SMOOTH COCYCLES OVER HOMOGENEOUS DYNAMICAL SYSTEMS

by

Felipe Alberto Ramírez

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2010

Doctoral Committee:

Professor Ralf J. Spatzier, Co-Chair
Assistant Professor Juan Souto, Co-Chair
Professor Richard D. Canary
Professor Lizhen Ji
Associate Professor Martin J. Strauss

This dissertation is dedicated with love and admiration to my parents.

ACKNOWLEDGEMENTS

I would first like to thank my advisor, Ralf Spatzier, for his patience, encouragement, and generosity. This work would not have been possible without his guidance.

I am also grateful to Joseph Conlon for helpful correspondence; to Livio Flaminio for providing the proof of an important lemma; to Gopal Prasad for discussions that clarified aspects of the theory of Lie groups and their representations; to Benjamin Schmidt, who has been a very helpful and encouraging mathematical older brother; to Juan Souto for taking the time to read a previous draft of this thesis and for making suggestions that improved it; to Karen Smith for welcoming me to the University of Michigan and for giving me invaluable advice on several matters; and to Alejandro Uribe for enlightening discussions on Fourier analysis and for giving me a head-start on a possible future project.

I would like to thank those people who have enriched the non-mathematical aspects of my life during my time in graduate school: my friends and colleagues. I am especially lucky to have had the opportunity to share these years with Drs. Bober, Fernández, Goldmakher, Gómez, Jiménez, Krawitz, Parra, and Vivas.

Finally, I thank my parents, to whom this work is dedicated; my little brothers, Alejandro and Basti, who have been my best friends and role models my whole life; and my wife, Yeni, who stands beside me through everything.

TABLE OF CONTENTS

DEDICATION ii

ACKNOWLEDGEMENTS iii

CHAPTER

I. Introduction 1

1.1 On cocycles in dynamics 1
1.1.1 Standard definitions; some mathematical context . . 2
1.1.2 Some natural ways in which cocycles arise 3
1.1.3 A recent application of cocycle rigidity 4
1.2 On homogeneous dynamics 5
1.2.1 A connection to representation theory and cocycles 6
1.3 Previous work on cocycles 7
1.4 Summary of results 8

II. Background and definitions 11

2.1 Basic dynamics 11
2.2 Lie groups and Lie algebras 12
2.2.1 Lie groups 13
2.2.2 Lie algebras 14
2.2.3 Haar measure and lattices 18
2.3 Ergodicity 18
2.4 Cocycles 20
2.5 Representation theory 25
2.5.1 Finite dimensional representations of split semisimple Lie algebras 25
2.5.2 Unitary representations and Sobolev spaces 28
2.5.3 Some remarks on the unitary dual of $\mathrm{SL}(2, \mathbb{R})$ . . . 31
2.6 Regularity theorems 32
2.6.1 Sobolev Embedding Theorem 32
2.6.2 Hypoelliptic operators 34

2.7 Hyperbolicity 35
2.7.1 Anosov group actions 35
2.7.2 Partial hyperbolicity 36

III. Problem and results 39

3.1 Problem statement 39
3.2 Main results 39
3.3 Restatement of problem in cocompact case 42

IV. Previous work 45

4.1 Work of Katok and Spatzier 45
4.1.1 Summary of the methods of proof of Theorem 4.1 . 46
4.2 Work of Flaminio and Forni 48
4.2.1 Preview of how we apply Theorem 4.2 49
4.3 Work of Mieczkowski 50

V. Cocycles over abelian actions generated by a unipotent flow and a semisimple flow 52

5.1 Strategy for the proof of Theorem A 52
5.2 Defining a useful subalgebra $\mathfrak{h}$ 54
5.3 Representations of $\mathfrak{h}$ 54
5.3.1 Representations of $\mathfrak{h}_1 = \mathfrak{sl}(2, \mathbb{R})$ 55
5.3.2 Representations of $\mathfrak{h}_2 = \mathbb{R}\mathcal{X}$ 55
5.4 Invariant distributions and vanishing of obstructions 56
5.5 Solutions in irreducible representations of $\mathfrak{h}$ 58
5.6 Global solution 59
5.7 Smoothness of global solution; proof of Theorem A 61

VI. Cocycles over abelian actions generated by two commuting unipotent flows 71

6.1 Strategy for the proof of Theorem B 71
6.2 Obtaining transfer functions 72
6.3 Different embeddings 73
6.4 Getting enough embeddings 73
6.5 Proofs of Theorems B and C 75
6.6 Proof of Corollary E 78

VII. Applications 81

7.1 Proof of Theorem F 81
7.2 Smooth time-changes; proof of Theorem G 82

BIBLIOGRAPHY 85

CHAPTER I

Introduction

This work is on smooth real-valued cocycles over certain homogeneous dynamical systems.[1] Precise statements of the results, as well as a statement of the general problem they address, can be found in Chapter III. Background and definitions for the mathematical content of the proofs can be found in Chapter II; the proofs themselves are in Chapters V and VI.

This chapter seeks to give some historical and mathematical motivation for the study of cocycles and cohomology in dynamics and, in particular, for the problems we treat in the later chapters. The mathematical content of this chapter—some definitions and descriptions of the tools we use, as well as descriptions of past and present results—is reiterated in Chapters II, III, and IV.

1.1 On cocycles in dynamics

A cocycle, defined below, is a type of map that is associated to a group action. Many commonly asked questions in dynamics reduce to determining whether or not two cocycles are equivalent. Such questions include:

- Given an action on a space, does there exist a measure on the space that is

[1]This material is based upon work supported by the National Science Foundation under Grant Number 0602191.

invariant under the action?

- Are two given group actions on a space conjugate?
- Is a given group action locally rigid? That is, can "small" perturbations be conjugated back to the original action?

These questions arise naturally in the study of dynamical systems, and for this reason, cocycles have become objects of study in their own right.

1.1.1 Standard definitions; some mathematical context

For some mathematical context, we begin with a brief overview of standard definitions.

Given a (left)-action by a group H on a probability measure space (X, μ), a cocycle with values in the group K is a measurable map $\alpha : H \times X \to K$ that satisfies the cocycle identity:

$$\alpha(h_1 h_2, x) = \alpha(h_1, h_2 x)\alpha(h_2, x) \quad \text{for all} \quad h_1, h_2 \in H \quad \text{and} \quad x \in X. \tag{1.1}$$

In this thesis, X will always be a homogeneous space G/Γ, with a G-invariant measure, and K will always be the real numbers, $\mathbb{R}$.

Two cocycles α and β are said to be equivalent, or cohomologous, if one can solve the cohomology equation:

$$\alpha(h, x) = P(hx)^{-1}\beta(h, x)P(x) \quad \text{for some} \quad P : X \to K. \tag{1.2}$$

Again, many questions reduce to solving a cohomology equation to show that some cocycle is either cohomologically trivial or cohomologically constant. (Constant cocycles are those that have no dependence on $x \in X$. They correspond to homomorphisms $H \to K$.)

Attention in our work is restricted to smooth $\mathbb{R}$-valued cocycles. We are interested in smooth solutions to Equation (1.2), or smooth cohomology. Here, one can define the first smooth cohomology group $H^1_\infty(X, H, \mathbb{R})$ as the set of smooth $\mathbb{R}$-valued cocycles over the action of H on X, modulo the equivalence relation from smooth cohomology. In this set up, no two constant cocycles are cohomologous, therefore, $\mathrm{Hom}(H, \mathbb{R}) \subset H^1_\infty(X, H, \mathbb{R})$. Much of the study of the first smooth cohomology of group actions focuses on determining the extent to which $H^1_\infty(X, H, \mathbb{R})$ differs from $\mathrm{Hom}(H, \mathbb{R})$. (The term "cocycle rigidity" refers to situations where the two sets are equal.) The main question in this thesis is:

Question. *Under what conditions is a given cocycle $\alpha : H \times X \to \mathbb{R}$ cohomologically constant? For which group actions do we have cocycle rigidity?*

Remark. One can ask the same question for $\mathbb{R}^k$-valued cocycles, but this just reduces to the above case by taking component functions.

1.1.2 Some natural ways in which cocycles arise

The derivative cocycle. One may notice the similarity between the cocycle identity (Equation (1.1)) and the chain rule for taking derivatives of composed functions. In fact, differentiation of diffeomorphisms $f \in \mathrm{Diff}(M^n)$ can be seen as a cocycle taking values in the group $\mathrm{GL}(n, \mathbb{R})$. Here, the cocycle identity is exactly the familiar chain rule which tells us that the differential of a composition of diffeomorphisms is $d(f \circ g)_x = df_{g(x)} \circ dg_x$.

Orbit equivalences and time changes. Given two group actions on a manifold X—one by the group H and the other by the group K—an orbit equivalence is a diffeomorphism $\tau : X \to X$ that takes orbits of H to orbits of K. Stated another way, for every $x \in X$ and $h \in H$, we have $\tau(hx) = k_{h,x}\tau(x)$, where $k_{h,x} \in K$ depends on h and x. One can easily check that in order to respect the group operations in H

and K, the element $k_{h,x}$ must satisfy the cocycle identity. That is, we can see $k_{h,x}$ as a cocycle, $k : H \times X \to K$.

In the simplest scenario, we have a flow on the manifold X. Here, an orbit equivalence that takes orbits to themselves is called a time change, and yields a cocycle $\alpha : \mathbb{R} \times X \to \mathbb{R}$. In this setting, cocycle rigidity (or the statement that all smooth cocycles are cohomologically constant) tells us that all time changes are of a particular type: they arise by taking some homomorphism $\beta : \mathbb{R} \to \mathbb{R}$, and a smooth function $P : X \to \mathbb{R}$, and defining the cocycle according to the cohomology equation (Equation (1.2)).

In this way, cocycles and the cohomology equation provide a way of studying orbit equivalences.

1.1.3 A recent application of cocycle rigidity

As mentioned above, cocycle rigidity lends itself as a tool for proving other rigidity results. Vaguely speaking, questions are of the following type: How many different ways are there for a given group to act on a given space? A "rigidity result" is one where the answer to this question is "few."

One such question is that of local rigidity. After endowing the space of actions of a Lie group H on a manifold X with the C^r-topology, one has a meaningful notion of what it means for two such actions to be near one another. Suppose σ is such an action. It is called locally rigid if any other sufficiently nearby action $\bar{\sigma}$ is conjugate to σ, by a conjugating diffeomorphism Φ of X that is close to the identitiy, and up to an automorphism ρ of H that is close to the identity; that is, Φ satisfies

$$\Phi \circ \sigma(\rho(h)) = \bar{\sigma}(h) \circ \Phi \quad \text{for all} \quad h \in H. \tag{1.3}$$

Using work of D. Mieczkowski [Mie07], D. Damjanovic and A. Katok have recently obtained a local rigidity-like result for unipotent actions by $\mathbb{R}^2$ or $\mathbb{R}^3$ on quotients of semisimple Lie groups [DK]. They showed local rigidity, restricted to certain parametric families of perturbations. That is, given an action σ, they define a family $\{\bar{\sigma}(\lambda)\}_{\lambda \in B \subset \mathbb{R}^{2,3}}$ of perturbations and conditions on the size of the parameter λ, so that there exists, for any such family, a value $\lambda_0 \in B$ such that $\bar{\sigma}(\lambda_0)$ can be conjugated back to σ in the sense of (1.3).

Since our work is motivated by that of Mieczkowski, we have elected to discuss the results from [Mie07] in Section 1.4.

1.2 On homogeneous dynamics

Since all of our work is on cocycles over actions on homogeneous spaces, we will use this section to briefly introduce the subject of homogeneous dynamics. For a detailed introduction to the subject, we refer the reader to [Sta00].

The main object of study in homogeneous dynamics is a Lie group G or, more commonly, a quotient of G by a lattice $\Gamma \subset G$, together with an action by some subgroup $H \subset G$. We denote the action of H on G/Γ by $H \curvearrowright G/\Gamma$.

One studies the structure of the H-orbits in G/Γ using notions such as ergodicity, mixing, and minimality. In the case that G is noncompact, simple, and with finite center, and $H \subset G$ is closed and noncompact, there is a well-known theorem proved by R. Howe and C. Moore which states that the action $H \curvearrowright G/\Gamma$ is ergodic [HM79]. This is Theorem 2.23 in Section 2.3; we make extensive use of it.

A natural concern in the homogeneous setting is the shape of Hx, for some $x \in G/\Gamma$. For example, is it a submanifold of G/Γ? Is its closure a submanifold?

In a series of landmark papers in the early 1990s, M. Ratner settled these questions for actions $H \curvearrowright G/\Gamma$ by groups H that are generated by unipotent elements [Rat90a, Rat90b, Rat91a, Rat91b, Rat91c]. (A unipotent element of a Lie group G is one that is the image of a nilpotent element of the Lie algebra $\mathfrak{g}$, under the exponential map.

These terms are defined in Section 2.2.) She proved, among other strong results, that the closure $\overline{Hx}$ of the orbit of any element $x \in G/\Gamma$ under the action of a unipotent subgroup $H \subset G$ is a very nice submanifold of G/Γ. In fact, it is a homogeneous space. In contrast, there are examples of semisimple flows on homogeneous spaces with orbit closures that are fractals [Mor05].

Such results have found connections to number theory. Most notably, in 1987, G. Margulis proved the Oppenheim Conjecture by establishing a special case of Ratner's Orbit Closure Theorem [Mar87]. This, and the theorems of Ratner, have inspired a great deal of research into the connections between homogeneous dynamics and number theory, ranging from quantitative versions of the Oppenheim Conjecture, to Diophantine approximations and the Littlewood conjecture.

1.2.1 A connection to representation theory and cocycles

Briefly, we recall that in a unitary representation $\pi : G \times \mathcal{H} \to \mathcal{H}$ of G, one calls the vector $v \in \mathcal{H}$ *smooth* if $g \mapsto \pi(g, v)$ defines a smooth map $G \to \mathcal{H}$. Here, one writes $v \in C^\infty(\mathcal{H})$. In the case of the left-regular representation of a semisimple Lie group G on $L^2(G/\Gamma)$, one has that $f \in C^\infty(L^2(G/\Gamma))$ if $\mathcal{V}^k f \in L^2(G/\Gamma)$ for all $\mathcal{V} \in \mathrm{Lie}(G) := \mathfrak{g}$ and $k \in \mathbb{Z}_+$. (If the lattice Γ is cocompact, then the smooth vectors are exactly the smooth functions on G/Γ.)

All of our work takes place on homogeneous systems $H \curvearrowright G/\Gamma$, where G is a semisimple Lie group. We restrict our attention to smooth cocycles $\alpha : H \times G/\Gamma \to \mathbb{R}$ that are also smooth vectors for the left-regular representation of G on $L^2(G/\Gamma)$. That is, we require that $\alpha(a, \cdot) \in C^\infty(L^2(G/\Gamma))$ for all $a \in A$. This allows us to study the cohomology equation by studying the unitary dual of G.

We summarize some of the representation theory of semisimple Lie groups in Section 2.5. A thorough treatment is found in [Kna01].

1.3 Previous work on cocycles

It is worth mentioning a theme that presents itself in several guises: that higher rank promotes rigidity. In the setting of cocycles, one sees that higher-rank actions tend to have fewer cohomology classes.

- **Anosov flows and diffeomorphisms.** Anosov flows and diffeomorphisms, defined in Section 2.7.1, can roughly be thought of as flows and diffeomorphisms that exhibit hyperbolic behavior. Here, there is an infinite-dimensional space of obstructions to solving the cohomology equation for smooth $\mathbb{R}$-valued cocycles, and A. Livsic famously showed in 1972 that these are exactly the obstructions coming from periodic orbits of the action [Liv72]; A. Katok and R. Spatzier showed in 1994 that these obstructions vanish for higher-rank abelian Anosov actions [KS94b]. The work of Katok and Spatzier is discussed in Chapter IV.

  For both results, the stable and unstable foliations of the space play a central role. (Stable and unstable foliations with respect to Anosov actions are discussed in Section 2.7.) In particular, the regularity of transfer functions is achieved by studying their behavior along leaves of these foliations. We employ similar methods in Section 5.7 to show that our transfer functions are smooth.

- **Unipotent actions.** Some of our main results are for unipotent actions, where we show cocycle rigidity for a broad class of higher-dimensional examples (Theorems B and C, Corollary D). This builds on work of D. Mieczkowski; In [Mie07], he treated the upper-triangular unipotent action on $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))/\Gamma$. Both of these rely on work of L. Flaminio and G. Forni, where they characterized the space of obstructions to solving the cohomology equation for horocycle flows on quotients of $\mathrm{PSL}(2,\mathbb{R})$, and showed that this space is infinite-dimensional [FF03].

  The results in [Mie07] and [FF03] are achieved by working closely with the unitary dual of $\mathrm{SL}(2,\mathbb{R})$. Our work is inspired by these results. The work of Mieczkowski and Flaminio–Forni is discussed in Chapter IV.

- **Solvable actions.** For solvable actions, we have Theorem A and Corollary E, which allows one to establish cocycle rigidity for certain higher-dimensional solvable actions. Recently, M. Asaoka has shown that there are nonhomogeneous actions of the upper-triangular solvable subgroup of $\mathrm{PSL}(2,\mathbb{R})$ on closed three-dimensional manifolds [Asa09], suggesting that one may not have cocycle rigidity for $\mathbb{R}$-valued cocycles here. M. Belliart has announced that Asaoka's result does not hold for codimension-one actions of a broad class of higher-dimensional solvable groups [Bel]. This seems to fit the theme, and may conceivably lead to cocycle rigidity for a broader class of solvable actions than the one treated by Theorem A.

Much of this thesis has to do with exploring the extent to which this theme holds for unipotent and solvable actions.

1.4 Summary of results

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ be an irreducible lattice. Consider the subgroups

$$A = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \mid r, t \in \mathbb{R} \right\}$$

and

$$U = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid r, s \in \mathbb{R} \right\}.$$

Mieczkowski showed that for smooth $\mathbb{R}$-valued cocycles over the actions of A and U on $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))/\Gamma$, the obstructions to solving the cohomology equation vanish [Mie07]; that is,

$$H^1_\infty((\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}))/\Gamma,H,\mathbb{R})=\mathrm{Hom}(H,\mathbb{R})\quad\text{for}\quad H=A\text{ or }U.$$

Notice that A is generated by two commuting elements of the Lie algebra of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$—one semisimple, and the other nilpotent. Similarly, the unipotent subgroup U is generated by two commuting nilpotent elements of the Lie algebra. Our work builds on Mieczkowski's by showing cocycle rigidity for actions on quotients of semisimple Lie groups by subgroups analogous to A and U.

Let $G=G_1\times\cdots\times G_k$ be the product of noncompact simple Lie groups, $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_k$ the Lie algebra, and $\Gamma\subset G$ an irreducible lattice. Let $A\subset G$ be a subroup that is generated by two commuting elements $\mathcal{X},\mathcal{U}\in\mathfrak{g}$ with $\mathcal{X}$ semisimple, and $\mathcal{U}$ nilpotent. One of the main results is:

Theorem A. *If Γ is cocompact and irreducible, and each $\mathfrak{g}_i$ contains stable and unstable vectors for the flow of $\mathcal{X}$, then $H^1_\infty(G/\Gamma,A,\mathbb{R})=\mathrm{Hom}(A,\mathbb{R})$.*

Now, suppose that G admits an embedding of $\overline{\mathrm{SL}(2,\mathbb{R})}^{l_1}\times\overline{\mathrm{SL}(2,\mathbb{R})}^{l_2}$, where $\overline{\mathrm{SL}(2,\mathbb{R})}^{m}$ denotes an m-sheeted cover of $\mathrm{SL}(2,\mathbb{R})$. Let $U\subset G$ denote the unipotent subgroup generated by the nilpotent elements $\mathcal{U}_1=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\times(0)$ and $\mathcal{U}_2=(0)\times\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\in\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{sl}(2,\mathbb{R})\subset\mathfrak{g}$. We have:

Theorem B. *If the projection of $\mathcal{U}_1+\mathcal{U}_2$ to $\mathfrak{g}_i$ is nonzero for all $i=1,\ldots,k$, then $H^1_\infty(G/\Gamma,U,\mathbb{R})=\mathrm{Hom}(U,\mathbb{R})$.*

Using tools from ergodic theory, we have bootstrapped Theorems A and B to other subgroups $H\subset G$ containing either A or U. That is, $H^1_\infty(G/\Gamma,H,\mathbb{R})=\mathrm{Hom}(H,\mathbb{R})$, where H is a subgroup of G containing A or U whose center acts ergodically on G/Γ. In particular, keeping G,Γ,U as in the statement for Theorem B, we have:

Theorem C. *Let $V\subset G$ be the maximal unipotent subgroup containing U. Then $H^1_\infty(G/\Gamma,V,\mathbb{R})=\mathrm{Hom}(V,\mathbb{R})$.*

For a concrete example, Theorem C can be applied to the case $G = \mathrm{SL}(n, \mathbb{R})$ to obtain the following:

Corollary D. *Let $G = \mathrm{SL}(n, \mathbb{R})$ where $n > 3$, and $\Gamma \subset G$ a lattice. Let $V \subset G$ be the subgroup of upper-triangular matrices with 1's along the diagonal. Then $H^1_\infty(G/\Gamma, V, \mathbb{R}) = \mathrm{Hom}(V, \mathbb{R})$.*

A similar result also holds for the upper-triangular matrices.

Corollary E. *Let $G = \mathrm{SL}(n, \mathbb{R})$ where $n > 2$, and $\Gamma \subset G$ a cocompact lattice. Let $W \subset G$ be the subgroup of upper-triangular matrices. Then $H^1_\infty(G/\Gamma, W, \mathbb{R}) = \mathrm{Hom}(W, \mathbb{R})$.*

Finally, we apply these theorems to time-changes of certain $\mathbb{R}^n$-actions on homogeneous spaces. First, we prove an extension of Theorems A and B for actions by $\mathbb{R}^n$ that contain actions of the type seen in Theorems A and B.

