Theorem 4.21, for each $(n+1)$-tuple

$$\mathbf{P}(m_1, \ldots, m_\rho) \; = \; (P_0(m_1, \ldots, m_\rho), \ldots, P_n(m_1, \ldots, m_\rho))$$

of numerical functions there exists a projective scheme parametrizing all flags of length $(n+1)$ whose Hilbert $(n+1)$-tuple is equal to $\mathbf{P}$. So, we have countably many flag families as in Theorem 4.21 with the property that any smooth and irreducible variety X of dimension n with Picard number ρ, and any admissible flag $Y_\bullet$ on X is a fiber in at least one of these families.

By the above, it suffices to show the countability of global Okounkov cones for one of these flat families. So fix one of them, say $\phi : \mathcal{X} \to T$, and let $\mathcal{Y}_\bullet$ be the given flag on $\mathcal{X}$. We will consider T and the $\mathcal{Y}_i$ with their reduced structures and will assume, without loss of generality, that T is irreducible. It is enough to show the countabiltity of global Okounkov cones on a non-empty open set $U \subseteq T$, as we can further argue inductively on the dimension of T. So we can assume that T is smooth and that there exists a closed point $t_0 \in T$ admitting an admissible fiber, which is embedded in Y as in (4.2). Now since each $\phi_i := \phi_i|_{\mathcal{Y}_i}$ is flat, by [16, Theorem 12.2.4] the set of all points $t \in T$ such that $Y_{i,t}$ is smooth and irreducible is open in T. So, by further restricting T, we can assume that ϕ is a family of admissible flags and that there is a closed point $t_0 \in T$ whose fiber is embedded in Y as in (4.2).

With the assumptions made above, notice that we have the natural embedding $\mathcal{X} \subseteq Y \times T$. On Y we have ρ Cartier divisors $D_1, \ldots, D_\rho$ which form a base for $N^1(Y)_\mathbb{R}$. As X_{t_0} is a smooth variety embedded in Y as in (4.2), the Cartier divisors $D_1|_{X_{t_0}}, \ldots, D_\rho|_{X_{t_0}}$ form an $\mathbb{R}$-base for $N^1(X_{t_0})_\mathbb{R}$. As each fiber of $\phi : \mathcal{X} \to T$ is smooth and of dimension n, by [24, Theorem III.10.2] the map ϕ is smooth and we can apply Proposition 4.23, which says that under our conditions the map

$$\rho_{X_t} \ : \ \mathbb{R}^\rho \ \to \ N^1(X_t)_\mathbb{R},$$

where $\rho_{X_t}(D_i) := D_i|_{X_t}$, is an injective integral linear morphism for all $t \in T$.

Under these assumptions, it remains to show that the set of all convex bodies

$$(\mathrm{id}_{\mathbb{R}^n} \times \rho_{X_t}^{-1})(\Delta_{Y_{t,\bullet}}(X_t)) \ \subseteq \ \mathbb{R}^n \times \mathbb{R}^\rho,$$

for all $t \in T$, is countable. It's actually sufficient to show that there exists a subset $F = \cup F_m \subseteq T$ consisting of a countable union of proper Zariski-closed subsets $F_m \subsetneqq T$ such that

$$(\mathrm{id}_{\mathbb{R}^n} \times \rho_{X_t}^{-1})(\Delta_{Y_{t,\bullet}}(X_t)) \text{ is independent of } t \in T \setminus F\ .$$

This reduction implies Theorem 4.19, because one can argue inductively on the dimension of T and apply this reduction for each family of flags $\phi : \phi^{-1}(F_m) \to F_m$ containing an admissible fiber.

With this in hand let's prove the reduction above. Denote by

$$B_t \ = \ (\rho_{X_t}^{-1})(\overline{\mathrm{Eff}}(X_t)_{\mathbb{R}}) \ \subseteq \ \mathbb{R}^\rho$$

and if B is the closed convex cone generated by $\cup_{t\in T} B_t$, then we need first to show that B is a pointed cone.

Before proving this, note that if $\xi \in B_{t_1}$ for some closed point $t_1 \in T$, then we have

$$(\xi \cdot D_1^{i_1} \cdot \ldots \cdot D_\rho^{i_\rho} \cdot X_t) \geq 0$$

for any $i_1 + \ldots + i_\rho = n-1$ and $t \in T$. It follows that for any $\xi \in B$ we have that $(\xi \cdot D_1^{i_1} \cdot \ldots \cdot D_\rho^{i_\rho} \cdot X_t) \geq 0$ for any collection $\{i_j\}$ such that $i_1 + \ldots + i_\rho = n-1$ and any $t \in T$. If B is not pointed there exists an $\xi \in B$ such that $-\xi \in B$, and hence

$$(\xi \cdot D_1^{i_1} \cdot \ldots \cdot D_\rho^{i_\rho} \cdot X_t) \ = \ 0$$

for any $i_1 + \ldots + i_\rho = n-1$ and $t \in T$. Now, set $H = D_1 + \ldots + D_\rho$ and notice that for any $t \in T$ the Cartier divisor $H|_{X_t}$ is ample. Thus, by the above, we have that

$$(\xi \cdot H^{n-1} \cdot X_t) \ = \ 0,$$

$$(\xi^2 \cdot H^{n-2} \cdot X_t) \;=\; 0.$$

By [27, Proposition I.4.3] it follows that $\xi|_{X_t} =_{\text{num}} 0$ for all $t \in T$, and this contradicts the condition that the divisors $D_1|_{X_{t_0}}, \ldots, D_\rho|_{X_{t_0}}$ are linearly independent.

That $B \subseteq \mathbb{R}^\rho$ is a pointed cone implies that there exists a $\mathbb{Q}$-base $\{H_1, \ldots, H_\rho\} \subseteq \mathbb{R}^\rho$ such that the positive convex cone generated by the $H_1, \ldots, H_\rho$ inside $\mathbb{R}^\rho$, call it A, has the property that $B_t \subseteq A$ for any $t \in T$, i.e. every effective divisor on X_t (taken to a large power), contained in the image of ρ_{X_t}, is numerically equivallent to a positive $\mathbb{N}$-linear combination of the $H_i|_{X_t}$.

Going back to our family of admissible flags, $\phi : \mathcal{X} \to T$ and the flag $\mathcal{Y}_\bullet$, notice that each morphism $\phi_i : \mathcal{Y}_i \to T$ has smooth and irreducible fibers, hence again by [24, Theorem III.10.2] each map ϕ_i is smooth. Now because the base T is assumed to be smooth, it implies that each $\mathcal{Y}_i$ is smooth, i.e. $\mathcal{Y}_{i+1} \subseteq \mathcal{Y}_i$ is Cartier. Thus our family of flags satisfies the conditions of [34, Theorem 5.1], whose proof says that for any divisor D on $\mathcal{X}$, there exists a non-empty open subset $U \subset T$ such that the finite sets

$$\text{(4.3)} \qquad \operatorname{Im}(\nu_{Y_{t,\bullet}}(X_t; D|_{X_t}) : \; H^0(X_t, \mathcal{O}_{X_t}(D)) \;\to\; \mathbb{Z}^n)$$

coincide for all $t \in U$. Now for each $t \in T$ the Okounkov cone $\Delta_{\mathcal{Y}_{t,\bullet}}(X_t)$ is the cone generated by the semigroup

$$\Gamma_{\mathcal{Y}_{t,\bullet}}(X_t) \;=\; \{(\nu(s), (\mathbf{m}H)|_{X_t}) \mid 0 \neq s \in H^0(X_t, \mathcal{O}_{X_t}(\mathbf{m}H), \forall \mathbf{m} \in \mathbb{N}^\rho\}$$

inside $\mathbb{R}^n \times N^1(X_t)_{\mathbb{R}}$, where $\mathbf{m}H = m_1H_1 + \ldots + m_\rho H_\rho$ is seen as an element in $\mathbb{R}^\rho$. As the semigroups generating the Okounkov cone are countable, applying (4.3) we deduce the existence of a subset $F = \cup F_m \subseteq\; T$ consisting of countably many Zariski closed sets $F_m \subsetneq T$ such that

$$\rho^{-1}_{X_t}\big(\Delta_{\mathcal{Y}_{t,\bullet}}(X_t)\big) \;\subseteq\; \mathbb{R}^n \times \mathbb{R}^\rho \text{ is independent of } t \in T \setminus F.$$

In the end, notice that if the fiber X_t is embedded as in (1), then the map ρ_X is an isomorphism and we finish the proof of the theorem. $\square$

4.4 Conditions on Okounkov bodies on surfaces

In this section our main goal is to characterize the convex bodies arising as Okounkov bodies of big $\mathbb{R}$-divisors on smooth surfaces, by proving Theorem D. Also we give some fairly strong restrictions on the set of Okounkov bodies for big $\mathbb{Q}$-divisors. The main technical tool is Zariski decomposition.

Fixing notation, let S be a smooth surface and D a pseudo-effective real (resp. rational) divisor on X. Fix is an admissible flag on S, consisting of a smooth curve $C \subseteq S$ and a point $x \in C$. The *Zariski decomposition* [4, Theorem 14.14] states that D can uniquely written as a sum

$$D \ = \ P(D) \ + \ N(D)$$

of $\mathbb{R}$ (resp. $\mathbb{Q}$)-divisors with the property that $P(D)$ is nef, $N(D)$ is effective with negative definite intersection matrix [1]and $(P(D).C) = 0$ for every irreducible component C of $N(D)$. $P(D)$ is called the *positive part* of D and $N(D)$ the *negative part*. Another important property of the Zariski decomposition is the minimality of the negative part [4, Lemma 14.10]. This states that if $D = M + N$, where M is nef and N effective, then $N - N(D)$ is effective.

We prove Theorem D using Lazarsfeld's and Mustaţă's description of the Okounkov body on a surface [34, Theorem 6.4] via Zariski decomposition. Let ν be the coefficient of C in the negative part $N(D)$ and let

$$\mu \ = \ \mu(D;C) \ = \ \sup\{ \ t > 0 \mid D - tC \text{ is big } \}.$$

[1]An effective divisor $\sum a_i C_i$ is said to have a negative definite intersection matrix if the matrix$((C_i \cdot C_j))_{i,j}$ is negative definite.