Theorem F. *Let G be as in the statements for Theorems A and B, with $\Gamma \subset G$ irreducible and cocompact. Suppose there is a locally free action by diffeomorphisms by $\mathbb{R}^n$, $n \geq 2$, on G/Γ, such that there is a subgroup $\mathbb{R}^2 \subset \mathbb{R}^n$ whose restricted action on G/Γ is one of the actions from either Theorem A or Theorem B. Then, for any $l \in \mathbb{Z}_+$, $H^1_\infty(G/\Gamma, \mathbb{R}^n, \mathbb{R}^l) = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^l)$.*

This theorem is proved using similar techniques to those used for Theorem C. The change of focus from $\mathbb{R}$-valued cocycles to $\mathbb{R}^l$-valued cocycles does not add any significant generality to the result. We prove Theorem F for $\mathbb{R}$-valued cocycles and then consider component functions to obtain the statement for $\mathbb{R}^l$-valued cocycles.

We then apply Theorem F to obtain the following result for smooth time-changes:

Theorem G. *Any smooth time-change of an $\mathbb{R}^n$-action of the type in Theorem F is smoothly conjugate to the original action, up to an automorphism of the acting group.*

CHAPTER II

Background and definitions

This chapter contains the background material and definitions that will be used in the proofs of the main theorems.

2.1 Basic dynamics

The following information is readily available in any introductory text on dynamics. We recommend [BS02, KH95].

The most basic setup in the study of dynamical systems is that of a group G acting on a set X.

Definition 2.1 (Group action)**.** A group G is said to *act (on the left)* on a set X if there is a map $\phi : G \times X \to X$ such that

- $\phi(gh, x) = \phi(g, \phi(h, x))$ for all $g, h \in G$, $x \in X$; and,
- $\phi(1, x) = x$ for the identity $1 \in G$ and for all $x \in X$.

In this case, ϕ is called a *group action*, and the fact that G acts on X is often denoted $G \curvearrowright X$. When there is no risk of confusion, $\phi(g, x)$ is denoted gx. In this way, the two properties above can be expressed as

- $(gh)x = g(hx)$ for all $g, h \in G$, $x \in X$; and,

- $1x = x$ for the identity $1 \in G$ and for all $x \in X$.

In practice, the set X usually has some additional structure, and the action by the group G is compatible with this structure. For example, if X is a topological space, one might study systems where G acts by continuous maps. Alternately, it is very common that one has a probability measure space (X, μ), and an action $\mathbb{Z} \curvearrowright X$ by iterations of some measure preserving transformation $T : X \to X$.

Definition 2.2 (Measure preserving)**.** A map $T : (X, \mu) \to (X, \mu)$ on a measure space is called *measure preserving* if $\mu(T^{-1}(A)) = \mu(A)$ for any measurable subset $A \subset X$.

An action $G \curvearrowright (X, \mu)$ is said to be *measure preserving* if all elements of G act by measure preserving transformations.

One is then interested in the orbits of the action, especially their long-term behavior. Qualitative notions, such as how well the orbit of a generic point "fills out" the space, are studied using rigorously defined concepts, such as *topological transitivity* and *minimality* in the topological category, and *ergodicity* and *mixing* in the measurable category. For the definitions of these terms, we refer the reader, again, to any introductory text on dynamics, such as [BS02, KH95]. The present work makes repeated use of ergodicity; consequently, we have devoted Section 2.3 to defining and studying this notion.

To conclude this section, we remark that all of the group actions in our work are on finite volume quotients of semisimple Lie groups, the definition of which will be given in Section 2.2.

2.2 Lie groups and Lie algebras

This section contains basic information about Lie groups and Lie algebras that can be found in [Hum78, Jac79, Kna02].

2.2.1 Lie groups

Definition 2.3 (Lie group)**.** A *Lie group* is a smooth manifold G together with smooth maps

$$\star : G \times G \to G$$

and

$$\mathrm{inv} : G \to G$$

such that G is a group with multiplication given by $\star$, and inverse given by inv. Usually, $g \star h$ is denoted gh, and $\mathrm{inv}(g)$ is denoted g^{-1}, for elements $g, h \in G$.

More succinctly, a Lie group is a smooth manifold G that is also a group, where the group operations are smooth maps with respect to the differentiable structure of G.

Example 2.4 ($\mathbb{R}^n$ and M_n are Lie groups)**.** Clearly, $\mathbb{R}^n$ is a smooth manifold. It is also a Lie group, where

- $\star : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y}$; and,
- $\mathrm{inv} : \mathbb{R}^n \to \mathbb{R}^n$ by $\mathbf{x} \mapsto -\mathbf{x}$.

It is easy to see that these maps are smooth.

One sees that the set of $n \times n$ matrices, denoted M_n, is also a Lie group by making the identification $\mathrm{M}_n \simeq \mathbb{R}^{n^2}$.

Example 2.5 (S^1 is a Lie group)**.** One can make the circle S^1 into a Lie group by identifying it with the set of complex numbers of norm equal to 1. The group operations are then just multiplication and inversion, inherited from $\mathbb{C}$.

Example 2.6 ($\mathrm{GL}(n, \mathbb{R})$ is a Lie group)**.** The set of $n \times n$ invertible matrices, denoted $\mathrm{GL}(n, \mathbb{R})$, is an open subset of M_n, and, therefore, is a smooth manifold. Furthermore, the operations of matrix multiplication and inversion in $\mathrm{GL}(n, \mathbb{R})$ are polynomial in each component, and so they are smooth maps. They make $\mathrm{GL}(n, \mathbb{R})$ into a Lie group.

Example 2.7 ($\mathrm{SL}(n, \mathbb{R})$ is a Lie group)**.** The set of $n \times n$ matrices of determinant 1, denoted $\mathrm{SL}(n, \mathbb{R})$, being the preimage of the regular value $1 \in \mathbb{R}$ under the determinant map $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$, is a smooth submanifold of $\mathrm{GL}(n, \mathbb{R})$. Since determinants are multiplicative, $\mathrm{SL}(n, \mathbb{R})$ is closed under matrix multiplication and inversion. Therefore, $\mathrm{SL}(n, \mathbb{R})$ is a Lie group.

We will be mostly concerned with semisimple Lie groups.

Definition 2.8 (Simple and semisimple Lie group)**.** A connected Lie group is called *simple* if it has no nontrivial, connected, proper, normal subgroups.

A connected Lie group is called *semisimple* if its Lie algebra is semisimple. (Lie algebras are defined and discussed in the next section.) Every simple Lie group is semisimple.

Remark 2.9. Note that a simple (and therefore also a semisimple) Lie group may have infinite center. For example, the universal cover of $\mathrm{SL}(2, \mathbb{R})$ is a simple Lie group with infinite center. Though it is explicitly stated in the theorems, it is worth mentioning here that we require all of our semisimple Lie groups to have *finite* center. This is in order to apply the Howe–Moore Ergodicity Theorem (Theorem 2.23, Section 2.3).

2.2.2 Lie algebras

Definition 2.10 (Lie algebra)**.** A *Lie algebra* over a field $\mathbb{F}$ is a vector space V over $\mathbb{F}$, together with a bilinear map $[\cdot, \cdot] : V \times V \to V$ such that

- $[v, v] = 0$ for all $v \in V$; and,
- $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ for all $u, v, w \in V$ (*Jacobi identity*).

The map $[\cdot, \cdot]$ is called the *Lie bracket* of the Lie algebra V.

The first property of $[\cdot, \cdot]$ in the above definition, together with the fact that it is bilinear, implies that the Lie bracket is antisymmetric; that is, $[u, v] = -[v, u]$ for all $u, v \in V$.

All of the Lie algebras in our work are over the base field $\mathbb{R}$.

To every Lie group, one can assign a real Lie algebra.

Definition 2.11 (Lie algebra of a Lie group)**.** Given a Lie group G, its *Lie algebra* is the space of all vector fields on G that are invariant under right translation by group elements. Equivalently, it is the tangent space to G at $1 \in G$. Notationally, we write $\mathrm{Lie}(G) := \mathfrak{g}$ for the Lie algebra of G.

Given $g \in G$, one can define a map $\mathrm{conj}(g) : G \to G$, where $\mathrm{conj}(g)$ is just conjugation by g. (That is, $\mathrm{conj}(g)(h) = ghg^{-1}$ for all $h \in G$.) Since the image of $1 \in G$ under this map is 1, the derivative maps $\mathfrak{g} \to \mathfrak{g}$. Notationally, $\mathrm{Ad}_g := \mathrm{d}(\mathrm{conj}(g))$.

Now, $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{g})$; that is, $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. The derivative of Ad,

$$\mathrm{ad} := \mathrm{d}(\mathrm{Ad}) : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),$$

is used to define the bracket operation on $\mathfrak{g}$. Namely,

$$[\mathcal{X}, \mathcal{Y}] := \mathrm{ad}(\mathcal{X})(\mathcal{Y})$$

for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{g}$. One can check that $\mathfrak{g}$ together with $[\cdot, \cdot]$ is indeed a Lie algebra.

Given $\mathcal{X} \in \mathfrak{g}$, there is a unique homomorphism $\gamma : \mathbb{R} \to G$ with $\gamma'(0) = \mathcal{X}$. We define the exponential map from $\mathfrak{g}$ to G by

$$\exp(\mathcal{X}) := \gamma(1).$$

Through the exponential map, the Lie algebra's structure encodes the local behavior of the underlying Lie group. The relationship between the Lie bracket and the group operation is expressed through the *Baker–Campbell–Hausdorff formula*, which states that

$$\log(\exp(\mathcal{X}) \cdot \exp(\mathcal{Y})) = \mathcal{X} + \mathcal{Y} + \frac{1}{2}[\mathcal{X}, \mathcal{Y}] + \frac{1}{12}[\mathcal{X}, [\mathcal{X}, \mathcal{Y}]] + \frac{1}{12}[\mathcal{Y}, [\mathcal{Y}, \mathcal{X}]] + \cdots$$

for all $\mathcal{X}, \mathcal{Y}$ sufficiently close to $0 \in \mathfrak{g}$. Here, log denotes a local inverse to exp. This exists because the exponential map is a local diffeomorphism. All the higher order terms in the Baker–Campbell–Hausdorff formula are nested brackets.

Example 2.12 ($\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{R})$)**.** The set $\mathfrak{gl}(n, \mathbb{R})$ of $n \times n$ matrices is the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$. The set $\mathfrak{sl}(n, \mathbb{R})$ of matrices with trace 0 is the Lie algebra of $\mathrm{SL}(n, \mathbb{R})$. In both of these cases, the exponential map happens to be given by the usual exponential power series: for $\mathcal{X} \in \mathfrak{gl}(n, \mathbb{R})$,

$$\exp(\mathcal{X}) = 1 + \mathcal{X} + \frac{\mathcal{X}^2}{2!} + \frac{\mathcal{X}^3}{3!} + \frac{\mathcal{X}^4}{4!} + \cdots .$$

Definition 2.13 (Simple and semisimple Lie algebras)**.** A Lie algebra $\mathfrak{g}$ is called *simple* if it has no ideals other than $\{0\}$ and $\mathfrak{g}$. It is called *semisimple* if it has no nontrivial abelian ideals. Equivalently, it is semisimple if it is the direct sum of simple Lie algebras.

Remark 2.14. In view of the above definition, one can define a simple (respectively, semisimple) Lie group as one whose Lie algebra is simple (respectively, semisimple). Every semisimple Lie algebra is the product of simple Lie algebras.

Definition 2.15 (Semisimple and nilpotent elements)**.** An element $\mathcal{X} \in \mathfrak{g}$ is called *semisimple* if $\mathrm{ad}(\mathcal{X})$ is diagonalizable over $\mathbb{C}$. An element $\mathcal{U} \in \mathfrak{g}$ is called *nilpotent* if $(\mathrm{ad}(\mathcal{U}))^k = 0$ for some $k \in \mathbb{N}$.

Example 2.16 ($\mathfrak{sl}(n, \mathbb{R})$)**.** Our most highly used example of a semisimple Lie algebra is $\mathfrak{sl}(n, \mathbb{R})$, especially the case where $n = 2$. Here, semisimple elements are easy to find, as they are just diagonal matrices and their conjugates. Similarly, examples of nilpotent elements can be found by simply taking matrices with zeroes along the diagonal, and their conjugates.

We now have enough definitions to state the following classical result of Jacobson and Morozov, concerning subalgebras of semisimple Lie algebras. Later, we work with semisimple Lie algebras that contain a nilpotent element $\mathcal{U}$ and a commuting semisimple element $\mathcal{X}$. The following generalization of the Jacobson–Morozov Lemma [Sta00] will guarantee that our Lie algebras contain certain Lie subalgebras whose representations are easy to manipulate.

Theorem 2.17 (Jacobson–Morozov)**.** *Let $\mathcal{U}$ be a nilpotent element in a semisimple Lie algebra $\mathfrak{g}$, commuting with a semisimple element $\mathcal{X} \in \mathfrak{g}$. Then there exist a semisimple element $\mathcal{Y} \in \mathfrak{g}$ and a nilpotent element $\mathcal{V} \in \mathfrak{g}$ such that $[\mathcal{Y}, \mathcal{U}] = \mathcal{U}$, $[\mathcal{Y}, \mathcal{V}] = -\mathcal{V}$, and $[\mathcal{U}, \mathcal{V}] = \mathcal{Y}$, where $\mathcal{Y}$ and $\mathcal{V}$ commute with $\mathcal{X}$.*

We use Theorem 2.17 to find conveniently embedded copies of $\mathfrak{sl}(2, \mathbb{R})$ in our Lie algebras.

As with Lie groups, we will mostly be concerned with semisimple Lie algebras. In some cases, we make use of certain properties enjoyed by what are called split semisimple Lie algebras, which we will now define.

Definition 2.18 (Splitting Cartan subalgebra; split Lie algebra)**.** A *splitting Cartan subalgebra* of the Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that the set $\{\mathrm{ad}(\mathcal{V})\}_{\mathcal{V} \in \mathfrak{t}}$ is simultaneously diagonalizable over $\mathbb{R}$. A *split Lie algebra* is a Lie algebra that contains a splitting Cartan subalgebra.

Finite dimensional representations of split semisimple Lie algebras have special decompositions that will be used in our work. These decompositions are discussed in Section 2.5.

2.2.3 Haar measure and lattices

Any Lie group can be equipped with a measure that is invariant under right-multiplication by group elements. That is, there exists a measure μ on G such that $\mu(Ug) = \mu(U)$ for all $g \in G$ and any measurable subset $U \subset G$. This measure is called *(right)-Haar measure*, and is unique up to scalar multiplication. There is also a left-Haar measure, and the two do not agree in general. A group in which they do agree is called *unimodular*.

Semisimple groups are unimodular.

All of our semisimple Lie groups will come equipped with Haar measure, and often, we will consider quotients of our Lie groups by lattices.

Definition 2.19 (Lattice)**.** Given a Lie group G, a *lattice* is a discrete subgroup $\Gamma \subset G$ such that the quotient G/Γ has finite measure, where the measure is inherited from the Haar measure on G.

The quotient G/Γ is a manifold, which inherits Haar measure from G, by right-invariance. Furthermore, this measure, which we also denote by μ, is left-invariant, since G is unimodular. Now, any subgroup $A \subset G$ acts on G/Γ on the left by measure preserving transformations.

We are interested in what are called irreducible lattices.

Definition 2.20 (Irreducible lattice)**.** Let $G = G_1 \times \cdots \times G_k$ be a semisimple Lie group that is the product of the simple Lie groups G_i, $i = 1, \ldots, k$. A lattice $\Gamma \subset G$ is *irreducible* if its projection to any partial product of the G_i is not a discrete subgroup.

2.3 Ergodicity

For a thorough background in ergodic theory, we refer the reader to [BS02, KH95, Wal82]. In this section we record the results from this theory that we will use.

Definition 2.21 (Ergodic transformation)**.** Let (X, μ) be a probability measure space. A measure preserving transformation $T : X \to X$ is called *ergodic* if all the T-invariant subsets have measure 0 or 1.

We use the following equivalent formulations of ergodicity repeatedly. A measure preserving transformation $T : X \to X$ is ergodic if and only if:

- Every $f \in L^1(X)$ with the property that $f(T(x)) = f(x)$ for almost every $x \in X$ is almost everywhere constant.

- Every $f \in L^2(X)$ with the property that $f(T(x)) = f(x)$ for almost every $x \in X$ is almost everywhere constant.

Qualitatively, ergodicity describes the tendency of a transformation to move points around to fill up the space. More precisely, one can consider the orbit of a point $x \in X$ under iterates of the transformation and ask whether the time average for a given function $f : X \to \mathbb{R}$ along this orbit is equal to the space average of the function. If the transformation is ergodic, it is the case that time averages coincide with space averages. This is known as the Birkhoff Ergodic Theorem.

Theorem 2.22 (Birkhoff Ergodic Theorem)**.** *Let $T : (X, \mu) \to (X, \mu)$ be a measure preserving transformation on a probability measure space, and let $f \in L^1(X)$. Then the time average $(1/n)\sum_{i=0}^{n-1} f(T^i(x))$ converges for almost every $x \in X$ to a function $f^* \in L^1(X)$. Also, $f^* \circ T = f^*$ almost everywhere, and $\int_X f^* \, d\mu = \int_X f \, d\mu$.*

Curiously, ergodicity is not mentioned in the statement of the Birkhoff Ergodic Theorem. However, if one has that the transformation T is ergodic, then, by one of our equivalent formulations of ergodicity, the function f^* is almost everywhere constant. Therefore, in the ergodic setting, Theorem 2.22 guarantees that for almost every $x \in X$,

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) \longrightarrow \int f;$$

that is, time averages (almost always) coincide with space averages.

One can also define ergodicity in the more general setting of a measure preserving group action $G \curvearrowright X$. Here, the action is said to be *ergodic* if the only G-invariant subsets of X are the empty set and X itself. (A subset $Y \subset X$ is G-invariant if $gY = Y$ for all $g \in G$.)

As we mentioned in Section 2.1, we will mostly work with spaces that are quotients of semisimple Lie groups, with actions by subgroups. In other words, we will consider actions such as $H \curvearrowright G/\Gamma$, where G is a semisimple Lie group, $\Gamma \subset G$ is a lattice, and $H \subset G$ is a subgroup. The following theorem, known as the Howe–Moore Ergodicity Theorem, is very useful. We quote from [FK02].

Theorem 2.23 (Howe–Moore)**.** *Let G be a noncompact simple Lie group with finite center and let $\Gamma \subset G$ be a lattice in G. Then any closed noncompact subgroup H of G acts ergodically on G/Γ by left translations.*

As an application of Theorem 2.23, let G and $\Gamma \subset G$ be as in the statement of the theorem. Let $\mathcal{V} \in \mathfrak{g}$. Then the flow $\phi_t^{\mathcal{V}}$ of $\mathcal{V}$ on G/Γ is ergodic unless $\mathcal{V}$ is a semisimple element such that the eigenvalues of $\operatorname{ad}(\mathcal{V}) \in \operatorname{End}(\mathfrak{g})$ are all purely imaginary. In fact, this is still true if G is semisimple and the lattice $\Gamma \subset G$ is irreducible [AS95]. In particular, the flow of a nilpotent element $\mathcal{U} \in \mathfrak{g}$ is ergodic, and the flow of a semisimple element $\mathcal{X} \in \mathfrak{g}$ is ergodic as long as $\operatorname{ad}(\mathcal{X})$ has some eigenvalue that is not purely imaginary.

2.4 Cocycles

The following definitions are standard, and can all be found in [Kat01] and [Mie06]. For a survey of the uses of cocycles in dynamics, see [Kat01].

Definition 2.24 ((Degree 1) cocycle)**.** For a measurable action of a group H on a measure space (X, μ), a *G-valued degree* 1 *cocycle* is defined to be a measurable map

$\alpha : H \times X \to G$ satisfying

$$\alpha(h_1h_2, x) = \alpha(h_1, h_2x)\alpha(h_2, x), \tag{2.1}$$

where G is a group. A G-cocycle whose image is the identity element in G is called a *trivial cocycle*. A homomorphism $\phi : H \to G$ satisfies the cocycle identity by setting $\phi(h, x) = \phi(h)$, and is called a *constant cocycle*.

Remark 2.25. It should be noted that all of the cocycles in our study will be $\mathbb{R}$-valued degree 1 cocycles, and so we will often refer to them simply as cocycles, or $\mathbb{R}$-valued cocycles, since there is no risk of confusion.

Equation (2.1) is called the *cocycle identity*. Occasionally, it will be convenient to think of an $\mathbb{R}$-valued cocycle as being a map $\alpha : H \to \mathcal{F}(X)$ where $\mathcal{F}(X)$ denotes the measurable functions on X. In this case, the cocycle identity is

$$\alpha(h_1h_2)(x) = \alpha(h_1)(h_2x) + \alpha(h_2)(x).$$

(In fact, this is the notation we use below to define the infinitesimal generator of a smooth cocycle.)

As the terminology suggests, there is a notion for when two cocycles over a given action are equivalent, or cohomologous. In fact, this is one of the central ideas in our work.

Definition 2.26 (Cohomology)**.** Two G-cocycles α and β are said to be *cohomologous* if there exists a measurable map $P : X \to G$ such that

$$\beta(h, x) = P(hx)^{-1}\alpha(h, x)P(x). \tag{2.2}$$

The map P is referred to as a *transfer function*, and (2.2) is called the *cohomology equation*. We say that a cocycle is a *coboundary* if it is cohomologous to the trivial cocycle. It is an *almost coboundary* if it is cohomologous to a constant cocycle.

Remark 2.27. Notice that if the group G is abelian, and P satisfies equation (2.2), then so does $g \cdot P$ for any fixed $g \in G$.

We will be concerned exclusively with smooth $\mathbb{R}$-valued cocycles over group actions on smooth manifolds. Specifically, the acting group will be a connected Lie subgroup H of a connected (semi)simple Lie group G, and the space X will be $X = G/\Gamma$, where $\Gamma \subset G$ is an irreducible lattice. For α to be a *smooth cocycle*, we require that it be a smooth map in the usual sense, and that $\alpha(h, \_)$ be a smooth vector in $L^2(G/\Gamma)$ for all $h \in H$. That is, $\alpha(h, \_) \in C^\infty(L^2(G/\Gamma))$. (The definition of smooth vectors of a unitary representation is given in Section 2.5.)

Definition 2.28 (Infinitesimal generator)**.** The *infinitesimal generator* of a smooth cocycle α is defined to be

$$\omega(\mathcal{V}) = \frac{d}{dt}\alpha(\exp t\mathcal{V})|_{t=0}.$$

It is a real-valued function on the manifold, whose value at some point $x \in X$ is

$$\omega_x(\mathcal{V}) = \frac{d}{dt}\alpha(\exp t\mathcal{V}, x)|_{t=0}.$$

Remark 2.29. As we will presently show, all the properties of a smooth cocycle that we are interested in are reflected in its infinitesimal generator. In fact, we prove our theorems about smooth cocycles by working directly with their infinitesimal generators, and even refer sometimes to the infinitesimal generator of a cocycle as the cocycle itself.

Consider the case where H is abelian. Then ω is a closed 1-form on the H-orbits in X. To see this, take $\mathcal{V}, \mathcal{W} \in \mathfrak{h}$. Since $[\mathcal{V}, \mathcal{W}] = 0$, the Baker–Campbell–Hausdorff formula implies that

$$\begin{aligned}\omega(\mathcal{V}+\mathcal{W}) &= \frac{d}{dt}\alpha(\exp(t\mathcal{V}+t\mathcal{W}))|_{t=0} \\ &= \frac{d}{dt}\alpha(\exp(t\mathcal{V})\exp(t\mathcal{W}))|_{t=0},\end{aligned}$$

which, by the cocycle identity, can be rewritten as

$$\begin{aligned}&= \frac{d}{dt}[\alpha(\exp(t\mathcal{V}))\circ m(\exp(t\mathcal{W}))+\alpha(\exp(t\mathcal{W}))]|_{t=0} \\ &= \frac{d}{dt}[\alpha(\exp(t\mathcal{V}))\circ m(\exp(t\mathcal{W}))]|_{t=0}+\omega(\mathcal{W}) \qquad (2.3)\end{aligned}$$

where we use $m(g) : G/\Gamma \to G/\Gamma$ to denote left multiplication by the group element $g \in G/\Gamma$. For convenience, set $\boldsymbol{\alpha}(t,x) = \alpha(\exp(t\mathcal{V}))(x)$ and $\boldsymbol{m}(t,x) = m(\exp(t\mathcal{W}))(x)$. Now, evaluating the function in expression (2.3) at some point $x_0 \in X$,

$$\begin{aligned}&= \frac{d}{dt}\boldsymbol{\alpha}(t,\boldsymbol{m}(t,x_0))|_{t=0}+\omega_{x_0}(\mathcal{W}) \\ &= \frac{d\boldsymbol{\alpha}}{dt}(0,x_0)+\frac{d\boldsymbol{\alpha}}{dx}(0,x_0)\cdot\frac{d\boldsymbol{m}}{dt}(0,x_0)+\omega_{x_0}(\mathcal{W}) \\ &= \omega_{x_0}(\mathcal{V})+\omega_{x_0}(\mathcal{W})+\frac{d\boldsymbol{\alpha}}{dx}(0,x_0)\cdot\frac{d\boldsymbol{m}}{dt}(0,x_0).\end{aligned}$$

Observing that $\boldsymbol{\alpha}(0,x) = 0$ for all $x \in X$, we see that

$$\omega(\mathcal{V}+\mathcal{W}) = \omega(\mathcal{V})+\omega(\mathcal{W}),$$

concluding our demonstration that ω is a 1-form on the H-orbits of X.

Next, we show that ω is closed. Take local coordinates $\{x_1,\ldots,x_n\}$ in a neighborhood of $x \in G/\Gamma$ such that $\{\mathcal{X}_1(x) = \frac{\partial}{\partial x_1}(x),\ldots,\mathcal{X}_k(x) = \frac{\partial}{\partial x_k}(x)\}$ is a basis for $\mathfrak{h}$. Now we have that

$$\omega_x = \sum_{i=1}^{k}\frac{d}{dt}\alpha(\exp(t\mathcal{X}_i),x)|_{t=0}\,\mathrm{d}x_i,$$

or, denoting $\alpha_i(x) := \frac{d}{dt}\alpha(\exp(t\mathcal{X}_i), x)|_{t=0}$,

$$\omega_x = \sum_{i=1}^{k} \alpha_i(x)\,\mathrm{d}x_i.$$

The exterior derivative is then

$$\mathrm{d}\omega = \sum_{i<j} (\frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j})\,\mathrm{d}x_i \wedge \mathrm{d}x_j.$$

But,

$$\begin{aligned}
(\frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j})(x) &= \frac{d}{ds}\frac{d}{dt}\,[\alpha(\exp(t\mathcal{X}_j), \exp(s\mathcal{X}_i)x) \\
&\quad -\alpha(\exp(t\mathcal{X}_i), \exp(s\mathcal{X}_j)x)]|_{t=s=0} \\
&= \frac{d}{ds}\frac{d}{dt}\,[\alpha(\exp(t\mathcal{X}_j)\exp(s\mathcal{X}_i), x) - \alpha(\exp(t\mathcal{X}_i)\exp(s\mathcal{X}_j), x) \\
&\quad -\alpha(\exp(s\mathcal{X}_i), x) + \alpha(\exp(s\mathcal{X}_j), x)]|_{t=s=0}.
\end{aligned}$$

As the last line has no dependence on t, we are left with

$$\begin{aligned}
&= \frac{d}{ds}\frac{d}{dt}\,[\alpha(\exp(t\mathcal{X}_j)\exp(s\mathcal{X}_i), x) - \alpha(\exp(t\mathcal{X}_i)\exp(s\mathcal{X}_j), x)]|_{t=s=0} \\
&= \frac{d}{ds}\frac{d}{dt}\,[\alpha(\exp(t\mathcal{X}_j + s\mathcal{X}_i), x) - \alpha(\exp(t\mathcal{X}_i + s\mathcal{X}_j), x)]|_{t=s=0} \\
&= 0,
\end{aligned}$$

where, again, we have made use of the fact that X_i and X_j commute. We have just shown that, in the case that H is abelian, the cocycle identity implies that ω is a closed 1-form on the H-orbits in X.

In the context of infinitesimal generators, the cohomology equation becomes $\omega = \eta - \mathrm{d}P$, where P is the transfer function, and η is another smooth cocycle. Therefore, in this context, a cocycle α is cohomologically trivial if its associated 1-form ω is exact. It should also be noted that if the cocycle α is cohomological to a constant cocycle, then that constant cocycle is given by

$$c(h) = \int\limits_{G/\Gamma} \alpha(h, g)\, d\mu,$$

where μ is Haar measure.

Given a closed 1-form on the H-orbit foliation of X, one can recover the cocycle α by $\alpha(\exp \mathcal{V}) = \int_0^1 \omega(\mathcal{V}) \circ m(\exp t\mathcal{V})\, dt$. Thus, the problem of determining which cocycles are cohomologically trivial can be translated to the problem of finding which closed 1-forms on the H-orbits of X are exact. In fact, this point of view is the most useful for our purposes.

2.5 Representation theory

The information in this section can be found in [BdlHV08, Hum78, Kna01].

One of our main tools is representation theory. On one hand, in Chapter VI we use knowledge of the finite dimensional representations of split semisimple Lie algebras to show that functions that are smooth in certain directions on a homogeneous space are actually smooth in all directions. On the other hand, the infinite dimensional unitary representations of semisimple Lie groups pervade our work. For example, the unitary dual of $\mathrm{SL}(2, \mathbb{R})$ is a central object of study in [FF03] and [Mie07], whose results we apply.

2.5.1 Finite dimensional representations of split semisimple Lie algebras

In Section 2.2 we introduced the notion of a split semisimple Lie algebra. These are useful in our work because their finite dimensional representations admit decompositions that make them especially easy to work with. We review the facts.

Definition 2.30 ((Irreducible) representation of a Lie algebra)**.** A *Lie algebra representation* of the Lie algebra $\mathfrak{g}$ on the finite dimensional vector space V is a representation in the usual sense (viewing $\mathfrak{g}$ as an additive group), that respects the Lie bracket. Equivalently, it is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$.

A finite dimensional representation V of $\mathfrak{g}$ is *irreducible* if it has no $\mathfrak{g}$-invariant subspaces other than $\{0\}$ and V.

In the above definition, the Lie bracket of $\mathrm{End}(V)$ is the commutator of matrices. Explicitly, saying that the Lie bracket is preserved means that for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{g}$,

$$\rho([\mathcal{X}, \mathcal{Y}]) = \rho(\mathcal{X})\rho(\mathcal{Y}) - \rho(\mathcal{Y})\rho(\mathcal{X}).$$

Now, let $\mathfrak{g}$ be split semisimple. There is a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that $\mathrm{ad}(\mathcal{T}) \in \mathrm{End}(\mathfrak{g})$ is diagonalizable for all $\mathcal{T} \in \mathfrak{t}$. The fact that these are *simultaneously* diagonalizable means that we can see $\mathfrak{g}$ as a direct sum of simultaneous one-dimensional eigenspaces for the elements of $\mathfrak{t}$. For $\mathcal{V} \in \mathfrak{g}$ in one of these eigenspaces, we have

$$\mathrm{ad}(\mathcal{T})(\mathcal{V}) = [\mathcal{T}, \mathcal{V}] = \alpha(\mathcal{T})\mathcal{V},$$

for all $\mathcal{T} \in \mathfrak{t}$, where $\alpha : \mathfrak{t} \to \mathbb{R}$ is a linear map.

Definition 2.31 (Roots and root spaces)**.** The linear map α described above, and all such linear maps that arise in the same way, are called *roots*. The set of roots is denoted Φ. For a root $\alpha \in \Phi$, a one-dimensional eigenspace $\mathfrak{g}_\alpha$ corresponding to α is called a *root space*.

Now we can express

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Notice that the trivial map $0 : \mathfrak{t} \to \mathbb{R}$ is the simultaneous eigenvalue for $\mathfrak{t}$, so $\mathfrak{g}_0 \neq \{0\}$. In fact, $\mathfrak{g}_0 = \mathfrak{t}$.

Above, we have decomposed $\mathfrak{g}$ with respect to the representation of $\mathfrak{g}$ on itself, acting by ad. A similar decomposition holds for finite dimensional representation V

of $\mathfrak{g}$. The following definition is used to define a natural ordering of the roots that will be helpful in defining this decomposition.

Definition 2.32 (Positive roots)**.** We call a subset $\Phi^+ \subset \Phi$ a set of *positive roots* if:

- For every $\alpha \in \Phi$, exactly one of $\{\alpha, -\alpha\}$ is in Φ^+; and,
- For every pair $\alpha, \beta \in \Phi^+$, we have that $\alpha + \beta \in \Phi^+$ if it is a root.

Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a finite dimensional irreducible representation. Then the $\rho(\mathcal{T}) \in \mathrm{End}(V)$ is diagonalizable for all $\mathcal{T} \in \mathfrak{t}$, simultaneously. That is, we can also see V as a direct sum of eigenspaces V_λ with eigenvalue $\lambda : \mathfrak{t} \to \mathbb{R}$ a linear map.

Definition 2.33 (Weights and weight spaces)**.** The linear map λ as described above is called a *weight* for the representation, and V_λ is called a *weight space.*

Remark 2.34. Roots and root spaces are the weights and weight spaces for the adjoint representation of $\mathfrak{g}$ on itself.

The following simple calculation shows that $\rho(\mathfrak{g}_\alpha)V_\lambda \subset V_{\lambda+\alpha}$. Let $\mathcal{V} \in V_\lambda$ and $\mathcal{X} \in \mathfrak{g}_\alpha$. Then, for any $\mathcal{T} \in \mathfrak{t}$,

$$\begin{aligned} \rho(\mathcal{T})\rho(\mathcal{X})\mathcal{V} &= \rho(\mathcal{X})\rho(\mathcal{T})\mathcal{V} + \rho([\mathcal{T}, \mathcal{X}])\mathcal{V} \\ &= \rho(\mathcal{X})(\lambda(\mathcal{T})\mathcal{V}) + \rho(\alpha(\mathcal{T})\mathcal{X})\mathcal{V} \\ &= (\lambda + \alpha)(\mathcal{T})\rho(\mathcal{X})\mathcal{V}. \end{aligned}$$

Thus, $\rho(\mathcal{X})\mathcal{V} \in V_{\lambda+\alpha}$. In fact, the difference of any two weights is a sum of roots. We can now introduce a partial ordering on the weights by decreeing that $\lambda_1 > \lambda_2$ if $\lambda_1 - \lambda_2$ is a sum of positive roots. Since V is finite dimensional, there are only finitely many weights, and so we can choose a maximal weight λ with respect to the ordering.

Definition 2.35 (Highest weights and highest weight spaces)**.** The maximal weight λ with respect to the ordering defined above is called a *highest weight* and V_λ is a *highest weight space*.

By irreducibility of the representation, V is generated by the highest weight space V_λ as a $\mathfrak{g}$-module. In other words, if $\mathcal{V} \in V_\lambda$, then V is spanned by elements of the form $\rho(\mathcal{X}_1)\rho(\mathcal{X}_2)\ldots\rho(\mathcal{X}_k)\mathcal{V}$ where the $\mathcal{X}_i \in \mathfrak{g}$.

It is a fact that the representation is completely determined by its highest weight.

Now, suppose that V is any finite dimensional representation of $\mathfrak{g}$, not necessarily irreducible. Then it is a direct sum of irreducible representations, each with its own highest weight. We denote the set of highest weights for a finite dimensional representation by Ψ. Then we have

$$V = \bigoplus_{\lambda \in \Psi} V^\lambda$$

where V^λ denotes the irreducible representation of $\mathfrak{g}$ with highest weight $\lambda : \mathfrak{t} \to \mathbb{R}$. As we saw above, each V^λ is in turn the direct sum of weight spaces.

Remark 2.36. A special case that will be of particular importance to us is that of an embedded split semisimple Lie algebra. That is, given a simple Lie algebra $\mathfrak{g}$, any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts on $\mathfrak{g}$ by ad, so we can consider $\mathfrak{g}$ as a representation of $\mathfrak{h}$. If $\mathfrak{h}$ is split semisimple, then we can decompose $\mathfrak{g}$ with respect to weights of a Cartan subalgebra of $\mathfrak{h}$. This point of view will be used in Chapter VI.

2.5.2 Unitary representations and Sobolev spaces

Recall the definition of a Hilbert space.

Definition 2.37 (Hilbert space)**.** A *Hilbert space* is a Banach space $\mathcal{H}$ with an inner product, denoted $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. When there is no risk of confusion, the subscript is omitted.

Remark 2.38. A more accessible description is that a Hilbert space is a complete vector space with an inner product. For example, any finite dimensional vector space is a Hilbert space, by identifying it with $\mathbb{R}^n$ and taking the usual inner product. A better example to keep in mind is $L^2(X, \mu)$, where X is a measure space. Here, the inner product is defined as

$$\langle f, g \rangle = \int_X f\overline{g}\, d\mu$$

for $f, g \in L^2(X, \mu)$. Notice, then, that Hilbert spaces can be infinite dimensional. All of our Hilbert spaces will be complex.

In the finite dimensional case, one is often interested in the group of automorphisms of a vector space that preserve its inner product. For real vector spaces one studies the orthogonal group, and for complex vector spaces one studies what is called the unitary group. We record the definition of the analogous object in the context of Hilbert spaces.

Definition 2.39 (Unitary group)**.** For a Hilbert space $\mathcal{H}$, the *unitary group* $\mathcal{U}(\mathcal{H})$ is the set of all invertible bounded linear operators $U : \mathcal{H} \to \mathcal{H}$ that preserve the inner product in the following sense: for all $f, g \in \mathcal{H}$, $\langle Uf, Ug \rangle = \langle f, g \rangle$.

We are now prepared to give the definition of a unitary representation.

Definition 2.40 ((Irreducible) unitary representation)**.** A *unitary representation* of the group G on a Hilbert space $\mathcal{H}$ is a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ such that the map $G \to \mathcal{H}$ given by $g \mapsto \pi(g)f$ is continuous for all $f \in \mathcal{H}$.

A unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is *irreducible* if there are no closed G-invariant subspaces of $\mathcal{H}$ other than $\{0\}$ and $\mathcal{H}$.

Unitary representations of semisimple Lie groups have been extensively studied. For example, the unitary dual of $\mathrm{SL}(2, \mathbb{R})$ is completely understood (a detailed description of it is available in [Lan75]) and is essential to the results of [Mie07] and [FF03]. Both of these provide theorems which are in turn essential to our work.

Given a Lie group G and a lattice $\Gamma \subset G$, one can define a representation of G on $L^2(G/\Gamma)$, called the left-regular representation. All of the unitary representations we work with are left-regular representations of some semisimple Lie group.

Definition 2.41 (Left-regular representation)**.** Let G be a semisimple Lie group, and $\Gamma \subset G$ a lattice. Define $\pi : G \times L^2(G/\Gamma) \to L^2(G/\Gamma)$ by $\pi(g, f)(x) = f(g^{-1}x)$ for all $g \in G$, $f \in L^2(G/\Gamma)$, and $x \in G/\Gamma$. Then π is a unitary representation and is called the *left-regular representation* of G on $L^2(G/\Gamma)$.

Remark 2.42*.* The fact that the left-regular representation of a semisimple Lie group G is unitary is a consequence of the fact that semisimple Lie groups are unimodular, meaning the Haar measure on G/Γ is both left- and right-invariant.

Definition 2.43 (Smooth vector)**.** Given a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, one says that $v \in \mathcal{H}$ is a *smooth vector* if the map $g \mapsto \pi(g)v$ is smooth in the usual sense. The set of smooth vectors in a representation $\mathcal{H}$ is denoted $C^\infty(\mathcal{H})$.

For the left-regular representation of a semisimple Lie group G on $L^2(G/\Gamma)$, where $\Gamma \subset G$ is a lattice, a smooth vector is a smooth function $f \in L^2(G/\Gamma)$ such that $\mathcal{V}^k f \in L^2(G/\Gamma)$ for all $\mathcal{V} \in \mathrm{Lie}(G)$ and $k \in \mathbb{N}$. In this case, we write $f \in C^\infty(L^2(G/\Gamma))$. If Γ is cocompact, then the smooth vectors are exactly the smooth functions on G/Γ.

It is often useful to consider a less restrictive subspace of the unitary representation $\mathcal{H}$ of G, called the *Sobolev space* of order $s \in \mathbb{Z}_+$, and denoted $W^s(\mathcal{H})$. It is defined as the maximal domain of the operator $(I - \Delta)^{s/2}$, where Δ denotes the Laplacian from G. $W^s(\mathcal{H})$ is a Hilbert space with inner product defined by

$$\langle f, g \rangle_s = \langle (I - \Delta)^s f, g \rangle_{\mathcal{H}}.$$

Sobolev spaces of representations of $\mathrm{SL}(2, \mathbb{R})$ are of particular importance to our work, insofar as it is necessary to consider them in order to apply Theorem 4.2 [FF03].

Finally, we recall the following theorem of Kolmogorov–Mautner [Sta00], which will allow us to restrict our attention to *irreducible* unitary representations for much of our study.

Theorem 2.44 (Kolmogorov–Mautner)**.** *Given any unitary representation π of a locally compact second countable group G in a separable Hilbert space $\mathcal{H}$, there exists a Lebesgue–Stieltjes measure $d\mu$ on $\mathbb{R}$ such that $\mathcal{H}$ is the direct integral $\mathcal{H} = \int_{\mathbb{R}} \mathcal{H}_\mu \, d\mu$ of Hilbert spaces $\mathcal{H}_\mu$ with unitary representations π_μ of the group G on $\mathcal{H}_\mu$, where $\pi(g)f = \int_{\mathbb{R}} \pi_\mu(g) f_\mu \, d\mu$. For $d\mu$-almost all $\mu \in \mathbb{R}$, the representation π_μ is irreducible.*

2.5.3 Some remarks on the unitary dual of $\mathrm{SL}(2, \mathbb{R})$

The irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$ are of great importance in this work, insofar as they are central to the results of Flaminio–Forni [FF03] and Mieczkowski [Mie07], both of which we use. This section summarizes the aspects of the unitary dual of $\mathrm{SL}(2,\mathbb{R})$ that we will need in later chapters, particularly when applying the results in [FF03]. There are many sources for a complete description, such as [Kna01, Lan75, HT92].

The Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ is the set of 2×2 matrices with trace 0. It is denoted by $\mathfrak{sl}(2,\mathbb{R})$. We fix the following generators for $\mathfrak{sl}(2,\mathbb{R})$:

$$\mathcal{X} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}.$$

The Casimir operator is then defined as $\Box = \mathcal{X}^2 + \mathcal{Y}^2 - \Theta^2$. It is in the center of the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$, and thus acts as a multiplicative scalar in each irreducible unitary representation. In fact, the irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$ are classified by this scalar. In accordance with the notation from [FF03] and [Mie07], we use $\mathcal{H}_\mu$ to denote the irreducible unitary representation of $\mathrm{SL}(2,\mathbb{R})$ where $\mu \in \mathbb{R}_+$ and $\Box$ acts by $-\mu$.

We are especially interested in unitary representations $\mathcal{H}$ of $\mathrm{SL}(2,\mathbb{R})$ that have a *spectral gap* for the Casimir operator. By this, we mean that there exists μ_0 such that $0 < \mu_0 < \mu$ for all μ that appear in the direct integral decomposition

$$\mathcal{H} = \int_{\oplus} \mathcal{H}_\mu \, ds(\mu).$$

This condition on unitary representations is what allows us to construct a solution to the cohomology equation in a representation after solving the cohomology equation in all of its irreducible components.