When there is no risk of confusion we will denote $\mu(D;C)$ by $\mu(D)$. For any $t \in [\nu, \mu)$ we set $D_t = D - tC$ and write $D_t = P_t + N_t$ for its Zariski decomposition. Then there exist two continuous functions $\alpha, \beta : [\nu, \mu] \to \mathbb{R}_+$ defined by

$$\alpha(t) = \mathrm{ord}_x(N_t|_C), \quad \beta(t) = \mathrm{ord}_x(N_t|_C) + (P_t \cdot C),$$

such that the Okounkov body $\Delta(D) \subseteq \mathbb{R}^2$ is the region bounded by the graph of α and β:

$$\Delta(D) = \{(t, y) \in \mathbb{R}^2 \mid \nu \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\}.$$

With this preparation in hand, set $D' = D - \mu C$, which is pseudo-effective by definition of μ. Now for any $t \in [\nu, \mu]$ write $s = \mu - t$ and set

$$D'_s = D' + sC = D' + (\mu - t)C = D - tC.$$

In comparison to the proof of [34, Theorem 6.4], it turns out that it is more fruitful to consider the set of the divisors D_t for $t \in [\nu, \mu]$ in the form D'_s for $s \in [0, \mu - \nu]$. If $D'_s = P'_s + N'_s$ is the Zariski decomposition of D'_s, then our aim is to study N'_s as a function of $s \in [0, \mu - \nu]$ and this is done in the following proposition:

Proposition 4.24. *The function $s \rightarrowtail N'_s$ is decreasing on the interval $[0, \mu - \nu]$, i.e. for each $0 \leq s' < s \leq \mu - \nu$ the divisor $N'_{s'} - N'_s$ is effective. Moreover, for some $k \geq 1$, there exists a partition $0 = p_0 < p_1 < \ldots < p_{k-1} < p_n = \mu - \nu$ such that there exist effective divisors A_i and B_i with B_i rational and $N'_s = A_i + sB_i$ for all $s \in [p_i, p_{i+1}]$.*

Proof. Let $C_1, \ldots, C_r$ be the irreducible components of $\mathrm{Supp}(N')$, where $N' = N'_0$. Choose real numbers s', s such that $0 \leq s' < s \leq \mu - \nu$. We can then write

$$P'_{s'} = D'_{s'} - N'_{s'} = (D'_s - (s - s')C) - N'_{s'} = D'_s - ((s - s')C + N'_{s'}).$$

As $P'_{s'}$ is nef and the negative part of the Zariski deomposition is minimal, the divisor $(s-s')C + N'_{s'} - N'_s$ is effective. It remains to show that C is not in the support of N'_s for any $s \in [0, \mu-\nu]$. If C were in the support of N'_s for some s, then for any $\lambda > 0$ the Zariski decomposition of $D'_{s+\lambda}$ would be $D'_{s+\lambda} = P'_s + (N'_s + \lambda C)$. In particular, C would be in the support of $N'_{\mu-\nu}$, contradicting the definition of ν.

Rearranging the C_i, we assume that the support of $N_{\mu-\nu}$ consists of $C_k, \ldots, C_r$. Set

$$p_j \overline{\text{def}} = \sup\{s \mid C_j \subseteq \operatorname{Supp}(N'_s)\} \text{ for all } j = 1, \ldots, k-1$$

and without loss of generality suppose that $0 = p_0 < p_1 < \ldots < p_{k-1} < p_k = \mu - \nu$. We will show that N'_s is linear on $[p_i, p_{i+1}]$ for this choice of the p_i. By the continuity of the Zariski decomposition (see [5]), it is enough to show that N'_s is linear on the open interval (p_i, p_{i+1}). If $s \in (p_i, p_{i+1})$ then the support of N'_s is precisely $\{C_{i+1}, \ldots, C_r\}$, and N'_s is determined uniquely by the equations

$$N'_s \cdot C_j \ = \ (D' + sC) \cdot C_j, \text{ for } i+1 \leq j \leq r \ .$$

As the intersection matrix defined by the curves $C_{i+1}, \ldots, C_r$ is non-degenerate, there exist uniquely defined divisors A_i and B_i supported on those curves such that

$$A_i \cdot C_j \ = \ D' \cdot C_j \text{ and } B_i \cdot C_j \ = \ C \cdot C_j \text{ for all } i+1 \leq j \leq r \ .$$

Notice that B_i is a rational divisor. By [4, Lemma 14.9], both A_i and B_i are effective and by the above we have $N'_s = A_i + sB_i$ for any $s \in (p_i, p_{i+1})$. □

Proof of Theorem D. By [34, Theorem 6.4] we already know that the function α is convex, β is concave and $\alpha \leq \beta$. The description given for α and β and Proposition 4.24 it follows that α and β are piecewise linear with only finitely many break-points. Moreover, α is an increasing function of t by Proposition 4.24, because $N_t = N'_{\mu-s}$ and $\alpha(t) = \mathrm{ord}_x(N_t|C)$. This proves that the Okounkov body of any real divisor on a surface has the required form.

Conversely, we show that a polygon as in Theorem D is the Okounkov body of a real T-invariant divisor on some toric surface. This section of the proof is based on Proposition 6.1 in [34], which characterizes the Okounkov body of a T-invariant divisor with respect to a T-invariant flag on a toric variety in terms of the polygon associated to T in the character lattice $M_{\mathbb{Z}}$ associated to the toric variety.

Let $\Delta \subseteq \mathbb{R}^2$ be a polygon of the form given in Theorem D. As α is increasing, we can assume after translation that $(0,0) \in \Delta \subseteq \mathbb{R}^2_+$. We identify $\mathbb{R}^2$ with the vector space $M_{\mathbb{R}}$ associated to a character lattice $M_{\mathbb{Z}} = \mathbb{Z}^2$ of the smooth toric surface S we want to find. Let $E_1, \ldots, E_m$ be the edges of Δ and for each edge E_i choose a primitive vector $v_i \in N_{\mathbb{R}}$ normal to E_i in the direction of the interior of Δ, where $N_{\mathbb{R}}$ is the dual of $M_{\mathbb{R}}$. We can then write

$$\Delta \;=\; \{\; u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle + a_i \geq 0 \text{ for all } i = 1 \ldots m \;\}$$

for some set of positive rational a_i. After adding vectors $v_{m+1}, \ldots, v_r$ we can assume the set of all vectors v_i has the following properties:

1. The toric surface S associated to the complete fan Σ which is defined by the rays $\{\mathbb{R}_+ \cdot v_1, \ldots, \mathbb{R}_+ \cdot v_r\}$ is smooth projective toric surface.
2. None of the vectors v_i lie in the interior of the first quadrant.
3. For some $i_1, i_2 \in \{1, \ldots, m\}$ we have $v_{i_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_{i_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Condition (2) is possible because α is increasing. Since Δ is compact there exists real numbers $a_{m+1}, \ldots, a_r \in \mathbb{Q}_+$ such that

$$\Delta \;=\; \{\; u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle + a_i \geq 0 \text{ for all } i = 1 \ldots r \;\}$$

Condition (3) implies that we can choose $a_{i_1} = a_{i_2} = 0$. The general theory of toric surfaces now tells us that each v_i represents a T-invariant divisor D_i on S and on setting $D = \Sigma a_i D_i$ the polytope $P(D) \subseteq M_{\mathbb{R}}$ associated to D is equal to Δ. We choose on S the flag consisting of the curve $C = D_{i_1}$ and the point $\{x\} = D_{i_1} \cap D_{i_2}$. The curve C is smooth and the intersection $D_{i_1} \cap D_{i_2}$ is a point because S is smooth and, besides v_{i_1} and v_{i_2}, none of the vectors v_i are contained in $\mathbb{R}^2_+$. By [34, Proposition 6.1], the Okounkov body $\Delta_{(C,x)}(S; D)$ of D with respect to the flag (C, x) is equal to $\psi_{\mathbb{R}}(P(D))$ where the map $\psi_{\mathbb{R}} : M_{\mathbb{R}} \to \mathbb{R}^2$ is defined as

$$\psi_{\mathbb{R}}(u) \;=\; (\langle u, v_{i_1} \rangle, \langle u, v_{i_2} \rangle) \text{ for any } u \in M_{\mathbb{R}}.$$

In our case $\psi_{\mathbb{R}} \equiv \mathrm{id}_{\mathbb{R}}$, so $\Delta_{(C,x)}(S; D) = P(D) = \Delta$ by construction. This completes the proof of Theorem D. □

It is now natural to ask, which of these polygons is the Okounkov body of a rational divisor. It follows from the above toric-surface construction that any polygon of the form considered in Theorem D which is given by rational data is the Okounkov body of a rational divisor. The following proposition gives a partial converse, showing the the rationality of the divisor implies strong rationality conditions on the data points of the Okounkov body.

Proposition 4.25. *Let S be a smooth projective surface, D a big rational divisor on S and (C, x) be an admissible flag on S. Then:*

1. *TAll the vertices of the polygon contained in the set $([\nu, \mu] \times \mathbb{R}$ have rational coordinates.*

2. *$\mu(D)$ is either rational or satisfies a quadratic equation over $\mathbb{Q}$.*

3. *if an irrational number $a > 0$ satisfies a quadratic equation over $\mathbb{Q}$ and the conjugate $\overline{a}$ of a over $\mathbb{Q}$ is strictly larger than a, then there exists a smooth, projective surface S, an ample $\mathbb{Q}$-divisor D and an admissible flag on S such that $\mu(D) = a$.*

Proof. As the positive and negative parts of the Zariski decomposition of a $\mathbb{Q}$-divisor remain rational, ν is a rational number. By the description of α and β, the starting point $(\alpha(\nu), \beta(\nu))$ has rational coordinates. According to Proposition 4.24, the breakpoints of α and β can only occur at $t_i = \mu - p_i$. As the functions α and β are piecewise linear with rational slope, we only need to prove that the t_i are rational. This follows from the Zariski chamber decomposition of the cone of big divisors [5, Theorem 1.1], which locally has a finite decomposition into rational locally polyhedral subcones, and the fact that $\mu - p_i = \inf\{t \mid C_i \subset \mathrm{Supp}(N_t)\}$.