2.6 Regularity theorems

Much of the effort in our work is in establishing differentiability for functions that may only be in $L^2(G/\Gamma)$, where G is a semisimple Lie group and $\Gamma \subset G$ is a lattice. In each case, our efforts will show that, for a particular $P \in L^2(G/\Gamma)$, the derivatives $\mathcal{V}_i^p P$ exist as L^2-functions for a collection of vectors $\mathcal{V}_i \in \mathfrak{g}$, and all $p \in \mathbb{Z}_+$. The hope, then, is that this is enough to conclude that P is actually a smooth function on the manifold G/Γ. The purpose of this section is to discuss some regularity theorems that deal with this kind of scenario.

2.6.1 Sobolev Embedding Theorem

We begin by returning to Sobolev spaces, this time defined over $\mathbb{R}^n$. Our definitions follow the exposition of the same material in [Zim90].

Let

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{Z}_+$ for $i = 1, \ldots, k$, and

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

For an open set $\Omega \subset \mathbb{R}^n$, let

$$C^r(\Omega) = \{f : \Omega \to \mathbb{R} \mid D^\alpha f \text{ exists and is continuous for all } \alpha \text{ with } |\alpha| \leq r\}$$

and

$$BC^r(\Omega) = \{f \in C^r \mid D^\alpha f \text{ is bounded on } \Omega, \text{ for all } |\alpha| \leq r\}.$$

Now $C^\infty(\Omega)$ and $BC^\infty(\Omega)$ are just the intersections of these spaces (respectively) with r going to ∞. Finally, let

$$W^{2,k}(\Omega) = \{f \mid D^\alpha_w f \text{ exists for all } \alpha \text{ with } |\alpha| \leq k \text{ and } D^\alpha_w f \in L^2(\Omega)\},$$

where D^α_w denotes weak differentiation.

We now state the Sobolev Embedding Theorem.

Theorem 2.45. *If $f \in W^{2,k}(\mathbb{R}^n)$ and $k > r + \frac{n}{2}$, then $f \in BC^r(\mathbb{R}^n)$. Furthermore, the inclusion map $W^{2,k}(\mathbb{R}^n) \to BC^r(\mathbb{R}^n)$ is bounded.*

This theorem shows that a function on $\mathbb{R}^n$ that has "enough" weak derivatives that are all L^2-functions is guaranteed to be a bounded continuous function with derivatives in the "usual" sense, up to a certain order. In particular, an L^2-function that has weak L^2-derivatives of all orders is guaranteed to be a smooth bounded function.

2.6.2 Hypoelliptic operators

A differential operator $\mathcal{D}$ on $\mathbb{R}^m$ is called *hypoelliptic* if for any smooth function f, the distributional solutions to $\mathcal{D}P = f$ are smooth. It is a fact that every elliptic operator is hypoelliptic.

Recall that if a function $f : \mathbb{R}^m \to \mathbb{R}$ has the property that $\frac{\partial^n f}{\partial x_i^n}$ exists for all $n \in \mathbb{Z}_+$ and all coordinate directions x_i $(i = 1, \ldots, m)$, then f is smooth. This is a consequence of the fact that $\sum_i \frac{\partial^n f}{\partial x_i^n}$ is an elliptic operator.

The following theorem by Katok and Spatzier generalizes this to functions on a manifold. It states that if a function on a manifold has all derivatives (as continuous or local L^2-functions) in "enough" directions, then the function is guaranteed to be smooth. (Of course, this is made precise in the statement of the theorem.) The proof makes use of results on hypoelliptic operators of L. Rothschild, B. Helffer and F. Nourrigat, and G. Metivier and C. Rockland.

Theorem 2.46 (Katok–Spatzier, [KS94a])**.** *let $D_1, \ldots, D_k$ be C^∞ plane fields on a manifold M such that their sum $\sum_{i=1}^k D_i$ is totally non-integrable and satisfies the following condition: For each j, the dimension of the space spanned by the commutators of length at most j at each point is constant in a neighborhood. Let P be a distribution on M. Assume that for any positive integer p and C^∞ vector field X tangent to any D_j, the p^{th} partial derivative $X^p(P)$ exists as a continuous or local L^2-function. Then P is C^∞ on M.*

Remark. As an application of Theorem 2.46, Katok and Spatzier proved a version of Theorem 4.1 for standard partially hyperbolic actions. (See Sections 2.7 and 4.1 for definitions.) Specifically, they proved that any smooth $\mathbb{R}$-valued cocycle over a standard partially hyperbolic action is smoothly cohomologous to a constant cocycle.

We use Theorem 2.46 in the context of functions on homogeneous spaces: An L^2-function f on a quotient G/Γ of a semisimple Lie group is smooth if there is a set of Lie algebra elements $\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_l\} \subset \mathfrak{g}$ such that $\mathcal{X}_i^p f$ exists (as an L^2-function) for all $i \in \{1, \dots, l\}$ and $p \in \mathbb{Z}_+$, and the elements $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_l$ span $\mathfrak{g}$ as a Lie algebra.

In later chapters, this is our main tool for proving that certain transfer functions are smooth.

2.7 Hyperbolicity

Information on hyperbolic and partially hyperbolic dynamical systems can be found in [KH95, HP06].

2.7.1 Anosov group actions

Definition 2.47 (Anosov diffeomorphism; (un)stable distribution)**.** Let M be a smooth manifold. A diffeomorphism $f : M \to M$ is *Anosov* if there is a df-invariant splitting

$$TM = E^- \oplus E^+$$

and constants $A, B, \Lambda_-, \Lambda_+ \in \mathbb{R}_+$ such that

$$\|df_x^n(\mathcal{V})\| \leq A \cdot e^{-n\Lambda_-} \cdot \|\mathcal{V}\|$$

and

$$\left\|df_x^{-n}(\mathcal{W})\right\| \leq B \cdot e^{-n\Lambda_+} \cdot \|\mathcal{W}\|$$

for all $x \in M$, $\mathcal{V} \in E_x^-$, $\mathcal{W} \in E_x^+$ and $n \in \mathbb{Z}_+$. E^- and E^+ are called the *stable* and *unstable* distributions for f.

Taking iterates, a diffeomorphism yields a group action by $\mathbb{Z}$. Thus, we have just defined what is often referred to as an *Anosov group action* by $\mathbb{Z}$. The corresponding definition for an action by $\mathbb{R}$ only differs from the above definition in that the splitting is now

$$TM = E^- \oplus E^0 \oplus E^+,$$

where E^0 is the tangent distribution to the flow.

The theory of Anosov flows is very developed, and there are many cocycle rigidity results. One such is the Livsic theorem, which states that the obstructions to solving the cohomology equation for an Anosov flow are exactly those coming from the periodic orbits. These obstructions do not usually vanish; however, in the case of Anosov group actions by higher-rank abelian groups ($\mathbb{Z}^k$ or $\mathbb{R}^k$ where $k \geq 2$), there is a result by Katok and Spatzier that guarantees the vanishing of obstructions. (See Theorem 4.1 in Chapter IV, [KS94b].)

2.7.2 Partial hyperbolicity

Often, one is interested in flows that exhibit hyperbolic behavior, but may not be Anosov. For example, a flow may have expanding and contracting directions, but perhaps these directions do not exhaust the tangent space as they do for an Anosov flow. We have the following definition.

Definition 2.48 (Partially hyperbolic diffeomorphism, [HP06])**.** A diffeomorphism $f : M \to M$ is *partially hyperbolic* if for every $x \in M$ there is a splitting

$$T_xM = E_x^- \oplus E_x^c \oplus E_x^+,$$

and there are constants $C > 0$ and

$$0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3$$

with $\mu_1 < 1 < \lambda_3$ such that for every $n \in \mathbb{Z}_+$ we have

- $C^{-1}\lambda_1^n \|\mathcal{V}\| \leq \|df_x^n(\mathcal{V})\| \leq C\mu_1^n \|\mathcal{V}\|$ for $\mathcal{V} \in E_x^-$,

- $C^{-1}\lambda_2^n \|\mathcal{V}\| \leq \|df_x^n(\mathcal{V})\| \leq C\mu_2^n \|\mathcal{V}\|$ for $\mathcal{V} \in E_x^c$,
- $C^{-1}\lambda_3^n \|\mathcal{V}\| \leq \|df_x^n(\mathcal{V})\| \leq C\mu_3^n \|\mathcal{V}\|$ for $\mathcal{V} \in E_x^+$.

E^-, E^c, and E^+ are called the *stable*, *center*, and *unstable* distributions for the flow ϕ_t. The stable and unstable distributions integrate to the stable and unstable foliations, W^- and W^+.

Remark 2.49. This definition is analogous to the definition of an Anosov diffeomorphism. Notice that vectors in the stable distribution contract exponentially as one applies the diffeomorphism, and vectors in the unstable distribution contract exponentially as one applies the inverse of the diffeomorphism. That is, stable vectors contract in "forward time," while unstable vectors contract in "backward time." The expansion and contraction of vectors in the center distribution are dominated by the rates of contraction in the other distributions. In familiar terms, vectors in the center distribution neither expand nor contract too much.

Again, one can make the same definition for flows. In this case, the flow direction is part of the center distribution.

In view of the analogy between our definitions for Anosov flows and partially hyperbolic flows, we will often refer to the constants $A, B, \Lambda_-, \Lambda_+ \in \mathbb{R}_+$ from the definition for Anosov flows when describing the expansion and contraction rates of (un)stable vectors for partially hyperbolic flows. That is, for a partially hyperbolic flow ϕ_t, we will write

$$\|d\phi_t(\mathcal{V})\| \leq A \cdot e^{-t\Lambda_-} \cdot \|\mathcal{V}\|$$

for all $\mathcal{V} \in E^-$ and $t > 0$, and

$$\|d\phi_{-t}(\mathcal{W})\| \leq B \cdot e^{-t\Lambda_+} \cdot \|\mathcal{W}\|$$

for all $\mathcal{W} \in E^+$ and $t > 0$. This is simply convenient notation for some of our calculations.

Our main example of partial hyperbolicity comes from flows on homogeneous spaces. If G is a noncompact semisimple Lie group, and $\Gamma \subset G$ is an irreducible lattice, then the flow $\phi_t^{\mathcal{X}}$ on G/Γ is partially hyperbolic for any semisimple $\mathcal{X} \in \mathfrak{g} := \mathrm{Lie}(G)$ whose roots are not all purely imaginary. The distributions E^- and E^+ are invariant under translation on the right by group elements, and so we can identify them with subspaces of the Lie algebra $\mathfrak{g}$ of right-invariant vector fields on G. We will often make this identification implicitly; that is, we will write

$$\mathfrak{g} = E^- \oplus E^c \oplus E^+,$$

and refer to elements of the distributions $E^{\pm}$ as though they are members of the Lie algebra $\mathfrak{g}$. It should be understood that we are really referring to the elements' images in $\mathfrak{g}$ under this identification by taking right-translates.

CHAPTER III

Problem and results

3.1 Problem statement

Consider a noncompact semisimple Lie group G with finite center, and an irreducible lattice $\Gamma \in G$. Any two linearly independent and commuting elements of the Lie algebra define an abelian Lie subalgebra. That is, if $[\mathcal{X}_1, \mathcal{X}_2] = 0$ for $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{g} := \mathrm{Lie}(G)$, then $\mathbb{R}\mathcal{X}_1 + \mathbb{R}\mathcal{X}_2 \subset \mathfrak{g}$ is a Lie subalgebra isomorphic to $\mathbb{R}^2$ through the identification $r\mathcal{X}_1 + s\mathcal{X}_2 \mapsto (r, s) \in \mathbb{R}^2$. Through the exponential map, this subalgebra defines an action on G/Γ by an abelian group $A \cong \mathbb{R}^2$. Namely, $(r, s)(x) = \exp(r\mathcal{X}_1 + s\mathcal{X}_2)x$ for $x \in G/\Gamma$.

Let $\alpha : A \times G/\Gamma \to \mathbb{R}$ be a smooth cocycle (defined in Section 2.4) over the action $A \curvearrowright G/\Gamma$. The main objective of our work is to determine the conditions under which α is smoothly cohomologous to a constant cocyle.

3.2 Main results

Let $G = G_1 \times \cdots \times G_k$ be a product of noncompact simple Lie groups, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ its Lie algebra, where $\mathfrak{g}_i := \mathrm{Lie}(G_i)$ for $i = 1, \ldots, k$. We prove the following theorems.

Theorem A. *Suppose* $\Gamma \subset G$ *is a cocompact irreducible lattice. Suppose* $\mathcal{U} \in \mathfrak{g}$ *is nilpotent and* $\mathcal{X} \in \mathfrak{g}$ *is semisimple such that* $[\mathcal{U}, \mathcal{X}] = 0$ *and each* $\mathfrak{g}_i$ *contains stable and unstable vectors for the flow* $\phi_t^{\mathcal{X}}$. *Then any smooth* $\mathbb{R}$*-valued cocycle over the action by* $\mathbb{R}^2$ *on* G/Γ *defined by the flows* $\phi_t^{\mathcal{U}}$ *and* $\phi_t^{\mathcal{X}}$ *is cohomologous to a constant cocycle, via a smooth transfer function.*

Theorem B. *Suppose* G *admits an embedding of* $\overline{\mathrm{SL}(2,\mathbb{R})}^{l_1} \times \overline{\mathrm{SL}(2,\mathbb{R})}^{l_2}$, *and* $\Gamma \subset G$ *is an irreducible lattice. Consider* $\mathcal{U}_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \times (0)$ *and* $\mathcal{U}_2 = (0) \times \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{g}$. *If the projection of* $\mathcal{U}_1 + \mathcal{U}_2$ *to* $\mathfrak{g}_i$ *is nonzero for all* $i = 1, \ldots, k$, *then any smooth* $\mathbb{R}$*-valued cocycle over the action by* $\mathbb{R}^2$ *on* G/Γ *defined by the flows* $\phi_t^{\mathcal{U}_1}$ *and* $\phi_t^{\mathcal{U}_2}$ *is cohomologous to a constant cocycle, via a smooth transfer function.*

Remark 3.1. Observe that in Theorem A, we require the lattice to be cocompact, whereas in Theorem B we do not. For Theorem A we use cocompactness in Section 5.7 to show that the transfer functions are smooth. In the proof of Theorem B, we use a different method—one that does not require compactness of the space—to establish the smoothness of transfer functions.

In [Mie07] and [Mie06], Mieczkowski proved Theorems A and B for the case where $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The result was achieved using tools from the unitary representation theory of $\mathrm{SL}(2,\mathbb{R})$. Mieczkowski's results are essential to our work.

It is also worth remarking that Mieczkowski proves that smooth cocycles over the action $\left\{\left(\begin{smallmatrix} 1 & a+bi \\ 0 & 1 \end{smallmatrix}\right)\right\} \curvearrowright \mathrm{SL}(2,\mathbb{C})/\Gamma$ are not always cohomologically constant. This shows that the assumption in Theorem B that there is an embedded $\overline{\mathrm{SL}(2,\mathbb{R})}^{l_1} \times \overline{\mathrm{SL}(2,\mathbb{R})}^{l_2}$ cannot be removed from the statement of the theorem without replacing it with other assumptions.

Given an embedding $\overline{\mathrm{SL}(2,\mathbb{R})}^{l_1} \times \overline{\mathrm{SL}(2,\mathbb{R})}^{l_2} \hookrightarrow G$, one can consider a maximal unipotent subgroup $V \subset G$ containing the unipotent elements of the embedded $\overline{\mathrm{SL}(2,\mathbb{R})}^{l_1} \times \overline{\mathrm{SL}(2,\mathbb{R})}^{l_2}$ obtained by exponentiating $\mathcal{U}_1$ and $\mathcal{U}_2$. Using Theorem B, we prove

Theorem C. *Let G, Γ and $\mathcal{U}_1$, $\mathcal{U}_2$ be as in Theorem B. Let $U \subset G$ be the rank-2 abelian subgroup generated by $\mathcal{U}_1$ and $\mathcal{U}_2$, and let $V \subset G$ be the maximal unipotent subroup containing U. Then a smooth $\mathbb{R}$-valued cocycle over the V-action on G/Γ is cohomologous to a constant cocycle via a smooth transfer function $P \in C^\infty(G/\Gamma)$.*

As an easy application of C, we obtain the following.

Corollary D. *Any smooth $\mathbb{R}$-valued cocycle over the action by the strictly upper triangular group $V \subset \mathrm{SL}(n, \mathbb{R})$ (with $n > 3$) on $\mathrm{SL}(n, \mathbb{R})/\Gamma$ is smoothly cohomologous to a constant cocycle.*

Proof. The result follows by considering the embedding of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ into the first two diagonal 2×2 blocks of $\mathrm{SL}(n, \mathbb{R})$, and applying Theorem C. $\square$

As a corollary to Theorem A, and using similar arguments to those in the proof for Theorem C, we obtain

Corollary E. *If $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ is a cocompact lattice and $n > 2$, then any smooth $\mathbb{R}$-valued cocycle over the action by the upper triangular group $W \subset \mathrm{SL}(n, \mathbb{R})$ on $\mathrm{SL}(n, \mathbb{R})/\Gamma$, with Γ is smoothly cohomologous to a constant cocycle.*

Finally, we have the following applications of Theorems A and B. The first is an extension to certain actions by $\mathbb{R}^n$ on homogeneous spaces.

Theorem F. *Let G be as in the statements for Theorems A and B, with $\Gamma \subset G$ cocompact. Suppose there is a locally free action by diffeomorphisms by $\mathbb{R}^n$, $n \geq 2$, on G/Γ, such that there is a subgroup $\mathbb{R}^2 \subset \mathbb{R}^n$ whose restricted action on G/Γ is one of the actions from either Theorem A or B. Then any smooth $\mathbb{R}^l$-valued cocycle over this action is smoothly cohomologous to a constant cocycle.*

The next theorem is an application of Theorem F to smooth time-changes.

Theorem G. *Any smooth time-change of an $\mathbb{R}^n$-action of the type in Theorem F is smoothly conjugate to the original action, up to an automorphism of the acting group.*

3.3 Restatement of problem in cocompact case

In the cocompact case, there is another point of view that is especially useful to us. As discussed in Section 2.4, instead of working with smooth cocycles, we can work with their infinitesimal generators.

Let $G, \Gamma, \mathcal{X}_1, \mathcal{X}_2, A, \alpha$ be as in the problem statement in Section 3.1, with the additional assumption that Γ is a cocompact lattice. First, recall that if α is cohomologically constant, then that constant is

$$c(a) = \int\limits_{G/\Gamma} \alpha(a, g)\, d\mu,$$

for all $a \in A$. Since we can always take the difference $\alpha(a, g) - c(a)$ as our cocycle, we may assume without loss of generality that

$$0 = \int\limits_{G/\Gamma} \alpha(a, g)\, d\mu,$$

for all $a \in A$.

Now, the infinitesimal generator ω of α is a closed 1-form on the A-orbits of G/Γ. The problem in this context is to determine the conditions under which ω is exact. More precisely, since we are interested in *smooth* cohomology, the problem is to determine when there exists a smooth $P : G/\Gamma \to \mathbb{R}$ satisfying $\mathrm{d}P = \omega$.

We work mostly with the component functions

$$f = \omega(\mathcal{X}_1) \qquad \text{and} \qquad g = \omega(\mathcal{X}_2).$$

The assumptions on α imply that $f, g \in C^\infty(L^2(G/\Gamma))$, and that both f and g

integrate to 0. To see this, compute

$$\begin{aligned}\int\limits_{G/\Gamma} f\,d\mu &= \int\limits_{G/\Gamma} \frac{d}{dt}\alpha(\exp(t\mathcal{X}_1))|_{t=0}\,d\mu \\ &= \int\limits_{G/\Gamma} \lim_{t\to 0}\frac{1}{t}(\alpha(\exp(t\mathcal{X}_1)) - \alpha(1))\,d\mu \\ &= \int\limits_{G/\Gamma} \lim_{t\to 0}\frac{1}{t}(\alpha(\exp(t\mathcal{X}_1)))\,d\mu.\end{aligned}$$

For each fixed t, the expression above is 0, by assumption. Since G/Γ is compact, we know that $\int_{G/\Gamma} f\,d\mu$ exists. Therefore, it must be 0.

For infinitesimal generators, the cocycle identity becomes $\mathrm{d}\omega = 0$. Recalling the calculations in Section 2.4, we have the following expression in local coordinates.

$$\mathrm{d}\omega = (\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_2})\,\mathrm{d}x_1 \wedge \mathrm{d}x_2,$$

where f and g are playing the roles that α_1 and α_2 played in Section 2.4. Now, in terms of f and g, the cocycle identity is

$$\mathcal{X}_2 f = \mathcal{X}_1 g.$$

Finally, if ω is exact, then we have that there exists a smooth function P satisfying $\mathrm{d}P = \omega$. In terms of f and g, this says that there exists a smooth function $P : G/\Gamma \to \mathbb{R}$ satisfying

$$\mathcal{X}_1 P = f \qquad \text{and} \qquad \mathcal{X}_2 P = g.$$

The problem can now be stated in the following way. Given $f, g \in C^\infty(L^2(G/\Gamma))$ with $\mathcal{X}_2 f = \mathcal{X}_1 g$, find the conditions under which there exists a smooth function $P : G/\Gamma \to \mathbb{R}$ such that $\mathcal{X}_1 P = f$ and $\mathcal{X}_2 P = g$.

Theorem A can now be restated as

Theorem A. *Suppose G is a noncompact semisimple Lie group with finite center, and $\Gamma \subset G$ is a cocompact irreducible lattice. Suppose $\mathcal{U} \in \mathfrak{g}$ is nilpotent and $\mathcal{X} \in \mathfrak{g}$ is a semisimple element such that $[\mathcal{U}, \mathcal{X}] = 0$, and assume the Lie algebra of each factor of G contains (un)stable vectors for the flow $\phi_t^{\mathcal{X}}$. Suppose $f, g \in C^\infty(L^2(G/\Gamma))$ satisfy $\mathcal{U}g = \mathcal{X}f$ and both f and g integrate to 0. Then there exists $P \in C^\infty(G/\Gamma)$ such that $\mathcal{U}P = f$ and $\mathcal{X}P = g$.*

Remark 3.2. This is the statement of Theorem A that we prove. One can state a similar version of Theorem B that is weaker than the other statements we have in that it requires cocompactness of the lattice Γ.