For (2), notice that the volume $\mathrm{vol}_X(D)$, the area of the Okounkov polygon $\Delta(D)$, is rational. As the slopes and intermediate breakpoints of $\Delta(D)$ are rational, the relation computing the area of $\Delta(D)$ gives a quadratic equation for $\mu(D)$ with rational coefficients. Notice that μ will be irrational when one edge of the polygon $\Delta(D)$ sits on the vertical line $t = \nu$.

For the last part we consider a construction of Morrison [39, Theorem 2.9] of K3 surfaces, which states that any even integral quadratic form of signature $(1, 2)$ occurs as the self intersection form of a K3 surface S, with Picard number 3. An argument of Cutkosky (p. 96 [11]) shows that if the coefficients of the form are all divisible by 4, then the pseudo-effective and nef cones of S coincide and are given by

$$\{\alpha \in N^1(S) \,|\, (\alpha^2) \geq 0\,,\, (h \cdot \alpha) > 0\}$$

for any ample divisor h on S. If D is an ample divisor and $C \subseteq S$ an irreducible curve, then the function $f(t) = (D - tC)^2$ has two positive roots and $\mu(D)$ with respect to C is equal to the smaller one, i.e.

$$\mu(D) = \frac{(D \cdot C) - \sqrt{(D \cdot C)^2 - (D^2)(C^2)}}{(C^2)} .$$

Since we are only interested in the roots of f, then we start with any integral quadratic form of signature $(1, 2)$ and multiply it by 4. Hence we can exhibit any number with the required properties as $\mu(D)$ for suitable choices of D, C, and the quadratic form. □

4.5 Non-polyhedral Okounkov bodies

In this section we will give two examples of non-polyhedral Okounkov bodies of divisors on Mori dream space varieties, showing in particular that divisors with finitely generated section rings can nevertheless have non-polyhedral Okounkov bodies. The first example is Fano. The second is not, but has the advantage that the existence of non-polyhedral Okounkov bodies is stable under perturbations of the flag.

Proposition 4.26. *Let X be a smooth projective variety of dimension n equipped with an admissible flag $Y_\bullet$. Suppose that D is a divisor such that $D - sY_1$ is ample. Then we have the following lifting property:*

$$\Delta_{Y_\bullet}(X; D) \cap \left(\{s\} \times \mathbb{R}^{n-1}\right) = \Delta_{Y_\bullet}(Y_1; (D - sY_1)|_{Y_1}).$$

In particular, if $\overline{\mathrm{Eff}}(X)_\mathbb{R} = \mathrm{Nef}(X)_\mathbb{R}$ then on setting

$$\mu(D; Y_1) = \sup\{\ t \mid D - tY_1 \ \textit{effective}\ \}$$

we have that the Okounkov body $\Delta_{Y_\bullet}(X; D)$ is the closure in $\mathbb{R}^n$ of the set

$$\{(s, \underline{v}) \mid 0 \leq s < \mu(D; Y_1), \underline{v} \in \Delta_{Y_\bullet}(Y_1; (D - sY_1)|_{Y_1})\}.$$

Proof. In order to prove the lifting property we will use [34, Theorem 4.26], which states that in our setting we have

$$\Delta_{Y_\bullet}(X;D)\cap\left(\{s\}\times\mathbb{R}^{n-1}\right) \;=\; \Delta_{Y_\bullet}(X|Y_1;D-sY_1)$$

where the second body is the restricted Okounkov body defined in [34, Section 2.4]. Hence it is enough to show that

$$\Delta_{Y_\bullet}(X|Y_1;D-sY_1) \;=\; \Delta_{Y_\bullet}(Y_1;(D-sY_1)|_{Y_1}), \tag{4.4}$$

We will prove this for rational $s\in\mathbb{Q}_+$, as the general case follows by the continuity property of slices of the Okounkov bodies. Combining [34, Theorem 4.26] and [34, Proposition 4.1] we obtain that the restricted Okounkov body satisfies the required homogeneity condition, i.e.

$$\Delta_{Y_\bullet}(X|Y_1;p(D-sY_1)) \;=\; p\Delta_{Y_\bullet}(X|Y_1;(D-sY_1)) \text{ for all } p\in\mathbb{N}.$$

Now, by the construction of restricted Okounkov bodies, to show (4.4) it is enough to prove that

$$H^1(X,m(p(D-sY_1)-Y_1)) \;=\; 0$$

for sufficiently large divisible $p,m\in\mathbb{N}$. As $D-sY_1$ is an ample divisor, this follows from Serre vanishing and so we complete the proof. □

Corollary 4.27. *Let X be a smooth three-fold and $Y_\bullet=(X,S,C,x)$ an admissible flag on X. Suppose that $\overline{\mathrm{Eff}}(X)_\mathbb{R} = \mathrm{Nef}(X)_\mathbb{R}$ and $\overline{\mathrm{Eff}}(S)_\mathbb{R} = \mathrm{Nef}(S)_\mathbb{R}$. Then for an ample divisor D on X, its Okounkov body with respect to the admissible flag $Y_\bullet$ can be described as*

$$\Delta_{Y_\bullet}(X;D) \;=\; \{\ (r,t,y)\in\mathbb{R}^3 \mid 0\le r\le \mu(D;S), 0\le t\le f(r), 0\le y\le g(r,t)\ \}$$

where $f(r)=\sup\{\ s>0 \mid (D-rS)|_S-sC \text{ is ample }\}$ and $g(r,t)=(C.(D-rS)|_S)-t(C^2)$ with the interesection numbers taking place on the surface S.

Remark 4.28. (1) This corollary follows from Proposition 4.26 in combination with the description on surfaces given by Lazarsfeld and Mustaţă [34, Theorem 6.4].
(2) Observe from Prosposition 4.27 that only the function $f : [0, \mu(D;S)] \to \mathbb{R}_+$ might force the Okounkov body to have a “weird” shape. Assume that the Picard number of S is at least three. Then $f(r)$ is the unique real number such that the divisor $(D - rS)|_S - f(r)C$ lies on the boundary of the pseudo-effective cone, which coincides under our assumption with the nef cone. Such classes define a curve in $N^1(S)_{\mathbb{R}}$, obtained by intersecting the boundary of $\mathrm{Nef}(S)_{\mathbb{R}}$ with the plane passing through the points defined by the class of D and supported by the directions determined by the vectors given by the classes $(D|_S - C)$ and $(D - S)|_S$ inside the vector space $N^1(S)_{\mathbb{R}}$. Hence, “worse” the shape of the nef cone of S, “worse” the shape of the Okounkov body.

Example 4.29 (Non-polyhedral Okounkov body on Fano varieties)**.** Set $X = \mathbb{P}^2 \times \mathbb{P}^2$ and let D be a divisor in the linear series $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(3,1)$. We set

$$Y_\bullet \;:\; Y_0 = \mathbb{P}^2 \times \mathbb{P}^2 \;\supseteq\; Y_1 = \mathbb{P}^2 \times E \;\supseteq\; Y_2 = E \times E \;\supseteq\; Y_3 = C \;\supseteq\; Y_4 = \{\mathrm{pt}\}$$

where E is a general elliptic curve. Since E is general we have that

$$\overline{\mathrm{Eff}}(E \times E)_{\mathbb{R}} = \mathrm{Nef}(E \times E)_{\mathbb{R}} = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z \geq 0, xy + xz + yz \geq 0\}$$

under the identification

$$\mathbb{R}^3 \to N^1(E \times E)_{\mathbb{R}},\; (x, y, z) \to xf_1 + yf_2 + z\Delta_E,$$

where $f_1 = \{\mathrm{pt}\} \times E$, $f_2 = E \times \{\mathrm{pt}\}$ and Δ_E is the diagonal divisor. We let $C \subseteq E \times E$ be a smooth general curve in the complete linear series $|f_1 + f_2 + \Delta_E|$ and Y_4 is a general point on C. We show that the Okounkov body $\Delta_{Y_\bullet}(X; D)$ is not polyhedral. For this it is actually enough to show that the slice $\Delta_{Y_\bullet}(X; D) \cap \big(\{0\} \times \mathbb{R}^3\big)$ is not polyhedral. Since $\overline{\mathrm{Eff}}(\mathbb{P}^2 \times \mathbb{P}^2)_{\mathbb{R}} = \mathrm{Nef}(\mathbb{P}^2 \times \mathbb{P}^2)_{\mathbb{R}}$, by Proposition 4.26 this slice is identified with the Okounkov body $\Delta_{Y_\bullet}(Y_1; \mathcal{O}_{Y_1}(D))$. In the following we give a full description of this body, whose shape it will turn out to be round.

The three-fold $Y_1 = \mathbb{P}^2 \times E$ is homogeneous, so its nef cone is equal to its pseudo-effective cone: this cone is bounded by the rays $\mathbb{R}_+[\text{line} \times E]$ and $\mathbb{R}_+[\mathbb{P}^2 \times \{\text{pt}\}]$. So we fall under the conditions of Corollary 4.27.

Using the description of $\mathrm{Nef}(Y_1)_{\mathbb{R}}$ we had above, it is not hard to show that $\mu(\mathcal{O}_{Y_1}(D), Y_2) = 1$. Also, a simple calculation gives us

$$g(r,t) \;=\; (C.(D|_{Y_1} - rY_2)|_{Y_2}) - t(C^2) \;=\; 24 - 18r - 6t.$$

Finally, from Corollary 4.27 we have

$$\begin{aligned} f(r) &= \sup\{s > 0 \mid (D - rY_2)|_{Y_2} - sC \text{ is ample}\} \\ &= \sup\{s > 0 \mid (9 - 9r - s)f_1 + (3 - s)f_2 - s\Delta_E \text{ is ample }\}. \end{aligned}$$

After calculation, we see that for positive s the divisor $(9 - 9r)f_1 + (3 - s)f_2 - s\Delta_E$ is ample if and only if $s < (4 - 3r - \sqrt{9r^2 - 15r + 7})$. Therefore, by Corollary 4.27 we can describe the Okounkov body $\Delta_{Y_\bullet}(Y_1; D)$ as

$$\{(r,t,y) \in \mathbb{R}^3 \mid 0 \le r \le 1, 0 \le t \le 4 - 3r - \sqrt{9r^2 - 15r + 7}, 0 \le y \le 24 - 18r - 6t\}$$

The Okounkov body $\Delta_{Y_\bullet}(X; D)$ is therefore non polyhedral.

Example 4.30. In the following, we give an example of a Mori dream space, such that under a linear deformation of the flag, the Okounkov body of a random divisor remains non-polyhedral. Our construction is based heavily on an example of Cutkosky's [11]. He considers a K3 surface S whose Néron-Severi space $N^1(S)_{\mathbb{R}}$ is isomorphic to $\mathbb{R}^3$ with the lattice $\mathbb{Z}^3$ and the intersection form $q(x,y,z) = 4x^2 - 4y^2 - 4z^2$. He shows that:

1. The divisor class on S represented by the vector $(1, 0, 0)$ corresponds to the class of a very ample line bundle L, which embedds S in $\mathbb{P}^3$ as a quartic surface.

2. The nef and pseudo-effective cones of S coincide, and a vector $(x, y, z) \in \mathbb{R}^3$ represents a nef (pseudo-effective) class if it satisfies the inequalities

$$4x^2 - 4y^2 - 4z^2 \geq 0\ ,\ x \geq 0,$$

where the first inequality is saying that the self interesection is positive, and the second that the intersection of the divisor with L is positive.

We consider $S \subset \mathbb{P}^3$, and the pseudo-effective Cartier divisors represented by the vectors $(1, 1, 0)$ and $(1, 0, 1)$, which we denote by α and β respectively. By Riemann-Roch we have that $H^0(Z, \alpha) \geq 2$ and $H^0(Z, \beta) \geq 2$, so both α and β, being extremal rays in the pseudo-effective cone, are classes of irreducible moving curves. Since $\alpha^2 = \beta^2 = 0$, both of these families are base-point free, and it follows from the base-point free Bertini theorem that there are smooth irreducible curves C_1 and C_2 representing α and β respectively, which are elliptic by the adjunction formula. We may assume that C_1 and C_2 meet transversally in $C_1 \cdot C_2 = 4$ points.

The threefold Z we are interested in is constructed as follows. Let $\pi_1 : Z_1 \to \mathbb{P}^3$ be the blow-up along the curve $C_1 \subseteq \mathbb{P}^3$. Then Z is the blow up of the strict transform $\overline{C}_2 \subseteq Z_1$ of the curve C_2. Denote by $\pi_2 : Z \to Z_1$ the second blow-up, and by π the composition $\pi_1 \circ \pi_2 : Z \to \mathbb{P}^3$. If E_2 is the exceptional divisor of π_2 and E_1 the strict transform of the exceptional divisor of π_1 under π_2 then the following takes place.

Proposition 4.31. *(i) The variety Z is a Mori dream space.*

(ii) Let $L_1 = -K_Z$, and let L_2 and D be two ample divisors on Z such that the classes $D, [L_1], [L_2]$ are linearly independent in $N^1(Z)_{\mathbb{R}}$. Then the Okounkov

body $\Delta_{Y_\bullet}(X;D)$ is non-polyhedral with respect to any admissible flag (Y_1,Y_2,Y_3) such that $\mathcal{O}_Z(Y_1)=L_1$ and $\mathcal{O}_{Y_1}(Y_2)=L_2|_{Y_1}$.

Remark 4.32. This example is more satisfying than the previous one, because the Picard number of a general choice of Y_1 is equal to that of Z. At the same time, the non-polyhedral shape of the Okounkov body persists under a linear change of the flag or a change of the divisor D. Moreover, it would be interesting to see if there exists an admissible flag on Z such that for any divisor D, the Okounkov body of D is polyhedral. This will check the veracity of [34, Problem 7.1] for Mori dream spaces, but not Fano.

Proof. (i) By Corollary 1.3.1 in [9], it will be enough to find an effective big dlt divisor Δ on Z such that $-K_Z-\Delta$ is ample. For this we only need to show that $-K_Z$ is big and nef. Indeed, if this turns to be true, then there exists an effective divisor E such that $-K_Z-\epsilon E$ is ample for any sufficiently small ϵ: we then set $\Delta=\delta(-K_Z)+\epsilon E$ for any sufficiently small δ and ϵ.

Let's show that $-K_Z$ is nef. We know that $-K_Z=4H-E_2-E_1$, where $H=\pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$. The idea is to prove first that any base point of $\mathcal{O}_Z(-K_Z)$, if it exists, is contained in $\pi^{-1}(C_1\cap C_2)$. Notice that $\pi_*\mathcal{O}_Z(-K_Z)=\mathcal{O}_{\mathbb{P}^3}(4)\otimes\mathcal{I}_{C_1+C_2}$. Now the idea is to show that both C_1 and C_2 are the complete intersection of a pair of quadrics. Since C_i is an elliptic curve embedded in $\mathbb{P}^3$ by a complete linear series of degree 4, the dimension of $H^0(\mathcal{O}_{\mathbb{P}^3}(2H)\otimes\mathcal{I}_{C_i})$ is at least two: let P_1^i,P_2^i be two of these quadrics. As $C_i\subseteq\mathbb{P}^3$ is nondegenerate and of degree 4, then it is the complete intersection of P_i and Q_i. Now the inverse image of each $P_1P_2,P_1Q_2,Q_1P_2,Q_1Q_2$ is a section of $\mathcal{O}_Z(-K_Z)$, implying that the base points of $\mathcal{O}_Z(-K_Z)$ are included in $\pi^{-1}(C_1\cap C_2)$.

In order to show that $-K_Z$ is nef, is enough to check that the intersection of $-K_Z$

with any curve contained in $\pi^{-1}(C_1 \cap C_2)$ is positive. Set $C_1 \cap C_2 = \{p_1, p_2, p_3, p_4\}$. Let R_1 and R_2 be the class of a curve in the ruling of E_1 respectively E_2. For any i the set $\pi^{-1}(p_i)$ is then the union of two irreducible curves, one of class R_2 and the other of class $R_1 - R_2$. We have that $R_1 \cdot H = R_2 \cdot H = R_1 \cdot E_2 = R_2 \cdot E_1 = 0$ and $R_1 \cdot E_1 = -1$, $R_2 \cdot E_2 = -1$. In particular, $-K_Z \cdot R_2 = 1$ and $-K_Z \cdot (R_1 - R_2) = 0$ and hence $-K_Z$ is nef but not ample.

It remains only to check that $-K_Z$ is big. More explicitly, we show that the image of $\mathbb{P}^3$ under the rational map

$$\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4 \ , \phi = [F : P_1P_2 : P_1Q_2 : Q_1P_2 : Q_1Q_2]$$

is three-dimensional. Here F is the polynomial defining the surface $S \subseteq \mathbb{P}^3$ and is hence an element of the set $H^0(\mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{I}_{C_1+C_2})$.

The idea is to show that the image of the restriction map $\phi|_S$ of ϕ to S has dimension two. For this notice that $\phi|_S$ can be factored as follows

$$f \circ (\phi_1 \times \phi_2) \ : \ S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^4$$

where $f([a:b],[c:d]) = [0 : ac : ad : bc : bd]$ and $\phi_i = [P_i : Q_i]$. Because the image of f has dimension 2, it is enough to show that $\phi_1 \times \phi_2$ is generically surjective. Since C_1 is the locus where both P_1 and Q_1 vanish on S, the general fiber of ϕ_1 is in the class $(2,0,0) - (1,1,0) = (1,-1,0)$ and likewise the general fiber of ϕ_2 is $(1,0,-1)$. But $(1,0,-1) \not\geq (1,-1,0)$ and $(1,-1,0) \not\geq (1,0,-1)$ in $N^1(S)$, and ϕ_1 and ϕ_2 are individually generically surjective, so $\phi_1 \times \phi_2$ is also generically surjective. Thus $\phi|_S$ has a two dimensional image.

It follows that either $\mathrm{Im}(\phi)$ is three dimensional or $\mathrm{Im}(\phi) \subset \overline{\mathrm{Im}(\phi_S)}$. But, if $p \notin S$, then $F(p) \neq 0$ so $\phi(p) \notin \overline{\mathrm{Im}(\phi_S)}$, so the image of ϕ is three dimensional and $-K_Z$ is big.