CHAPTER IV

Previous work

This chapter contains essential (and relatively recent) results by other authors.

4.1 Work of Katok and Spatzier

Let $A \cong \mathbb{R}^k$ or $\mathbb{Z}^k$, where $k \geq 2$, and let M be a smooth manifold. An action $A \curvearrowright M$ is called *Anosov* if A contains an element that acts partially hyperbolically (see Section 2.7.2) in such a way that E^c is the tangent distribution of the orbits of A. Such an element is called *normally hyperbolic*.

Given Anosov actions, there are basic constructions from which one can obtain other Anosov actions. These were listed by Anatole Katok and Ralf Spatzier in [KS94b] as:

- products of Anosov actions,
- quotients or covers,
- restrictions to subgroups, and
- suspensions.

They then defined classes of examples, collectively called *standard* Anosov actions, that cannot be obtained by applying these constructions to Anosov flows and diffeomorphisms. (We refer the reader to [KS94b] for a more detailed description of these constructions and examples.) They proved

Theorem 4.1 (Katok–Spatzier, [KS94b]). *Consider a standard Anosov A-action on a manifold M where A is isomorphic to $\mathbb{R}^k$ or $\mathbb{Z}^k$ with $k \geq 2$. Then*

- *Any C^∞-cocycle $\beta : A \times M \to \mathbb{R}^l$ is C^∞-cohomologous to a constant cocycle.*
- *Any Hölder cocycle into $\mathbb{R}^l$ is Hölder cohomologous to a constant cocycle.*

Among the standard actions is the action on a compact quotient of a semisimple Lie group by a split Cartan subgroup. For example, one can take a cocompact lattice $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$, and act on $\mathrm{SL}(2, \mathbb{R})/\Gamma$ by the diagonal subgroup. This is a standard Anosov action, therefore, any smooth cocycle into $\mathbb{R}^l$ over this action is smoothly cohomologous to a constant cocycle.

4.1.1 Summary of the methods of proof of Theorem 4.1

The method of proof of Theorem 4.1 is of particular interest, because we use similar ideas in parts of the proof of Theorem A. Suppose A, M, β are as in the statement of the theorem. By taking component functions, we can assume that β takes its values in $\mathbb{R}$. We may also assume β has 0 averages (for the same reasons that are explained in Section 3.3). Fix a normally hyperbolic element $a \in A$. One can easily check that

$$P^+(x) = \sum_{k=0}^{\infty} \beta(a, a^k x) \qquad \text{and} \qquad P^-(x) = -\sum_{k=-\infty}^{-1} \beta(a, a^k x)$$

are formal solutions to the cohomology equation; that is, they satisfy

$$\beta(b, x) = P^\pm(x) - P^\pm(bx)$$

for all $b \in A$ and $x \in M$. The theorem is proved by showing that P^+ and P^- coincide, and that they are smooth functions on M.

The first step toward this is showing that the formal sums P^+ and P^- are actually distributions on M. Given a function $f \in L^2(M)$, one would like to define

$$P^+(f) = \sum_{k=0}^{\infty} \langle \beta(a, a^k x), f \rangle$$

and

$$P^-(f) = -\sum_{k=-\infty}^{-1} \langle \beta(a, a^k x), f \rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $L^2(M)$. Of course, for this to make sense, the sums must converge. In the case of cocycles defined by Hölder functions, Katok and Spatzier use results on the exponential decay of matrix coefficients to establish this convergence. Notice that the summands are matrix coefficients for $\beta(a, \cdot)$ and $f \in L^2(M)$. In fact, decay of matrix coefficients is also used to show that P^+ and P^- coincide, in the Hölder case.

Next, one shows that P^+ and P^- are smooth functions on M. This is done by first showing that they can be differentiated in directions tangent to the stable and unstable foliations for a. These foliations constitute a totally non-integrable system of plane fields on M, therefore, Theorem 2.46 implies that $P^+ = P^-$ is smooth on M.

We briefly discuss here a heuristic for how it is established that P^+ can be differentiated along the stable foliation. The same argument can be used to show that P^- (which is equal to P^+) can be differentiated along the unstable foliation of M. We use these types of methods in Chapter V, so the arguments are phrased more precisely there.

Suppose $x, y \in M$ are points on a common stable submanifold, and consider the difference $P^+(x) - P^+(y)$. It can be rewritten as

$$\begin{aligned} P^+(x) - P^+(y) &= P^+(x) - P^+(a^l x) + P^+(a^l x) - P^+(a^l y) + P^+(a^l y) - P^+(y) \\ &= \beta(a^l, x) - \beta(a^l, y) + P^+(a^l x) - P^+(a^l y) \\ &= \beta(a^l, x) - \beta(a^l, y) + \sum_{k=l}^{\infty} (\beta(a, a^k x) - \beta(a, a^k y)), \end{aligned} \tag{4.1}$$

for any $l \in \mathbb{Z}_+$. Now, since x and y lie on the same stable submanifold, $a^k x$ and $a^k y$ approach each other exponentially with k. This, together with the fact that β is assumed to be smooth, gives us sufficient control over the last term in line (4.1) to conclude that the limit

$$\lim_{y \to x} \frac{P^+(x) - P^+(y)}{\|x - y\|}$$

is continuous. Similar considerations allow one to obtain higher derivatives, also.

4.2 Work of Flaminio and Forni

Let $\mathcal{U} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2, \mathbb{R})$, and let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a lattice. The flow $\phi_t^{\mathcal{U}}$ of $\mathcal{U}$ on $\mathrm{SL}(2, \mathbb{R})/\Gamma$ is called the horocycle flow. It is the action on $\mathrm{SL}(2, \mathbb{R})/\Gamma$ by the one-parameter subgroup of $\mathrm{SL}(2, \mathbb{R})$ defined by $\mathcal{U}$. A smooth cocycle over this action is determined by its infinitesimal generator, which in turn is determined by the value it takes on $\mathcal{U}$. Hence, the cohomology problem here is to determine when a given function $f \in C^\infty(L^2(\mathrm{SL}(2, \mathbb{R})/\Gamma))$ can be realized as $\mathcal{U}P = f$ for some $P \in C^\infty(L^2(\mathrm{SL}(2, \mathbb{R})/\Gamma))$.

Livio Flaminio and Giovanni Forni treated this problem in irreducible unitary representations of $\mathrm{PSL}(2, \mathbb{R})$. They showed that for horocycle flows on quotients of $\mathrm{PSL}(2, \mathbb{R})$, the $\mathcal{U}$-invariant distributions are the only obstructions to solving the cohomology equation for a given Sobolev vector in an irreducible unitary representation of $\mathrm{PSL}(2, \mathbb{R})$. If these obstructions vanish, then there is a solution to the cohomology equation, and it comes with a fixed loss of Sobolev order. Their theorem, which we now state, is applied in Chapter V.

Theorem 4.2 (Flaminio–Forni, [FF03])**.** *Let $s > 1$. If $\mu > \mu_0 > 0$, then there exists a constant $C_{\mu_0,s,t}$ such that for all $f \in W^s(\mathcal{H}_\mu)$,*

- *if $t < -1$, or*
- *if $t < s - 1$ and $D(f) = 0$ for all $D \in \mathcal{I}_{\mathcal{U}}(W^s(\mathcal{H}_\mu))$,*

then the equation $\mathcal{U}P = f$ has a solution $P \in W^t(\mathcal{H}_\mu)$, which satisfies the Sobolev estimate $\|P\|_t \leq C_{\mu_0,s,t}\,\|f\|_s$. Solutions are unique modulo the trivial subrepresentation if $t > 0$.

Though Theorem 4.2 is stated for irreducible unitary representations, it also works in any unitary representation which has a spectral gap for the Casimir operator. That is, if there is a $\mu_0 > 0$ that works for every irreducible component of a unitary representation of $\mathrm{PSL}(2,\mathbb{R})$, then one can apply Theorem 4.2 in each component. In particular, the regular representation of $\mathrm{PSL}(2,\mathbb{R})$ on $L^2(\mathrm{PSL}(2,\mathbb{R})/\Gamma)$ has a spectral gap, so we can solve the cohomology equation for any $f \in C^\infty(L^2(\mathrm{PSL}(2,\mathbb{R})/\Gamma))$ on which all $\mathcal{U}$-invariant distributions vanish.

4.2.1 Preview of how we apply Theorem 4.2

In much of our work, we are interested in copies of $\mathrm{SL}(2,\mathbb{R})$ embedded in other Lie groups. (In fact, we are really interested in embedded copies of finite-sheeted covers, but the representations are the same.) This can be seen in the statement of Theorem B, as well as in the proof of Theorem A, where we use the Jacobson–Morozov Lemma (Theorem 2.17) to guarantee that such embedded copies exist. These embeddings are convenient because we can then look at $L^2(G/\Gamma)$ (where G is the group in question and $\Gamma \subset G$ is a lattice) as a unitary representation of $\mathrm{SL}(2,\mathbb{R})$, where the action is restricted from the regular representation of G on the same space.

The general format of proof in our theorems, especially for Theorem A, is to consider the direct integral decomposition of $L^2(G/\Gamma)$ with respect to the unitary action from our embedded $\mathrm{SL}(2,\mathbb{R})$. In each irreducible component, we show that the obstructions from Theorem 4.2 vanish, so we can apply the theorem to obtain a solution in each irreducible for the cohomology equation for $\mathcal{U} \in \mathfrak{sl}(2,\mathbb{R})$. Since Theorem 4.2 also provides estimates for the Sobolev norms of solutions, we can patch our solutions together across the irreducibles to obtain a solution $P \in C^\infty(L^2(G/\Gamma))$.

Subsequent steps in our proofs consist of first showing that P is actually a smooth function on G/Γ, not just a smooth vector for the representation of $\mathrm{SL}(2,\mathbb{R})$ on $L^2(G/\Gamma)$; and, second, that P solves the cohomology equation for the other Lie algebra element. (Recall that Theorem A refers to a nilpotent $\mathcal{U}$ and semisimple $\mathcal{X}$, while Theorem B refers to two unipotent flows, along $\mathcal{U}_1$ and $\mathcal{U}_2$.)

4.3 Work of Mieczkowski

In [Mie07], David Mieczkowski worked with actions on $(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}))/\Gamma$ and $(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{C}))/\Gamma$ by subgroups analogous to the ones we consider in Theorems A and B . In fact, we apply the following theorem in our proof of Theorem B.

Theorem 4.3 (Mieczkowski, [Mie07])**.** *In a unitary representation $\mathcal{H}$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, if there exists a $\mu_0 > 0$ such that the spectrum of each Casimir satisfies $\sigma(\Box_i) \cap (0,\mu_0) = \emptyset$, then we have the following. Let $f, g \in W^{2s}(\mathcal{H})$, $(s > 1)$, and satisfy the equation $\mathcal{U}_2 f = \mathcal{U}_1 g$. If $t < s-1$, then there exist solutions $P, P' \in W^t(\mathcal{H})$ such that $\mathcal{U}_1 P = f$ and $\mathcal{U}_2 P' = g$. Furthermore, the norms of P, P' must satisfy $\|P\|_t \le C_{\mu_0,s,t}\|f\|_{2s}$, and $\|P'\|_t \le C_{\mu_0,s,t}\|g\|_{2s}$. If $t > 1$, then P and P' must coincide, so that there is a true simultaneous solution.*

Notice that Theorem 4.3 has some of the same features as Theorem 4.2. For example, there is a requirement that the Casimir operators $\Box_1$ and $\Box_2$ from the two factors have a spectral gap. Also, the estimates on the Sobolev norms of solutions come from Theorem 4.2. This is because Mieczkowski obtains solutions by showing that the obstructions coming from Theorem 4.2 vanish in each irreducible component of the representation.

Notice, also, that if one applies Theorem 4.3 to smooth vectors in the left-regular representation of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ on $L^2((\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))/\Gamma)$, then one gets exactly an analog of Theorem B. In fact, our proof of Theorem B works by applying Theorem 4.3 to obtain a solution to the cohomology equation that is smooth in directions tangent to the embedded $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \subset G$. As with Theorem A, the proof then proceeds by showing that this function is smooth on the manifold.

Remark. Theorem B is actually stated for groups that admit embeddings of $\overline{\mathrm{SL}(2,\mathbb{R})}^{l_1} \times \overline{\mathrm{SL}(2,\mathbb{R})}^{l_2}$. This is not a problem, because the unitary representations of a finite-sheeted cover of $\mathrm{SL}(2,\mathbb{R})$ are unitarily equivalent to representations of $\mathrm{SL}(2,\mathbb{R})$.

CHAPTER V

Cocycles over abelian actions generated by a unipotent flow and a semisimple flow

5.1 Strategy for the proof of Theorem A

As in the statement of Theorem A, we let $G = G_1 \times \cdots \times G_k$ be a product of noncompact simple Lie groups with finite center, and $\Gamma \subset G$ a cocompact irreducible lattice. Suppose $\mathcal{U} \in \mathfrak{g}$ is nilpotent and $\mathcal{X} \in \mathfrak{g}$ is a semisimple element such that $[\mathcal{U}, \mathcal{X}] = 0$ and such that there are stable and unstable vectors in each $\mathfrak{g}_i := \mathrm{Lie}(G_i)$. Then the commuting flows $\phi_t^{\mathcal{U}}$ and $\phi_t^{\mathcal{X}}$ of $\mathcal{U}$ and $\mathcal{X}$ on G/Γ form a group action by $\mathbb{R}^2$. Suppose α is a smooth cocycle over this action. Then its infinitesimal generator ω is determined by the smooth functions

$$f = \omega(\mathcal{U}) \quad \text{and} \quad g = \omega(\mathcal{X}),$$

and these satisfy the relation $\mathcal{U}g = \mathcal{X}f$. Now, finding a smooth solution to the cohomology equation for α is equivalent to finding a smooth function $P : G/\Gamma \to \mathbb{R}$ such that

$$\mathcal{U}P = f \quad \text{and} \quad \mathcal{X}P = g.$$

Our strategy is to choose a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ containing $\mathcal{U}$ and $\mathcal{X}$, and consider its corresponding subgroup $H \subset G$. We have the left-regular unitary representation of H on $L^2(G/\Gamma)$, so there is a direct integral decomposition

$$L^2(G/\Gamma) = \int_{\oplus} \mathcal{H}_\nu \, ds(\nu),$$

where ds-almost all $\mathcal{H}_\nu$ are irreducible. Naturally, the corresponding decomposition of $f \in L^2(G/\Gamma)$ is denoted

$$f = \int_{\oplus} f_\nu \, ds(\nu), \quad f_\nu \in \mathcal{H}_\nu.$$

(This decomposition also holds for the Sobolev spaces, $W^s(L^2(G/\Gamma))$.) The idea will be to choose an $\mathfrak{h}$ whose representations are well-enough understood that we can find solutions P_ν to the cohomology equation in each irreducible $\mathcal{H}_\nu$. This, together with estimates on the Sobolev norms of the P_ν, will allow us to glue these solutions together to get a global solution $P \in L^2(G/\Gamma)$.

Next, we must show that the solution P is smooth on G/Γ. For this, we consider the stable and unstable submanifolds of G/Γ with respect to the flow $\phi_t^{\mathcal{X}}$ along the semisimple element $\mathcal{X} \in \mathfrak{g}$. A Livsic type argument will show that P is smooth along these foliations. (It is worth noting that this is the only place where the cocompactness of $\Gamma \subset G$ is used.) Then, since these directions span $\mathfrak{g}$ as a Lie algebra, we can use Theorem 2.46 to show that P is smooth on G/Γ.

The following sections are devoted to proving these claims.

First, we introduce a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ containing $\mathcal{U}$ and $\mathcal{X}$ whose representations will be useful to our treatment of the problem.

5.2 Defining a useful subalgebra $\mathfrak{h}$

By Theorem 2.17, we can find a subalgebra $\mathfrak{h}_1 \in \mathfrak{g}$ such that $\mathfrak{h}_1 \cong \mathfrak{sl}(2,\mathbb{R})$, $\mathcal{U} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R})$, and $[\mathcal{X}, \mathfrak{sl}(2,\mathbb{R})] = 0$.

Now we can consider the subalgebra $\mathfrak{h} := \mathfrak{h}_1 \times \mathfrak{h}_2 = \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}\mathcal{X}$. The subgroup $H \subset G$ corresponding to $\mathfrak{h}$ is a product, $H = H_1 \times H_2$ where $H_1 = \overline{\mathrm{SL}(2,\mathbb{R})}^k$ is a k-sheeted cover of $\mathrm{SL}(2,\mathbb{R})$, and $H_2 = \mathbb{R}_+$. The advantage of this is that the unitary representations of H are easy to work with. Our ultimate goal is to find $P \in C^\infty(G/\Gamma)$ such that $\mathcal{U}P = f$ and $\mathcal{X}P = g$. The first step toward achieving this is to prove the following lemma and apply it to the left-regular representation of H on $L^2(G/\Gamma)$.

Lemma 5.1. *Suppose $\mathcal{H}$ is a unitary representation of $H_1 \times H_2$, and suppose there is a spectral gap for the Casimir operator from H_1. If f, $g \in C^\infty(\mathcal{H})$ satisfy $\mathcal{U}g = \mathcal{X}f$, then there exists $P \in \mathcal{H}$ satisfying $\mathcal{U}P = f$.*

The next few sections will be devoted to proving Lemma 5.1. The full proof is stated in Section 5.6. First, we will summarize some of the details of the representation theory of $\mathfrak{h}$.

5.3 Representations of $\mathfrak{h}$

The subgroup of G corresponding to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is $H = H_1 \times H_2$, where $\mathfrak{h}_i$ is the Lie algebra H_i. Irreducible unitary representations of H are of the form $\mathcal{H}_\mu \otimes \mathcal{H}_\theta$, where $\mathcal{H}_\mu$ is an irreducible unitary representation of H_1 and $\mathcal{H}_\theta$ is an irreducible unitary representation of H_2. (This is a consequence of the semisimplicity of H_1. See Lemma 1 in Chapter 3, Section 3 of [GGPS69], and Theorem 4.5.7.3 in [War72].) The subscripts μ and θ are explained in the following sections.

5.3.1 Representations of $\mathfrak{h}_1 = \mathfrak{sl}(2, \mathbb{R})$

We will be concerned with the irreducible unitarizable representations of $\mathfrak{sl}(2, \mathbb{R})$; that is, those representations that arise as the derivatives of irreducible unitary representations of some Lie group whose Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$ (in our study, this Lie group is being denoted H_1). In fact, all such representations can be realized from irreducible unitary representations of some finite cover of $\mathrm{SL}(2, \mathbb{R})$. In turn, all of these are unitarily equivalent to irreducible representations of $\mathrm{SL}(2, \mathbb{R})$, itself [HT92].

We fix the following generators for $\mathfrak{sl}(2, \mathbb{R})$:

$$\mathcal{X} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}.$$

Then we have the Laplacian operator defined by $\Delta = \mathcal{X}^2 + \mathcal{Y}^2 + \Theta^2$, and the Casimir operator defined by $\Box = \mathcal{X}^2 + \mathcal{Y}^2 - \Theta^2$. The Casimir operator is in the center of the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{R})$, and so it acts as a multiplicative scalar in each irreducible representation. The value of this scalar classifies the irreducible representations of $\mathrm{SL}(2, \mathbb{R})$, so we will denote by $\mathcal{H}_\mu$ the representation where $\Box$ acts by $-\mu$.

Any unitary representation of H_1 decomposes as a direct integral of $\mathcal{H}_\mu$'s. If there exists a μ_0 such that $0 < \mu_0 < \mu$ for all $\mathcal{H}_\mu$ appearing in this decomposition, we say that the unitary representation has a *spectral gap* for the Casimir operator.

5.3.2 Representations of $\mathfrak{h}_2 = \mathbb{R}\mathcal{X}$

Since H_2 is abelian, any element $\exp(t\mathcal{X})$ acts as the multiplicative scalar $e^{it\theta}$ in an irreducible unitary representation $\mathcal{H}_\theta$, for some real θ. For our purposes, the most important feature of $\mathcal{H}_\theta$ is that it is one dimensional. As such, we can pick a smooth vector $v_\theta \in \mathcal{H}_\theta$ of norm 1 as a basis.

5.4 Invariant distributions and vanishing of obstructions

In general, given an irreducible unitary representation of a Lie algebra $\mathfrak{h}$ on a Hilbert space $\mathcal{H}$, one has obstructions to solving the cohomology equation coming from distributions that are invariant under the flow. For example, given $\mathcal{U} \in \mathfrak{h}$ and $w \in C^\infty(\mathcal{H})$, in order to solve the equation $\mathcal{U}v = w$, one must have that $D(w) = 0$ for every $\mathcal{U}$-invariant distribution D. The set of $\mathcal{U}$-invariant distributions on a representation $\mathcal{H}$ is denoted $\mathcal{I}_{\mathcal{U}}(\mathcal{H})$.

Now, we recall our situation, where $\mathfrak{h} = \mathfrak{h}_1 \times \mathfrak{h}_2$, $\mathfrak{h}_1 = \mathfrak{sl}(2, \mathbb{R})$, and $\mathcal{U}$,$\mathcal{X}$ are in $\mathfrak{h}_1$,$\mathfrak{h}_2$, respectively. (See Section 5.2.) Consider an irreducible representation $\mathcal{H}_{\mu,\theta} = \mathcal{H}_\mu \otimes \mathcal{H}_\theta$, and a cocycle given by $f_{\mu,\theta}$, $g_{\mu,\theta} \in C^\infty(\mathcal{H}_{\mu,\theta})$ satisfying $\mathcal{U}g_{\mu,\theta} = \mathcal{X}f_{\mu,\theta}$. We write $f_{\mu,\theta} = f_\mu \otimes v_\theta$ and $g_{\mu,\theta} = g_\mu \otimes v_\theta$ for some f_μ and g_μ in $\mathcal{H}_\mu$, recalling that $v_\theta \in \mathcal{H}_\theta$ is the norm 1 basis discussed in Section 5.3.2

The goal of this section is to show that the obstructions coming from the first factor vanish. More precisely, we show that if D is a $\mathcal{U}$-invariant distribution on $W^s(\mathcal{H}_\mu)$, the Sobolev space of order $s \geq 0$, then $D(f_\mu) = 0$. (It will turn out that this is enough to write down a solution to the cohomology equation in $\mathcal{H}_\mu \otimes \mathcal{H}_\theta$.)