(*ii*) For the second part notice that Y_1 is the strict transform of a smooth K3 surface containing both C_1 and C_2. Since Y_1 is linearly equivalent to S on Z, the intersection numbers involving C_1, C_2 and H are the same on both S and Y_1. Thus C_1, C_2, and H are linearly independent in $N^1(Y_1)_{\mathbb{R}}$, and if V is the subspace they generate, then for the same reason and from the fact that the cones $\mathrm{Nef}(S)_{\mathbb{R}}$ and $\overline{\mathrm{Eff}}(S)_{\mathbb{R}}$ are equal and dual we have

$$V \cap \mathrm{Nef}(Y_1)_{\mathbb{R}} = V \cap \overline{\mathrm{Eff}}(Y_1)_{\mathbb{R}} = \{xH + yC_1 + zC_2 \mid 4(x+y+z)^2 - 4y^2 - 4z^2 \geq 0\}.$$

Using this and the fact that $D - \epsilon Y_1$ is ample for any small $\epsilon > 0$, it follows that the intersection $\Delta_{Y_\bullet}(Z;D) \cap \big([0,\mu] \times \mathbb{R}^2\big)$, for some small $\mu > 0$, has the same description as the one given in Corollary 4.27. By Remark 4.28, the Okounkov body $\Delta_{Y_\bullet}(Z;D)$ is not polyhedral if the function

$$f(r) = \sup\{\ s > 0 \mid (D - rY_1)|_{Y_1} - sY_2 \in V \cap \overline{\mathrm{Eff}}(Y_1)_{\mathbb{R}}\},$$

where $r \in [0,\mu]$, is not piecewise linear. Since D, L_1 and L_2 are linearly independent in $N^1(Z)_{\mathbb{R}}$ and Y_1 is a section of a big and nef divisor, then the restrictions $D|_{Y_1}$, $(-K_Z)|_{Y_1}$ and $L|_{Y_1}$ remain linear independent in $N^1(Y_1)_{\mathbb{R}}$. By Remark 4.28, because of the shape of $V \cap \overline{\mathrm{Eff}}(Y_1)_{\mathbb{R}}$, the function f is not piecewise linear and this finishes the proof. $\square$

CHAPTER V

Volume Functions

5.1 Introduction

In the last part of this thesis, our main concern is the volume function associated to a (multigraded) linear series. We start by studying the classical case, i.e. the volume function associated to a complex projective variety. In §5.2, we give the definition and present the basic properties of the volume function in this setup, following [33, Section 2.2.C]. In §5.3 we prove the countability of the volume functions in the classical case. In the same section we construct a four-fold whose volume function is locally given by a transcendental formula. Using the same ideas as in Theorem 4.19, we also show the countability of ample, nef, big, and pseudo-effective cones for all complex projective varieties. In the last section, §5.4, we deal with the volume function of non-complete linear series. Here our main focus is to show that any continuous, log-concave and homogeneous function is the volume function of some multigraded linear series. This is in contrast with the classical case where we only have countably many of them. The material in §5.3 and §5.4 is adapted from [28].

5.2 The volume function

In this section we give the definition of the volume of a Cartier divisor and discuss its main properties.

In the following, X will be a complex projective variety of dimension n and let D be a Cartier divisor on X. Our main concern in this section will be an invariant which originates in the Riemann-Roch problem. Recall that the Riemann-Roch problem asks for the computations of the dimensions

$$h^0(X, \mathcal{O}_X(kD)) \ =_{\text{def}} \ \dim_{\mathbb{C}}(H^0(X, \mathcal{O}_X(kD)))$$

as a function of k. Our main focus lies on the asymptotic behavior of these dimensions. In the most interesting cases they grow like k^n. We introduce the following invariant which measures this growth.

Definition 5.1 (Volume of a line bundle)**.** Let X be a complex projective variety of dimension n, and let D be a Cartier divisor on X. The *volume* of D is defined to be the non-negative real number

$$\operatorname{vol}_X(D) \ = \ \limsup_{k\to\infty} \frac{h^0(X, \mathcal{O}_X(kD))}{k^n/n!}.$$

Sometimes it will be preferable to use the language of line bundles. If $\mathcal{L}$ is a line bundle on X, we write $\operatorname{vol}_X(\mathcal{L})$ for the volume of a divisor D with $\mathcal{O}_X(D) \simeq \mathcal{L}$.

Remark 5.2 (Volume of ample Cartier divisors). Suppose D is an ample Cartier divisor on X. Then by Asymptotic Riemann-Roch (see [33, Example 1.2.19]) we have

$$h^0(X, \mathcal{O}_X(kD)) \ = \ \frac{(D^n)}{n!} \cdot k^n + O(k^{n-1}),$$

implying that whenever D is ample, $\operatorname{vol}_X(D) = (D^n)$.

Remark 5.3 (Characterization of bigness). By the definition of bigness, the integral divisor D is big if and only if $\operatorname{vol}_X(D) > 0$.

Example 5.4 (Positive integers as volumes)**.** If $m \in \mathbb{N}$ is a positive natural number, then it is not hard to find a variety X and an integral divisor D satisfying the property that $\operatorname{vol}_X(D) = m$.

For this let $f : X \to \mathbb{P}^2$ be a cyclic cover of degree m, ramified along some smooth plane curve. Define $\mathcal{O}_X(D) \simeq f^*(\mathcal{O}_{\mathbb{P}^2}(1))$, so that then the Cartier divisor D is ample because the map f is finite. In this case, using Remark 5.2, we deduce

$$\mathrm{vol}_X(D) \;=\; (D^2) \;=\; \deg(f) \cdot (\mathcal{O}_{\mathbb{P}^2}(1)^2) \;=\; m.$$

Example 5.5 (Integral divisors with small volume)**.** If $a \in \mathbb{N}$ is a positive natural number, then we give an example of a surface S and an integral divisor D on S with the property $\mathrm{vol}_S(D) = \frac{1}{a}$.

For this, let $C = \mathbb{P}^1$, $\mathcal{L}_1 = \mathcal{O}_{\mathbb{P}^1}(1-a)$ and $\mathcal{L}_2 = \mathcal{O}_{\mathbb{P}^1}(1)$, and set

$$S = \mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2) \text{ and } \mathcal{O}_S(D) \simeq \mathcal{O}_S(1).$$

To compute $\mathrm{vol}_S(D)$, one reduces the problem to $\mathbb{P}^1$ as follows:

$$h^0(S, \mathcal{O}_S(k)) \;=\; \sum_{a_1+a_2=k} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_1(1-a)+a_2)) \;=\; \sum_{a_1=0}^{a_1=k} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k - a_1 a)).$$

If $[\frac{k}{a}]$ is the round-down of k/a, then we have

$$h^0(S, \mathcal{O}_S(k)) \;=\; \sum_{a_1=0}^{a_1=[\frac{k}{a}]} (k - a_1 a + 1) \;=\; \frac{k^2}{2a} + O(k)$$

and hence $\mathrm{vol}_S(\mathcal{O}_S(1)) = \frac{1}{a}$.

We turn next to the variational properties of the volume. The following result can be found [33, Section 2.2.C].

Theorem 5.6 (Variational properties of volume)**.** *Let X be a complex projective variety of dimension n and let D be a big Cartier divisor.*

(i) For any natural number $p > 0$,

$$\mathrm{vol}_X(p \cdot D) \;=\; p^n \cdot \mathrm{vol}_X(D).$$

(ii) The volume of D depends only on the numerical class of D.

Remark 5.7. Although the original proof of these properties of the volume are algebraic in nature, the emergence of Okounkov bodies highlighted their convexity. Since the volume of an integral divisor is (up to a factor) the Euclidean volume of its Okounkov body, by Theorem 4.12, these two properties follow from Theorem 4.10, which tells us that similar properties hold for Okounkov bodies.

An important consequence of Theorem 5.6 is that one can extend the volume to rational classes in $N^1(X)_{\mathbb{Q}}$. Hence one obtains a homogeneous function of degree n

$$\mathrm{vol}_X : N^1(X)_{\mathbb{Q}} \to \mathbb{R},$$

vanishing outside the big cone, $\mathrm{Big}(X)_{\mathbb{Q}}$. It turns out that this function is continuous and log-concave (see [33, Theorem 2.2.44] and [33, Theorem 11.4.9]).

Theorem 5.8. *Let X be an irreducible complex projective variety of dimension n.*

(i) The function $\xi \mapsto \mathrm{vol}_X(\xi)$ on $N^1(X)_{\mathbb{Q}}$ extends uniquely to a continuous function

$$\mathrm{vol}_X : N^1(X)_{\mathbb{R}} \to \mathbb{R}.$$

(ii) The volume function satisfies the log-concavity relation

$$\mathrm{vol}_X(\xi + \xi')^{1/n} \ \geq \ \mathrm{vol}_X(\xi)^{1/n} \ + \ \mathrm{vol}_X(\xi')^{1/n}$$

for any two big classes $\xi, \xi' \in N^1(X)_{\mathbb{R}}$.

Remark 5.9. The volume function is actually differentiable inside the big cone. This follows as an application of the restriction theorem of Okounkov bodies, the proof of which was given by Lazarsfeld and Mustaţă in [34, Theorem 4.26]. It was also proved independently in [10], where Boucksom-Favre-Jonsson found a nice formula for the derivative of vol_X in any direction, and used it to answer some question of Teissier.

Remark 5.10. In this general setup, this theorem can be found in [33]. On the other hand, continuity of vol_X inside $\mathrm{Big}(X)_{\mathbb{Q}}$, follows easily from the existence of global Okunkov cone, Theorem 4.18. The same theorem and the Brunn-Minkowski inequality imply also the log-concavity property for volumes of big classes.

5.3 Countability of volume functions for complete linear series.

In the previous section we noticed that almost all the features known about the volume function follow from convex geometry arguments. This is in contrast with the fact that the volume function is an algebraic geometry invariant. In this section we want to discuss an aspect which happens only in algebraic geometry. Essentially we prove that for all irreducible varieties there are only countably many volume functions.

Theorem 5.11 (Countability of volume functions)**.** *There exist countably many functions* $(f_j : \mathbb{R}^\rho \to \mathbb{R})_{j\in\mathbb{N}}$ *such that for any complex smooth and projective variety* X *of dimension* n *with Picard number* ρ*, there is an integral linear isomorphism*

$$\rho_X : \mathbb{R}^\rho \to \mathrm{N}^1(X)_{\mathbb{R}}$$

with the property that $\mathrm{vol}_X \circ \rho_X = f_j$ *for some* $j \in \mathbb{N}$*.*

Remark 5.12. The countability of volume functions for all possibly singular irreducible varieties follows easily from Theorem 5.11. Let X be an irreducible projective variety and suppose $\mu : X' \to X$ is a resolution of singularities of X. In this case it is not hard to see that the pullback map

$$\mu^* : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathrm{N}^1(X')_{\mathbb{R}}$$

is linear and injective. Additionally, by [33, Example 2.2.49], we have

$$\mathrm{vol}_X \equiv \mathrm{vol}_{X'} \circ \mu^*.$$

Since the map μ^* is defined by choosing $\dim(\mathrm{N}^1(X)_{\mathbb{R}})$ integral vectors inside $\mathrm{N}^1(X')_{\mathbb{R}}$, then the same statement as on Theorem 5.11 takes place for only complex varieties.

Remark 5.13. Since the function vol_X is the volume function of the global Okounkov cone $\Delta_{Y_\bullet}(X) \subseteq \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$, the countability of volume functions follows from Theorem 4.19, where we prove the countability of global Okounkov cones.

An interesting consequence of Theorem 5.11 is the fact that the set of all volumes is countable. For this let's introduce the following definition.

Definition 5.14 (The semigroup of volumes)**.** Denote by

$$\mathbb{V} \;=\; \{a \in \mathbb{R}_+ \mid a = \mathrm{vol}_X(D) \text{ for some pair } (X, D)\}$$

where X is a complex irreducible projective variety and D a Cartier divisor on X. We call this set $\mathbb{V}$ *the semigroup of volumes.*

Corollary 5.15. *The set* $\mathbb{V} \subseteq \mathbb{R}_+$ *is a countable multiplicative semigroup with respect to the product.*

Remark 5.16. The fact that the set $\mathbb{V}$ is a multiplicative semigroup follows from the Künneth formula.

For this take two pairs (X_1, D_1) and (X_2, D_2). Also, take a surface S as in Example 5.5 and a Cartier divisor D_3 such that

$$\mathrm{vol}_S(D_3) \;=\; \frac{n_1! \cdot n_2! \cdot 2!}{(n_1 + n_2 + 2)!},$$

where n_1 and n_2 are the dimensions of X_1 and X_2 respectively.

Now consider the pair

$$(X, A) \;=\; (X_1 \times X_2 \times S, p_1^*(D_1) \otimes p_2^*(D_2) \otimes p_3^*(D_3)),$$

where p_1, p_2, and p_3 are the projection to each factor. Then by the Künneth formula one has

$$h^0(X, mA) \ = \ h^0(X_1, mD_1) \cdot h^0(X_2, mD_2) \cdot h^0(S, mD_3)$$

for all $m \in \mathbb{N}$. This implies

$$\mathrm{vol}_X(A) \ = \ \mathrm{vol}_{X_1}(D_1) \cdot \mathrm{vol}_{X_2}(D_2)$$

and one deduces that the product puts a multiplicative structure on $\mathbb{V}$.

Remark 5.17. It is not hard to see that all the positive rational numbers are contained in $\mathbb{V}$, i.e. $\mathbb{Q}_+ \ \subseteq \ \mathbb{V}$. This follows by combining Example 5.4, Example 5.5 and the multiplicative structure of $\mathbb{V}$ given by the product.

As one might suspect, the set of volumes $\mathbb{V}$ does not consist only of positive rational numbers. In [33, Section 2.3.B] one finds an example of a quadratic irrational volume. Inspired by this, we construct in the following an example whose volume function is given locally by a transcendental function. Additionally, this example shows that the set of volumes $\mathbb{V}$ also contains transcendental numbers, enhancing the mystery surrounding this set.

Example 5.18 (Transcendental volume)**.** Inspired by a construction of Cutkosky, as explained in [33, Chapter 2.3], we give an example of a four-fold X whose volume function is locally given by a transcendental formula.

Suppose E is a general elliptic curve and set $Y = E \times E$. Let f_1, f_2 be the fibers of Y and Δ its diagonal. Then by [33, Lemma 1.5.4] we have a full description of all the cones on Y, i.e.

$$\mathrm{Nef}(Y)_{\mathbb{R}} = \overline{\mathrm{Eff}(Y)}_{\mathbb{R}} = \{x \cdot f_1 + y \cdot f_2 + z \cdot \Delta \mid xy + xz + yz \geq 0, x + y + z \geq 0\}.$$

Let $H_1 = f_1 + f_2 + \Delta$, $H_2 = -f_1$ and $H_3 = -f_2$ and define the vector bundle

$$V \ = \ \mathcal{O}_{E\times E}(H_1) \oplus \mathcal{O}_{E\times E}(H_2) \oplus \mathcal{O}_{E\times E}(H_3).$$

If $X = \mathbb{P}(V)$ and $\pi : X \to Y$ is the projection map then we have the following proposition:

Proposition 5.19 (Transcendental volume function)**.** *With the above notation there exists an open set in* $\mathrm{Big}(X)_{\mathbb{R}}$*, where the volume is given by a transcendental formula.*

Remark 5.20. Let D be a Cartier divisor on X such that $\mathcal{O}_X(D) \simeq \mathcal{O}_X(1)$. Then the proof of Proposition 5.19 implies also that $\mathrm{vol}_X(D)$ is a transcendental number, hence the semigroup of volumes $\mathbb{V}$ contains transcendental numbers.

Proof. As before let D be a Cartier divisor such that $\mathcal{O}_X(D) \simeq \mathcal{O}_X(1)$. The characterization of Cartier divisors on X and the fact that the function vol_X is continuous and homogeneous on $\mathrm{Big}(X)_{\mathbb{R}}$, imply that it is enough to consider integral divisors of the form

$$A \;=\; D + \pi^*(L')$$

where $L' = c_1 f_1 + c_2 f_2 + c_3 \Delta$ is a Cartier divisor on Y with $(c_1, c_2, c_3) \in \mathbb{N}^3$. Now, we can describe the volume as

$$\mathrm{vol}_X(A) \;=\; \frac{\sum_{a_1+a_2+a_3=m} h^0(Y, mL' + a_1H_1 + a_2H_2 + a_3H_3)}{m^4/24}$$

where the sum is taken over all natural numbers a_i.

The idea is to use Riemann-Roch on Y, which by Kodaira vanishing says that for an ample Cartier divisor L we have

$$h^0(Y, L) \;=\; \frac{1}{2}(L^2).$$

We claim that in the sum above only the ample divisors $L = mL' + a_1H_1 + a_2H_2 + a_3H_3$ count. If L is not nef, then it has no sections and hence doesn't count in the sum. When L is nef but not ample, the description of the nef cone of Y implies that

$$(L^2) \;=\; (mL' + a_1H_1 + a_2H_2 + a_3H_3)^2 = 0.$$

Hence we have at most $2m$-tuples $(a_1, a_2, a_3) \in \mathbb{N}^3$ with $a_1 + a_2 + a_3 = m$, for which the integral divisor is nef but not ample. Now for each one of them we have the bound:

$$\begin{aligned} h^0(Y, L) &= h^0(Y, (mc_1 + a_1 - a_2)f_1 + (mc_2 + a_1 - a_3)f_2 + (mc_3 + a_1)\Delta) \\ &\leq h^0(Y, (mc_1 + a_1)f_1 + (mc_2 + a_1)f_2 + (mc_3 + a_1)\Delta) \simeq Cm^2 \end{aligned}$$

for large $m \in \mathbb{N}$, where the latter part follows from Riemann-Roch as the divisor is ample, because $(c_1, c_2, c_3) \in \mathbb{N}^3$. This tells us that nef but not ample divisors do not count in the computation of the volume, and by Riemann-Roch we have

$$\mathrm{vol}_X(A) = \lim_{m\to\infty} \frac{4!}{2m^4} \sum_{a_1+a_2+a_3=m} ((mc_1+a_1-a_2)f_1+(mc_2+a_1-a_3)f_2+(mc_3+a_1)\Delta))^2,$$

where the sum is taken over all ample divisors. Now write $x_i = a_i/m$ and make the following change of coordinates $T : \mathbb{R}^3 \to \mathbb{R}^3$:

$$W(c_1) = c_1 + x_1 - x_2, W(c_2) = c_2 + x_1 - x_3, W(c_3) = c_3 + x_1$$

Then we can write our volume as

$$\mathrm{vol}_X(A) = \int_{\Gamma(c_1,c_2,c_3)} (W(c_1)f_1 + W(c_2)f_2 + W(c_3)\Delta)^2$$

where the set $\Gamma(c_1, c_2, c_3)$ is the intersection of the image of the triangle with the vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ through the map T and the nef cone $\mathrm{Nef}(Y)_{\mathbb{R}}$. For example, when $c_1 = c_2 = c_3 = 1/4$ this set is represented in Figure 5.1. The shape of $\Gamma(c_1, c_2, c_3)$ and the use of Maple enables one to show easily that $\mathrm{vol}(A)$ is given by a transcendental formula in the c_i. □

Countability of ample and nef cones. The question of countability, which was proved for volume functions and Okounkov bodies, can be asked also for all the cones

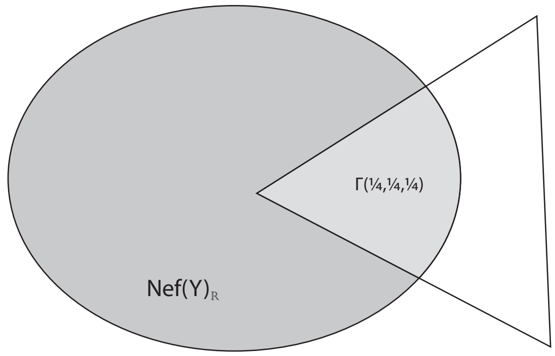

Figure 5.1: Intersection of $\Gamma(c_1, c_2, c_3)$ with the nef cone $\mathrm{Nef}(Y)_{\mathbb{R}}$

of projective varieties. We end this section, by proving the counability of nef cones for all such varieties. For this we will use the same construction we applied to the proof of the countability of Okounkov cones in Theorem 4.19.

Theorem 5.21 (Countability of nef cones)**.** *There exists countably many closed convex cones $A_i \subseteq \mathbb{R}^\rho$ for $i \in \mathbb{N}$, such that for any complex smooth and projective variety X of dimension n with Picard number ρ, there is an integral linear isomorphism*

$$\rho_X : \mathbb{R}^\rho \to \mathrm{N}^1(X)_{\mathbb{R}}$$

with the property that $\rho_X^{-1}(\mathrm{Nef}(X)_{\mathbb{R}}) \;=\; A_i$ for some $i \in \mathbb{N}$.

Remark 5.22 (Countability of nef cones for irreducible varieties)*.* As in Remark 5.12, the countability of nef cones for possibly singular varieties follows from Theorem 5.21.