The following lemma was communicated to us by L. Flaminio in a more general form than the one in which we present it; we give a statement and proof that applies specifically to our setup. Keeping the same notation as above, $\mathcal{U} \in \mathfrak{h}_1$, $\mathcal{X} \in \mathfrak{h}_2$, and $\mathcal{H}_\mu \otimes \mathcal{H}_\theta$ is an irreducible representation of H.

Lemma 5.2 (Flaminio)**.** *Let $D \in \mathcal{I}_{\mathcal{U}}(W^s(\mathcal{H}_\mu))$, where $s \geq 0$. Define $\bar{D} : W^s(\mathcal{H}_\mu) \otimes \mathcal{H}_\theta \to \mathcal{H}_\theta$ by*

$$\bar{D} = D \otimes 1.$$

That is, for all $u \in W^s(\mathcal{H}_\mu)$ and $v \in \mathcal{H}_\theta$, $\bar{D}(u \otimes v) = D(u)v$. Suppose that $f, g \in C^\infty(\mathcal{H}_\mu \otimes \mathcal{H}_\theta)$ satisfy $\mathcal{U}g = \mathcal{X}f$. Furthermore, suppose that the equation $\mathcal{X}w = 0$ implies that $w = 0$. Then $\bar{D}(f) = 0$.

Proof. By the $\mathcal{U}$-invariance of D, we see that for any $u \otimes v \in W^s(\mathcal{H}_\mu) \otimes \mathcal{H}_\theta$,

$$\bar{D}(\mathcal{U}(u \otimes v)) = \bar{D}((\mathcal{U}u) \otimes v) = D(\mathcal{U}u)v = 0.$$

Therefore, we have $\bar{D}(\mathcal{U}w) = 0$ for all $w \in W^s(\mathcal{H}_\mu) \otimes \mathcal{H}_\theta$. Now, the diagram

$$\begin{tikzcd} W^s(\mathcal{H}_\mu) \otimes \mathcal{H}_\theta \arrow[r, "\bar{D}"] \arrow[d, "1\otimes\pi_\theta"'] & \mathcal{H}_\theta \arrow[d, "\pi_\theta"] \\ W^s(\mathcal{H}_\mu) \otimes \mathcal{H}_\theta \arrow[r, "\bar{D}"] & \mathcal{H}_\theta \end{tikzcd}$$

commutes, where π_θ denotes the representation of $\mathfrak{h}_2$ on $\mathcal{H}_\theta$, and the vertical arrows correspond to the map obtained by choosing some element of $\mathfrak{h}_2$. So, we have that

$$\mathcal{X}\bar{D}(f) = \bar{D}(\mathcal{X}f) = \bar{D}(\mathcal{U}g) = 0.$$

By the last assumption in the Lemma, this implies that $\bar{D}(f) = 0$. □

In our situation we indeed have that the equation $\mathcal{X}w = 0$ implies $w = 0$. (This follows from ergodicity of the flow of $\mathcal{X}$ on G/Γ.) Therefore, we can apply Lemma 5.2 to see that $\bar{D}(f_{\mu,\theta}) = 0$ for any $D \in \mathcal{I}_\mathcal{U}(W^s(\mathcal{H}_\mu))$. But,

$$\begin{aligned} \bar{D}(f_{\mu,\theta}) &= \bar{D}(f_\mu \otimes v_\theta) \\ &= D(f_\mu)v_\theta \\ &= 0, \end{aligned}$$

therefore,

$$D(f_\mu) = 0.$$

In the next section, we will use this to write down a solution $P_{\mu,\theta} \in \mathcal{H}_\mu \otimes \mathcal{H}_\theta$.

5.5 Solutions in irreducible representations of $\mathfrak{h}$

In Section 5.4, we saw that for any $D \in \mathcal{I}_{\mathcal{U}}(W^s(\mathcal{H}_\mu))$, $D(f_\mu) = 0$. Therefore, we can apply Theorem 4.2 to obtain a solution $P_\mu \in \mathcal{H}_\mu$ to the equation $\mathcal{U}P_\mu = f_\mu$, satisfying the estimate $\|P_\mu\| \leq C_{\mu_0,1+\epsilon,0}\,\|f_\mu\|_{1+\epsilon}$, where $0 < \mu_0 < \mu$ and $\epsilon > 0$. Set $P_{\mu,\theta} = P_\mu \otimes v_\theta$.

Lemma 5.3. *$P_{\mu,\theta}$ as defined above is a solution to $\mathcal{U}P_{\mu,\theta} = f_{\mu,\theta}$ in the irreducible representation $\mathcal{H}_\mu \otimes \mathcal{H}_\theta$, and it satisfies the estimate*

$$\|P_{\mu,\theta}\| \leq C_{\mu_0,1+\epsilon,0}\,\|f_{\mu,\theta}\|_{1+\epsilon}$$

for any $\epsilon > 0$.

Proof. The first assertion follows from

$$\begin{aligned}
\mathcal{U}P_{\mu,\theta} &= \mathcal{U}(P_\mu \otimes v_\theta) \\
&= (\mathcal{U}P_\mu) \otimes v_\theta \\
&= f_\mu \otimes v_\theta \\
&= f_{\mu,\theta}.
\end{aligned}$$

The norm estimate comes from combining

$$\|P_\mu\| = \|P_{\mu,\theta}\|$$

and

$$\|f_\mu\|_{1+\epsilon} \leq \|f_{\mu,\theta}\|_{1+\epsilon}$$

with the estimate on P_μ obtained from applying Theorem 4.2. $\square$

5.6 Global solution

We are now prepared to build a global solution P in any unitary representation $\mathcal{H}$ of $H_1 \times H_2$ that has a spectral gap for the Casimir operator from H_1. That is, we can prove Lemma 5.1.

Proof of Lemma 5.1. We have the decomposition

$$\mathcal{H} = \int_{\oplus} \mathcal{H}_\mu \otimes \mathcal{H}_\theta \, ds(\mu, \theta)$$

where each irreducible $\mathcal{H}_\mu \otimes \mathcal{H}_\theta$ appears with some multiplicity $m(\mu, \theta)$. We then decompose f and g as

$$f = \int_{\oplus} f_{\mu,\theta} \, ds(\mu, \theta), \quad f_{\mu,\theta} \in \mathcal{H}_\mu \otimes \mathcal{H}_\theta$$

and

$$g = \int_{\oplus} g_{\mu,\theta} \, ds(\mu, \theta), \quad g_{\mu,\theta} \in \mathcal{H}_\mu \otimes \mathcal{H}_\theta.$$

From Lemma 5.3, we have solutions $P_{\mu,\theta}$ in each irreducible representation $\mathcal{H}_\mu \otimes \mathcal{H}_\theta$. Set

$$P = \int_{\oplus} P_{\mu,\theta} \, ds(\mu, \theta).$$

Then it is clear that $\mathcal{U}P = f$, formally.

To see that $P \in \mathcal{H}$, we use the estimates on the norms of the $P_{\mu,\theta}$. We have

$$\begin{aligned} \|P\|^2 &= \int_{\oplus} \|P_{\mu,\theta}\|^2 \, ds(\mu, \theta) \\ &\leq \int_{\oplus} C_{\mu_0, 1+\epsilon, 0} \, \|f_{\mu,\theta}\|^2_{1+\epsilon} \, ds(\mu, \theta) \\ &= C_{\mu_0, 1+\epsilon, 0} \, \|f\|^2_{1+\epsilon} \end{aligned}$$

where $0 < \mu_0 < \mu$ for all μ that appear in the decomposition of $\mathcal{H}$, and $\epsilon > 0$. This proves that $P \in \mathcal{H}$. $\square$

The restriction of $L^2(G/\Gamma)$ to H_1 has a spectral gap. This follows from work of D. Kleinbock and G. Margulis in [KM99] which, when combined with a theorem of Y. Shalom in [Sha00], yields the following theorem, quoted from [Mie07].

Theorem 5.4. *Let $G = G_1 \times \cdots \times G_k$ be a product of noncompact simple Lie groups, $\Gamma \subset G$ an irreducible lattice, and $H \subset G$ a non-amenable closed subgroup. Then the restriction of $L^2(G/\Gamma)$ to H has a spectral gap.*

Therefore, we can apply Lemma 5.1 to obtain a solution $P \in L^2(G/\Gamma)$ to the equation $\mathcal{U}P = f$. Now, for a fixed $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{U}(P(\phi_t^{\mathcal{X}}x) - P(x)) &= \frac{d}{ds}[P(\phi_t^{\mathcal{X}}\phi_s^{\mathcal{U}}x) - P(\phi_s^{\mathcal{U}}x)]_{s=o} \\ &= \frac{d}{ds}[P(\phi_s^{\mathcal{U}}\phi_t^{\mathcal{X}}x)]_{s=0} - \frac{d}{ds}[P(\phi_s^{\mathcal{U}}x)]_{s=o},\end{aligned}$$

since the flows of $\mathcal{X}$ and $\mathcal{U}$ commute. Then, because $\mathcal{U}P = f$,

$$\begin{aligned} &= f(\phi_t^{\mathcal{X}}x) - f(x) \\ &= \int_0^t \frac{d}{d\tau}[f(\phi_\tau^{\mathcal{X}}x)]\,d\tau \\ &= \int_0^t \mathcal{X}f(\phi_\tau^{\mathcal{X}}x)\,d\tau,\end{aligned}$$

which, by the identity $\mathcal{U}g = \mathcal{X}f$,

$$\begin{aligned} &= \int_0^t \mathcal{U}g(\phi_\tau^{\mathcal{X}}x)\,d\tau \\ &= \mathcal{U}(\int_0^t g(\phi_\tau^{\mathcal{X}}x)\,d\tau).\end{aligned}$$

Since the flow of $\mathcal{U}$ on G/Γ is ergodic, this implies that

$$P(\phi_t^{\mathcal{X}} x) - P(x) = \int_0^t g(\phi_\tau^{\mathcal{X}} x)\, d\tau.$$

One sees that the right hand side is differentiable in t. Differentiating, we obtain $\mathcal{X}P = g$. Thus, P simultaneously solves $\mathcal{U}P = f$ and $\mathcal{X}P = g$.

Our task in the next section is to show that P is smooth.

5.7 Smoothness of global solution; proof of Theorem A

By the assumption on $\mathcal{X} \in \mathfrak{g}$, we have a splitting of the tangent bundle of G/Γ,

$$T(G/\Gamma) = E^- \oplus E^c \oplus E^+,$$

where E^- and E^+ are the stable and unstable distributions with respect to the flow of $\mathcal{X}$; that is, there exist constants $A, B, \Lambda_-, \Lambda_+ \in \mathbb{R}_+$ such that

$$\left\| d\phi_t^{\mathcal{X}}(\mathcal{V}) \right\| \leq A \cdot e^{-t\Lambda_-} \cdot \|\mathcal{V}\|$$

for all $\mathcal{V} \in E^-$ and $t > 0$, and

$$\left\| d\phi_{-t}^{\mathcal{X}}(\mathcal{W}) \right\| \leq B \cdot e^{-t\Lambda_+} \cdot \|\mathcal{W}\|$$

for all $\mathcal{W} \in E^+$ and $t > 0$. Furthermore, we have assumed that the intersections $E^- \cap \mathfrak{g}_i$ and $E^+ \cap \mathfrak{g}_i$ are nontrivial for all $i = 1, \ldots, k$. The distributions E^- and E^+ integrate to the stable and unstable foliations for the flow $\phi_t^{\mathcal{X}}$ on G/Γ, denoted W^- and W^+, respectively. For $y \in W^-(x)$ and $z \in W^+(x)$, we have

$$\mathrm{dist}(\exp(t\mathcal{X})x, \exp(t\mathcal{X})y) \leq A \cdot e^{-t\Lambda_-} \cdot \mathrm{dist}(x, y),$$

and

$$\mathrm{dist}(\exp(-t\mathcal{X})x, \exp(-t\mathcal{X})z) \leq B \cdot e^{-t\Lambda_+} \cdot \mathrm{dist}(x, z),$$

for all $t > 0$.

We will begin our proof that the solution $P \in L^2(G/\Gamma)$ is smooth by examining how P behaves along leaves of the foliations W^- and W^+. The following lemma will establish that P satisfies a Lipschitz continuity condition locally on these leaves.

Lemma 5.5. *For almost every $x \in G/\Gamma$, there is a neighborhood $V_x \subset W^-(x)$ containing x such that for almost every $y \in V_x$, the following holds:*

$$|P(x) - P(y)| \leq K_- \cdot \mathrm{dist}(x, y),$$

where $K_- > 0$ is a constant. Similarly, there is a neighborhood $V'_x \subset W^+(x)$ such that for almost every $y \in V'_x$, the following holds:

$$|P(x) - P(y)| \leq K_+ \cdot \mathrm{dist}(x, y),$$

where $K_+ > 0$ is a constant.

Proof. To begin, note that for any $x, y \in G/\Gamma$,

$$|P(y) - P(x)| =$$

$$|P(y) - P(\exp(t\mathcal{X})y) \tag{5.1}$$

$$+ \quad P(\exp(t\mathcal{X})y) - P(\exp(t\mathcal{X})x) \tag{5.2}$$

$$+ \quad P(\exp(t\mathcal{X})x) - P(x)|. \tag{5.3}$$

Combining lines (5.1) and (5.3), we have

$$
\begin{aligned}
|P(y) - P(x)| &= \Big| \int_0^t (\mathcal{X}P(\exp(\tau\mathcal{X})x) - \mathcal{X}P(\exp(\tau\mathcal{X})y))\, d\tau \\
&+ P(\exp(t\mathcal{X})y) - P(\exp(t\mathcal{X})x)\Big|. \\
&= \Big| \int_0^t (g(\exp(\tau\mathcal{X})x) - g(\exp(\tau\mathcal{X})y))\, d\tau \\
&+ P(\exp(t\mathcal{X})y) - P(\exp(t\mathcal{X})x)\Big|.
\end{aligned}
$$

We will show that for almost every $x \in G/\Gamma$ and almost every y in some neighborhood $V_x \subset W^-(x)$ containing x, there is an increasing divergent sequence $\{t_k\}$ such that

$$|P(\exp(t_k\mathcal{X})y) - P(\exp(t_k\mathcal{X})x)| \longrightarrow 0.$$

We begin by noting that, since $\mathcal{X}P = g$ is smooth and G/Γ is compact, g is Lipschitz continuous on G/Γ. That is, for all $x, y \in G/\Gamma$, we have

$$|g(x) - g(y)| \leq C \cdot \mathrm{dist}(x, y)$$

for some $C > 0$.

We cover G/Γ by a collection of coordinate charts of the form $U \times V$, where $\{z\} \times V$ is a neighborhood of a stable leaf of W^- for every $z \in U$. Since the foliation is absolutely continuous, this can be done in such a way that Fubini's theorem holds in each of these charts, with respect to Lebesgue measures on U and V.

Let $E \subset G/\Gamma$ be a Lusin set for P of measure 0.99. Then for almost every $x \in G/\Gamma$,

$$\frac{1}{T} \int_0^T \chi_E(\exp(t\mathcal{X})x)\, dt \longrightarrow 0.99,$$

as $T \to \infty$, where χ_E is the characteristic function for E. Suppose $U_x \times V_x$ is a coordinate chart containing x. By Fubini's Theorem, we also have that for almost every $x \in G/\Gamma$, and almost every $y \in \{p_1(x)\} \times V_x$,

$$\frac{1}{T}\int_0^T \chi_E(\exp(t\mathcal{X})y)\,dt \longrightarrow 0.99.$$

(Here, $p_1 : U_x \times V_x \to U_x$ is projection onto the first coordinate.) For such x and y, there is an increasing divergent sequence $\{t_k\} \subset \mathbb{R}_+$ such that $\exp(t_k\mathcal{X})x$ and $\exp(t_k\mathcal{X})y$ are in the Lusin set E for all k. Thus, for almost every $x \in G/\Gamma$ and almost every $y \in \{p_1(x)\} \times V_x$,

$$|P(\exp(t_k\mathcal{X})y) - P(\exp(t_k\mathcal{X})x)| \longrightarrow 0.$$

Now, for these $x \in G/\Gamma$ and $y \in \{p_1(x)\} \times V_x$,

$$\begin{aligned}
|P(y) - P(x)| &= \left|\int_0^\infty (g(\exp(\tau\mathcal{X})x) - g(\exp(\tau\mathcal{X})y))\,d\tau\right| \\
&\leq \int_0^\infty |(g(\exp(\tau\mathcal{X})x) - g(\exp(\tau\mathcal{X})y))|\,d\tau \\
&\leq \int_0^\infty C \cdot \mathrm{dist}(\exp(\tau\mathcal{X})y, \exp(\tau\mathcal{X})x)\,d\tau \\
&\leq \int_0^\infty C \cdot A \cdot \mathrm{dist}(y, x) \cdot e^{-\tau\Lambda_-}\,d\tau \\
&= \frac{C \cdot A}{\Lambda_-} \cdot \mathrm{dist}(x, y).
\end{aligned}$$

This is the desired local Lipschitz condition along stable leaves for the flow of $\mathcal{X}$, with $K_- = \frac{C \cdot A}{\Lambda_-}$.

The preceding argument holds *mutatis mutandis* for the unstable foliation, W^+.

□

We use this Lipschitz condition in the following lemma, which establishes that P can be differentiated in stable and unstable directions.

Lemma 5.6. *Suppose $P \in L^2(G/\Gamma)$ satisfies $\mathcal{X}P = g$, where $\mathcal{X} \in \mathfrak{g}$ is semisimple and $g \in C^\infty(L^2(G/\Gamma))$. Let $\mathcal{V}$ be a stable or unstable vector for the flow of $\mathcal{X}$. Then $\mathcal{V}^k P \in L^2(G/\Gamma)$ for all $k \in \mathbb{N}$.*

Proof. Without loss of generality, we assume $\mathcal{V}$ is a stable unit vector for $\mathcal{X}$; that is, $\mathcal{V} \in E^-$ and $\|\mathcal{V}\| = 1$. The following argument can be carried out for unstable vectors by considering negative time.

We now compute

$$
\begin{aligned}
\mathcal{V}P(x) &= \lim_{s\to 0} \frac{P(\exp(s\mathcal{V})x) - P(x)}{s} \\
&= \lim_{s\to 0} \frac{1}{s}(P(\exp(s\mathcal{V})x) - P(\exp(t\mathcal{X})\exp(s\mathcal{V})x)) && (5.4) \\
&+ \lim_{s\to 0} \frac{1}{s}(P(\exp(t\mathcal{X})\exp(s\mathcal{V})x) - P(\exp(t\mathcal{X})x)) && (5.5) \\
&+ \lim_{s\to 0} \frac{1}{s}(P(\exp(t\mathcal{X})x) - P(x)) && (5.6)
\end{aligned}
$$

where $t \in \mathbb{R}_+$. Combining lines (5.4) and (5.6), we have

$$
\begin{aligned}
\mathcal{V}P(x) &= \lim_{s\to 0} \frac{1}{s} \int_0^t (\mathcal{X}P(\exp(\tau\mathcal{X})x) - \mathcal{X}P(\exp(\tau\mathcal{X})\exp(s\mathcal{V})x))\, d\tau \\
&+ \lim_{s\to 0} \frac{1}{s}(P(\exp(t\mathcal{X})\exp(s\mathcal{V})x) - P(\exp(t\mathcal{X})x)).
\end{aligned}
$$

Setting $g_\tau(x) := g(\exp(\tau\mathcal{X})x)$,

$$\begin{aligned}
\mathcal{V}P(x) &= \lim_{s\to 0}\frac{1}{s}\int_0^t \left(g_\tau(x) - g_\tau(\exp(s\mathcal{V})x)\right)d\tau \\
&+ \lim_{s\to 0}\frac{1}{s}(P(\exp(t\mathcal{X})\exp(s\mathcal{V})x) - P(\exp(t\mathcal{X})x)) \\
&= -\lim_{s\to 0}\frac{1}{s}\int_0^t\int_0^s \mathcal{V}g_\tau(\exp(\sigma\mathcal{V})x)\,d\sigma\,d\tau \\
&+ \lim_{s\to 0}\frac{1}{s}(P(\exp(t\mathcal{X})\exp(s\mathcal{V})x) - P(\exp(t\mathcal{X})x)) \\
&= -\int_0^t \lim_{s\to 0}\frac{1}{s}\int_0^s \mathcal{V}g_\tau(\exp(\sigma\mathcal{V})x)\,d\sigma\,d\tau \\
&+ \lim_{s\to 0}\frac{1}{s}(P(\exp(t\mathcal{X})\exp(s\mathcal{V})x) - P(\exp(t\mathcal{X})x)) \\
&= -\int_0^t \mathcal{V}g_\tau(x)\,d\tau \\
&+ \lim_{s\to 0}\frac{1}{s}(P(\exp(t\mathcal{X})\exp(s\mathcal{V})x) - P(\exp(t\mathcal{X})x)).
\end{aligned}$$

Since this expression is constant in t, we can take a limit,

$$\begin{aligned}
\mathcal{V}P(x) &= -\lim_{t\to\infty}\int_0^t \mathcal{V}g_\tau(x)\,d\tau \\
&+ \lim_{t\to\infty}\lim_{s\to 0}\frac{1}{s}(P(\exp(t\mathcal{X})\exp(s\mathcal{V})x) - P(\exp(t\mathcal{X})x)). \qquad (5.7)
\end{aligned}$$

By Lemma 5.5, we have control over line (5.7) for almost every x in the following way:

$$\begin{aligned}
&\left|\lim_{t\to\infty}\lim_{s\to 0}\tfrac{1}{s}(P(\exp(t\mathcal{X})\exp(s\mathcal{V})x) - P(\exp(t\mathcal{X})x))\right| \\
\leq\ & \lim_{t\to\infty}\lim_{s\to 0}\frac{K_-\cdot A\cdot e^{-t\Lambda_-}}{s}\cdot \operatorname{dist}(\exp(s\mathcal{V})x, x) \\
\leq\ & \lim_{t\to\infty}\lim_{s\to 0}\frac{K_-\cdot A\cdot e^{-t\Lambda_-}}{s}\cdot s \\
=\ & 0.
\end{aligned}$$

So we are left with

$$\mathcal{V}P(x) = -\int_0^\infty \mathcal{V}g_\tau(x)\, d\tau. \tag{5.8}$$

The following calculations will show that (5.8) defines an L^2-function on G/Γ. Since $\mathcal{V} \in E^-$,

$$\left| \int_0^t \mathcal{V}g_\tau(x)\, d\tau \right| \;\leq\; \int_0^t A \cdot e^{-\tau\Lambda_-} \cdot |\mathcal{V}g(\exp(\tau\mathcal{X})x)|\; d\tau.$$

We define the functions

$$h_t(x) = \int_0^t A \cdot e^{-\tau\Lambda_-} \cdot |\mathcal{V}g(\exp(\tau\mathcal{X})x)|\; d\tau$$

and

$$H_t(x) = -\int_0^t \mathcal{V}g_\tau(x)\, d\tau$$

for $t \in \mathbb{R}_+$. Then we have that $|H_n(x)| \leq h_n(x)$ for all $n \in \mathbb{N}$. Denoting Haar measure on G/Γ by μ, we have

$$\begin{aligned}
\|h_t\|_{L^2}^2 &= \int_{G/\Gamma} \left| \int_0^t A \cdot e^{-\tau\Lambda_-} \cdot \mathcal{V}g(\exp(\tau\mathcal{X})x)\, d\tau \right|^2 d\mu \\
&\leq \int_{G/\Gamma} \int_0^t \left| A \cdot e^{-\tau\Lambda_-} \cdot \mathcal{V}g(\exp(\tau\mathcal{X})x)\right|^2 d\tau\, d\mu \\
&= \int_0^t \int_{G/\Gamma} A^2 \cdot e^{-2\tau\Lambda_-} \cdot |\mathcal{V}g(\exp(\tau\mathcal{X})x)|^2\; d\mu\, d\tau \\
&= \int_0^t A^2 \cdot e^{-2\tau\Lambda_-} \cdot \|\mathcal{V}g\|_{L^2}^2\; d\tau.
\end{aligned}$$

It is easy to see that the sequence $\{h_n\}_{n\in\mathbb{N}} \subset L^2(G/\Gamma)$ is Cauchy, so converges in $L^2(G/\Gamma)$. Now, the sequence $\{H_n\}$ is dominated by $\{h_n\}$, therefore, by the Dominated Convergence Theorem, $\mathcal{V}P \in L^2(G/\Gamma)$.