Let X be a projective variety and suppose $\mu : X' \to X$ is a resolution of singularities of X. Then the pullback map

$$\mu^* \;:\; \mathrm{N}^1(X)_{\mathbb{R}} \to \mathrm{N}^1(X')_{\mathbb{R}}$$

is linear and injective and has the property that $\text{Nef}(X)_{\mathbb{R}} = (\mu^*)^{-1}(\text{Nef}(X')_{\mathbb{R}})$. Since the map μ^* is defined by choosing $\dim(\text{N}^1(X)_{\mathbb{R}})$ integral vectors inside $\text{N}^1(X')_{\mathbb{R}}$, we deduce the countability of nef cones for all projective varieties.

Remark 5.23 (Countability of ample, big and pseudo-effective cones). If one consider the finite real vector space $N^1(X)_{\mathbb{R}}$, for some irreducible projective variety X, with the usual topology then we know that the ample cone $\text{Amp}(X)_{\mathbb{R}}$ is the interior of $\text{Nef}(X)_{\mathbb{R}}$. Thus the same statement as Theorem E is valid for ample cones. Also, the big cone has the description

$$\text{Big}(X)_{\mathbb{R}} = \{\xi \in N^1(X)_{\mathbb{R}} \mid \text{vol}_X(\xi) > 0\}.$$

Thus using Theorem 5.11 the same statement can be deduced for big cones. As for the pseudo-effective cone, notice that it is the closure of the big cone.

Proof of Theorem 5.21. As in the proof of Theorem 4.19, let Y be the product of ρ projective spaces $\mathbb{P}^{2n+1}$. Then any smooth projective variety X of dimension n with Picard number ρ can be embedded in Y as in (4.2) such that this embedding $X \subseteq Y$ has the property that the restriction map $\rho_X : \mathbb{R}^\rho \to \text{N}^1(X)_{\mathbb{R}}$ is an integral linear isomorphism. This embedding forces the multigraded Hilbert function of X, P_X, to be a polynomial with rational coefficients and degree equal to $\dim(X)$.

Thus we have countably many multigraded functions which appear as the multigraded Hilbert function of a smooth projective variety of dimension n and Picard number ρ. Using the representability of the multigraded Hilbert functor (see [22, Corollary 1.2]), there exist countably many flat families such that any smooth and irreducible variety X of dimension n with Picard number ρ is a fiber in at least one of these families.

Fix one of these families: $\phi : \mathcal{X} \to T$. To prove the countability of nef cones in this family we will use the same ideas as in the proof of Theorem 4.19. So we can shrink T such that for all $t \in T$ the fiber X_t is smooth, irreducible and reduced. Also we can assume that the map ϕ is smooth. Finally, using Proposition 4.23, the restriction map

$$\rho_{X_t} \ : \ \mathbb{R}^\rho \ \to \ \mathrm{N}^1(X_t)_\mathbb{R}$$

is an injective integral linear morphism for all $t \in T$.

Now, we can proceed and prove Theorem 5.21. For this denote by

$$A_t := \rho_{X_t}^{-1}(\mathrm{Nef}(X_t)_\mathbb{R}),$$

for each $t \in T$. We have to show that the set $(A_t)_{t\in T}$ is countable. It is enough to show that there exists a subset $F = \cup F_m \subseteq T$ consisting of a countable union of proper Zariski-closed subsets $F_m \subsetneqq T$, such that A_t is independent of $t \in T \setminus F$. This reduction will imply Theorem 5.21, as one can argue inductively on $\dim(T)$.

The set of all cones $(A_t)_{t\in T}$ has the following property: if $t_o \in T$, there exists a subset $\cup F_{t_0}^m \subsetneqq T$, which does not contain t_0, and consists of a countable union of proper Zariski-closed sets such that

$$A_{t_0} \ \subseteq \ A_t, \ \text{for all } t \in T \setminus \cup F_{t_0}^m. \tag{5.1}$$

To verify this, choose any element $D \in A_{t_0} \cap \mathbb{Z}^\rho$. By [33, Theorem 1.2.17] on the behaviour of nefness in families, there exists a countable union $F_{t_0,D} \subseteq T$ of proper subvarieties of T, not containing t_0, such that $D \in A_t$, for all t outside of $F_{t_0,D}$. As A_{t_o} is a closed pointed cone, the set $A_{t_o} \cap \mathbb{Z}^\rho$ is countable and generates A_{t_o} as a cone. Thus the cone A_{t_0} is contained in A_t for all t not included in any of the subsets $F_{t_0,D}$ with $D \in A_{t_0} \cap \mathbb{Z}^\rho$. Since the base field is the complex numbers, the union of all the $F_{t_0,D}$ remains a countable union of proper subvarieties of T.

Now, denote by $A := \cup_{t \in T} A_t$. To finish the proof is enough to find a closed point $t \in T$ with $A_t = A$. For this, first note that $A \subseteq \mathbb{R}^\rho$ satisfies the second countability axiom, i.e. it has a countable base. So by [42, Theorem 30.3] there exists a countable set $\{t_i \in T |\ i \in \mathbb{N}\}$ such that $A = \cup_{i \in \mathbb{N}} A_{t_i}$. By (5.1), for each $i \in \mathbb{N}$ there exists a countable union of proper Zariski-closed subsets $F_i \subsetneq T$ with the property

$$A_{t_i} \ \subseteq \ A_t, \text{ for all } t \in T \setminus F_i,$$

and as before $\cup F_i$ remains a countable union of proper Zariski-closed subsets. This proves Theorem 5.11 in the case of nef cones because for each $t \in T \setminus \cup F_i$ and $i \in \mathbb{N}$ we have $A_{t_i} \subseteq A_t$ and hence $A_t = A$. □

5.4 Volume functions of non-complete linear series

In this section we will study the volume function of a multigraded linear series.

For this, let X be an irreducible projective variety and let $D_1, \ldots, D_p$ be Cartier divisors on X. As before write $\mathbf{m}D = m_1 D_1 + \ldots + m_p D_p$ for $\mathbf{m} = (m_1, \ldots, m_p) \in \mathbb{N}^p$ and suppose we are given a multigraded linear series $W_\bullet$ on X with respect to $D_1, \ldots, D_p$, as in Definition 4.14. In this case we can define the volume function of $W_\bullet$ as follows:

Definition 5.24 (Multigraded volume function)**.** With the notations above, then define the *volume function* of $W_\bullet$, $\mathrm{vol}_{W_\bullet} : \mathbb{N}^p \to \mathbb{R}_+$, where

$$\mathrm{vol}_{W_\bullet}(\mathbf{m}) \ = \ \limsup_{k \to \infty} \frac{\dim_{\mathbb{C}}(W_{k \cdot \mathbf{m}})}{k^n / n!}$$

for all $\mathbf{m} \in \mathbb{N}^p$.

Remark 5.25. This definition is a natural extension of the one given in the classical case. By [34, Lemma 4.6] the pseudo-effective cone $\overline{\mathrm{Eff}}(X)_{\mathbb{R}} \subseteq N^1(X)_{\mathbb{R}}$ is pointed, so one can choose Cartier divisors $D_1, \ldots, D_\rho$ on X such that they form a $\mathbb{Z}$-basis of $N^1(X)$ and every effective line bundle is numerically equivalent to an $\mathbb{N}$-linear combination of the D_i-s. Define the integral linear isomorphism

$$\rho_X \ : \ \mathbb{R}^\rho \to N^1(X)_\mathbb{R}, \text{ where } \rho_X(e_i) = D_i.$$

With this in hand, consider the complete multigraded linear series $V_\bullet$, where we define $V_\mathbf{m} = H^0(X, \mathcal{O}_X(\mathbf{m}D))$ for each $\mathbf{m} \in \mathbb{N}^\rho$. Then by definition for any $\mathbf{m} \in \mathbb{N}^\rho$ we have $(\mathrm{vol}_X \circ \rho_X)(\mathbf{m}) \equiv \mathrm{vol}_{W_\bullet}(\mathbf{m})$ and this defines the volume function of X as vol_X vanishes outside the cone $\rho_X(\mathbb{R}^\rho_+)$.

In the following we will prove Theorem F, and show that in fact any function as in the theorem appears (modulo compressing) as the volume function of some multigraded linear series on $Y = \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ (p times).

Theorem 5.26. *Let $K \subseteq \mathbb{R}^p_+$ be a closed convex cone with nonempty interior and suppose $f : K \to \mathbb{R}_+$ is a continuous function that is non-zero, log-concave and homogeneous of degree p in the interior of K. Then there exists a multigraded linear series $W_\bullet$ on Y,*

$$W_\mathbf{m} \ \subseteq \ H^0(Y, \mathcal{O}_Y(\mathbf{m}))$$

for any $\mathbf{m} \in \mathbb{N}^p$ having the property that $\mathrm{supp}(W_\bullet) = K$ *and* $\mathrm{vol}_{W_\bullet} \equiv k \cdot f$ *in the interior of K, for some positive constant k.*

Remark 5.27. Notice that each function as in Theorem 5.26 can be constructed from a continuous concave function $g : B \to \mathbb{R}_+$, where $B \subseteq \mathbb{R}^{p-1}_+$ is a bounded convex body. If we choose an affine hyperplane $H \subseteq \mathbb{R}^p_+$, not containing the origin, with the property that $H \cap C = B$ is a bounded convex body, then the function

$$g \ := \ \sqrt[p]{f} \ : \ B \to \mathbb{R}_+$$

has the properties needed. The same construction works backwards and in order to find wild functions as in Theorem 5.26, one can focus on continuous and concave functions. For example, one can take a negative, bounded function defined on a closed interval, which is nowhere differentable, and integrate it twice to obtain a continuous, concave and nowhere three times differentiable function.

The main idea of the proof of Theorem 5.26 is to use Okounkov cones for multigraded linear series introduced in Definition 4.16. For this, we will use the construction from Example 4.17. So let $Y = \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ (p-times) and pick on Y the following flag

$$(5.2)\ \ Y_\bullet \ :\ Y_0 = Y \ \supseteq \ Y_1 = [0:1]\times\mathbb{P}^1\times\ldots\times\mathbb{P}^1 \ \supseteq \ldots \supseteq \ Y_p = [0:1]\times\ldots\times[0:1].$$

Let $V_\bullet$ be the complete multigraded linear series, with $V_{\mathbf{m}} = H^0(Y, \mathcal{O}_Y(\mathbf{m}))$ for all $\mathbf{m} \in \mathbb{N}^p$. We showed in Example 4.17 that the Okounkov cone of $V_\bullet$ with respect to $Y_\bullet$ has the following description

$$\Delta(V_\bullet) \ = \ \{(z_1, \ldots, z_{2p}) \mid 0 \leq z_i \leq z_{p+i} \text{ for all } i = 1, \ldots, p\} \ \subseteq \ \mathbb{R}^p_+ \times \mathbb{R}^p_+.$$

With this in hand we can proceed to prove Theorem 5.26. First we show that any nonempty closed convex cone $\Delta' \subseteq \Delta(V_\bullet)$ is the Okounkov cone of some multigraded linear subseries $W_\bullet \subseteq V_\bullet$. This will be done in the following proposition:

Proposition 5.28. *If $\Delta' \subseteq \Delta(V_\bullet)$ is a closed convex cone with non-empty interior, then there exists a multigraded linear subseries $W_\bullet \subseteq V_\bullet$ whose Okounkov cone with respect to $Y_\bullet$ is Δ'. Moreover, if $\pi_2 : \mathbb{R}^p_+ \times \mathbb{R}^p_+ \to \mathbb{R}^p_+$ is the projection on the second factor, then $\pi_2(\Delta') = \mathrm{supp}(W_\bullet)$, and for any $\mathbf{m} \in \mathrm{int}(\pi_2(\Delta')) \cap \mathbb{N}^p_+$ we have*

$$\mathrm{vol}_{\mathbb{R}^p}\big(\Delta' \cap (\mathbb{R}^p_+ \times \{\mathbf{m}\})\big) \ = \ \frac{1}{p!} \cdot \mathrm{vol}_{W_\bullet}(\mathbf{m}),$$

where the left side is the standard Euclidean volume in $\mathbb{R}^p$.

Remark 5.29. Proposition 5.28, says that the function

$$\mathrm{vol}_{W_\bullet} : \mathrm{int}(\mathrm{supp}(W_\bullet)) \to \mathbb{R}_+$$

is the volume of slices of the cone Δ', hence it is continuous, log-concave and homogeneous of degree p in the interior of $\mathrm{supp}(W_\bullet)$. On the other hand, since $W_\bullet$ is included in $V_\bullet$, the function is bounded, i.e. for some $k_1 > 0$

$$\mathrm{vol}_{W_\bullet}(v) \ \leq \ k_1||v||^{1/p}, \text{ for all } v \in \mathrm{supp}(W_\bullet).$$

The concavity of $(\mathrm{vol}_{W_\bullet})^{1/p}$, implies that that the function $\mathrm{vol}_{W_\bullet}$ satisfies a Hölder condition of degree p, [49, Theorem 1.5.1]:

$$|\mathrm{vol}_{W_\bullet}(v) - \mathrm{vol}_{W_\bullet}(w)| \ \leq \ k_2||v-w||^{1/p},$$

for all $v, w \in \mathrm{int}(\mathrm{supp}(W_\bullet))$. Hence the boundedness and the Hölder condition imply that the function $\mathrm{vol}_{W_\bullet}$ can be extended continuously on the whole support of $W_\bullet$.

Proof. For any $\mathbf{m} \in \mathbb{N}^p$, let

$$\Gamma_{\mathbf{m}}(V_\bullet) = \Gamma(V_\bullet) \cap (\mathbb{N}^p \times \{\mathbf{m}\})$$

and set

$$\Gamma'_{\mathbf{m}} = \Delta' \cap \Gamma_{\mathbf{m}}(V_\bullet).$$

Now for each $\mathbf{m} \in \mathbb{N}^p$ let $W_{\mathbf{m}}$ be the vector space generated by the set of monomials

$$\{x_1^{n_1} \cdot y_1^{m_1 - n_1} \cdot \ldots \cdot x_p^{n_p} \cdot y_p^{m_p - n_p} \mid (n_1, \ldots, n_p, m_1, \ldots, m_p) \in \Gamma'_{\mathbf{m}} \}$$

where $\{x_i, y_i\}$ represent the coordinates of the i-th factor in Y. As Δ' is a cone and $\Gamma(V_\bullet)$ is a semigroup, it follows that

$$\Gamma' \ = \ \bigcup \Gamma'_{\mathbf{m}} \ \subseteq \ \mathbb{N}^p \times \mathbb{N}^p$$

is also a semigroup. Furthermore, by construction, we have

$$W_{\mathbf{m}} \cdot W_{\mathbf{n}} \subseteq W_{\mathbf{m}+\mathbf{n}},$$

for all $\mathbf{m}, \mathbf{n} \in \mathbb{N}^p$. Hence $W_\bullet = (W_{\mathbf{m}})_{\mathbf{m}\in\mathbb{N}^p}$ is a multigraded linear series. It remains to show that Δ' is the Okounkov cone of $W_\bullet$ with respect to $Y_\bullet$. By construction, the multigraded semigroup of $W_\bullet$ is Γ', hence it is enough to prove that Γ' generates Δ' as a closed convex cone. For this, it suffices to show that Δ' is generated by the set $\Delta' \cap (\mathbb{N}^p \times \mathbb{N}^p)$. Since Δ' is a pointed cone, one can choose an affine hyperplane H, not containing the origin, such that the set $\Delta' \cap H$ is compact and generates Δ'. Because Δ' has a nonempty interior, then the same can be said about $\Delta' \cap H$. Therefore the set of rational points is dense inside it and generates Δ' as a closed convex cone. Thus Δ' coincides with the Okounkov cone of the multigraded linear series $W_\bullet$ with respect to the admissible flag $Y_\bullet$.

We end the proof by noticing that because Δ' is the Okounkov body of the multigraded linear series $W_\bullet$, Lemma 4.6 yields

$$\dim(W_{\mathbf{m}}) = \#\big(\Delta' \cap (\mathbb{N}^p \times \{\mathbf{m}\})\big)$$

and so by the definition of the Euclidean volume the last statement also follows. □

By the previous proposition, in order to finish the proof of Theorem 5.26, it remains to show that any function as in the statement of the theorem is the volume function of some cone $\Delta' \subseteq \mathbb{R}^p_+ \times \mathbb{R}^p_+$, defined as the Euclidean volume of the slice $\Delta' \cap \big(\mathbb{R}^p_+ \times \{\underline{m}\}\big)$ for any $\underline{m} \in \mathbb{R}^p_+$. For lack of a suitable reference we give a proof of this fact.

Proposition 5.30. *If $f : K \longrightarrow \mathbb{R}_+$ is a function as in Theorem 5.26, then there exists a closed convex cone in $C \subseteq \mathbb{R}^p_+ \times K$ with a nonempty interior such that*

$$f(v) = \mathrm{vol}_{\mathbb{R}^p}(\{w \in \mathbb{R}^p_+ \mid (w, v) \in C\})$$

for all $v \in \mathrm{int}(K)$.

Remark 5.31. In the statement of Theorem 5.26 we say that a function f is proportional to the volume function of a multigraded linear series because the cones in Proposition 5.28 are included in $\Delta(V_\bullet)$ and therefore any cone $C \subseteq \mathbb{R}^p_+ \times \mathbb{R}^p_+$ has to be scaled in order to satisfy this condition. This in turn scales the volume function of the original cone.

Proof. In order to ease the presentation, for any $v \in K$ we define

$$r(v) \;:=\; \sqrt[p]{f(v)/C_p}$$

where C_p is a positive constant chosen such that the volume of a ball with this radius will be $f(v)$. The idea is to find first a linear map $g : \mathbb{R}^p \to \mathbb{R}^p$ with the property that for any vector $v \in K$ the ball

$$B_{g(v)}(r(v)) \;=\; \{w \in \mathbb{R}^p \mid ||w - g(v)|| \leq r(v)\ \}$$

is contained in $\mathbb{R}^p_+$. To find this map, use the fact that K is a pointed cone and first choose a linear form l, strictly positive on $K \setminus \{0\}$. Now the function $\sqrt[p]{f(v)}/l(v)$ is homogeneous of degree 0 and continuous on K and hence bounded above. So choosing an appropriate positive constant k, the linear map $g(v) = k(l(v), \ldots, l(v))$ satisfies the property needed. Now the cone C, we are looking for, is the closure of the following open set

$$C' \;=\; \{(w, v) \in \mathbb{R}^p_+ \times \mathrm{int}(K) \mid ||w - g(v)|| < r(v)\ \}.$$

It remains to show that C' is an open convex cone. As f is homogenuous of degree p in the interior of K, then C' is an open cone contained in $\mathbb{R}^p_+ \times \mathbb{R}^p_+$ such that

$$\mathrm{vol}_{\mathbb{R}^p}\{w \in \mathbb{R}^p_+ \mid (w, v) \in C\} = \mathrm{vol}_{\mathbb{R}^p}(B_{g(v)}(r(v))) = f(v)$$

for any $v \in \text{int}(K)$. To prove the convexity of the cone C', let $(w_1, v_1), (w_2, v_2) \in C'$ be two points and denote $(w_3, v_3) = (w_1 + w_2, v_1 + v_2)$. Using the fact that g is linear we obtain

$$\begin{aligned} ||w_3 - g(v_3)|| &\leq ||(w_1 - g(v_1))|| + ||(w_2 - g(v_2))|| \leq \\ &\leq \sqrt[p]{\frac{f(v_1)}{C_p}} + \sqrt[p]{\frac{f(v_2)}{C_p}} \leq \sqrt[p]{\frac{f(v_3)}{C_p}} \end{aligned}$$

where the first inequality is the triangle inequality, the second follows from the fact that $(w_i, v_i) \in C'$ for $i = 1, 2$, and the last follows from the log-concavity of f. This tells us that $(w_3, v_3) \in C'$ and therefore C' is convex. This completes the proof of the proposition. $\square$

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BIBLIOGRAPHY

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