We now show that $\mathcal{V}^2 P(x) \in L^2(G/\Gamma)$. It will be apparent that one can apply $\mathcal{V}$ successively with the same procedure. First, we apply $\mathcal{V}$ to expression (5.8) to yield

$$\begin{aligned}
\left|\mathcal{V}^2 P(x)\right| &= \left| -\lim_{s\to 0}\frac{1}{s}\int_0^\infty \left(\mathcal{V}g_\tau(\exp(s\mathcal{V})x) - \mathcal{V}g_\tau(x)\right) d\tau \right| \\
&\le \lim_{s\to 0}\int_0^\infty \frac{1}{s}\left|\mathcal{V}g_\tau(\exp(s\mathcal{V})x) - \mathcal{V}g_\tau(x)\right| d\tau \\
&\le \lim_{s\to 0}\int_0^\infty \frac{1}{s}\cdot A\cdot e^{-\tau\Lambda_-}\left|\mathcal{V}g(\exp(s\mathcal{V})\exp(\tau\mathcal{X})x) - \mathcal{V}g(\exp(\tau\mathcal{X})x)\right| d\tau
\end{aligned}$$

Since $\mathcal{V}g$ is smooth on G/Γ, we have that

$$\frac{1}{s}\left|\mathcal{V}g(\exp(s\mathcal{V})\exp(\tau\mathcal{X})x) - \mathcal{V}g(\exp(\tau\mathcal{X})x)\right| \le M$$

for all $s > 0$, and some $M > 0$. Therefore, the integrand is dominated by $M(\tau) = M\cdot A\cdot e^{-\tau\Lambda_-}$. Thus, by the Dominated Convergence Theorem, we can bring the limit inside to see that $\mathcal{V}^2 P \in L^2(G/\Gamma)$. Furthermore, one can repeat this procedure, applying $\mathcal{V}$ to (5.8), to see that $\mathcal{V}^k P \in L^2(G/\Gamma)$ for all k. $\square$

We will use the following lemma to show that the stable and unstable directions span $\mathfrak{g}$ as a Lie algebra, that is, by taking successive brackets. By Theorem 2.46, this will imply that P is smooth on G/Γ.

Lemma 5.7. *Suppose $\mathfrak{g}$ is a simple Lie algebra, and $\mathcal{X} \in \mathfrak{g}$ is a semisimple element with nonzero stable and unstable vectors in $\mathfrak{g}$. Consider the splitting*

$$\mathfrak{g} = E^- \oplus E^c \oplus E^+$$

into stable and unstable directions. Let $\mathfrak{L} \subset \mathfrak{g}$ be the subalgebra generated by E^- and E^+. Then $\mathfrak{L} = \mathfrak{g}$.

Proof. We will show that $\mathfrak{L} \subset \mathfrak{g}$ is an ideal. Note that every element of $\mathfrak{L}$ is a sum of elements of the form

$$\mathcal{V} = [\mathcal{V}_1, [\mathcal{V}_2, [\mathcal{V}_3, \cdots, [\mathcal{V}_{k-1}, \mathcal{V}_k] \cdots]]]$$

where $\mathcal{V}_i$ is either in E^- or E^+. Suppose $\mathcal{W} \in E^c$. By repeatedly applying the Jacobi identity, we can express $[\mathcal{V}, \mathcal{W}]$ as a sum of terms of the form

$$\mathcal{W}_\sigma = \pm[\mathcal{V}_{\sigma(1)}, [\mathcal{V}_{\sigma(2)}, [\mathcal{V}_{\sigma(3)}, \cdots, [\mathcal{V}_{\sigma(k)}, \mathcal{W}] \cdots]]]$$

where σ is a permutation on the set $\{1, 2, 3, \ldots, k\}$. It is easy to see that if $\mathcal{V}_{\sigma(k)}$ is stable, then so is $[\mathcal{V}_{\sigma(k)}, \mathcal{W}]$; similarly, if $\mathcal{V}_{\sigma(k)}$ is unstable, then so is $[\mathcal{V}_{\sigma(k)}, \mathcal{W}]$. Therefore, $\mathcal{W}_\sigma \in \mathfrak{L}$ and $[\mathcal{V}, \mathcal{W}] \in \mathfrak{L}$. This proves that $\mathfrak{L}$ is an ideal in $\mathfrak{g}$. $\mathfrak{L}$ contains nonzero elements, therefore, $\mathfrak{L} = \mathfrak{g}$. □

We are now ready to state the proof of the first main theorem.

Proof of Theorem A. We have a semisimple Lie group G with finite center, $\Gamma \subset G$ a lattice, $\mathcal{U} \in \mathfrak{g}$ nilpotent , and $\mathcal{X} \in \mathfrak{g}$ semisimple and commuting with $\mathcal{U}$, such that the flow $\phi_t^{\mathcal{X}}$ has stable and unstable directions in the Lie algebra of each factor of G. We have $f, g \in C^\infty(L^2(G/\Gamma))$ satisfying $\mathcal{U}g = \mathcal{X}f$, and $\int_{G/\Gamma} f = \int_{G/\Gamma} g = 0$.

By the Jacobson–Morozov Lemma (Theorem 2.17), we can find the subalgebra $\mathfrak{h} := \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}\mathcal{X} \subset \mathfrak{g}$ such that $\mathcal{U} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \times (0) \in \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}\mathcal{X}$. The corresponding subgroup of $\mathfrak{h}$ is $H = H_1 \times H_2 \subset G$.

The left-regular unitary representation of H on $L^2(G/\Gamma)$ decomposes as

$$L^2(G/\Gamma) = \int_\oplus \mathcal{H}_\mu \otimes \mathcal{H}_\theta \, ds(\mu, \theta),$$

where ds-almost every $\mathcal{H}_\mu \times \mathcal{H}_\theta$ is irreducible, so we restrict our attention to an irreducible $\mathcal{H}_\mu \otimes \mathcal{H}_\theta$. By Lemma 5.2, the obstructions to solving $\mathcal{U}P = f_{\mu,\theta}$ coming from $\mathcal{U}$-invariant distributions vanish in each irreducible $\mathcal{H}_\mu \otimes \mathcal{H}_\theta$. With this, we apply Theorem 4.2 to find a solution $P_\mu \in \mathcal{H}_\mu$. By Lemma 5.3, $P_{\mu,\theta} = P_\mu \otimes v_\theta$ is a solution to $\mathcal{U}P_{\mu,\theta} = f_{\mu,\theta}$ in $\mathcal{H}_\mu \otimes \mathcal{H}_\theta$, and it satisfies the estimate

$$\|P_{\mu,\theta}\| \leq C_{\mu_0,1+\epsilon,0} \|f_{\mu,\theta}\|_{1+\epsilon},$$

where $0 < \mu_0 < \mu$.

Now, Theorem 5.4 guarantees that the regular representation of H on $L^2(G/\Gamma)$ has a spectral gap for the Casimir operator from H_1. Therefore, by Lemma 5.1 we can glue the $P_{\mu,\theta}$'s together to get a solution $P \in L^2(G/\Gamma)$ to the equation $\mathcal{U}P = f$. By ergodicity of the flow of $\mathcal{U}$ on G/Γ, we also get that $\mathcal{X}P = g$ (see the discussion at the end of Section 5.6).

By Lemma 5.6, $\mathcal{V}^k P \in L^2(G/\Gamma)$ for any $\mathcal{V} \in \mathfrak{g}$ that is stable or unstable with respect to $\mathcal{X}$. By assumption on $\mathcal{X}$, for each $i = 1, \ldots, k$, we have the decomposition

$$\mathfrak{g}_i = E_i^- \oplus E_i^0 \oplus E_i^+$$

into stable and unstable directions for the flow $\phi_t^{\mathcal{X}}$. By Lemma 5.7, these directions span each $\mathfrak{g}_i$ as a Lie algebra. Therefore the distributions E^- and E^+ span $\mathfrak{g}$ as a Lie algebra, so we can apply Theorem 2.46 to see that P is smooth. This proves the theorem. $\square$

CHAPTER VI

Cocycles over abelian actions generated by two commuting unipotent flows

6.1 Strategy for the proof of Theorem B

Let $H = \overline{\mathrm{SL}(2,\mathbb{R})}^{l_1} \times \overline{\mathrm{SL}(2,\mathbb{R})}^{l_2}$ be the product of two finite-sheeted covers of $\mathrm{SL}(2,\mathbb{R})$, and let $U \in H$ be the unipotent subgroup obtained by exponentiating $\mathcal{U}_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \times (0)$ and $\mathcal{U}_2 = (0) \times \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$. Given an embedding $i : H \hookrightarrow G$ into a noncompact semisimple Lie group with finite center, and a smooth cocycle α over the U-action on G/Γ, Mieczkowski's results imply a solution $P \in L^2(G/\Gamma)$ to the cohomology equation that is smooth in directions tangent to the H-orbits in G. Our ultimate goal is to show that P is actually smooth in all directions.

Suppose $i' : H \hookrightarrow G$ is a different embedding, and that $i|_U = i'|_U$. Then there is another transfer function $Q \in L^2(G/\Gamma)$ that is smooth in directions tangent to the H-orbits corresponding to this new embedding. An ergodicity argument will show that P and Q differ by a constant, which can be chosen to be zero. Finally, we will show that there are enough embeddings of H into G that coincide on U to prove that P is smooth in all directions.

The following sections are devoted to proving these assertions.

6.2 Obtaining transfer functions

In this section we show that the results in [Mie07] and [Mie06] can be applied to show that there are transfer functions that are smooth in the H-orbit directions of G.

Let α be a smooth cocycle over the action of U on G/Γ. Its infinitesimal generator ω is completely determined by where it sends the generators $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathfrak{u}$. In other words, it is determined by the functions

$$f = \omega(\mathcal{U}_1) \qquad g = \omega(\mathcal{U}_2).$$

Now the cocycle identity is

$$\mathcal{U}_1 g = \mathcal{U}_2 f$$

and the cohomology equation is

$$\mathcal{U}_1 P = f \quad \text{and} \quad \mathcal{U}_2 P = g.$$

Suppose we have a unitary representation of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ on the Hilbert space $\mathcal{H}$. Theorem 4.3 shows that if the Casimir element for both factors has a spectral gap, then there is a smooth vector $P \in C^\infty(\mathcal{H})$ that is a solution to the cohomology equation.

Since the unitary representations of a finite sheeted cover of $\mathrm{SL}(2,\mathbb{R})$ are unitarily equivalent to those for $\mathrm{SL}(2,\mathbb{R})$, Theorem 4.3 holds for representations of $H = \overline{\mathrm{SL}(2,\mathbb{R})}^{l_1} \times \overline{\mathrm{SL}(2,\mathbb{R})}^{l_2}$.

An embedding $H \hookrightarrow G$ induces a unitary representation of $\overline{\mathrm{SL}(2,\mathbb{R})}^{l_1} \times \overline{\mathrm{SL}(2,\mathbb{R})}^{l_2}$ on $L^2(G/\Gamma)$. In order to apply the previous theorem, we need to show that the Casimir elements for both factors have spectral gaps. But this is immediate from Theorem 5.4. Therefore, we can apply Theorem 4.3. Our smooth cocycle α is determined by the smooth functions $f, g \in C^\infty(L^2(G/\Gamma))$, and Theorem 4.3 guarantees the existence of the transfer function $P \in C^\infty(L^2(G/\Gamma))$ that is a smooth vector with respect to the representation of $\overline{\mathrm{SL}(2,\mathbb{R})}^{l_1} \times \overline{\mathrm{SL}(2,\mathbb{R})}^{l_2}$ on $L^2(G/\Gamma)$.

6.3 Different embeddings

We point out that if there are two different embeddings $i : H \hookrightarrow G$ and $i' : H \hookrightarrow G$ that coincide on $U \subset H$, then the corresponding transfer functions P and Q differ by a constant. This is a simple consequence of the ergodicity of the flow of $\mathcal{U}$ on G/Γ. We can choose the constant to be 0, so the transfer functions P and Q that we get from the embeddings i and i' agree almost everywhere. Furthermore, they are smooth along their respective H-orbits. Therefore, the partial derivatives of P in directions tangent to the $i'(H)$-orbits also exist, as L^2 functions. Our next goal is to show that there are enough embeddings of H into G to span all directions with the orbits.

6.4 Getting enough embeddings

In this section it will be convenient to denote H as being a subgroup, $H \subset G$. Different embeddings that coincide on U will be achieved by conjugating H by elements of the centralizer $Z(U)$ of U in G. We will look at the images of the Lie algebra $\mathfrak{h}$ under these conjugations and show that the Lie algebra generated by the union of these is all of $\mathfrak{g}$, the Lie algebra of G. Theorem 2.46 will then imply that the solution P is smooth.

Proposition 6.1. *Suppose H is a finite-dimensional split semisimple Lie group, $U \subset H$ is a unipotent subgroup, and G is a simple Lie group into which H embeds. Let $\mathfrak{u}$, $\mathfrak{h}$, and $\mathfrak{g}$ be their respective Lie algebras. Denote by $Z(U)$ the centralizer of U in G.*

Let $\mathfrak{L} = \langle \mathrm{Ad}(Z(U))\mathfrak{h} \rangle$ *be the Lie algebra generated by*

$$\mathrm{Ad}(Z(U))\mathfrak{h} = \left\{ g\mathcal{X}g^{-1} | g \in Z(U) \quad \text{and} \quad \mathcal{X} \in \mathfrak{h} \right\}.$$

Then $\mathfrak{L} = \mathfrak{g}$.

Proof. The centralizer of $\mathfrak{u}$ in $\mathfrak{g}$, denoted $\mathfrak{z}(\mathfrak{u})$, is the Lie algebra of $Z(U)$. Notice that for all $\mathcal{X} \in \mathfrak{z}(\mathfrak{u})$ and $\mathcal{Y} \in \mathfrak{h}$,

$$t \mapsto \exp(t\mathcal{X}) \cdot \mathcal{Y} \cdot \exp(-t\mathcal{X})$$

is a curve in $\mathfrak{L}$ with velocity $[\mathcal{X}, \mathcal{Y}]$ at $t = 0$. Therefore, $[\mathfrak{z}(\mathfrak{u}), \mathfrak{h}] \subset \mathfrak{L}$.

Since $\mathfrak{h}$ is split, there is a splitting Cartan subalgebra $\mathfrak{t} \subset \mathfrak{h}$ that acts diagonally on $\mathfrak{g}$, and we can order its roots so that $\mathfrak{u}$ is spanned by the positive root spaces. Then we have a decomposition of $\mathfrak{g}$ into the sum

$$\mathfrak{g} = \bigoplus_{\lambda \in \Psi} \mathfrak{g}^{\lambda}$$

where $\mathfrak{g}^{\lambda}$ is the sum of all $\mathrm{ad}(\mathfrak{h})$-invariant subspaces of $\mathfrak{g}$ with highest weight λ, and Ψ is a finite set of highest weights. For $\lambda \in \Psi$,

$$\mathfrak{g}_{\lambda} = \left\{ \mathcal{X} \in \mathfrak{g}^{\lambda} | [\mathcal{T}, \mathcal{X}] = \lambda(\mathcal{T})\mathcal{X} \quad \text{for all } \mathcal{T} \in \mathfrak{t} \right\}.$$

Since $\mathfrak{u}$ is in the positive root spaces, any element of $\mathfrak{u}$ annihilates any highest weight vector, so $\mathfrak{g}_{\lambda} \subset \mathfrak{z}(\mathfrak{u})$ for all $\lambda \in \Psi$. Now, for any $\mathcal{X} \in \mathfrak{g}_{\lambda}$ and $\mathcal{T} \in \mathfrak{t}$, we have that $[\mathcal{X}, \mathcal{T}] \in [\mathfrak{z}(\mathfrak{u}), \mathfrak{h}] \subset \mathfrak{L}$. But $[\mathcal{X}, \mathcal{T}] = -\lambda(\mathcal{T})\mathcal{X}$, so if $\lambda \neq 0$, then $\mathcal{X} \in \mathfrak{L}$. This shows that for $\lambda \neq 0$, $\mathfrak{g}_{\lambda} \subset \mathfrak{L}$. Since for any $\lambda \in \Psi$, $\mathfrak{g}_{\lambda}$ generates $\mathfrak{g}^{\lambda}$ as an $\mathfrak{h}$-module,

$$\bigoplus_{\lambda \in \Psi \setminus \{0\}} \mathfrak{g}^{\lambda} \subset \mathfrak{L}.$$

Let $\mathfrak{i}$ be the Lie algebra generated by $\bigoplus_{\lambda \in \Psi \setminus \{0\}} \mathfrak{g}^\lambda$. Then it is clear that $\mathfrak{i} \subset \mathfrak{L}$, and that

$$\mathfrak{g} = \mathfrak{i} + \mathfrak{g}^0 = \mathfrak{i} + \mathfrak{z}(\mathfrak{t}).$$

We claim that $\mathfrak{i}$ is $\mathrm{ad}(\mathfrak{z}(\mathfrak{t}))$-invariant. Let $\mathcal{X}$ be a non-zero (not necessarily highest) weight vector with weight λ, and let $\mathcal{Z} \in \mathfrak{z}(\mathfrak{t})$. Then, for any $\mathcal{T} \in \mathfrak{t}$,

$$[[\mathcal{X}, \mathcal{Z}], \mathcal{T}] = [[\mathcal{X}, \mathcal{T}], \mathcal{Z}] = \lambda(\mathcal{T})[\mathcal{X}, \mathcal{Z}].$$

Thus, $[\mathcal{X}, \mathcal{Z}]$ is a weight vector with weight λ. This shows that the non-zero weight spaces are $\mathrm{ad}(\mathfrak{z}(\mathfrak{t}))$-invariant, and since $\mathfrak{i}$ is the Lie algebra generated by these, it is also $\mathrm{ad}(\mathfrak{z}(\mathfrak{t}))$-invariant.

Obviously, $\mathfrak{i}$ is also $\mathrm{ad}(\mathfrak{i})$-invariant, hence it is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, and $\mathfrak{i}$ contains more than just 0, we see that $\mathfrak{i}$ must equal $\mathfrak{g}$. Finally, since $\mathfrak{i} \subset \mathfrak{L}$, we get the desired result that $\mathfrak{L} = \mathfrak{g}$. □

Our H is split semisimple. We will use this lemma to show that there are enough conjugates of $\mathfrak{h}$ in $\mathfrak{g}$ by elements of $Z(U)$ to generate $\mathfrak{g}$ as a Lie algebra. This is all that is needed to prove Theorem B; the proof will be stated in the following section.

6.5 Proofs of Theorems B and C

Here we present the proofs of Theorems B and C. We will keep the same notation for H and U throughout.

Proof of Theorem B. We have a product $G = G_1 \times \cdots \times G_2$ of noncompact simple Lie groups with finite center. We have assumed that G admits an embedding of H such that the projection U_i of U to G_i is nontrivial for all $i = 1, \ldots, k$. Suppose we are given a smooth cocycle $\alpha : U \times G/\Gamma \to \mathbb{R}$.

By the discussion following Theorem 4.3 [Mie07], there exists a transfer function $P \in L^2(G/\Gamma)$ for the given smooth cocycle α, and P is smooth in directions tangent to the H-orbits corresponding to the given embedding $i : H \hookrightarrow G$. We obtain different embeddings of H into G by conjugating the image of i by elements of the centralizer $Z(U)$ of U in G. Such embeddings will clearly all agree on U. P is differentiable, in the L^2 sense, in directions that are tangent to the H-orbits corresponding to any of these embeddings.

To see that there are enough such embeddings to span $\mathfrak{g}$ as a Lie algebra, observe that the projection H_i of H to G_i is a split semisimple Lie subgroup of G_i, for all i. Proposition 6.1 then shows that there are enough conjugates of $\mathfrak{h}_i := \mathrm{Lie}(H_i)$ by elements of $Z_{G_i}(U_i) \subset Z(U)$ to span $\mathfrak{g}_i := \mathrm{Lie}(G_i)$. Thus, there are enough conjugates of $\mathfrak{h}$ by elements of $Z(U)$ to span $\mathfrak{g}$. Therefore, by Theorem 2.46, P is smooth on G/Γ. This completes the proof. $\square$

Proof of Theorem C. Let α be a cocycle over the V-action on G/Γ. Then it restricts to a cocycle over the U-action on G/Γ, so by the previous theorem there is a smooth transfer function P that satisfies

$$\alpha(u, x) = -P(ux) + c(u) + P(x)$$

for all $u \in U$ and $x \in G/\Gamma$, where $c : U \to \mathbb{R}$ is a constant cocycle. Let V' be the center of V. Then for $v \in V'$,

$$\begin{aligned}
\alpha(v,x) &= \alpha(uvu^{-1},x) \\
&= \alpha(u^{-1},x)+\alpha(v,u^{-1}x)+\alpha(u,vu^{-1}x) \\
&= -P(u^{-1}x)+c(u^{-1})+P(x) \\
&\quad -P(vx)+c(u)+P(vu^{-1}x) \\
&\quad +\alpha(v,u^{-1}x) \\
&= -P(vx)+P(x) \\
&\quad -P(u^{-1}x)+P(vu^{-1}x)+\alpha(v,u^{-1}x)
\end{aligned}$$

Regrouping terms, we see that

$$\alpha(v,x)+P(vx)-P(x) = -P(u^{-1}x)+P(vu^{-1}x)+\alpha(v,u^{-1}x)$$

is a U-invariant smooth function on G/Γ for every $v \in V'$. By ergodicity of the U-action on G/Γ, it is constant. Therefore, setting $c'(v) = -P(u^{-1}x)+P(vu^{-1}x)+\alpha(v,u^{-1}x)$, we have shown that P satisfies

$$\alpha(v,x) = -P(vx)+c'(v)+P(x)$$

for all $v \in V'$ and $x \in G/\Gamma$. It is clear that $c' = c$ on $U \cap V'$.

Now, V' is closed and noncompact in G and hence, by Theorem 2.23, acts ergodically on G/Γ. Therefore, we can carry out the same calculation as above, where V' will play the role that U played, and V will play the role that V' played. This shows that P satisfies

$$\alpha(v,x) = -P(vx)+c(v)+P(x)$$

for all $v \in V$ and $x \in G/\Gamma$, and completes the proof of the theorem. □

6.6 Proof of Corollary E

Theorem A is applied in the proof of Corollary E, as well as similar arguments to those used in the proof of Theorem C. For simplicity, we state the following lemma which follows from the calculations done in the proof of Theorem C.

Lemma 6.2. *Suppose the group $A \cong \mathbb{R}^2$ is generated by two one-parameter subgroups, $\{g_t\}_{t\in\mathbb{R}}$ and $\{h_t\}_{t\in\mathbb{R}}$. Suppose A acts on the compact manifold M and assume that the action on M by g_t is ergodic. Let $\alpha : A \times M \to \mathbb{R}$ be a smooth cocycle. Then any smooth function $P : M \to \mathbb{R}$ that solves the cohomology equation for the restricted cocycle $\bar{\alpha} = \alpha|_{\{g_t\}_{t\in\mathbb{R}}\times M}$ is also a solution to the cohomology equation for α.*

Proof. The proof follows the same calculations used in the proof of Theorem C. We have that

$$\alpha(g_t, x) = -P(g_t x) + c(g_t) + P(x)$$

for all $t \in \mathbb{R}$ and $x \in M$. Now, for $s \in \mathbb{R}$,

$$\begin{aligned}
\alpha(h_s, x) &= \alpha(g_t h_s g_t^{-1}, x) \\
&= \alpha(g_t^{-1}, x) + \alpha(h_s, g_t^{-1}x) + \alpha(g_t, h_s g_t^{-1}x) \\
&= -P(g_t^{-1}x) + c(g_t^{-1}) + P(x) \\
&\quad -P(h_s x) + c(g_t) + P(h_s g_t^{-1}x) \\
&\quad +\alpha(h_s, g_t^{-1}x) \\
&= -P(h_s x) + P(x) \\
&\quad -P(g_t^{-1}x) + P(h_s g_t^{-1}x) + \alpha(h_s, g_t^{-1}x)
\end{aligned}$$

for all $t \in \mathbb{R}$ and $x \in M$.

By regrouping terms, we see that $\alpha(h_s, x)+P(h_s x)-P(x)$ is a g_t-invariant function on M for every fixed $s \in \mathbb{R}$. Since g_t is ergodic, we see that

$$\alpha(h_s, x) + P(h_s x) - P(x) = c'(h_s)$$

where $c'(h_s)$ is a constant depending on h_s. This shows that P is also a transfer function for α restricted to the h_s-action, and so it solves the cohomology equation for $\alpha : A \times M \to \mathbb{R}$. $\square$

Proof of Corollary E. Let $n > 2$, and $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a cocompact lattice. Denote by $W \subset \mathrm{SL}(n, \mathbb{R})$ the solvable subgroup of upper triangular matrices, and let α be a smooth cocycle over the action $W \curvearrowright \mathrm{SL}(n, \mathbb{R})/\Gamma$. Consider the following elements of the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ of $\mathrm{SL}(n, \mathbb{R})$:

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mathcal{X} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & -2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Restricting α to the action by the subgroup $H \subset \mathrm{SL}(n, \mathbb{R})$ generated by $\mathcal{U}$ and $\mathcal{X}$, we can apply Theorem A to get a smooth function $P : \mathrm{SL}(n, \mathbb{R})/\Gamma \to \mathbb{R}$ satisfying

$$\alpha(h, x) = -P(hx) + c(h) + P(x) \quad \text{for all} \quad h \in H, x \in \mathrm{SL}(n, \mathbb{R})/\Gamma,$$

where $c : H \to \mathbb{R}$ is some homomorphism. In particular, P solves the cohomology equation for α, restricted to the action by the subgroup $\{\exp(t\mathcal{X})\}_{t\in\mathbb{R}}$. Since $\{\exp(t\mathcal{X})\}_{t\in\mathbb{R}}$ is a subgroup of the diagonal subgroup $D \subset \mathrm{SL}(n, \mathbb{R})$, and D is abelian, Lemma 6.2 shows that P satisfies

$$\alpha(g, x) = -P(gx) + c'(g) + P(x),$$

for all $g \in D$ and $x \in \mathrm{SL}(n, \mathbb{R})/\Gamma$. Here, $c' : D \to \mathbb{R}$ is a homomorphism that agrees with c on $\{\exp(t\mathcal{X})\}_{t \in \mathbb{R}} = D \cap H$.

Now, for $1 < i < j \leq n$, let $\mathcal{U}_{ij}$ denote the element of the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ that has a 1 in the ij^{th} matrix entry, and zeroes elsewhere. (For example, $\mathcal{U} = \mathcal{U}_{12}$.) Choose a diagonal element of $\mathfrak{sl}(n, \mathbb{R})$, to be denoted $\mathcal{X}_{ij}$, such that $[\mathcal{U}_{ij}, \mathcal{X}_{ij}] = 0$. (For example $\mathcal{X} = \mathcal{X}_{12}$ is a suitable choice in the case $i = 1$ and $j = 2$.) Then $\{\exp(t\mathcal{X}_{ij})\}_{t \in \mathbb{R}}$ and $\{\exp(t\mathcal{U}_{ij})\}_{t \in \mathbb{R}}$ generate an abelian subgoup of $\mathrm{SL}(n, \mathbb{R})$. Furthermore, P solves the cohomology equation for the cocycle α, restricted to the action on $\mathrm{SL}(n, \mathbb{R})/\Gamma$ by $\{\exp(t\mathcal{X}_{ij})\}_{t \in \mathbb{R}}$. Again, Lemma 6.2 shows that P also solves the cohomology equation for α restricted to the subgroup $\{\exp(t\mathcal{U}_{ij})\}_{t \in \mathbb{R}}$.

Finally, since the upper triangular subgroup $W \subset \mathrm{SL}(n, \mathbb{R})$ is generated by the diagonal subgroup D and the unipotent subgroups $\{\exp(t\mathcal{U}_{ij})\}_{t \in \mathbb{R}}$ $(1 < i < j \leq n)$, we see that P solves the cohomology equation for the cocycle $\alpha : W \times \mathrm{SL}(n, \mathbb{R})/\Gamma \to \mathbb{R}$, as desired. $\square$

CHAPTER VII

Applications

In this final chapter we discuss applications of our theorems, as well as future directions of study. We prove Theorems F and G.

7.1 Proof of Theorem F

The proof of Theorem F follows the same scheme as the proofs for the corollaries to Theorems A and B. The idea is to take advantage of commutativity in the acting group, and ergodicity. In short, it is repeated application of Lemma 6.2.

Proof of Theorem F. Let α be any $\mathbb{R}^l$-valued smooth cocycle over the action $\mathbb{R}^k \curvearrowright G/\Gamma$ and let α_i be the i^{th} component function, $i = 1, \dots, l$. Then α_i is an $\mathbb{R}$-valued cocycle for all i. Let $\bar{\alpha}_i$ be the smooth cocycle obtained by restricting α_i to the subgroup of $\mathbb{R}^n$ whose action on G/Γ is one of the actions from Theorem A or Theorem B. By those theorems, there is a smooth transfer function $P_i : G/\Gamma \to \mathbb{R}$ solving the cohomology equation for $\bar{\alpha}_i$. Since the actions from Theorems A and B are ergodic, we can apply Lemma 6.2 repeatedly to show that P_i is a transfer function for the unrestricted cocycle, α_i. The smooth function $P = (P_1, P_2, \dots, P_l) : G/\Gamma \to \mathbb{R}^l$ is the desired transfer function for the original cocycle, α.

□

7.2 Smooth time-changes; proof of Theorem G

Given a locally free action $\rho : A \times X \to X$ by diffeomorphisms on a closed manifold X, a *smooth time-change* is another action $\rho^* : A \times X \to X$ by diffeomorphisms such that both actions have the same orbits. Denoting $\rho(a, x) = ax$ and $\rho^*(a, x) = a^*x$ for all $a \in A$ and $x \in X$, the fact that ρ^* is a time-change of ρ is simply the statement that for any $a \in A$, there exists an element $b \in A$ such that $ax = b^*x$. The following proposition is standard, and can be found in [KS94b] and [Mie07]; it illustrates the importance of cocycles in the study of time-changes.

Proposition 7.1. *Let X be a closed manifold. Suppose the locally free action $\rho : \mathbb{R}^n \to \mathrm{Diff}(X)$ has the property that any smooth $\mathbb{R}^n$-valued cocycle is smoothly cohomologous to a constant cocycle. Let ρ^* be a smooth time-change of ρ. If there exists a point $x_0 \in X$ whose isotropy with respect to ρ is trivial, then ρ and ρ^* are conjugate up to an automorphism of $\mathbb{R}^n$.*

Proof. The first claim is that there is a smooth cocycle $\beta : \mathbb{R}^n \times X \to \mathbb{R}^n$ satisfying $ax = \beta(a, x)^*x$ for all $a \in \mathbb{R}^n$ and $x \in X$. To see this, let $B \subset \mathbb{R}^n$ be a ball centered at the origin, small enough that $B \cap \mathrm{stab}(x) = \{0\}$, for all $x \in X$, where

$$\mathrm{stab}(x) = \{a \in \mathbb{R}^n \mid ax = x\}.$$

(Such a B is guaranteed to exist because ρ is locally free.) For each $x \in X$, there is a unique neighborhood $U_x \subset \mathbb{R}^n$ of the origin such that ρ^* maps $U_x \times \{x\}$ diffeomorphically onto $\rho(B \times \{x\})$. Thus, for every $x \in X$, we have that the composite map

$$B \times \{x\} \longrightarrow \rho(B \times \{x\}) \longrightarrow U_x \times \{x\} \longrightarrow \mathbb{R}^n$$

is smooth, where the first arrow is ρ, the second arrow is $(\rho^*|_{(U_x \times \{x\})})^{-1}$, and the third arrow is projection onto the U_x coordinate. We use this composite map to define our cocycle on B. That is, $\beta : B \times X \to \mathbb{R}^n$ by sending $(a, x) \in B \times X$ to the unique element $b \in U_x$ that satisfies $ax = b^*x$. We can then use the cocycle identity to extend β smoothly to all of $\mathbb{R}^n \times X$. The following calculation shows that for all $a \in \mathbb{R}^n$ and $x \in X$, $ax = \beta(a, x)^*x$. For a given $a \in \mathbb{R}^n$, choose $m \in \mathbb{Z}_+$ large enough that $a/m \in B$. Then, applying the cocycle identity, we can write

$$\begin{aligned}
\beta(a, x)^*x &= \beta(\frac{a}{m} + \cdots + \frac{a}{m}, x)^*x \\
&= [\beta(\frac{a}{m}, \frac{m-1}{m}ax) + \beta(\frac{a}{m}, \frac{m-2}{m}ax) + \cdots + \beta(\frac{a}{m}, x)]^*x \\
&= [\beta(\frac{a}{m}, \frac{m-1}{m}ax) + \beta(\frac{a}{m}, \frac{m-2}{m}ax) + \cdots + \beta(\frac{a}{m}, \frac{a}{m}x)]^*(\frac{a}{m})x \\
&= [\beta(\frac{a}{m}, \frac{m-1}{m}ax) + \beta(\frac{a}{m}, \frac{m-2}{m}ax) + \cdots + \beta(\frac{a}{m}, \frac{2a}{m}x)]^*(\frac{2a}{m})x \\
&\vdots \\
&= ax.
\end{aligned}$$

This proves the first claim.

Since we have assumed that the action ρ is C^∞-cocycle-rigid, we can assert that there is a smooth function $P : X \to \mathbb{R}^n$ satisfying $\beta(a, x) = -P(ax) + c(a) + P(x)$, for all $a \in \mathbb{R}^n$ and $x \in X$, where $c \in \mathrm{End}(\mathbb{R}^n)$.

Our next claim is that c is an automorphism. For this, take the element $x_0 \in X$ with $\mathrm{stab}(x_0) = \{0\}$, which we have assumed to exist. We then have that $\{\beta(a, x_0)\}_{a\in\mathbb{R}^n} = \mathbb{R}^n$. Since P has bounded range, it is clear that the image of c is all of $\mathbb{R}^n$. Since c is linear, it must be an automorphism.

Finally, set $\psi(x) = P(x)^*(x)$. We show that ψ is the desired conjugating diffeomorphism. First, it is easy to check that $\psi(ax) = c(a)^*\psi(x)$, which is the equivariance property that ψ must satisfy. It is only left to check that ψ is a diffeomorphism.

By the equivariance property, we see that ψ takes each ρ-orbit to a ρ^*-orbit. In fact, since the actions are orbit-equivalent, each orbit is taken to itself. Since the range of c is $\mathbb{R}^n$, the equivariance property implies that ψ maps every orbit *onto* itself, showing that ψ is surjective.

To prove injectivity, suppose $\psi(x) = \psi(y)$. Then x and y are on the same ρ-orbit, so $y = ax$ for some $a \in \mathbb{R}^n$. Therefore, $c(a)^*\psi(x) = \psi(x)$; that is, $c(a) \in \mathrm{stab}^*(\psi(x))$, where we have used stab* to denote istotropy for ρ^*. Setting $\psi_t(x) = [(1-t)P(x)]^*(x)$, for $t \in [0,1]$, we see that ψ is homotopic to the identity, which is the orbit equivalence corresponding to a trivial time-change. This shows that c maps $\mathrm{stab}(x)$ isomorphically onto $\mathrm{stab}^*(\psi(x))$. Therefore, $a \in \mathrm{stab}(x)$, and $x = y$, proving that ψ is injective.

The equivariance property implies that ψ is smooth, which completes the proof of the proposition. $\square$

The proof of Theorem G is an application of Theorem F and of Proposition 7.1.

Proof of Theorem G. We let $\mathbb{R}^n$ act on G/Γ in such a way that the action contains one of the actions from Theorem A or Theorem B. That is, $\mathbb{R}^n \curvearrowright G/\Gamma$ is an action of the type described in the statement of Theorem F. This action satisfies the assumptions in Proposition 7.1, therefore any smooth time-change is smooth conjugate to the original action, up to an automorphism of $\mathbb{R}^n$. $\square$

BIBLIOGRAPHY

BIBLIOGRAPHY

[AS95] D. V. Anosov and V. V. Solodov, *Hyperbolic sets*, Dynamical systems, IX, Encyclopaedia Math. Sci., vol. 66, Springer, Berlin, 1995, pp. 10–92.

[Asa09] Masayuki Asaoka, *Non-homogeneous locally free actions of the affine group*, Preprint, 2009.

[BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.

[Bel] Michel Belliart, *A differentiable classification of certain locally free actions of solvable Lie groups*, Preprint.

[BS02] Michael Brin and Garrett Stuck, *Introduction to dynamical systems*, Cambridge University Press, Cambridge, 2002.

[DK] Danijela Damjanovic and Anatole Katok, *Rigidity of families of some homogeneous parabolic abelian actions*, Preprint.

[FF03] Livio Flaminio and Giovanni Forni, *Invariant distributions and time averages for horocycle flows*, Duke Math. J. **119** (2003), no. 3, 465–526.

[FK02] Renato Feres and Anatole Katok, *Ergodic theory and dynamics of G-spaces (with special emphasis on rigidity phenomena)*, Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, 2002, pp. 665–763.

[GGPS69] I. M. Gel′fand, M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, Translated from the Russian by K. A. Hirsch, W. B. Saunders Co., Philadelphia, Pa., 1969. MR MR0233772 (38 #2093)

[HM79] Roger E. Howe and Calvin C. Moore, *Asymptotic properties of unitary representations*, J. Funct. Anal. **32** (1979), no. 1, 72–96.

[HP06] Boris Hasselblatt and Yakov Pesin, *Partially hyperbolic dynamical systems*, Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, pp. 1–55.

[HT92] Roger Howe and Eng-Chye Tan, *Nonabelian harmonic analysis*, Universitext, Springer-Verlag, New York, 1992, Applications of $SL(2, \mathbf{R})$.

[Hum78] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1978, Second printing, revised.

[Jac79] Nathan Jacobson, *Lie algebras*, Dover Publications Inc., New York, 1979, Republication of the 1962 original.

[Kat01] Anatole Katok, *Cocycles, cohomology and combinatorial constructions in ergodic theory*, Smooth ergodic theory and its applications (Seattle, WA, 1999), Proc. Sympos. Pure Math., vol. 69, Amer. Math. Soc., Providence, RI, 2001, In collaboration with E. A. Robinson, Jr., pp. 107–173.

[KH95] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza.

[KM99] D. Y. Kleinbock and G. A. Margulis, *Logarithm laws for flows on homogeneous spaces*, Invent. Math. **138** (1999), no. 3, 451–494.

[Kna01] Anthony W. Knapp, *Representation theory of semisimple groups*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001, An overview based on examples, Reprint of the 1986 original.

[Kna02] ______, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002.

[KS94a] A. Katok and R. J. Spatzier, *Subelliptic estimates of polynomial differential operators and applications to rigidity of abelian actions*, Math. Res. Lett. **1** (1994), no. 2, 193–202.

[KS94b] Anatole Katok and Ralf J. Spatzier, *First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 131–156.

[Lan75] Serge Lang, $\mathrm{SL}_2(\mathbf{R})$, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1975.

[Liv72] A. N. Livšic, *Cohomology of dynamical systems*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1296–1320.

[Mar87] Gregori Aleksandrovitch Margulis, *Formes quadratriques indéfinies et flots unipotents sur les espaces homogènes*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 10, 249–253.

[Mie06] David Mieczkowski, *The cohomological equation and representation theory*, Ph.D. thesis, The Pennsylvania State University, 2006.

[Mie07] ______, *The first cohomology of parabolic actions for some higher-rank abelian groups and representation theory*, J. Mod. Dyn. **1** (2007), no. 1, 61–92.

[Mor05] Dave Witte Morris, *Ratner's theorems on unipotent flows*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2005.

[Rat90a] Marina Ratner, *On measure rigidity of unipotent subgroups of semisimple groups*, Acta Math. **165** (1990), no. 3-4, 229–309.

[Rat90b] ______, *Strict measure rigidity for unipotent subgroups of solvable groups*, Invent. Math. **101** (1990), no. 2, 449–482.

[Rat91a] ______, *Distribution rigidity for unipotent actions on homogeneous spaces*, Bull. Amer. Math. Soc. (N.S.) **24** (1991), no. 2, 321–325.

[Rat91b] ______, *On Raghunathan's measure conjecture*, Ann. of Math. (2) **134** (1991), no. 3, 545–607.

[Rat91c] ______, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke Math. J. **63** (1991), no. 1, 235–280.

[Sha00] Yehuda Shalom, *Explicit Kazhdan constants for representations of semisimple and arithmetic groups*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 3, 833–863.

[Sta00] Alexander N. Starkov, *Dynamical systems on homogeneous spaces*, Translations of Mathematical Monographs, vol. 190, American Mathematical Society, Providence, RI, 2000, Translated from the 1999 Russian original by the author.

[Wal82] Peter Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York, 1982.

[War72] Garth Warner, *Harmonic analysis on semi-simple Lie groups. I*, Springer-Verlag, New York, 1972, Die Grundlehren der mathematischen Wissenschaften, Band 188.

[Zim90] Robert J. Zimmer, *Essential results of functional analysis*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1990. MR MR1045444 (91h:46002)